Long zero-sum free and n-zero-sum free sequences over finite cyclic groups

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Content

Introduction and Notation

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Definition 1.1

Let $G = \mathbb{Z}_n$, *for a sequence* $S = (n_1 g) \cdot \ldots \cdot (n_l g)$, $1 \leq n_1, \ldots, n_l \leq n$, *the index of* S *is defined by* $ind(S) = \min\{\|S\|_g | g \in \mathbb{Z}_n, \text{ with } \mathbb{Z}_n = \langle g \rangle\}$, *where* $\|S\|_g = \frac{n_1 + \cdots + n_l}{ord(g)}$.

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The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. Recently, there are considerable publications on this subject (see for example, [4, 5, 6, 7, 8, 10, 11, 12]).

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Let S be a minimal zero-sum (resp. zero-sum free) sequence of elements over abelian group G. An element g_0 in S is called splittable if there exist two elements $x, y \in G$ such that $x + y = g_0$ and $Sg_0^{-1}xy$ is a minimal zero-sum (resp. zero-sum free) sequence as well; otherwise, g_0 is called unsplittable. S is called splittable if at least one of the elements of S is splittable; otherwise, it is called unsplittable.

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long zero-sum free sequences

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We now consider the general structure of the zero-sum free sequences S in $\mathbb{Z}_n$ of length $\frac{n}{3} + 1 < |S| \leq \frac{n}{2}$, and our first main result (Theorem 2.2) shows that by removing at most two elements from S, the index of the remaining sequence is not more than $1 - \frac{4}{n}$.

long zero-sum free sequences

The following lemma is essential to approach our main results.

Lemma 2.1

[11, 12] Let $n \geq 13$ be an odd integer, and S be an unsplittable minimal zero-sum sequence of length $|S| > \lfloor \frac{n}{3} \rfloor + 2$ over $\mathbb{Z}_n$. If ind$(S) \geq 2$, then

$$S \sim g^{\alpha}(\frac{n+s}{2}g)^{2t}((\frac{n-s}{2}+1)g),$$

where g is a generator of $\mathbb{Z}_n$, $t \geq 1$, s is odd with $s \geq 3$ and $2\alpha + 2ts + 2 - s = n$. Moreover, ind$(S) = 2$.

The First Main Result.

Theorem 2.2

[3] Let $n \geq 50$ be an odd integer, and S be a zero-sum free sequence of length $\lfloor \frac{n}{3} \rfloor + 1 < |S| \leq \lfloor \frac{n}{2} \rfloor$ over $\mathbb{Z}_n$. Then, by removing at most two elements from S, the remaining sequence is equivalent to a sequence whose index is not more than $1 - 4/n$.

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1. **Case 1:** If $\operatorname{ind}(S') \leq 1$, then $\operatorname{ind}(S) < 1$ by (1). Thus $S \sim (n_1 h) \cdot \ldots \cdot (n_l h)$ with $ord(h) = n$ and $\operatorname{ind}(S) = \|S\|_h = \frac{n_1 + \cdots + n_l}{n}$. If $n_i + n_j \geq 4$ for some $1 \leq i \neq j \leq l$, then $\frac{\sum_{k=1}^{l} n_k - n_i - n_j}{n} \leq \frac{\sum_{k=1}^{l} n_k - 4}{n} < 1 - \frac{4}{n}$. If $n_i + n_j \leq 3$ for all i, j, then there is at most one i such that $n_i = 2$, and for all other $j \neq i$, $n_j = 1$. So $\sum_{k=1}^{l} n_k \leq \lfloor \frac{n}{2} \rfloor + 1$ (as $l \leq \lfloor \frac{n}{2} \rfloor$), and thus $\operatorname{ind}(S) \leq \frac{\lfloor \frac{n}{2} \rfloor + 1}{n} < 1 - \frac{4}{n}$.

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2. **Case 2:**. If $\mathrm{ind}(S') > 1$, then by Lemma 2.1, $S' \sim g^{\alpha}(\frac{n+s}{2} g)^{2t}((\frac{n-s}{2} + 1)g) \sim (2g)^{\alpha}(sg)^{2t}((n - s + 2)g) = T$.

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Lower bound on $h(S)$

Let S be a zero-sum free sequence over $\mathbb{Z}_n$. An extensively used result of Bovey et al [1] states that if $|S| = l > n/2$, then $h(S) \geq 2l - n + 1$. S. Savchev and F. Chen [8], and P. Yuan [10] gave a more precisely lower bound on $h(S)$ for zero-sum free sequences of length $l > n/2$ independently. We now give a lower bound for $h(S)$ when the length of S is between $\lfloor \frac{n}{3} \rfloor + 1$ and $\lfloor \frac{n}{2} \rfloor$.

