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On the structure of sets with a small doubling property in torsion free groups

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1. Introduction

The *sumset* (also called the *Minkowski sum*) of two subsets A and B of an abelian group $\mathbf{G}$ is the set of sums

$$A + B = \{a + b : \ a \in A,\, b \in B\}.$$

Some Classical Examples:

- Lagrange's theorem: $\mathbf{G} = \mathbb{Z}$ and A is the set of all *squares*. Then $4A = A + A + A + A = \mathbb{N}$.
- Goldbach's conjecture: $\mathbf{G} = \mathbb{Z}$ and P is the set of all *primes*. Then $3(P \cup \{0\}) = \mathbb{Z}_{\geq 2} \cup \{0\}$.
- The Brunn–Minkowski inequality: $\mathbf{G} = \mathbb{R}^d$ and A, B are *convex bodies*. Then: $|A{+}B| \geq \left(|A|^{1/d} + |B|^{1/d}\right)^d$.

We will examine in detail the **exact structure** of a finite set

$$A \subseteq G,$$

in the case of a *torsion free Abelian group*

$$G = \mathbb{Z}^n$$

assuming a **small doubling** property:

$$\sigma = \frac{|A + A|}{|A|} < c_0.$$

2. Tight Lower bounds for sets of lattice points

Lower bounds for the cardinality of the sumset of finite d–dimensional sets in $\mathbb{Z}^d$ have been given by Freiman [1] and Rusza [2]:

Theorem 1 (Freiman-Ruzsa)**.** *If* $A, B \subseteq \mathbb{Z}^d$ *are finite* $d-$*dimensional sets with* $|A| \geq |B|$, *then*

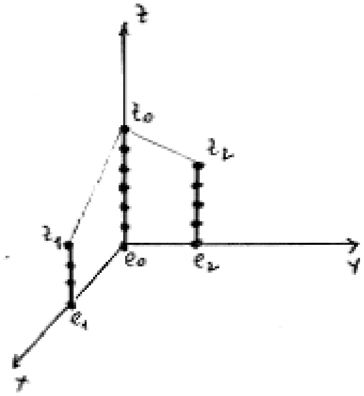
$$|A+A| \geq (d+1)|A| - \frac{d(d+1)}{2} \tag{1}$$

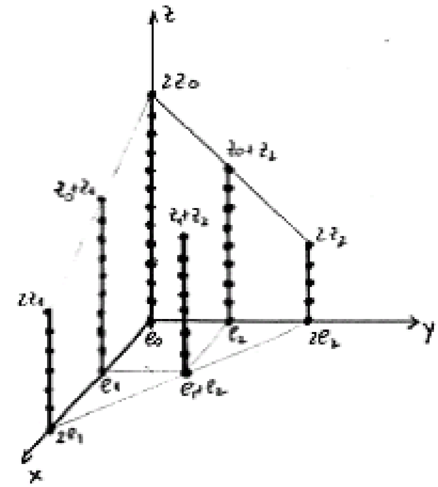
and

$$|A+B| \geq |A| + d|B| - \frac{d(d+1)}{2}. \tag{2}$$

Here $|X|$ denotes the cardinality of X.

Example in $G = \mathbb{Z}^3$





Assume $A = B$, A consists of three *arithmetic progressions* that lie on three parallel vertical lines. Thus

$$A = B, |A| = |B| = \sum_{0 \le i \le 2} (z_i + 1) = z_0 + z_1 + z_2 + 3,$$

and

$$|A + B| = |2A| = \sum_{0 \le i \le j \le 2} (z_i + z_j + 1) = 4|A| - 6.$$

The discrete case in the plane. In particular, if $A \subseteq \mathbb{Z}^2$ is a finite two-dimensional set, then Freiman's inequality (1) implies

$$|2A| \geq 3|A| - 3.$$

Although this lower bound is tight, Freiman [1] proved that if $A \subseteq \mathbb{Z}^2$ is contained in exactly $m \geq 1$ parallel lines, then:

$$|2A| \geq \left(4 - \frac{2}{m}\right)|A| - (2m-1), \tag{3}$$

i.e.

$$|2A| \geq \left(2\frac{|A|}{m} - 1\right)(2m-1). \tag{4}$$

The case of addition of two different sets in the plane was considered by Grynkiewicz and Serra in [3]:

Theorem 2 (Grynkiewicz-Serra)**.** *Let A and B be finite sets in $\mathbb{Z}^2$. Suppose that, for some line ℓ, A and B are covered by exactly m and n lines parallel to ℓ, respectively. Then*

$$|A+B| \geq \left(\frac{|A|}{m} + \frac{|B|}{n} - 1\right)(m+n-1). \qquad (5)$$

Note that if $A = B$, then $m = n$ and we get Freiman's inequality: $|2A| \geq \left(2\frac{|A|}{m} - 1\right)(2m-1)$.

