The characterization of minimal zero-sum sequences over finite cyclic groups

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Main Content

Notations

• G is a cyclic group of order n.
• $S = g_1 \cdot g_2 \cdots g_k$ is a sequence over G.
• S is called zero-sum if $g_1 + g_2 + \cdots + g_k = 0$.
• S is called minimal zero-sum if S is a zero-sum sequence without nonempty proper zero-sum subsequences.

Introduction

Problem

If S is a minimal zero-sum sequence over G, what is the structure of S?

A simple fact

If $S = (n_1 g) \cdot (n_2 g) \cdots (n_k g)$, where g is a generator of G, $1 \le n_i \le n$ for all $i \in [1, k]$ and $n_1 + \cdots + n_k = n$, then S is a minimal zero-sum sequence. In this case, we say this sequence has index 1 and denote by $\mathrm{Ind}(S) = 1$.

Problem

Whether do all minimal zero-sum sequences have the above structure? If not, what are they?

Long sequences

S is called long if $|S| \geq cn$ for some constant c.

Ponomarenko, (Integers 2004)

- If S is a minimal zero-sum sequence of length ≤ 3, then $\mathrm{Ind}(S) = 1$.
- If $\gcd(n, 6) \neq 1$, there are minimal zero-sum sequences of length 4 such that $\mathrm{Ind}(S) \neq 1$.

$$S = \begin{cases} g \cdot \left(\frac{n}{2}g\right) \cdot \left(\frac{n+2}{2}g\right) \cdot ((n-2)g), 2|n \\ g \cdot \left(\frac{n+3}{3}g\right) \cdot \left(\frac{2n+3}{3}g\right) \cdot ((n-3)g), 3|n \end{cases}$$

Conjecture

Conjecture

If $\gcd(n, 6) = 1$ and S is a minimal zero-sum sequence, then $\mathrm{Ind}(S) = 1$.

Let $S = (t_1 g) \cdot (t_2 g) \cdot (t_3 g) \cdot (t_4 g)$ and h be another generator. Let $g = kh$ for some $k \in \mathbb{Z}$ with $\gcd(k, n) = 1$. Then $S = (|kt_1|_n h) \cdot (|kt_2|_n h) \cdot (|kt_3|_n h) \cdot (|kt_4|_n h)$, where $|x|_n$ denotes the minimal positive residue of x modulo n.

Conjecture

If $\gcd(n, 6) = 1$ and $S = (t_1 g) \cdot (t_2 g) \cdot (t_3 g) \cdot (t_4 g)$ be a minimal zero-sum sequence, then there exist some $s \in \mathbb{Z}$ such that

$$\gcd(s, n) = 1 \text{ and } |st_1|_n + |st_2|_n + |st_3|_n + |st_4|_n = n.$$

Partial results

- Li, Plyley, Yuan and Zeng (JNT 2010) n is a prime power.
- Li and Peng (IJNT 2013)+Xia and Shen (JNT 2013) n is the product of two prime powers
- Xia (IJNT 2013); Shen and Xia (IJNT 2014) Some very special cases.

Sketched Idea

- Find $s \in [1, n-1]$ such that $|st_1|_n + |st_2|_n + |st_3|_n + |st_4|_n = n$.
- Prove that some s found in the above step is coprime with n.

Partial results

Shen, Xia and Li (CM 2014) $\langle S \rangle = G$ and $\gcd(t_i, n) \neq 1$ for some $i \in [1,4]$.

Sketched Idea

• Choose a suitable family of integers which are coprime with n. For example, when a prime $p | \gcd(t_1, n)$ but $p \nmid t_2, t_3, t_4$, they choose $1 + \frac{kn}{p},\ k \in [1,p]$.
• Prove that some s chosen in the above step satisfies $|st_1|_n + |st_2|_n + |st_3|_n + |st_4|_n = n$.

Main result

Zeng and Qi (2015)

If $\gcd(n, 30) = 1$ and $\gcd(t_i, n) = 1$ for all $i \in [1, 4]$, then the conjecture is true.

Corollary

Let $\gcd(n, 30) = 1$ and S be a minimal zero-sum sequence of length 4, then $\mathrm{Ind}(S) = 1$.

Idea

(1) We may assume that $S = (g) \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$, where $1 < a \leq b < c < n/2$ and $a + b = c + 1$. (By multiplying by -1 or 2)
(2) If we can find $s \in \mathbb{N}$ and $q \in \mathbb{N}$ such that $\gcd(s, n) = 1$, $sa < n$ and $sb < qn < sc$, that is

$$s < \frac{n}{a}, \ \frac{b}{n} < \frac{q}{s} < \frac{c}{n},$$

Then

$$\begin{aligned} & |s|_n + |sc|_n + |sn - sb|_n + |sn - sa|_n \\ = \quad & s + (sc - qn) + (qn - sb) + (n - sa) \\ = \quad & n + s(1 + c - b - a) \\ = \quad & n \end{aligned}$$

Hence it suffices to find a rational number $\frac{q}{s}$ separating $\frac{b}{n}$ and $\frac{c}{n}$, where $q \in \mathbb{N}$, $s < \frac{n}{a}$ and $\gcd(s, n) = 1$.

Proof

The gaps between the consecutive terms of the following two sequences of integers are small.

$$3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, \ldots \quad \frac{\ell_{i+1}}{\ell_i} \leq \frac{7}{5}$$

$$3, 5, 9, 15, 25, 27, 45, 75, 81, \ldots \quad \frac{\ell_{i+1}}{\ell_i} \leq \frac{9}{5} < 2$$

Proof

Hence $\{\frac{q}{s}:\ q\in\mathbb{N}, s<\frac{n}{a}, s=2^\alpha 3^\beta 5^\gamma\}$ is dense in some sense. Indeed the rational numbers in the set

$$\left\{\frac{q}{s}:\ q\in\mathbb{N}, s<\frac{n}{a}, s=2^\alpha 3^\beta 5^\gamma\right\}\cap[0,1/2]$$

Partition $[0,1/2]$ into the union of some intervals $I_0, I_1, \ldots, I_t$ such that the length of the interval I_i is not greater that $\frac{0.9a}{n}$ for all $i\in[1,t]$. However $\frac{c}{n}-\frac{b}{n}=\frac{a-1}{n}>\frac{0.9a}{n}$ provided that a is not too small. Hence a desired rational number exists.

Example

$$\frac{n}{a}\approx 8.8$$

$$\{\frac{1}{8},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{3}{8},\mathbf{\frac{2}{5}},\mathbf{\frac{1}{2}}\}$$

$$\frac{1}{2}-\frac{2}{5}=\frac{1}{10}\le\frac{0.9}{8.8}$$

Remark

This method fails in the case $5|n$. For example, let's replace 5 by 7 in the proof.

$$\mathbf{3,5,}6, 8, 9, 10, \ldots$$

$$\mathbf{3,6,}7, 8, 9, 12, \ldots$$

$$\mathbf{3,5,}9, 15, 25, 27, 45, 75, 81, \ldots$$

$$\mathbf{3,7,}9, 21, 27, 49, 63, 81 \ldots$$

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Thank You!