On some Multiplicative Problems of Erdős

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- How dense can a set $A \subseteq \mathbb{N}$ be, if none of them divides the product of k others?

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$a_1, \dots, a_k, b_1, \dots, b_k$ are distinct
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element $\longrightarrow$ edge

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How to choose $g(n)$?

condition (3) $\longrightarrow g(n) = e^{c \log n / \log \log n}$

Maximal number of edges of C_6-free graphs

Füredi, Naor, Verstraëte, Győri

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- $ex(u, v, C_6) < 2u + v^2/2$

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Theorem (P.)

$$\max |A| \leq \pi(n) + \pi(n/2) + cn^{2/3} \frac{\log n}{\log \log n}$$

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Lower bound

$$\pi(n) + \pi(n/2) + cn^{2/3} (\log n)^{-4/3} \leq \max |A|$$

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$a_0 \nmid a_1 a_2 \dots a_k$ and multiplicative bases

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If B is a multiplicative basis of order k for $\{1, 2, \dots, n\}$, then $F_k(n) \leq |B|$.

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Multiplicative bases

Lemma

Let $B_0 = \{\text{primes} \le n\} \cup \left\{ x : x \le \frac{n^{\frac{2}{k+1}}}{(\log n)^2} \right\}$.

If $a \le n$ is not in B_0^k, then

$$a = p_1 p_2 \dots p_{k+1} a',$$

where $p_1 \ge p_2 \ge \dots \ge p_{k+1}$ are primes such that $p_k p_{k+1} > \frac{n^{\frac{2}{k+1}}}{(\log n)^2}$.

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We need at least one edge in each $\{p_1, p_2, \dots, p_{k+1}\}$.

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$a_0 \nmid a_1 \dots a_k$-problem

The sets $\{p_1, p_2, \dots, p_{k+1}\}$ intersect each other in at most one element.

Infinite multiplicative bases

Raikov (1938)

B is a MB of order $k \implies \limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} \geq \Gamma\left(\frac{1}{k}\right)^{-1}$.

For every $k \geq 2$ $\exists$ a MB of order k such that $\limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} < \infty$.

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Theorem (P., Sándor)

B is a MB of order $k \implies \limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists C > 0$: For every $k \geq 2$ $\exists$ a MB of order k such that $\limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} < C$.

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B is a MB of order $k \implies \limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} \geq \Gamma\left(\frac{1}{k}\right)^{-1}$.

For every $k \geq 2$ $\exists$ a MB of order k such that $\limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} < \infty$.

Theorem (P., Sándor)

B is a MB of order $k \implies \limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists C > 0$: For every $k \geq 2$ $\exists$ a MB of order k such that $\limsup_{n\to\infty} \frac{|B(n)|}{n/\log^{\frac{k-1}{k}} n} < C$.

Theorem (P., Sándor)

B is a MB of order $k \implies \liminf_{n\to\infty} \frac{|B(n)|}{\frac{n}{\log n}} > 1$.

But it can be $< 1 + \varepsilon$.

$a_0 \nmid a_1 \dots a_k$-problem, infinite case

Theorem (P., Sándor)

$\forall k \geq 2 \ \exists \ A \subseteq \mathbb{Z}^+$ such that $\limsup_{n\to\infty} \frac{|A(n)|-\pi(n)}{\frac{n^{2/(k+1)}}{\log^2 n}} > 0$.

$a_0 \nmid a_1 \dots a_k$-problem, infinite case

Theorem (P., Sándor)

$\forall k \geq 2 \ \exists \ A \subseteq \mathbb{Z}^+$ such that $\limsup_{n\to\infty} \frac{|A(n)|-\pi(n)}{\frac{n^{2/(k+1)}}{\log^2 n}} > 0.$

Theorem (P., Sándor)

$\forall \varepsilon > 0 \ \liminf_{n\to\infty} \frac{A(n)-\pi(n)}{n^{\varepsilon}} = 0.$

But $\exists c > 0$ such that $\forall k \geq 2 \ \exists A \subseteq \mathbb{Z}^+$ such that

$|A(n)| \geq \pi(n) + e^{(\log n)^{1-\frac{c\sqrt{\log k}}{\sqrt{\log\log n}}}}$ holds for every $n \geq 10$.

Logarithmic density

Theorem (P., Sándor)

B is a MB of order k $\Longrightarrow$ $\liminf_{n\to\infty} \frac{\sum_{b\in B, a\leq n} \frac{1}{b}}{k\sqrt[k]{\log n}} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists\ C \forall k \geq 2\ \exists B$ MB of order k such that $\limsup_{n\to\infty} \frac{\sum_{b\in B, a\leq n} \frac{1}{b}}{k\sqrt[k]{\log n}} < C$.

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$a_0 \nmid a_1 \ldots a_k$-problem, logarithmic density

$\log\log n + O(1)$