Very Small Product Sets

Matt DeVos

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- $A, B \subseteq G$ are finite and nonempty,
- $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$.

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- $AB = \{ab \mid a \in A \text{ and } b \in B\}$.

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- **(IV)** A, B arithmetic progressions with a common difference. (progression)

G abelian

1. Lower Bound (Kneser)

Let A, B be finite nonempty subsets of an additive abelian group G. Then there exists $H \leq G$ so that

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2. Structure (Kemperman)

The nontrivial critical pairs can be constructed using a recursive process involving small sets and progressions.

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Lower Bound (D.)

Let A, B be finite nonempty subsets of an arbitrary multiplicative group G. Then there exists $H \leq G$ so that

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- a transform due to Kemperman.

Kemperman's Transform

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2. $(A \cup Ag)(B \cap g^{-1}B) \subseteq AB$
3. $|A \cap Ag| + |A \cup Ag| + |B \cap g^{-1}B| + |B \cup g^{-1}B| = 2|A| + 2|B|$

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Observe

1. $1 \notin ABC$,
2. $|A| + |B| + |C| = |A| + |B| + |G| - |AB| > |G|$.
3. (B, C) is critical. (since BC is disjoint from A^{-1})
4. (C, A) is critical. (since CA is disjoint from B^{-1})

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Theorem (Vosper)

If p is prime and (A, B, C) is a nontrivial trio in $\mathbb{Z}_p$ then one of the following holds:

(I) $\min\{|A|, |B|, |C|\} = 1$

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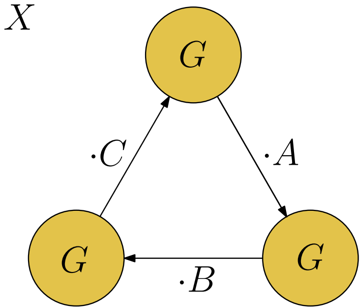

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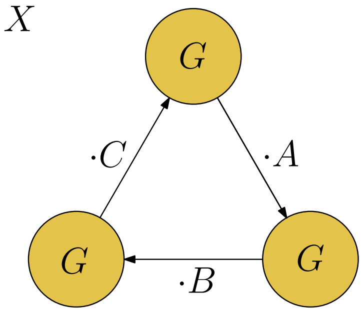

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1. X has no triangle

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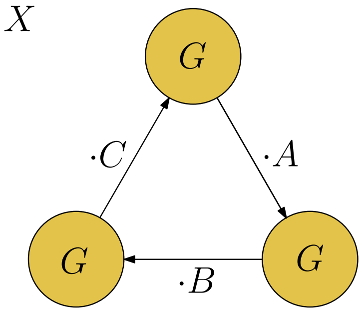

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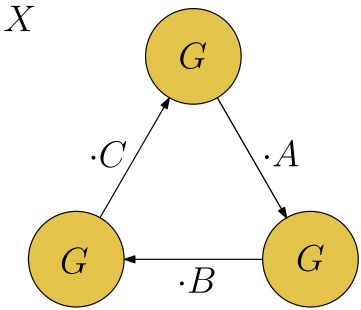

Observe

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2. The sum of the densities of the three bipartite subgraphs between blocks is $\frac{|A|}{|G|} + \frac{|B|}{|G|} + \frac{|C|}{|G|} > 1$.
3. G has a natural action on X which is transitive on each block.

New problem

Classify all tripartite graphs X which satisfy

1. X has no triangle
2. The sum of the densities of the three bipartite subgraphs between blocks is > 1.
3. The subgroup of $Aut(X)$ which fixes each block setwise still acts transitively on each block.

We call these **special** graphs.

Example

Consider a graph embedded in a surface with vertices V, edges E, and faces F for which the automorphism group acts transitively on V, E, and F.

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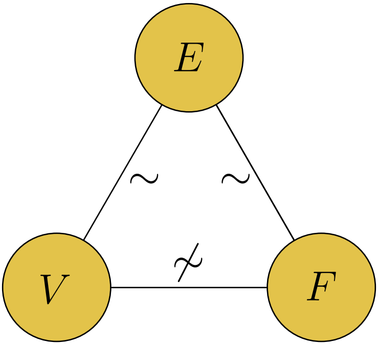

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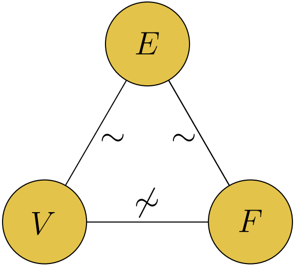

