

On some Multiplicative Problems of Erdős

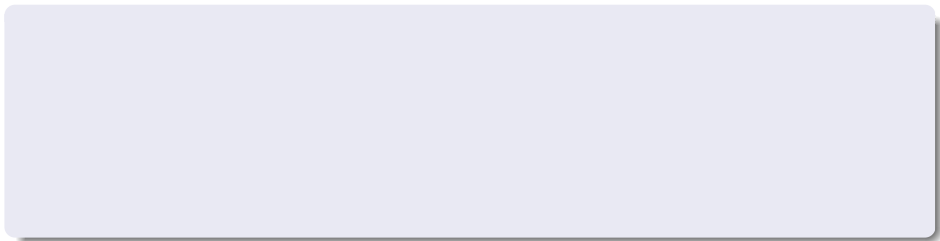
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Questions



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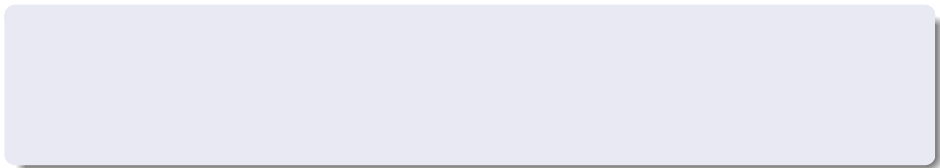
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- How dense can a set $A \subseteq \mathbb{N}$ be, if the equation

$$x^2 = a_1 a_2 \dots a_{2k} \quad (a_1, a_2, \dots, a_{2k} \in A)$$

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$a_1, \dots, a_k, b_1, \dots, b_k$ are distinct

$$a_1 a_2 \dots a_k = b_1 b_2 \dots b_k \implies a_1 \dots a_k b_1 \dots b_k = x^2$$

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$x^2 = a_1 \dots a_{2k}$ has no solution in $A \implies a_1 a_2 \dots a_k = b_1 b_2 \dots b_k$ has no solution in A

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$A \subseteq \{1, 2, \dots, n\} \implies \max |A| \sim \pi(n)$, moreover:

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element \longrightarrow edge

$m = uv \longrightarrow uv$ edge

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How to choose $g(n)$?

condition (3) $\longrightarrow g(n) = e^{c \log n / \log \log n}$

Maximal number of edges of C_6 -free graphs

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- $ex(u, v, C_6) < 2u + v^2/2$

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Theorem (P.)

$$\max |A| \leq \pi(n) + \pi(n/2) + cn^{2/3} \frac{\log n}{\log \log n}$$

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Lower bound

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$a_0 \nmid a_1 a_2 \dots a_k$ and multiplicative bases

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If B is a multiplicative basis of order k for $\{1, 2, \dots, n\}$, then $F_k(n) \leq |B|$.

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Multiplicative bases

Lemma

Let $B_0 = \{\text{primes} \leq n\} \cup \left\{x : x \leq \frac{n^{\frac{2}{k+1}}}{(\log n)^2}\right\}$.

If $a \leq n$ is not in B_0^k , then

$$a = p_1 p_2 \dots p_{k+1} a',$$

where $p_1 \geq p_2 \geq \dots \geq p_{k+1}$ are primes such that $p_k p_{k+1} > \frac{n^{\frac{2}{k+1}}}{(\log n)^2}$.

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We need at least one edge in each $\{p_1, p_2, \dots, p_{k+1}\}$.

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$a_0 \nmid a_1 \dots a_k$ -problem

The sets $\{p_1, p_2, \dots, p_{k+1}\}$ intersect each other in at most one element.

Raikov (1938)

B is a MB of order $k \implies \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} \geq \Gamma\left(\frac{1}{k}\right)^{-1}$.

For every $k \geq 2 \exists$ a MB of order k such that $\limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} < \infty$.

Infinite multiplicative bases

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Theorem (P., Sándor)

B is a MB of order $k \implies \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists C > 0$: For every $k \geq 2 \exists$ a MB of order k such that

$\limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} < C$.

Infinite multiplicative bases

Raikov (1938)

B is a MB of order $k \implies \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} \geq \Gamma\left(\frac{1}{k}\right)^{-1}$.

For every $k \geq 2 \exists$ a MB of order k such that $\limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} < \infty$.

Theorem (P., Sándor)

B is a MB of order $k \implies \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists C > 0$: For every $k \geq 2 \exists$ a MB of order k such that

$\limsup_{n \rightarrow \infty} \frac{|B(n)|}{n / \log^{\frac{k-1}{k}} n} < C$.

Theorem (P., Sándor)

B is a MB of order $k \implies \liminf_{n \rightarrow \infty} \frac{|B(n)|}{\log n} > 1$.

But it can be $< 1 + \varepsilon$.

Theorem (P., Sándor)

$$\forall k \geq 2 \exists A \subseteq \mathbb{Z}^+ \text{ such that } \limsup_{n \rightarrow \infty} \frac{|A(n)| - \pi(n)}{\frac{n^{2/(k+1)}}{\log^2 n}} > 0.$$

Theorem (P., Sándor)

$\forall k \geq 2 \exists A \subseteq \mathbb{Z}^+$ such that $\limsup_{n \rightarrow \infty} \frac{|A(n)| - \pi(n)}{\frac{n^{2/(k+1)}}{\log^2 n}} > 0$.

Theorem (P., Sándor)

$\forall \varepsilon > 0 \liminf_{n \rightarrow \infty} \frac{A(n) - \pi(n)}{n^\varepsilon} = 0$.

But $\exists c > 0$ such that $\forall k \geq 2 \exists A \subseteq \mathbb{Z}^+$ such that

$|A(n)| \geq \pi(n) + e^{(\log n)^{1 - \frac{c\sqrt{\log k}}{\sqrt{\log \log n}}}}$ holds for every $n \geq 10$.

Theorem (P., Sándor)

$$B \text{ is a MB of order } k \implies \liminf_{n \rightarrow \infty} \frac{\sum_{b \in B, a \leq n} \frac{1}{b}}{k \sqrt[k]{\log n}} \geq \frac{\sqrt{6}}{e\pi}.$$

$$\exists C \forall k \geq 2 \exists B \text{ MB of order } k \text{ such that } \limsup_{n \rightarrow \infty} \frac{\sum_{b \in B, a \leq n} \frac{1}{b}}{k \sqrt[k]{\log n}} < C.$$

Logarithmic density

Theorem (P., Sándor)

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$a_0 \nmid a_1 \dots a_k$ -problem, logarithmic density

$$\log \log n + O(1)$$