The use of additive tools in solving arithmetic anti-Ramsey problems

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Overview

1. Arithmetic anti-Ramsey Theory
2. Additive Number Theory
3. How to use additive tools in solving arithmetic anti-Ramsey problems

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Joint work with:

Oriol Serra

Mario Huicochea

Ramsey and anti-Ramsey Theory

Ramsey Theory studies the existence of monochromatic sets in colored universes.

anti-Ramsey Theory studies the existence of rainbow sets in colored universes.

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- Universes: $[n] = \{1, 2, ..., n\}$, $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$, G.
- Sets: $AP(t)$, solutions of linear equations.

Example of a Ramsey result and its anti-Ramsey forms

Theorem (B. L. Van der Waerden, 1927)

For any positive integers k and t there exists $W(k,t)$, such that every k-coloring of the set $[n]$, $n \geq W(k,t)$, contains a monochromatic $AP(t)$.

Example of a Ramsey result and its anti-Ramsey forms

Van der Waerden's Theorem ($k = t = 3$)

If n is large enough then any 3-coloring of $[n]$ contains a monochromatic $AP(3)$.

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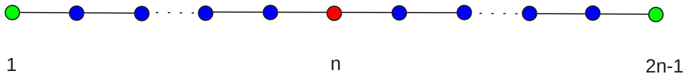

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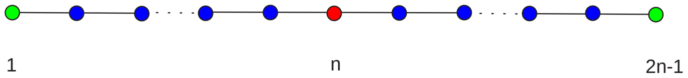

What conditions on the coloring guarantees a rainbow $AP(3)$?

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Theorem (Axenovich, Fon-Der-Flass, 2004)

Every partition of $[n]$ into 3 color classes with $r(n) < \min\{|A|, |B|, |C|\}$, where:

$$r(n) := \begin{cases} \left\lfloor \frac{n+2}{6} \right\rfloor & \text{if } n \neq 2 \pmod 6 \\ \frac{n+4}{6} & \text{otherwise} \end{cases}$$

contains a rainbow $AP(3)$.

More examples of anti–Ramsey results

Theorem (Jungić, Fox, Mahdian, Nešetřil, Radoičić, 2003)

Every 3*–coloring of* $\mathbb{Z}/n\mathbb{Z}$*, such that* $\frac{n}{6} < |A| \leq |B| \leq |C|$ *contains a rainbow solution of* $x + y = 2z$*.*

Theorem (Jungić, Fox, Mahdian, Nešetřil, Radoičić, 2003)

Let p *be a prime number. Every* 3*–coloring of* $\mathbb{Z}/p\mathbb{Z}$*, such that* $3 < |A| \leq |B| \leq |C|$ *contains a rainbow solution of* $ax + by + cz = d$ *with the only possible exception of* $x + y + z = d$*.*

Terminology and notation

- A coloring is rainbow-free with respect to a certain equation, if it contains no rainbow solution of the same.
- Let $m = m(\mathcal{U}, Eq)$ be the largest integer for which there is a rainbow-free coloring with the size of the smallest color class equal to m.

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Our approach

Rainbow Ramsey Theory studies the existence of rainbow sets under density conditions on the color classes.

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Study the structure of rainbow-free colorings

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Rainbow Ramsey Theory studies the existence of rainbow sets under <u>density conditions</u> on the color classes.

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Study the structure of rainbow-free colorings

by means of strong inverse theorems from Additive Number Theory.

Additive Number Theory

Let G be an additive abelian group.

For subsets $A, B \subseteq G$ we define the sumset of A and B as:

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$$

- How small can the sumset $A + B$ be?
- If $A + B$ is "small," how is the structure of A and B?