Remarks:

1. By choosing $A = B = A_0 \cup A_1 \cup ... \cup A_{m-1}$ and and letting each set A_i be an arithmetic progression with first term $(i, 0)$, difference $\mathbf{d} = (0, 1) \in \mathbb{Z}^2$ and length $|A_i| = |A|/m \in \mathbb{Z}$ one can check that inequalities (4) and (5) become tight.

2. The lower bound $|A+B| \geq \left(\frac{|A|}{m} + \frac{|B|}{n} - 1\right)(m+n-1)$ is **similar** to Bonnesen's strengthening of the Brunn-Minkowski inequality

$$\mu(A+B) \geq \left(\frac{\mu(A)}{M} + \frac{\mu(B)}{N}\right)(M+N) \geq \left(\sqrt{\mu(A)} + \sqrt{\mu(B)}\right)^2.$$

Moreover, for every $d \geq 2$ we have:

Addition of convex bodies in $\mathbb{R}^d$	
$G=\mathbb{R}^d$	•$\lvert A+A\rvert \geq 2^d \lvert A\rvert$ *trivial* •$\lvert A+B\rvert \geq \left(\lvert A\rvert^{1/d}+\lvert B\rvert^{1/d}\right)^d$ *Brunn-Minkowski* •$\lvert A+B\rvert \geq \left(M^{\frac{1}{d-1}}+N^{\frac{1}{d-1}}\right)^{d-1}\left(\frac{\lvert A\rvert}{M}+\frac{\lvert B\rvert}{N}\right)$ *Bonnesen*

Addition of finite sets in $\mathbb{Z}^d$	
$G=\mathbb{Z}^d$	•$\lvert A+A\rvert \geq (d+1)\lvert A\rvert - \frac{d(d+1)}{2}$ *Freiman* •$\lvert A+B\rvert \geq \lvert A\rvert + d\lvert B\rvert - \frac{d(d+1)}{2}$, if $\lvert A\rvert \geq \lvert B\rvert$ *Ruzsa* •$\lvert A+B\rvert \geq \left(\frac{\lvert A\rvert}{m}+\frac{\lvert B\rvert}{n}-1\right)(m+n-1)$. *Grynkiewicz-Serra* $d=2$

3. $\mathbb{Z}^2$ vs. $\mathbb{Z}^d$

Addition of finite sets in the plane	
$G = \mathbb{Z}^2$	$\bullet |A+B| \geq |A| + 2|B| - 3, \text{if} |A| \geq |B|$ *Freiman-Ruzsa* $\bullet |A+A| \geq \left(2\frac{|A|}{m} - 1\right)(2m-1) \cong (4-\epsilon)|A|$ *Freiman* $\bullet |A+B| \geq \left(\frac{|A|}{m} + \frac{|B|}{n} - 1\right)(m+n-1)$ *Grynkiewicz-Serra*

Addition of finite sets in $\mathbb{Z}^d$, with $d \geq 3$	
$G = \mathbb{Z}^d$	$\bullet |A+B| \geq |A| + d|B| - \frac{d(d+1)}{2}, \text{if} |A| \geq |B|$ *Freiman-Ruzsa* $\bullet |A+A| \geq (2^d - \epsilon)|A|.$ *Freiman* $\bullet |A+B| \geq .....$ *a Bonnesen-type Conjecture. See [8]*

