On a conjecture of Sárközy and Szemerédi

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1. Backgrounds

In 1957, P. Erdős mentioned a conjecture due to H. Hanani:

Conjecture (H. Hanani) Let $A = \{a_n\}$ and $B = \{b_n\}$ be two infinite sequences of increasing integers. If every sufficiently large integer can be represented in the form $a_i + b_j$, then

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 1,$$

where $A(x)$ and $B(x)$ are the counting functions of A and B, respectively.

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Hanani's conjecture is equivalent to

Conjecture (H. Hanani) Let $A = \{a_n\}$ and $B = \{b_n\}$ be two sequences of increasing integers such that every sufficiently large integer can be represented in the form $a_i + b_j$. If

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \le 1,$$

then one of $\{a_n\}$ and $\{b_n\}$ must be finite.

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Let $f(n)$ be the number of representations of the integer n in the form $a_i + b_j$. In 1960, Narkiewicz proved the following result.

Theorem 1 *(Narkiewicz) If $f(n) \geq k$ for all sufficiently large integers n and*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \leq k,$$

then

(i) $f(n) = k$ for all sufficiently large integers n;

(ii)

$$\lim_{x\to\infty} \frac{A(2x)}{A(x)} = 1$$

or

$$\lim_{x\to\infty} \frac{B(2x)}{B(x)} = 1.$$

[1] W. Narkiewicz, Remarks on a conjecture of Hanani in additive number theory, Colloq. Math. 7 (1959/60), 161-165.

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2. The additive complements

Two sequences A, B of non-negative integers are called *additive complements*, if their sum $A + B = \{a + b : a \in A, b \in B\}$ contains all sufficiently large integers.

It is easy to prove that, for additive complements A, B, we have

$$\liminf_{x\to\infty} \frac{A(x)B(x)}{x} \geq 1.$$

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3. Infinite additive complements

Two sequences A, B of non-negative integers are called *infinite additive complements*, if A, B are additive complements with $|A| = +\infty$ and $|B| = +\infty$.

Hanani's conjecture is equivalent to

Conjecture (H. Hanani) If A, B are infinite additive complements, then

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 1.$$

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4. Danzer's Theorem

In 1964, Danzer disproved Hanani's conjecture:

Theorem 2 *(Danzer, 1964) There exist infinite additive complements A, B such that*

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1.$$

Ref: L. Danzer, Über eine Frage von G. Hanani aus der additiven Zahlentheorie, J. Reine Angew. Math. 214/215 (1964), 392-394.

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5. Danzer's conjecture

In 1964, Danzer (with Erdős) also posed the following conjecture:

Conjecture 1 *(Danzer-Erdős Conjecture, 1964) If A, B are infinite additive complements such that*

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1,$$

then

$$A(x)B(x) - x \to +\infty \text{ as } x \to +\infty.$$

Ref: L. Danzer, Über eine Frage von G. Hanani aus der additiven Zahlentheorie, J. Reine Angew. Math. 214/215 (1964), 392-394.

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6. Sárközy-Szemerédi Theorem

In 1994, Sárközy and Szemerédi confirmed Danzer-Erdős conjecture. This result was announced in the book "Sequences" (1966) but published in 1994.

Theorem 3 *(Sárközy-Szemerédi Theorem, 1994) If A, B are infinite additive complements such that*

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1,$$

then

$$A(x)B(x) - x \to +\infty \text{ as } x \to +\infty.$$

Ref: A. Sárközy, E. Szemerédi, On a problem in additive number theory, Acta Math. Hungar. 64 (1994), 237-245.

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7. Two problems on additive complements

In this talk, we introduce our works on additive complements. We mainly concern the following two problems:

Problem 1 *Is it true that*

$$A(x)B(x) - x \to +\infty \text{ as } x \to +\infty$$

for any infinite additive complements A, B*?*

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Problem 2 *For any infinite additive complements A, B with*

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1,$$

is there an explicit function $h_{A,B}(x)$ with

$$h_{A,B}(x) \to +\infty \text{ as } x \to +\infty$$

such that

$$A(x)B(x) - x \geq h_{A,B}(x)$$

for all sufficiently large x?

In the book "Sequences" (1966), it is commented that "the precise rate of growth of $A(n)B(n) - n$ as $n \to \infty$ has yet to be settled."

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8. A generalization of the Sárközy-Szemerédi theorem

Related to the first problem, we have the following two related results.

In 2010, we proved the following result:

Theorem 4 *(Fang-C., 2010) For infinite additive complements A, B, if*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} < \frac{5}{4} \quad \textit{or} \quad \limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 2,$$

then

$$A(x)B(x) - x \to +\infty \textit{ as } x \to +\infty.$$

Ref: Jin-Hui Fang, Yong-Gao Chen, On additive complements, Proc. Amer. Math. Soc. 138 (2010), 1923-1927.

