# On some properties of the generalised multinomial measure

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# The multinomial measure

The multinomial measure on the unit interval (Sekiguchi, Okada, Shiota; 1996):
Let $q \geq 2$ be a positive integer. Denote $I = I_{0,0} = [0,1]$ and

$$I_{n,j} = \left[\frac{j}{q^n}, \frac{j+1}{q^n}\right), \text{ for } j = 0, 1, \ldots, q^n - 2, \qquad I_{n,q^n-1} = \left[\frac{q^n-1}{q^n}, 1\right],$$

for $n = 1, 2, 3, \ldots$. Let $\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$ with $0 \leq r_i \leq 1$ and $\sum_{k=0}^{q-1} r_k = 1$.

The *multinomial measure* $\mu_{q,\mathbf{r}}$ is the **probability measure on** $I$ defined by

$$\mu_{q,\mathbf{r}}(I_{n+1,qj+k}) = r_k \cdot \mu_{q,\mathbf{r}}(I_{n,j})$$

for $n = 0, 1, 2, \ldots$, $j = 0, 1, \ldots, q^n - 1$, $k = 0, 1, \ldots, q - 1$.

# Introducing the **generalised** multinomial measure

Let $\mathcal{A}$ be a denumerable set $\{a_1, a_2, \dots\}$, called *alphabet*.
Assume, without loss of generality, $\mathcal{A} = \{0, 1, \dots\} = \mathbb{N}_0$.

**Notations:**

- $\mathcal{W}$ the set of all (finite and infinite) words over the alphabet $\mathcal{A}$
- $\mathcal{W}_m$ the set of all words of length $m$ ($m \geq 1$) over the alphabet $\mathcal{A}$. Obviously $\mathcal{W}_1 = \mathcal{A}$
- for $l \geq m \geq 1$ (integers), and a word $\omega \in \mathcal{W}$, $\omega = \omega_1\omega_2\dots$ of length $l$ or $\infty$ let $\omega^{(m)}$ denote the word $\omega_1 \dots \omega_m$
- $\mathcal{W}_\infty$ the set of all words of infinite length over $\mathcal{A}$

# Introducing the generalised multinomial measure

Let $\mathbf{r} = \{r_0, r_1, \dots\}$ be an arbitrarily fixed sequence of real numbers such that $r_j > 0$ for all $j \geq 0$ and $\sum_{j=0}^{\infty} r_j = 1$. We introduce a probability measure on $\mathcal{W}$ in an inductive manner.

### Definition

We define, for any $k \in \mathbb{N}_0$, and for any $\omega, \omega' \in \mathcal{W}$, $\omega = \omega_1\omega_2 \dots$,

$$\mathbb{P}_{\mathbf{r}}(\omega_1 = k) := r_k \quad \text{and} \quad \mathbb{P}_{\mathbf{r}}(\omega = k\omega') := r_k \cdot \mathbb{P}_{\mathbf{r}}(\omega_2\omega_3 \dots = \omega'), \tag{1}$$

where $k\omega'$ denotes the (usual) concatenation of the letter $k$ with the word $\omega'$.

# Introducing the generalised multinomial measure

Now we construct a **function that assigns a real value to every word of $\mathcal{W}$.**

We proceed inductively: let $q \in (0,1)$ be an arbitrarily fixed real number and let $p = 1 - q$. We define, for any $m \geq 1$ the function $\mathsf{value}_m : \mathcal{W}_m \to [0,1)$, by

$$\mathsf{value}_1(k) = 1 - q^k \quad \mathsf{value}_m(k\omega) = \mathsf{value}_1(k) + pq^k \cdot \mathsf{value}_{m-1}(\omega), \tag{2}$$

for $\omega \in \mathcal{W}_{m-1}$.

### Definition

The function $\mathsf{value} : \mathcal{W} \to [0,1)$ is the (uniqe) real function with the property that for any $m \geq 1$ its restriction to $\mathcal{W}_m$ coincides with $\mathsf{value}_m$.

**Remark:** the closure (with respect to the canonic topology on $\mathbb{R}$) of the set $\mathsf{value}(\mathcal{W})$ is the interval $[0,1]$.