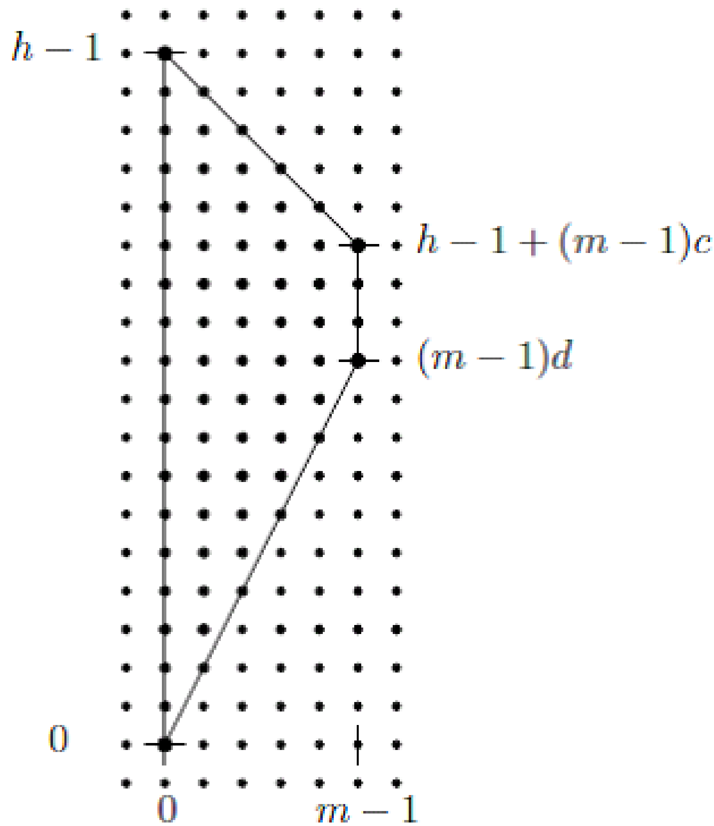
4. The structure of extremal sets

Our first result (Freiman-Grynkiewicz-Serra-S. [4]) shows that for two-dimensional sets equality holds in

$$|A+B| \geq \left(\frac{|A|}{m}+\frac{|B|}{n}-1\right)(m+n-1) \qquad (6)$$

if and only if A and B are both obtained by a planar linear transformation of two parallel *standard trapezoids* $T = T_m(h,c,d)$ and $T' = T_n(h',c,d)$ with common slopes c and d.

Here m and n are the minimum number of parallel lines covering A and B respectively.



EXAMPLE: A standard trapezoid $T = T_m(h, c, d)$ for $m = 6, h = 19, c = -1, d = 2$

Definition. A standard trapezoid $T = T_m(h, c, d)$ in $\mathbb{Z}^2$ is a translate of a bounded set T included in the lattice

$$\Lambda = \mathbb{Z}(0,1) \oplus \mathbb{Z}(1,d) \cong \mathbb{Z}^2$$

defined by the following inequality constraints:

- $0 \leq x \leq m - 1$
- $dx \leq y \leq cx + h - 1.$

The continuation of our study can be described using **an unifying "algorithm"** proposed by Freiman for solving inverse additive problems. Consider some (usually numerical) characteristic of the set under study (cardinality of a finite set, Lebesgue measure of a convex set..).

- Find an extremal value of this characteristic within the framework of the problem that we are studying.
- Study the structure of the set when its characteristic is equal to its extremal value.
- Study the structure of the set when its characteristic is near to its extremal value.
- ...Continue, taking larger neighborhoods for the characteristic.

5. The simplest inverse problem for sums of sets in several dimensions

We investigated the *exact structure* of multi-dimensional sets having the *smallest cardinality* of the sumset in [5] and obtained the following result:

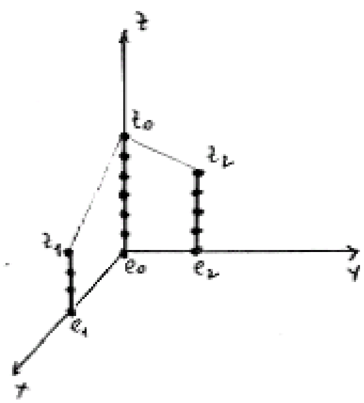
Theorem 3 (S. 1998). *Let* $\mathcal{A} \subseteq \mathbb{R}^n$ *be a finite set such that* $\dim \mathcal{A} \geq d$ *and*

$$|\mathcal{A} + \mathcal{A}| = (d+1)|\mathcal{A}| - \frac{1}{2}d(d+1).$$

If $|\mathcal{A}| \neq d+4$, *then* $\mathcal{A}$ *is a* d*-dimensional set and* $\mathcal{A}$ *consists of* d *parallel arithmetic progressions with the same common difference.*

