Additive properties of sequences on semigroups

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Two starting additive researches in group theory

For any finite abelian group G, let $D(G)$ be the smallest $\ell \in \mathbb{N}$ s.t., every sequence over G of length at least ℓ contains a nonempty zero-sum subsequence.

(H. Davenport, 1966)

Two starting additive researches in group theory

Any sequence T of terms from a finite cyclic group G of length $2|G|-1$ contains a zero-sum subsequence of length $|G|$.

(Erdős, Ginzburg and Ziv, 1961)

Additive Group Theory

The arithmetic properties of sequences, sets, or other combinatorial objects from groups come into the domain of Additive Group Theory

Number of distinct semigroups

Order	Groups	Semigroups	Commutative semigroups
2	1	4	3
3	1	18	12
4	2	126	58
5	1	1160	325
6	2	15,973	2143
7	1	836,021	17,291
8	5	1,843,120,128	221,805

Additively irreducible sequence

A sequence T on a commutative semigroup is called **additively reducible** if T contains a proper subsequence T' with $\sigma(T') = \sigma(T)$, and **additively irreducible** if otherwise.

Davenport constant for semigroups

Definition. Define the Davenport constant of a commutative semigroup $\mathcal{S}$, denoted $\mathrm{D}(\mathcal{S})$, to be the smallest $\ell \in \mathbb{N} \cup \{\infty\}$, s.t., every sequence T of length at least ℓ of terms from $\mathcal{S}$ is reducible.

(G.Q. Wang, W.D. Gao, Semigroup Forum, 2007)

Small Davenport constant for semigroups

Definition. For a commutative semigroup $\mathcal{S}$, let $\mathrm{d}(\mathcal{S})$ denote the smallest $\ell \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:
For any $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in \mathcal{S}$ there exists a subset $I \subset [1, m]$ such that $|I| \le \ell$ and

$$\sum_{i=1}^{m} a_i = \sum_{i \in I} a_i .$$

(A. Geroldinger, F. Halter-Koch, Non-Unique Factorizations, 2006.)

Proposition. Let $\mathcal{S}$ be a commutative semigroup. Then $\mathrm{D}(\mathcal{S})$ is finite if and only if $\mathrm{d}(\mathcal{S})$ is finite. Moreover, in case that $\mathrm{D}(\mathcal{S})$ is finite, we have

$$\mathrm{D}(\mathcal{S}) = \mathrm{d}(\mathcal{S}) + 1.$$

(G.Q. Wang, Additively irreducible sequences in commutative semigroups, arXiv:1504.06818.)

On polynomial rings $\mathbb{F}_q[x]$

Theorem. *Let $q > 2$ be a prime power, and let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$. Let R be a quotient ring of $\mathbb{F}_q[x]$ with $0 \neq R \neq \mathbb{F}_q[x]$. Then*

$$\mathrm{D}(\mathcal{S}_R) = \mathrm{D}(U(\mathcal{S}_R)).$$

(G.Q. Wang, Journal of Number Theory, 2015)

Problem 1. Let R be a quotient ring of $\mathbb{F}_2[x]$ with $0 \neq R \neq \mathbb{F}_2[x]$. Determine $\mathrm{D}(\mathcal{S}_R) - \mathrm{D}(U(\mathcal{S}_R))$.

Theorem. *Let* $\mathbb{F}_2[x]$ *be the ring of polynomials over the finite field* $\mathbb{F}_2$, *and let* $R = \frac{\mathbb{F}_2[x]}{(f)}$ *be a quotient ring of* $\mathbb{F}_2[x]$ *where* $f \in \mathbb{F}_2[x]$ *and* $0 \neq R \neq \mathbb{F}_2[x]$. *Then*

$$\mathrm{D}(\mathrm{U}(\mathcal{S}_R)) \leq \mathrm{D}(\mathcal{S}_R) \leq \mathrm{D}(\mathrm{U}(\mathcal{S}_R)) + \delta_f,$$

where

$$\delta_f = \begin{cases} 0 & \textit{if} \gcd(x*(x+1_{\mathbb{F}_2}),\ f) = 1_{\mathbb{F}_2}; \\ 1 & \textit{if} \gcd(x*(x+1_{\mathbb{F}_2}),\ f) \in \{x,\ x+1_{\mathbb{F}_2}\}; \\ 2 & \textit{if } gcd(x*(x+1_{\mathbb{F}_2}), f) = x*(x+1_{\mathbb{F}_2}). \end{cases}$$

L.Z. Zhang, H.L. Wang, Y.K. Qu, A problem of Wang on Davenport constant for the multiplicative semigroup of the quotient ring of $\mathbb{F}_2[x]$, arXiv:1507.03182.