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In 2014, the upper bound $5/4$ has been improved by us to $3-\sqrt{3}$.

Theorem 5 *(Fang-C., 2014) For infinite additive complements A, B, if*

$$\limsup_{x\to\infty}\frac{A(x)B(x)}{x}<3-\sqrt{3},$$

then

$$A(x)B(x)-x\to+\infty \text{ as } x\to+\infty.$$

Ref: Jin-Hui Fang, Yong-Gao Chen, On additive complements.III, J. Number Theory 141 (2014), 83-91.

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In 2011, we proved the following result:

Theorem 6 *(C.-Fang, 2011) For any integer a with $a \geq 2$, there exist infinite additive complements A and B such that*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} = \frac{2a+2}{a+2}$$

and $A(x)B(x) - x = 1$ for infinitely many positive integers x.

Ref: Yong-Gao Chen, Jin-Hui Fang, On additive complements.II, Proc. Amer. Math. Soc. 139 (2011), 881-883.

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By taking $a = 2$, we have the following corollary:

Corollary 1 *(C.-Fang, 2011) There exist infinite additive complements A and B such that*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} = \frac{3}{2}$$

and $A(x)B(x) - x = 1$ for infinitely many positive integers x.

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Letting $a \to +\infty$, we have the following corollary:

Corollary 2 *(C.-Fang, 2011) For any $\varepsilon > 0$, there exist infinite additive complements A and B such that*

$$2 - \varepsilon < \limsup_{x \to \infty} \frac{A(x)B(x)}{x} < 2$$

and $A(x)B(x) - x = 1$ for infinitely many positive integers x.

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I pose a conjecture here:

Conjecture 2 *For infinite additive complements A, B, if*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} < \frac{3}{2},$$

then

$$A(x)B(x) - x \to +\infty \text{ as } x \to +\infty.$$

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9. The Sárközy - Szemerédi conjecture

In 1994, Sárközy and Szemerédi also posed the following conjecture.

Conjecture 3 *(Sárközy - Szemerédi Conjecture, 1994) There exist infinite additive complements A, B with*

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1,$$

such that

$$A(x)B(x) - x = O(\min\{A(x), B(x)\}).$$

Ref: A. Sárközy, E. Szemerédi, On a problem in additive number theory, Acta Math. Hungar. 64 (1994), 237-245.

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10. Disproof of the Sárközy - Szemerédi conjecture

Recently, we disprove this conjecture. The following stronger result is proved.

Theorem 7 *(C.-Fang, 2015) For infinite additive complements A, B, if*

$$\lim_{x\to\infty}\frac{A(x)B(x)}{x}=1,$$

then, for any given $M>1$, we have

$$A(x)B(x)-x\geq(\min\{A(x),B(x)\})^M$$

for all sufficiently large integers x.

Ref: Yong-Gao Chen, Jin-Hui Fang, On a conjecture of Sárközy and Szemerédi, Acta Arith. 169 (2015), 47–58.

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In this talk, for simplicity, we assume that A and B are strictly increasing sequences. Without loss of generality, we may assume that M is a positive integer with $M > 1$.

By Narkiewicz's theorem, we may assume that

$$\lim_{x\to+\infty} \frac{A(2x)}{A(x)} = 1.$$

Since

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1,$$

it follows that

$$\lim_{x\to+\infty} \frac{B(2x)}{B(x)} = \lim_{x\to+\infty} \frac{B(2x)A(2x)}{2x} \frac{2x}{A(x)B(x)} \frac{A(x)}{A(2x)} = 2.$$

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By

$$\lim_{x\to+\infty}\frac{A(2x)}{A(x)}=1,\quad \lim_{x\to+\infty}\frac{B(2x)}{B(x)}=2,$$

we have

$$A(x)<x^{1/(4M)}<x^{1/4},\quad B(x)>x^{(4M-1)/(4M)}>x^{3/4}$$

for all sufficiently large x. Then

$$\min\{A(x),B(x)\}=A(x)$$

for all sufficiently large x. It is clear that

$$A(x)^M<x^{1/4}.$$

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The theorem says that $A(x)B(x) - x \geq A(x)^M$ for all sufficiently large positive integers x.

Suppose that the theorem is false. Then

$$A(x)B(x) - x < A(x)^M$$

for infinitely positive integers x.

Let $x_1 < x_2 < \cdots$ be all positive integers with

$$A(x_k)B(x_k) - x_k < A(x_k)^M.$$

We will derive a contradiction.

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Let w_k be the least integer with

$$B(x_k) - B(x_k - w_k) = 2A(x_k)^M.$$

We can prove that

$$w_k = o(x_k), \quad w_k \to \infty \text{ as } k \to \infty$$

and $A(x_k) = A(w_k)$ for all sufficiently large integers k.