Example in $G = \mathbb{Z}^3$

The general case: if $|\mathcal{A}| \neq d + 4$, then $\mathcal{A}$ consists of 3 parallel arithmetic progressions with the same common difference



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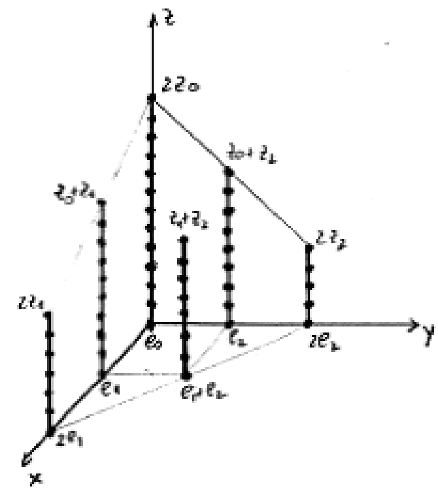
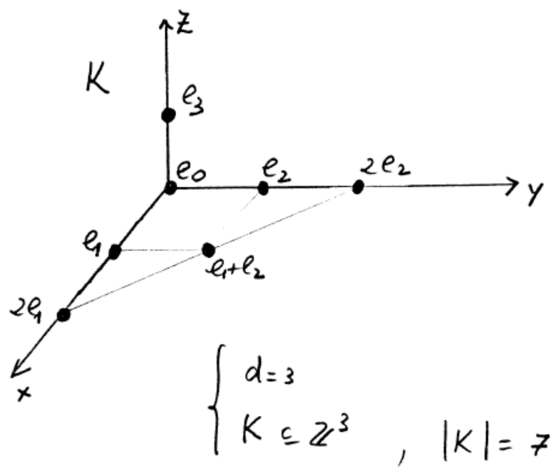


$A + A$

If $|\mathcal{A}| = d + 4$, then

$$\mathcal{A} = \{v_0, v_1, ..., v_d\} \cup \{2v_1, v_1 + v_2, 2v_2\},$$

where v_i are the vertices of a d-dimensional simplex.



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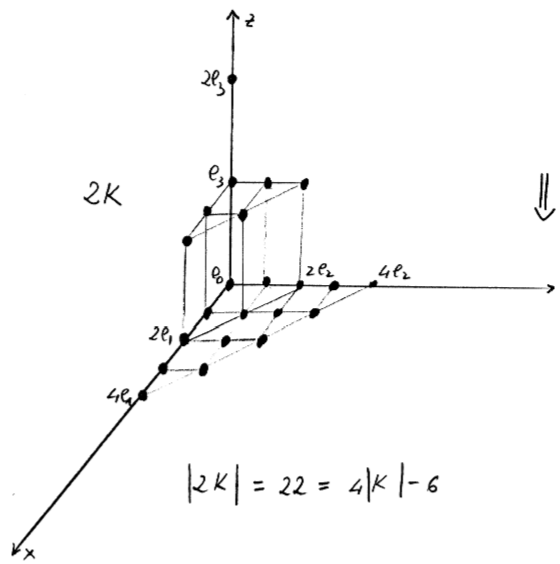


$A + A$

6. Planar sets with small sum set

The two dimensional case is well understood. Let us mention two results (see [9], [10]) valid for every integer $s \geq 2$:

Theorem 4 (S. 1998, 2008). *Let $\mathcal{K}$ be a finite set of $\mathbb{Z}^2$. If $|\mathcal{K}| \geq O(s^3)$, and*

$$|\mathcal{K}+\mathcal{K}| < \left(4 - \frac{2}{s+1}\right)|\mathcal{K}| - (2s+1) \ , \qquad (7)$$

then

(i) There exist s parallel lines which cover the set $\mathcal{K}$.

(ii) The set $\mathcal{K}$ is located inside a parallelogram that lies on a few lines which are well filled.