Irreducible sequences for groups

Definition. For any element $g \in G^\bullet$, let $\mathrm{D}_g(G)$ be the largest length of irreducible sequences T with $\sigma(T) = g$, which is called the relative Davenport constant of G with respect to the element $g \in G^\bullet$.

(M. Skałba, Acta Arith., 1993.)

Theorem. **If G is a finite abelian group and $g \in G^{\bullet}$, then**

$$\frac{1}{2}\mathrm{D}(G) \le \mathrm{D}_g(G) \le \mathrm{D}(G) - 1.$$

(M. Skałba, Acta Arith., 1993.)

Theorem. Let $\mathcal{S}$ be a commutative semigroup. Let a be an element of $\mathcal{S}^{\bullet}$ with $\Psi(a)$ being finite. If $|H_a|$ is infinite then $\mathrm{D}_a(\mathcal{S})$ is infinite, and if $|H_a|$ is finite then $\mathrm{D}_a(\mathcal{S})$ is finite and

$$\epsilon\, \mathrm{D}(\Gamma(H_a)) \leq \mathrm{D}_a(\mathcal{S}) \leq \Psi(a) + \mathrm{D}(\Gamma(H_a)) - 1$$

where

$$\epsilon = \begin{cases} \frac{1}{2}, \text{ if } & (a+a)\ \mathcal{H}\ a; \\ 1, \text{ if } & otherwise, \end{cases}$$

and both the lower and upper bounds are sharp.

(G.Q.Wang, Additively irreducible sequences in commutative semigroups, arxiv, 2015)

Theorem. Let R be a commutative unitary ring. Let a be an element of $\mathcal{S}_R^\bullet$ with $\Psi(a)$ being finite. Then

$$\Gamma(H_a) \cong \mathrm{U}(R_a),$$

where $R_a = R\diagup\mathrm{Ann}(a)$ be the quotient ring of R modulo the annihilator of a. If $\mathrm{U}(R_a)$ is infinite then $\mathrm{D}_a(\mathcal{S}_R)$ is infinite, and if $\mathrm{U}(R_a)$ is finite then $\mathrm{D}_a(\mathcal{S}_R)$ is finite and

$$\epsilon\,\mathrm{D}(\mathrm{U}(R_a)) \leq \mathrm{D}_a(\mathcal{S}_R) \leq \Psi(a) + \mathrm{D}(\mathrm{U}(R_a)) - 1.$$

In particular, if R is a finite commutative principal ideal unitary ring and $a \notin \mathrm{U}(R)$, then the above equality

$$\mathrm{D}_a(\mathcal{S}_R) = \Psi(a) + \mathrm{D}(\mathrm{U}(R_a)) - 1$$

holds.

Theorem. *Let* $R = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$. *Let* $\mathbf{a} = (\overline{a_1}, \ldots, \overline{a_r})$ *be an element of* $\mathcal{S}_R$, *where* $\overline{a_i} = a_i + n_i\mathbb{Z} \in \mathbb{Z}/n_i\mathbb{Z}$ *for* $i \in [1, r]$. *Let* $R' = \mathbb{Z}/\frac{n_1}{t_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\frac{n_r}{t_r}\mathbb{Z}$, *where* $t_i = \gcd(a_i, n_i)$ *for* $i \in [1, r]$. *Then*

$$\mathrm{D}_{\mathbf{a}}(\mathcal{S}_R) = \begin{cases} \mathrm{D}_{\mathbf{a}}(\mathrm{U}(R)), & \textit{if} \quad a \in \mathrm{U}(R); \\ \sum_{i=1}^{r} \Omega(t_i) + \mathrm{D}(\mathrm{U}(R')) - 1, & \textit{if otherwise}, \end{cases}$$

where $\Omega(t_i)$ *denotes the number of prime factors (repeat prime factors are also calculated) of the integer* t_i.