Abelian groups of prime order

Theorem (Cauchy, 1813; Davenport, 1935)

Let p be a prime number, and let A and B be nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$. Then $|A+B| \geq \min\{p, |A|+|B|-1\}$

Theorem (Vosper, 1956)

If (A, B) is a critical pair of nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$ then one of the following holds true:

- $|A|+|B| > p$ *and* $A+B = \mathbb{Z}/p\mathbb{Z}$.
- $|A|+|B| = p$ *and* $|A+B| = p-1$.
- $\min\{|A|, |B|\} = 1$.
- *A and B are arithmetic progressions with a common difference.*

General abelian groups

Theorem (Kneser, 1953)

Let G be an additive abelian group. If A and B are finite nonempty subsets of G, then

$$|A + B| \geq |A + H| + |B + H| - |H|$$

where $H = G_{A+B}$

- the stabilizer of $X \subseteq G$ is $G_X = \{g \in G \mid g + X = X\}$.

Theorem (Kemperman, 1960)

Structural characterization of critical pairs in G.

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$$|A + B| \leq |A| + |B|$$

$\mathbb{Z}_p = A \cup B \cup C$

Cauchy-Davenport   rainbow-free

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$$|A| + |B| - 1 \leq |A + B| \leq |A| + |B|$$

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Cauchy-Davenport rainbow-free

$$\Downarrow \qquad \Downarrow$$

$$|A| + |B| - 1 \leq |A + B| \leq |A| + |B|$$

- $|A + B| = |A| + |B| - 1 \quad \leftarrow$ Vosper's Theorem
- $|A + B| = |A| + |B| \quad \leftarrow$ Hamidoune-Rødseth's Theorem

General abelian groups

- Kneser's Theorem
- Kemperman's Theorem
- Grynkiewicz's Theorem

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[JFMNR]: $$m(\mathbb{Z}_p, x + y = 2z) = \begin{cases} 0 & \text{if } p \in \mathcal{P}_0 \\ 1 & \text{otherwise} \end{cases}$$

- $\mathcal{P}_0$ is the set of primes p for which 2 has either multiplicative order $p-1$, or multiplicative order $(p-1)/2$ with $(p-1)/2$ odd.

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Conjecture (Jungić, Fox, Mahdian, Nešetřil, Radoičić, 2003)

Let p denote the smallest prime factor of n in $\mathcal{P}_0$, and let q be the smallest prime factor of n in $\mathcal{P}_1$. Then:

$$m(\mathbb{Z}_n, x+y=2z) = \left\lfloor \frac{n}{\min\{2p, q\}} \right\rfloor$$

Theorem (M, Serra, 2012)

Let G be an abelian group of odd order n. A 3–coloring $G = A \cup B \cup C$ with $1 \le |A| \le |B| \le |C|$ is rainbow–free, if and only if, up to translation, there is a proper subgroup $H < G$, such that:

(i) $A \subseteq H$ and the 3–coloring induced in H is rainbow–free,

(ii) both $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$ are H–periodic sets, and

(iii) $\widetilde{B} = -\widetilde{B} = 2 \cdot \widetilde{B}$ and $\widetilde{C} = -\widetilde{C} = 2 \cdot \widetilde{C}$.

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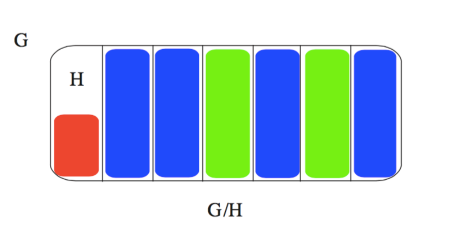

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Theorem (Huicochea, M, 2015)

A 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $1 \leq |A| \leq |B| \leq |C|$ is rainbow–free for equation:

$$a_1 x + a_2 y + a_3 z = b \ \ with \ \ some \ \ a_i \neq a_j, \tag{1}$$

if and only if:

(i) $A = \{s\}$ with $s(a_1 + a_2 + a_3) = b$ and

(ii) both B and C are sets invariant up to T_i for every $i \in \{1, 2, \ldots, 6\}$.

Corollary (Huicochea, M, 2015)

$$m(\mathbb{Z}_p, (1)) = \begin{cases} 0 & \text{if } a_1 + a_2 + a_3 = 0 \neq b \text{ or } |\langle d_1, d_2, ..., d_6 \rangle| = p - 1 \\ 1 & \text{otherwise} \end{cases}$$

Thank you for your attention!