The exact statement (ii) is:

Theorem 5 (S. 2008)**.** *Let* $s \geq 19$ *be an integer and let* $\mathcal{K}$ *be a finite subset of* $\mathbb{Z}^2$ *that lies on exactly* s *parallel lines. If*

$$|2\mathcal{K}| < (4 - \frac{2}{s+1})|\mathcal{K}| - (2s+1),$$

then there is a lattice $\mathcal{L} \subseteq \mathbb{Z}^2$ *and a parallelogram* $\mathcal{P}$ *such that*

$$\mathcal{K} \subseteq (\mathcal{P} \cap \mathcal{L}) + v$$

and

$$|\mathcal{P} \cap \mathcal{L}| \leq 24\Big(|\mathcal{K} + \mathcal{K}| - 2|\mathcal{K}| + 1\Big),$$

for some $v \in \mathbb{Z}^2$.

This gives an accurate description for the structure of planar sets having *doubling coefficient*

$$\sigma = \frac{|2K|}{|K|} < 4.$$

The set $\mathcal{K}$ can be covered by s parallel lines (the best possible result). Moreover, a suitable affine isomorphism maps $\mathcal{K}$ into a set of lattice points that lies inside a parallelogram of *bounded area:*

$$|\mathcal{P} \cap \mathcal{L}| \leq 24\Big(|\mathcal{K} + \mathcal{K}| - 2|\mathcal{K}| + 1\Big).$$

This estimate is nearly sharp and it is far better than the bounds arising from the known proofs of Freiman's main theorem.

We believe that for a best possible result, the constant factor 24 of Theorem 5 should be replaced by $\frac{1}{2}(1+\frac{1}{s-1})$, if instead of a covering parallelogram $\mathcal{P}$ we consider the convex hull of $\mathcal{K}$. We suggest the following

Conjecture (S.) *Let $\mathcal{K}$ be a finite subset of $\mathbb{Z}^2$ that lies on exactly $s \geq 2$ parallel lines. If*

$$|2\mathcal{K}| < (4 - \frac{2}{s+1})|\mathcal{K}| - (2s+1),$$

then the convex hull of $\mathcal{K}$ is covered by $2s-2$ compatible arithmetic progressions having together no more than

$$v = \frac{s}{2(s-1)}\Big(|\mathcal{K}+\mathcal{K}| - 2|\mathcal{K}| + 2s - 1\Big) \quad terms.$$

So far inequality this estimate has been proved only for $s = 2$ (Freiman 1966) and $s = 3$ (S. [11]).

The covering statement (i) for the case of different sets:

Theorem 6 (Grynkiewicz and Serra 2010). *Let* A, B *be finite subsets of* $\mathbb{Z}^2$ *with* $|A| \geq |B| \geq O(s^2)$. *If*

$$|A+B| < |A| + (3 - \frac{2}{s+1})|B| - 2s - 1,$$

then there is a line ℓ *such that each of* A *and* B *can be covered by at most* s *parallel translates of* ℓ.

Recently (2016) Grynkiewicz studied the case $s = 2$ for the threshold $|A+B| < |A| + \frac{19}{7}|B| - 5$ and obtained that there is a two-dimensional progression P that simultaneously contains A and B with few holes.

7. Linear structure for Multidimensional Inverse Additive Problems

Assume that the doubling coefficient of the sum set $2\mathcal{A}$ is not much exceeding the minimal one, i.e.

$$d+1 \leq \sigma = \frac{|2\mathcal{A}|}{|\mathcal{A}|} < \rho_d.$$

What can be said about the *exact structure* of $\mathcal{A}$? The expected result is: if

$$\rho_d \ is \ sufficiently \ small,$$

then the set A is contained in d "short" arithmetical progressions.

We obtained in [6] the following result:

Theorem 7 (S). *Let $\mathcal{K} \subseteq \mathbb{Z}^d$ be a finite set of dimension* $d \geq 2$.

(i) If $k > 3 \cdot 4^d$ and

$$|\mathcal{K} + \mathcal{K}| < (d + \frac{4}{3})|\mathcal{K}| - c_d,$$

where $c_d = \frac{1}{6}(3d^2 + 5d + 8)$, then $\mathcal{K}$ lies on d parallel lines.

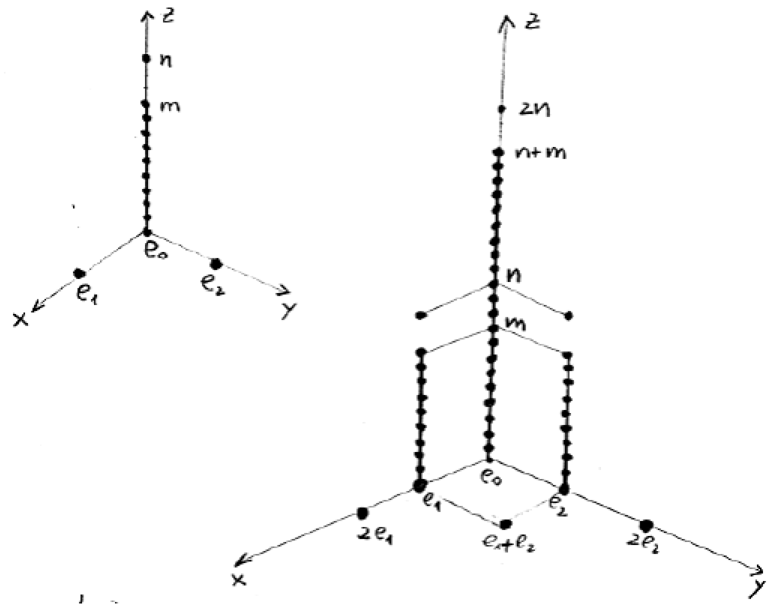
(ii) If $\mathcal{K}$ lies on d parallel lines and

$$|\mathcal{K} + \mathcal{K}| < (d + 2)|\mathcal{K}| - \frac{1}{2}(d + 1)(d + 2),$$

then $\mathcal{K}$ is contained in d parallel arithmetic progressions with the same common difference, having together no more than

$$v = |\mathcal{K} + \mathcal{K}| - d|\mathcal{K}| + \frac{1}{2}d(d + 1) \quad terms.$$

Remark: These results are best possible and cannot be sharpened by reducing the quantity v or by increasing the upper bounds for $|\mathcal{K}+\mathcal{K}|$:



$$|\mathcal{K}| = m+4, m < n \le 2m,$$

$$|2\mathcal{K}| = 3m+n+9, v = n+3 = |2\mathcal{K}| - 3|\mathcal{K}| + 6.$$

Next natural step:

We found that a similar result can be formulated and proved for $d-$dimensional sets that have a doubling coefficient less than

$$\rho_d = d + 2 - \epsilon.$$

These results make Freiman's Main Theorem more precise. More precisely, we studied in [7] the *exact structure* of d-dimensional sets satisfying the small doubling property

$$\rho_d = d + 2 - \frac{2}{s - d + 3},$$

where $s \geq d$ is a positive integer. In this case we prove that $\mathcal{K}$ lies on no more than s parallel lines.

Theorem 8 (S. 2016). *Let $\mathcal{K} \subseteq \mathbb{Z}^d$ be a finite set of dimension $d \geq 2$. If $|\mathcal{K}|$ is sufficiently large and*

$$|\mathcal{K} + \mathcal{K}| < (d + 2 - \frac{2}{s - d + 3})|\mathcal{K}| - constant,$$

then $\mathcal{K}$ lies on s parallel lines.

8. Proof Overview for the three-dimensional case

We study three dimensional sets having a small doubling coefficient less than

$$\rho = 5 - \frac{2}{s}, s \geq 3$$

Our first step is the following:

Theorem 9. *Let* $0 < \epsilon < 1$. *Let* $\mathcal{K} \subseteq \mathbb{Z}^3$ *be a finite set. If*

$$|2\mathcal{K}| < (5 - \epsilon)|\mathcal{K}|, \tag{8}$$

then there is a line $\ell \subseteq \mathbb{R}^3$ *such that*

$$|\ell \cap \mathcal{K}| > \delta|\mathcal{K}|,$$

where $\delta = \delta(\epsilon)$ *depends only on* ϵ.

Remark: This is a best possible result because the *doubling constant* $\sigma(\mathcal{K}) = \frac{|2\mathcal{K}|}{|\mathcal{K}|} < 5 - \epsilon$ cannot be replaced by $\sigma(\mathcal{K}) = 5$:

Example: The set $L \subseteq \mathbb{Z}^3$ defined by

$$L = \{(x, y, 0) : 0 \le x, y \le n - 1\} \cup \{e_3 = (0, 0, 1)\}$$

satisfies

$$|L| = n^2 + 1, |2L| = (2n - 1)^2 + n^2 + 1 = 5n^2 - 4n + 2,$$

has a doubling coefficient

$$\sigma(L) = \frac{|2L|}{|L|} = 5 - o(1),$$

AND for any line ℓ we have $|L \cap \ell| \le n \le \sqrt{|L|}$. Therefore, if $|L|$ is sufficiently large, the conclusion of Theorem 5 does not hold for the set L.

The second step of the proof uses an exact lower bound for three dimensional sets:

Lemma 10. *Let* $\mathcal{K} \subseteq \mathbb{Z}^3$ *be a finite set of affine dimension* $d = \dim \mathcal{K} = 3$. *Assume that there are* r *parallel lines* $\ell_1, ..., \ell_r$ *such that* $|\mathcal{K} \cap \ell_i| = k_i \geq 1$ *for every* $1 \leq i \leq r$ *and* $k = |\mathcal{K}| = k_1 + ... + k_r$. *If* $k_{\max} = \max\{k_1, ..., k_r\}$, *then*

(a) $|\mathcal{K} + \mathcal{K}| \geq (5 - \frac{2}{r-1})(|\mathcal{K}| - 1) - 2r + 4$.

(b) $|\mathcal{K} + \mathcal{K}| \geq (3|\mathcal{K}| - 3) + (r - 2)(k_{\max} - 1)$.

As a direct corollary of Lemma 10, we obtain:

Lemma 11. *Let* $\mathcal{K} \subseteq \mathbb{Z}^3$ *be a finite set of affine dimension* $\dim \mathcal{K} = 3$. *Assume that there is an integer* $s \geq 3$ *such that*

$$|\mathcal{K} + \mathcal{K}| < (5 - \frac{2}{s})(|\mathcal{K}| - 1) - 2s + 2.$$

If there is a line ℓ *such that* $|\mathcal{K} \cap \ell| \geq 2s - 1$, *then* $\mathcal{K}$ *lies on no more than* s *parallel lines.*

We conclude with the following:

Theorem 12. *Let $\mathcal{K} \subseteq \mathbb{Z}^3$ be a finite set of affine dimension* $\dim \mathcal{K} = 3$*. Assume that there is an integer* $s \geq 3$ *such that*

$$|\mathcal{K} + \mathcal{K}| < (5 - \frac{2}{s})(|\mathcal{K}| - 1) - 2s + 2.$$

If the cardinality of $\mathcal{K}$ is sufficiently large, $|\mathcal{K}| > k_0(s)$, then $\mathcal{K}$ lies on no more than s parallel lines.

9. Final remarks

For $s = 4$ and $d = 3$ we have

$$|\mathcal{K} + \mathcal{K}| < 4.5|\mathcal{K}| - 10.5 \tag{9}$$

then the set $\mathcal{K}$ lies on no more than four parallel lines.

For $s = 5$ and $d = 3$ we have

$$|\mathcal{K} + \mathcal{K}| < 4.6|\mathcal{K}| - 12.6 \tag{10}$$

then the set $\mathcal{K}$ lies on no more than five parallel lines.

Problem: Generalize the results for the case $\mathcal{K}+\mathcal{K}$ and

$$\rho_d = d + 2 - \frac{2}{s-d+3},$$

two the case of two distinct sets A, B.

(a) $d = 2$: the threshold for $\mathcal{K}+\mathcal{K}$ is $\rho_2 = 4 - \frac{2}{s+1}$

(b) $d = 3$: the threshold for $\mathcal{K}+\mathcal{K}$ is $\rho_2 = 5 - \frac{2}{s}$

(c) $d = 4$: the threshold for $\mathcal{K}+\mathcal{K}$ is $\rho_2 = 6 - \frac{2}{s+1}$

(d) $d = 5$: the threshold for $\mathcal{K}+\mathcal{K}$ is $\rho_2 = 7 - \frac{2}{s+2}$

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THANK YOU FOR YOUR ATTENTION.