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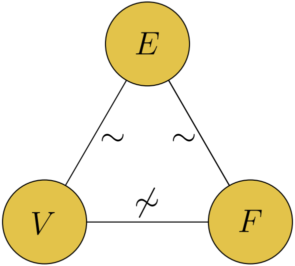

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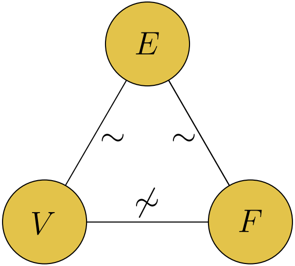
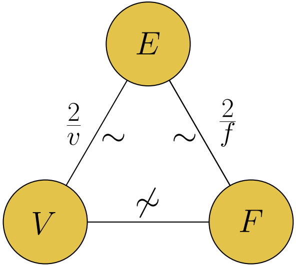


Observe:

- X has no triangle.
- The automorphism group is transitive on V, E, and F.
- next we compute compute densities..

Example

densities



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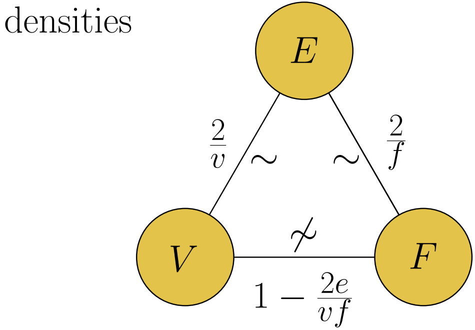
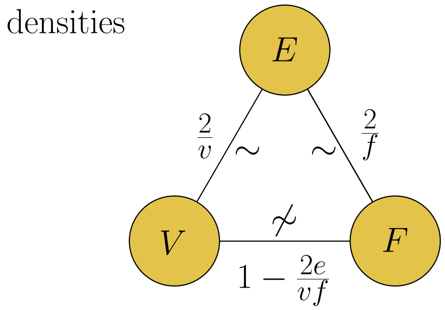


- The number of vertex-face incidences is $2e$
- The density of the vertex-face incidence bipartite graph is $\frac{2e}{vf}$
- The density of the vertex-face nonincidence bipartite graph is $1 - \frac{2e}{vf}$

Example



So the sum of the densities of the three bipartite graphs is

$$\frac{2}{f} + \left(1 - \frac{2e}{vf}\right) + \frac{2}{v} = 1 + \frac{2}{vf}\,(v - e + f)$$

so X is special precisely when $v - e + f > 0$

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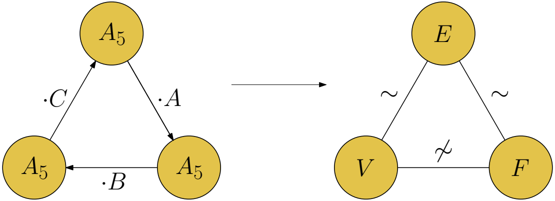

Dodecahedron

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The special graph based on Dodecahedron comes from critical trios in $A_5 \times \mathbb{Z}_2$ and A_5 (and more generally any group with a quotient isomorphic to one of these).

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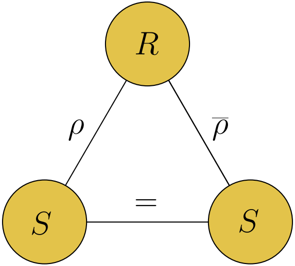
Note

To classify critical trios, we will first classify special graphs, then determine their automorphism groups

Describing the Classification

Every maximal special graph may be obtained by a sequential construction involving three recursive structures, terminating in one of three types of elementary structure.

Elementary Structure 1



Associated critical trios (A, B, C)

Trios with $|B| = 1$

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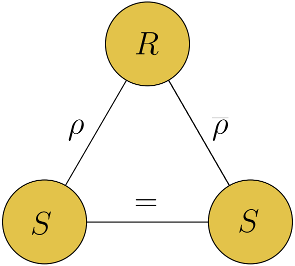
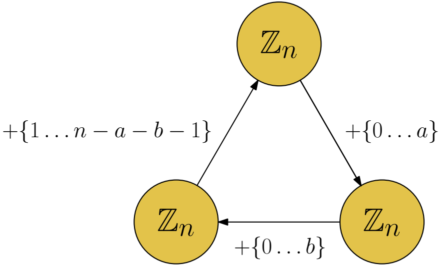


Associated critical trios (A, B, C)

Trios with $|B| = 1$ and more generally to those for which $AH = A$ and $B = Hx$ for some $H < G$.