Theorem 2.3

[3] Let $n \geq 13$ be an odd integer and l be an integer satisfying $\lfloor \frac{n}{3} \rfloor + 1 < l \leq \lfloor \frac{n}{2} \rfloor$. Let S be a zero-sum free sequence of length l over $\mathbb{Z}_n$, and i be an positive integer such that $1 \leq i \leq 3$ and $4l - 4 - n \equiv i \pmod 3$. If ind$(S) > 1$, then

(a) $h(S) \geq 3l - 2 - n$, if $\frac{2n+5-i}{5} \leq l \leq \lfloor \frac{n}{2} \rfloor$;

(b) $h(S) \geq \lfloor \frac{4l-4-n}{3} \rfloor + 1$, if $\lfloor \frac{n}{3} \rfloor + 1 < l < \frac{2n+5-i}{5}$.

Moreover, these estimates are best possible.

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Proof. By the assumption and Theorem 2.2, we may assume

$$S = (2g)^u(sg)^v W g_1 g_2,\ ||S(g_1g_2)^{-1}||_g \leq 1-\frac{4}{n}, \mathrm{ind}(S) = \|S\|_g,$$

where $s \geq 3$, $W|S,\ 1 \geq |W| \geq 0$ and $2g, sg \notin W$. We claim that $g \notin S$.

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$$n-4 \geq n||S(g_1g_2)^{-1}||_g \geq 2u + 3(l-2-u) = 3l-6-u. \quad (2)$$

$$n-4 \geq n||S(g_1g_2)^{-1}||_g \geq 2u+3v+4(l-2-u-v) = 4l-8-2u-v \quad (3)$$

The above inequalities yield that $u \geq 3l-2-n$ and $2u+v \geq 4l-4-n$, respectively. Since $h(S) \geq \max\{u,v\}$, we have $h(S) \geq \max\{3l-2-n, \lceil \frac{4l-4-n}{3} \rceil\}$.

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We next show that the above estimates are best possible by finding some extremal sequences S with $h(S) = 3l-2-n$ and $h(S) = \lceil \frac{4l-4-n}{3} \rceil$, respectively. This can be done by the case by case argument on $i = 1, 2, 3$.

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The n-zero-sum free sequences in $\mathbb{Z}_n$ play an important role in the investigation of the structure of zero-sum free sequences. Savchev and Chen [9] characterized $n-$zero-sum free sequences of length $|S| \geq \frac{3n}{2} - 1$. Our main theorem below characterizes those sequences S with $|S| = n + l$ where $l \geq n/p + p - 2$ and p is the smallest prime divisor of $|G| = n$.

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Theorem 3.1

Let G be a finite abelian group of order n, and $l \geq n/p + p - 2$ be an integer where p is the smallest prime divisor of n. Let S be a n-zero-sum free sequence over G of length $|S| = n + \ell$. Then, there is an element $g \in G$ such that

$$-g + S = 0^h T S'$$

with $h = h(S) \geq \ell + 1$, T is a zero-sum sequence of length $|T| \leq |G| - h - 1$, and S' is zero-sum free of length $|S'| \geq \ell + 1$.

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Corollary 3.2

Let p be the smallest prime dividing n, and S be an n-zero-sum free sequence over $\mathbb{Z}_n$ of length $|S| = n + \ell$, where $\ell \geq n/p + p - 2$ is an integer. Then, there exists $g \in \mathbb{Z}_n$ such that

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with $h \geq \ell + 1$, T is a zero-sum sequence of length $|T| \leq n - h - 1$, and S' is zero-sum free with $|S'| \geq \ell + 1$.

Subsums

Let S be a long zero-sum free sequence in $\mathbb{Z}_n$. We investigate when subsums of S (with $|S| \geq (n+2)/3$) form an interval: **Question:** Let S be a long zero-sum free sequence over $\mathbb{Z}_n$. Does there exist a sequence T, such that $T \sim S$ and subsums of T form an interval ? (such T is referred as to a smooth sequence).

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In [8], S. Savchev and F. Chen showed that if S is a zero-sum free sequence of length $|S| > \frac{n}{2}$ over $\mathbb{Z}_n$, then $ind(S) < 1$. Thus there exists a sequence T, such that $T \sim S$ and $\sigma(T) < n$ (as positive integers)(clearly, $1 \in T$ as $|T| = |S| > n/2$). Moreover, $\sum(T) = [1, \sigma(T)]$ is an interval (i.e., T is smooth). Next we will show that if S is a zero-sum free sequence over $\mathbb{Z}_n$ of length $|S| \geq \frac{n+2}{3}$ such that $1 \in S$ and $\sigma(S) < n$ (as positive integers), then $\sum(S)$ is an interval except for one special case.

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Theorem 4.1

[3] Let S be a sequence with positive integer terms of length $|S| = t \geq \frac{n+2}{3}$, and $\sigma(S) < n$. If $1 \in S$, then $\sum(S)$ is an interval except for the case when $S = S_0 n_l$, where $\sum(S_0)$ is an interval and $n_l > \sigma(S_0) + 1$.

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Thank you!

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