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Let $f_x(n)$ be the number of solutions of

$$a+b=n,\ a\in A,\ a\le x,\ b\in B,\ b\le x.$$

First we give an upper bound of

$$\sum_{f_{x_k}(n)\ge 1}(f_{x_k}(n)-1).$$

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Since

$$\sum_{n=0}^{n_0} f_{x_k}(n) + \sum_{n=n_0+1}^{x_k} (f_{x_k}(n) - 1) + \sum_{n=x_k+1}^{2x_k} f_{x_k}(n)$$
$$= \sum_{n=0}^{2x_k} f_{x_k}(n) - x_k + n_0 = A(x_k)B(x_k) - x_k + n_0,$$

it follows from $A(x_k) = A(w_k)$ that

$$\begin{aligned} \sum_{f_{x_k}(n)\geq 1} (f_{x_k}(n) - 1) &\leq A(x_k)B(x_k) - x_k + n_0 \\ &< A(x_k)^M + n_0 \\ &= A(w_k)^M + n_0 \\ &\leq w_k^{1/4} + n_0. \end{aligned}$$

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Next we give a lower bound of

$$\sum_{f_{x_k}(n)\geq 1} (f_{x_k}(n) - 1).$$

Let

$$g(n) = \sum_{\substack{(b,a)\in D \\ b-a=n}} 1,$$

where

$$D = \{(b,a) \in B \times A : a + w_k < b \leq x_k - w_k\}.$$

We can prove that

$$\left(\sum_{f_{x_k}(n)\geq 1} (f_{x_k}(n) - 1) \right)^2 \geq \sum_{g(n)\geq 1} (g(n) - 1).$$

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Noting that $w_k < b - a \leq x_k - w_k$ for all $(b, a) \in D$, we have

$$\sum_{g(n)\geq 1} 1 \leq \sum_{w_k < n \leq x_k - w_k} 1 = x_k - 2w_k.$$

It follows that

$$\begin{aligned} \sum_{g(n)\geq 1} (g(n) - 1) &= |D| - \sum_{g(n)\geq 1} 1 \qquad (1) \\ &\geq |D| - (x_k - 2w_k). \end{aligned}$$

So we need a lower bound of $|D|$.

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Recall that

$$D = \{(b, a) \in B \times A : a + w_k < b \le x_k - w_k\}.$$

Define D_1 and D_2 as follows:

$$D_1 = \{(b, a) \in B \times A : 2w_k < b \le x_k - w_k, b - a > w_k\},$$

$$D_2 = \{(b, a) \in B \times A : \frac{3}{2}w_k < b \le 2w_k, b - a > w_k\}.$$

Then

$$D_1 \cap D_2 = \emptyset, \quad D_1 \cup D_2 \subset D.$$

Hence

$$|D| \ge |D_1| + |D_2|.$$

In the following, we will give a lower bound of $|D|$ by giving lower bounds of $|D_1|$ and $|D_2|$.

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We can prove that

$$|D_1| \geq x_k - 2w_k + o(w_k), \quad |D_2| \geq \frac{1}{2}w_k + o(w_k).$$

Thus

$$|D| \;\geq\; |D_1| + |D_2| \geq x_k - 2w_k + \frac{1}{2}w_k + o(w_k). \tag{2}$$

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It follows from (1) and (2) that

$$\begin{aligned}\sum_{g(n)\ge 1}(g(n)-1) &= |D| - \sum_{g(n)\ge 1} 1\\ &\ge x_k - 2w_k + \frac{1}{2}w_k + o(w_k) - (x_k - 2w_k)\\ &= \frac{1}{2}w_k + o(w_k).\end{aligned}$$

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Thus

$$\sum_{f_{x_k}(n)\geq 1} (f_{x_k}(n)-1) \geq \sqrt{\sum_{g(n)\geq 1}(g(n)-1)} \geq \frac{\sqrt{2}}{2}\sqrt{w_k}(1+o(1)).$$

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Recall that

$$\sum_{f_{x_k}(n)\geq 1} (f_{x_k}(n)-1) \geq \frac{\sqrt{2}}{2}\sqrt{w_k}(1+o(1))$$

and

$$\sum_{f_{x_k}(n)\geq 1} (f_{x_k}(n)-1) \leq w_k^{1/4}+n_0,$$

we have

$$\frac{\sqrt{2}}{2}\sqrt{w_k}(1+o(1)) < w_k^{1/4}+n_0$$

for all sufficiently large integers k, a contradiction.

This completes the theorem.

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11. Ruzsa's theorem

Ruzsa's Theorem. *For any given function $\omega(x)$ such that*

$$\omega(x) \to \infty \quad \textit{as} \quad x \to \infty,$$

there exist additive complements A, B such that

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1$$

and $A(x)B(x) - x < \omega(x)$ for all sufficiently large x.

Ref: Imre Z. