Structural sum-product estimates

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Note that all of these estimates are tight, up to constant and logarithmic factors, as we see by considering the case when $A = \{1, 2, \ldots, N\}$.

Proof that $\left|\frac{A+A}{A+A}\right| \geq 2|A|^2 - 1$

The proofs of these results are primarily geometric. In order to give a taste of what is going on, we will sketch the proof of the following:

Theorem (Balog-ORN)

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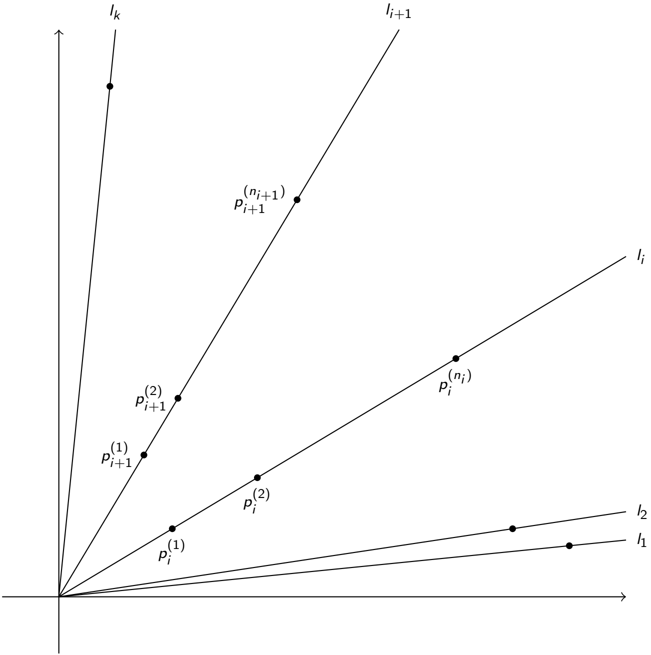
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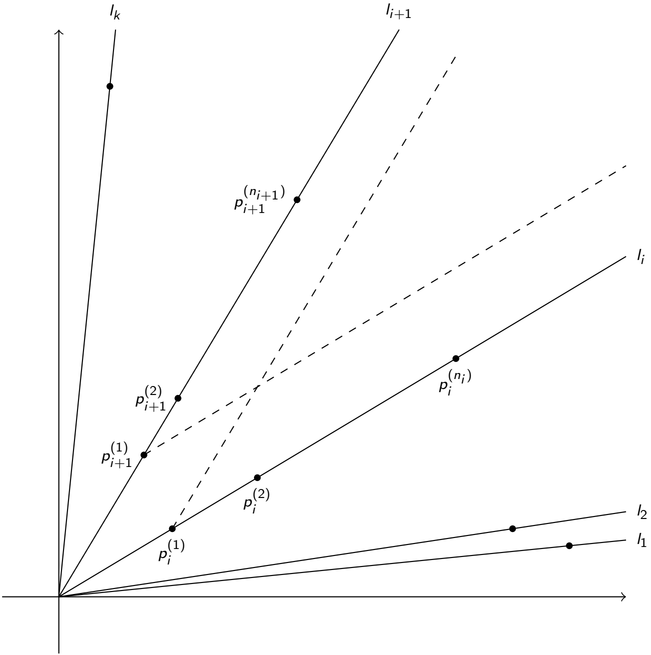
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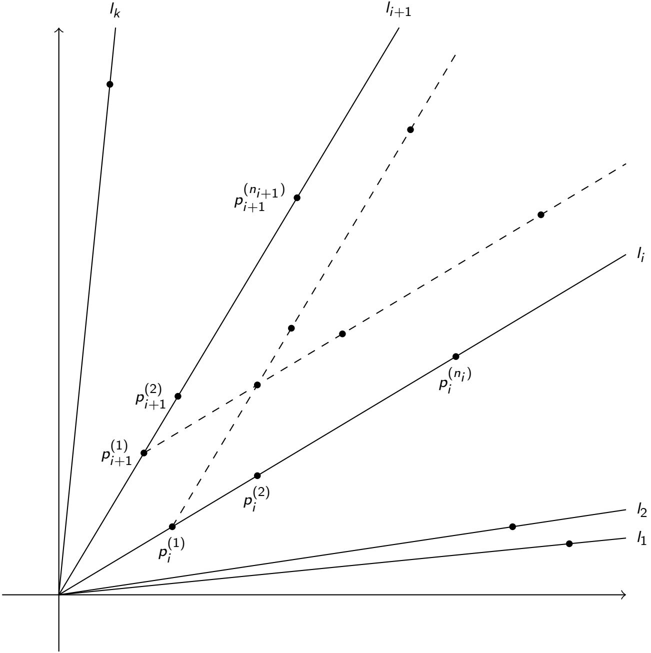
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Proof continued

This tells us that

$$\begin{aligned}\left|\frac{A+A}{A+A}\right| &\geq |A/A| + \sum_{i=1}^{|A/A|-1} (|l_i \cap P| + |l_{i+1} \cap P| - 1) \\ &= 1 + 2\sum_{i=1}^{|A/A|} (|l_i \cap P|) - |l_1 \cap P| - |l_{|A/A|} \cap P| \\ &= 1 + 2|A|^2 - 1 - 1 \\ &= 2|A|^2 - 1,\end{aligned}$$

as required.

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Similar conjectures and result were given for other well understood sets, e.g. $(A - A)(A - A)$.

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