# Introducing the generalised multinomial measure

An **order relation on $\mathcal{W}$** denoted by $\leq^*$ can be introduced as follows:

1. On $\mathcal{W}_1 = \mathcal{A} = \mathbb{N}_0$, $\leq^*$ coincides with the canonical order relation on $\mathbb{N}_0$.
2. For $m \geq 2$ and $\omega, \omega' \in \mathcal{W}_m$, $\omega = \omega_1 \dots \omega_m$, $\omega' = \omega'_1 \dots \omega'_m$ we have $\omega \leq^* \omega'$ either if $\omega_1 \leq^* \omega'_1$ or if there exists a $j \in \{1, \dots, m-1\}$ such that $\omega_i = \omega'_i$, for all $1 \leq i \leq j$ and $\omega_{j+1} \leq^* \omega'_{j+1}$.
3. For $\omega, \omega' \in \mathcal{W}$ we have $\omega \leq^* \omega'$ if there exists an integer $m \geq 1$ such that $\omega^{(m)} \leq^* \omega'^{(m)}$.

One can easily verify that the function value is strictly increasing with respect to $\leq^*$ and to the canonical order relation of real numbers.

# The generalised multinomial measure

The probability measure $\mathbb{P}_{\mathbf{r}}$ on $\mathcal{W}$ induces a probability measure $\mu_{\mathbf{r},q}$ on $[0,1]$, given as follows.

### Definition

We call *generalised multinomial measure* (of parameters $\mathbf{r}$ and $q$) the measure $\mu_{\mathbf{r},q}$ defined by

$$\mu_{\mathbf{r},q}\big([0,a)\big) := \mathbb{P}_{\mathbf{r}}\Big(\{\omega \in \mathcal{W} \mid \text{value}(\omega) \leq a\}\Big), \tag{3}$$

for any $a \in [0,1]$.

**Remarks:**

1. $\mu_{\mathbf{r},q}\big([1-q^k, 1-q^{k+1})\big) = r_k$
2. In the special case $r_k = q^k \cdot p$, for all $l \in \mathbb{N}_0$ one can show that $\mu_{\mathbf{r},q}$ coincides with the uniform measure on the unit interval.

# Order statistics of the generalised multinomial measure. The minimum

Here we consider $\mu_{\mathbf{r},q}$, for $r_j = \lambda\nu^j$, $j = 0, 1, \ldots$, $0 < \nu < 1$, $\nu = 1 - \lambda$ (**notation:** $\mu_{\nu,q}$)
**The problem setting:** We pick at random (with respect to $\mathbb{P}_{\mathbf{r}}$ on $\mathcal{W}$ defined above), independently, $n$ words from $\mathcal{W}_m$, for $n \geq 1$. We apply the function value to each of the chosen words and look for the **minimum among these $n$ values**. The same can be done with all random choices of $n$ words of $\mathcal{W}_\infty$. We denote by $a_n^{(m)}$ the average minimal value among all possible choices of $n$ words of length $m$. By taking the limit $a_n := \lim_{m\to\infty} a_n^{(m)}$ we obtain the average minimal value among all choices of $n$ words of $\mathcal{W}_\infty$. We are interested in the study of the **asymptotic behaviour of $a_n$, for $n \to \infty$**.

# The minimum. Finding the recursion.

The first step is to establish the recursion

$$a_n^{(m)} = \sum_{k=1}^{n} \binom{n}{k} \sum_{j=0}^{\infty} (\lambda \nu^j)^k (\nu^{j+1})^{n-k} \left(1 - q^j + pq^j \cdot a_k^{(m-1)}\right).$$

This is obtained from the relations

$$\text{value}_1(k) = 1 - q^k, \qquad \text{value}_m(k\omega) = \text{value}_1(k) + pq^k \cdot \text{value}_{m-1}(\omega),$$

based on the following **idea:** let $j$ be the minimum among the first letters of the $n$ words, i.e., there is an integer $k$, $1 \leq k \leq n$ such that $k$ words start with $j$, and the other $n-k$ words start with a letter greater than $j$.
$(\nu^{j+1} = \lambda \nu^{j+1} + \lambda \nu^{j+2} + \dots)$

# The minimum. Finding the recursion

By taking the limit for $m \to \infty$ in the above recursion we obtain

$$a_n = \sum_{k=1}^{n} \binom{n}{k} \lambda^k \nu^{n-k} \sum_{j=0}^{\infty} \nu^{jn} \left(1 - q^j + pq^j \cdot a_k\right).$$

This yields

$$a_n = \sum_{k=1}^{n} \binom{n}{k} \lambda^k \nu^{n-k} \left( \frac{1}{1-\nu^n} - \frac{1}{1-q\nu^n} + \frac{p}{1-q\nu^n} a_k \right),$$

and thus

$$a_n = 1 - \frac{1-\nu^n}{1-q\nu^n} + \frac{p}{1-q\nu^n} \sum_{k=1}^{n} \binom{n}{k} \lambda^k \nu^{n-k} a_k.$$

We obtain

$$a_n = \frac{p\nu^n}{1-q\nu^n} + \frac{p}{1-q\nu^n} \sum_{k=1}^{n} \binom{n}{k} \lambda^k \nu^{n-k} a_k.$$

Thus we have proven the following result.

## The average minimum. The recursion

### Proposition

*The average minimum value among $n$ words over $\mathbb{N}_0$ with respect to the generalised multinomial measure $\mu_{\nu,q}$ satisfies the recursion*

$$a_n = \frac{p\nu^n}{1-q\nu^n} + \frac{p}{1-q\nu^n}\sum_{k=1}^{n}\binom{n}{k}\lambda^k\nu^{n-k}a_k, \quad \textit{for all integers} \quad n \geq 1. \tag{4}$$

We set $a_0 = 0$, which is convenient for computational reasons. One can rewrite the above equation as

$$a_n = \frac{p\nu^n}{1-p\lambda^n-q\nu^n} + \frac{p}{1-p\lambda^n-q\nu^n}\sum_{k=0}^{n-1}\binom{n}{k}\lambda^k\nu^{n-k}a_k \tag{5}$$

in order to compute the elements $a_n$ inductively, for $n = 1, 2, \ldots$.

# The asymptotics of the average minimum

Putting everything together, we have obtained the following result.

**Theorem**

*The average $a_n$ of the minimum value among n random words with respect to the generalised multinomial measure $\mu_{\nu,q}$ admits the* **asymptotic estimate**

$$a_n = \Phi(-\log_\lambda n) n^{-\log_\lambda p}\Big(1 + \mathcal{O}\big(\frac{1}{n}\big)\Big), \tag{6}$$

*for $n \to \infty$, where $\Phi(x)$ is a periodic function having period 1 and known Fourier coefficients. The mean (zeroth Fourier coefficient) of $\Phi$ is given by the expression*

$$\frac{1}{\log \frac{1}{\lambda}} \int_0^\infty \Big(q e^{-\lambda z} \widehat{A}(\nu z) + p(e^{-\lambda z} - e^{-z})\Big) z^{\frac{\log p}{\log \lambda} - 1} dz. \tag{7}$$

# Remarks

**1.** One can rewrite the zeroth Fourier coefficient above as

$$\frac{q}{\log\frac{1}{\lambda}}\left(\Gamma\left(\frac{\log p}{\log\lambda}\right)+\sum_{n\geq 0}a_n\frac{\nu^n}{n!}\Gamma\left(n+\frac{\log p}{\log\lambda}\right)\right).$$

**2.** For the **special case** $\lambda = p$, $\mu_{\nu,q}$ is the uniform distribution on the unit interval, and we obtain $a_n = \frac{1}{n+1}$, for $n \geq 1$. This can be shown by induction.

# The maximum

**The problem setting**

We pick at random (with respect to $\mathbb{P}_{\mathbf{r}}$ on $\mathcal{W}$ defined above), independently, $n$ words from $\mathcal{W}_m$, for $n \geq 1$. We apply the function value defined above to each of the chosen words and look for the **maximum among these $n$ values**. The same can be done with all random choices of $n$ words of $\mathcal{W}_\infty$. We denote by $b_n^{(m)}$ the average maximal value among all possible choices of $n$ words of length $m$. By taking the limit $b_n := \lim_{m\to\infty} b_n^{(m)}$ we obtain the **average maximal value** among all choices of $n$ words of $\mathcal{W}_\infty$. We are also interested in the study of the **asymptotics of $b_n$, for $n \to \infty$**.

First, we establish the recursion

$$b_n^{(m)} = \sum_{k=1}^{n} \binom{n}{k} \sum_{j=0}^{\infty} (\lambda\nu^j)^k (1-\nu^j)^{n-k} \left(1-q^j+pq^j \cdot b_k^{(m-1)}\right), \text{ for } n \geq 1.$$

This is obtained from the definition of the function value based on the following **idea**:
let $j$ be the maximum among the first letters of the $n$ words, i.e., there is an integer $k$, $1 \leq k \leq n$ such that $k$ words start with $j$, and the other $n-k$ words start with a letter less than $j$.
For $m \to \infty$ in the above recursion we obtain

$$b_n = \sum_{k=1}^{n} \binom{n}{k} \sum_{j \geq 0} (\lambda\nu^j)^k (1-\nu^j)^{n-k} (1-q^j+pq^j b_k), \text{ for } n \geq 1.$$

$$(\lambda + \lambda\nu + \dots \lambda\nu^{j-1} = 1 - \nu^j)$$

# The average maximum value

Since $b_n$ is expected to be close to 1, we set $c_n = 1 - b_n$ for $n \geq 1$ and look for a recursion for $c_n$. Then, we study the asymptotic behavior of $c_n$. The recursion for $b_n$ can be rewritten as

$$1 - c_n = \sum_{k=1}^{n} \binom{n}{k} \sum_{j \geq 0} (\lambda \nu^j)^k (1 - \nu^j)^{n-k} \big(1 - q^j + pq^j (1 - c_k)\big), \quad \text{for } n \geq 1. \tag{8}$$

## Proposition

*If $b_n$ is the average maximum value among $n$ words over $\mathbb{N}_0$ with respect to the generalised multinomial measure $\mu_{\nu,q}$ and $c_n = 1 - b_n$, for $n \geq 1$, then $c_n$ satisfies the recursion*

$$c_n = \sum_{j \geq 0} \Big( (1 - \nu^{j+1})^n - (1 - \nu^j)^n \Big) q^{j+1} + \sum_{k=1}^{n} \binom{n}{k} \sum_{j \geq 0} (\lambda \nu^j)^k (1 - \nu^j)^{n-k} pq^j c_k, \quad \textit{for } n \geq 1. \tag{9}$$

## The average maximum. Aymptotics

### Theorem

*The average* $b_n$ *of the maximum value among n random words with respect to the generalised multinomial measure admits the asymptotic estimate*

$$b_n = 1 - \Phi(-\log_\nu n) n^{-\log_\nu q}\left(1 + \mathcal{O}(\frac{1}{n})\right), \tag{10}$$

*for* $n \to \infty$, *where* $\Phi(x)$ *is a periodic function having period 1 and known Fourier coefficients. The mean (zeroth Fourier coefficient) of* $\Phi$ *is given by the expression*

$$\frac{p}{\log \frac{1}{\nu}}\left(\Gamma\left(\frac{\log q}{\log \nu}\right) + \sum_{n \geq 0} c_n \frac{\lambda^n}{n!}\Gamma\left(n + \frac{\log q}{\log \nu}\right)\right). \tag{11}$$

**Remark.** For $\lambda = p$ we expect to get $c_n = \frac{1}{n+1}$, which indeed can be proven by induction.

# comparison / “duality”

average **minimum** value:

$$a_n = \Phi(-\log_\lambda n) n^{-\log_\lambda p}\Big(1 + \mathcal{O}(\frac{1}{n})\Big),$$

$$\frac{q}{\log\frac{1}{\lambda}}\left(\Gamma\Big(\frac{\log p}{\log\lambda}\Big) + \sum_{n\geq 0} a_n \frac{\nu^n}{n!}\Gamma\Big(n + \frac{\log p}{\log\lambda}\Big)\right).$$

average **maximum** value:

$$b_n = 1 - \Phi(-\log_\nu n) n^{-\log_\nu q}(1 + \mathcal{O}(\frac{1}{n})),$$

$$\frac{p}{\log\frac{1}{\nu}}\left(\Gamma\Big(\frac{\log q}{\log\nu}\Big) + \sum_{n\geq 0} c_n \frac{\lambda^n}{n!}\Gamma\Big(n + \frac{\log q}{\log\nu}\Big)\right).$$

**Thank you for your attention!**

**Danke!**