Elementary Structure 2



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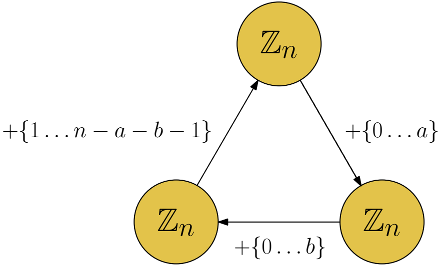

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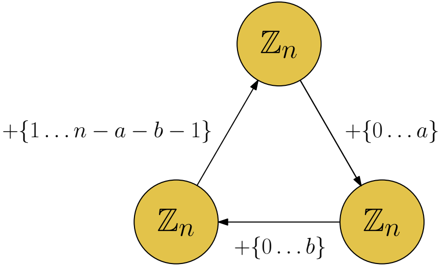

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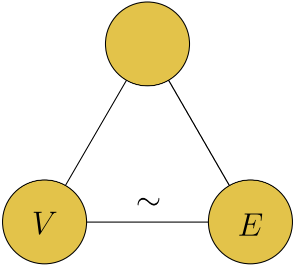
Those in cyclic groups where A, B, C are all progressions with a common difference. Trios in dihedral groups where A, B, C are all "dihedral progressions" with a common difference. More generally, this yields critical trios in groups G which cyclic or dihedral quotients.

Elementary Structure 3



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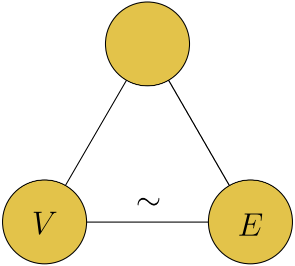

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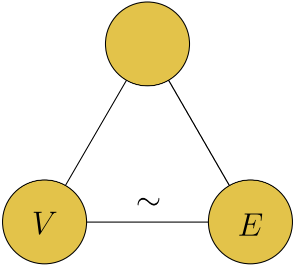

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Example

Let $A = H \cup xH$ and $B = H \cup Hx$. Then
$AB = H \cup xH \cup Hx \cup xHx$ has size $< 4|H| = |A| + |B|$

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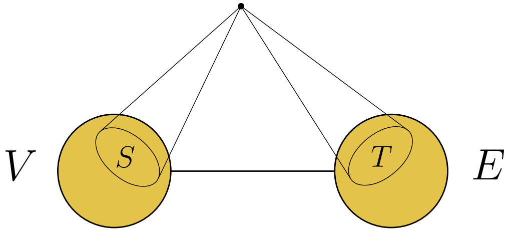

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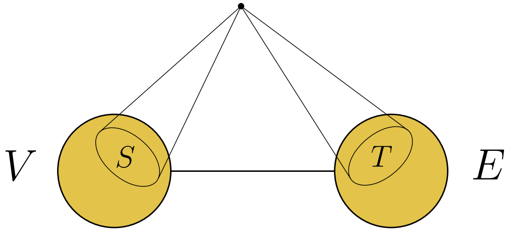

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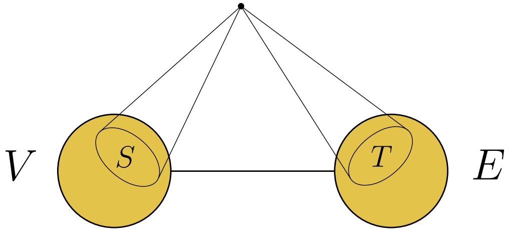

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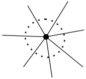
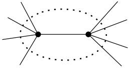
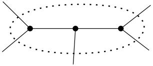
To find these very small product sets, we classify all edge-cuts of size $< 2d$ in d-regular vertex and edge transitive graphs.

Small Edge-Cuts

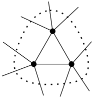
Edge cuts of size $< 2d$ in d-regular vertex and edge transitive graphs (Assume connected and $d \geq 3$)

1.  size d

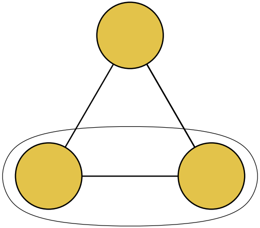
2.  size $2d - 1$

3. $d = 3$  size 5

4. G is one of Cube, Octahedron, Dodecahedron, Icosahedron, K_6, or Petersen and one side is a shortest cycle

   ex. 