(G.Q. Wang and W.D. Gao, Davenport constant for semigroups, Semigroup Forum, 2007)

Theorem. Let $\mathcal{S}$ be a commutative semigroup satisfying the a.c.c. for principal ideals, and let a be an element of $\mathcal{S}^\bullet$. If $|H_a|$ is infinite then $\mathrm{D}_a(\mathcal{S})$ is infinite, and if $|H_a|$ is finite then $\mathrm{D}_a(\mathcal{S})$ is finite and

$$\epsilon\, \mathrm{D}(\Gamma(H_a)) \leq \mathrm{D}_a(\mathcal{S}) \leq \Psi(a) + \mathrm{D}(\Gamma(H_a)) - 1.$$

Proposition. Let $\mathcal{S}$ be a commutative semigroup. Then $\mathrm{D}(\mathcal{S})$ is finite if and only if $\mathrm{D}_a(\mathcal{S})$ is bounded for all $a \in \mathcal{S}$, i.e., there exists a given large integer $\mathcal{M}$ such that $\mathrm{D}_a(\mathcal{S}) \leq \mathcal{M}$ for all $a \in \mathcal{S}$. In particular, if $\mathrm{D}(\mathcal{S})$ is finite then

$$\mathrm{D}(\mathcal{S}) = \max_{a \in \mathcal{S}} \{\mathrm{D}_a(\mathcal{S})\} + 1.$$

Proposition. Let $\mathcal{S}$ be a commutative Noetherian semigroup. Then $\mathrm{D}(\mathcal{S})$ and $\mathrm{d}(\mathcal{S})$ is finite if, and only if, $|H_a|$ is bounded for all $a \in \mathcal{S}$, i.e., there exists an integer $\mathcal{M}$ such that $|H_a| < \mathcal{M}$ for all $a \in \mathcal{S}$.

Problem 2. From the point of view of semigroup's structure, does there exists a sufficient and necessary condition to decide whether $\mathrm{D}_a(\mathcal{S})$ is finite or infinite?

Problem 3. From the point of view of semigroup's structure, does there exists a sufficient and necessary condition to decide whether $\mathrm{D}(\mathcal{S})$ is finite or infinite?

An Erdős Problem

"Any sequence T of terms from a commutative semigroup S of length at least $|S|$ contains a nonempty subsequence of sum equaling some idempotent."

(Proposed by Erdős to Burgess)

In 1969, confirmed by D.A. Burgess for finite commutative semigroup with only one idempotent.

Gillam-Hall-Williams Theorem

Theorem. Any sequence $T = (a_1, a_2, \ldots, a_t)$ on a semigroup $\mathcal{S}$ of length $t \geq |\mathcal{S}| - |E(\mathcal{S})| + 1$ contains several terms whose product (in their natural orders) is idempotent, i.e., there exists $1 \leq i_1 < i_2 < \ldots < i_k \leq t$ with $a_{i_1} * \cdots * a_{i_k} \in E(\mathcal{S})$.

(D.W.H. Gillam, T.E. Hall, N.H. Williams, Bull. London Math. Soc., 1972.)

Theorem A. Let $\mathcal{S}$ be a finite semigroup, and let $T \in \mathcal{F}(\mathcal{S})$ be a sequence with length $|T| = |\mathcal{S}| - |E(\mathcal{S})|$ and $\prod(T) \cap E(\mathcal{S}) = \emptyset$. Let $\mathcal{R} = \langle \mathrm{supp}(T) \rangle$. Then $\mathcal{R}$ is commutative with $\mathcal{S} \setminus \mathcal{R} \subseteq E(\mathcal{S})$ and the universal semilattice $Y(\mathcal{R})$ is a chain such that $x_1 * x_2 = x_1$ for any elements $x_1, x_2 \in \mathcal{R}$ with $x_1 \lneqq_{\mathcal{N}_{\mathcal{R}}} x_2$. Moreover,

(i) each archimedean component of $\mathcal{R}$ is, either a finite cyclic semigroup $\langle x \rangle$ with $x \in \mathrm{supp}(T)$ and $\mathcal{I}(x) \equiv 1 \pmod{\mathcal{P}(x)}$, or an ideal extension of a nontrivial finite cyclic group $\langle x_2 \rangle$ by a nontrivial finite cyclic nilsemigroup $\langle x_1 \rangle$ with $x_1, x_2 \in \mathrm{supp}(T)$ and the partial homomorphism $\varphi_{\langle x_2 \rangle}^{\langle x_1 \rangle}$ being trivial, i.e., $\varphi_{\langle x_2 \rangle}^{\langle x_1 \rangle}(x_1) = e_{\langle x_2 \rangle}$ where $e_{\langle x_2 \rangle}$ denotes the identity element of the subgroup $\langle x_2 \rangle$.