Recursive Structure 1



disconnected bipartite graph

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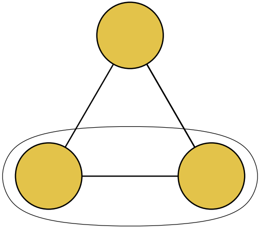

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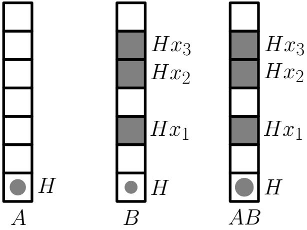
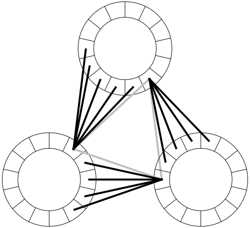


Recursive Structure 2



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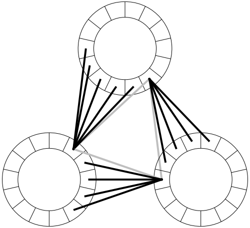
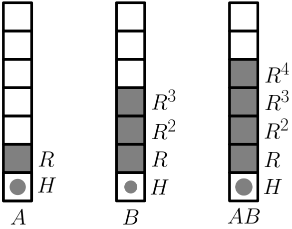
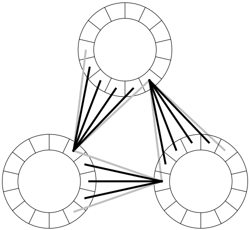


Below $N \lhd G$, and G/N is cyclic and generated by $R \in G/N$.



Recursive Structure 3



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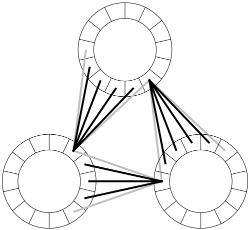
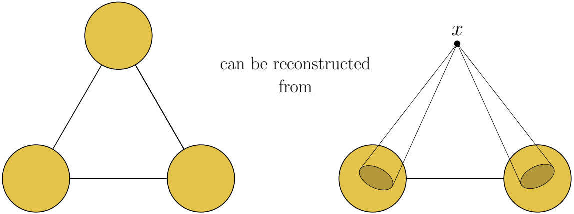
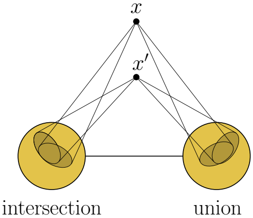


This double-fringed structure gives rise to critical trios in dihedral groups (and more generally groups with dihedral quotients).

Ideas in the Proof





Proving Vosper's Theorem

Theorem (Vosper)

If p is prime and (A, B, C) is a nontrivial trio in $\mathbb{Z}_p$ then one of the following holds:

(I) $\min\{|A|, |B|, |C|\} = 1$

(II) A, B, C are all arithmetic progressions with a common difference.

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Proof of Vosper: Suppose (for a contradiction) that the result is false and choose a counterexample (A, B, C) so that

(i) $|C|$ is minimum.

(ii) $|B|$ is minimum (subject to (i))

Proving Vosper's Theorem (continued)

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First suppose B is a Sidon set, so $|B \cap (B + g)| \leq 1$ for every $g \neq 0$. Then we may choose $c_1, c_2, c_3 \in C$ distinct and we have the contradiction:

$$|B+C| \geq |(B+c_1) \cup (B+c_2) \cup (B+c_3)| \geq 3|B| - 3 \geq 2|B| \geq |B| + |C|$$

Proving Vosper's Theorem (continued)

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We may choose $g \neq 0$ so that $|B \cap (B - g)| \geq 2$.

Proving Vosper's Theorem (continued)

Note: by the Cauchy-Davenport theorem $|A| + |B| + |C| = p + 1$.
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We may choose $g \neq 0$ so that $|B \cap (B - g)| \ge 2$. Consider

$$(A \cap (A + g), B \cup (B - g), C)$$
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If $A \cap (A + g) = \emptyset$ then we have the contradiction:

$$|A + B| \geq |(A \cup (A + g) + (B \cap (B - g))| \geq 2|A| + 1 > |A| + |B|$$

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So $A \cap (A + g) \neq \emptyset$ and both of our new trios are nontrivial. It now follows from Cauchy-Davenport that the sum of the sizes of the three sets in both of these new trios is $p + 1$. By our choice of counterexample, the theorem holds true for the trio $(A \cup (A + g), B \cap (B - g), C)$. It follows that C is an arithmetic progression, and now the result follows from our lemma. $\square$

The End

Thanks for your attention!