(ii) $\mathrm{v}_x(T) = \mathcal{I}(x) + \mathcal{P}(x) - 2$ for each element $x \in \mathrm{supp}(T)$.

(G.Q.Wang, Structure of the largest idempotent-free sequences in finite semigroups, arXiv, 2014.)

Erdős-Burgess constants

Define $\mathrm{I}(\mathcal{S})$, the **Erdős-Burgess constant** of $\mathcal{S}$, to be the least m s.t., every $T \in \mathcal{F}(\mathcal{S})$ of length at least m satisfies $\prod(T) \cap E(\mathcal{S}) \neq \emptyset$.

Define $\mathrm{SI}(\mathcal{S})$, the **strong Erdős-Burgess constant** of $\mathcal{S}$, to be the least ℓ s.t., every $T \in \mathcal{F}(\mathcal{S})$ of length at least ℓ contains several terms whose product (in their natural order) is idempotent.

Relation between two constants

(i). $\mathrm{I}(\mathcal{S}) \leq \mathrm{SI}(\mathcal{S}) \leq |\mathcal{S}| - |E(\mathcal{S})| + 1$, and the equality $\mathrm{I}(\mathcal{S}) = \mathrm{SI}(\mathcal{S}) = |\mathcal{S}| - |E(\mathcal{S})| + 1$ holds if and only if the semigroup $\mathcal{S}$ is given as in Theorem A;

(ii). For any finite commutative semigroup $\mathcal{S}$, $\mathrm{I}(\mathcal{S}) = \mathrm{SI}(\mathcal{S})$.

Problem 4. *Let $\mathcal{S}$ be a finite semigroup. Does there exist a sufficient and necessary condition to decide whether $\mathrm{I}(\mathcal{S}) = \mathrm{SI}(\mathcal{S})$ or not?*

Problem 5. *Let* S *be a finite semigroup. Find the sufficient and necessary condition to decide whether* $\mathrm{SI}(\mathcal{S}) = |\mathcal{S}| - |E(\mathcal{S})| + 1$. *Moreover, in case that* $\mathrm{SI}(\mathcal{S}) = |\mathcal{S}| - |E(\mathcal{S})| + 1$, *for any sequence* $T \in \mathcal{F}(\mathcal{S})$ *of length exactly* $|\mathcal{S}| - |E(\mathcal{S})|$ *such that* T *contains no several terms whose product (in their natural order in this sequence) is idempotent, determine the structure of the sequence* T.

Problem 6. *Let $\mathcal{S}$ be a finite commutative semigroup. Does there exist any relationship between the Erdős-Burgess constant* $\mathrm{I}(\mathcal{S})$ *and the Davenport constant* $\mathsf{D}(\mathcal{S})$*?*

A connection between Davenport constant and EGZ Theorem

For any finite abelian group G,

$$E(G) = D(G) + |G| - 1.$$

(W.D. Gao, A combinatorial problem of finite Abelian group, J. Number Theory, 58 (1996) 100 – 103.)

EGZ constant for semigroups

Definition. Define $E(\mathcal{S})$ of any finite commutative semigroup $\mathcal{S}$ as the smallest positive integer ℓ such that, every sequence $A \in \mathcal{F}(\mathcal{S})$ of length ℓ contains a subsequence B with $\sigma(B) = \sigma(A)$ and $|A| - |B| = \kappa(G)$, where

$$\kappa(\mathcal{S}) = \left\lceil \frac{|\mathcal{S}|}{\exp(\mathcal{S})} \right\rceil \exp(\mathcal{S}).$$

Results on EGZ Theorm in semigroups

Conjecture A. For any finite commutative semigroup $\mathcal{S}$,

$$E(\mathcal{S}) \leq D(\mathcal{S}) + \kappa(\mathcal{S}) - 1.$$

Conjecture B. For any finite commutative monoid $\mathcal{S}$,

$$E(\mathcal{S}) = D(\mathcal{S}) + \kappa(\mathcal{S}) - 1.$$

Obtained results on EGZ theorem for finite commutative semigroups

We confirmed Conjecture A holds true for **Group-free semigroups, Subdirectly irreducible semigroups, Archimedean semigroups with some constraint.**

(Adhikari, Gao, Wang, Erdős-Ginzburg-Ziv theorem for finite commutative semigroups, Semigroup Forum, 2014).

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Thank you!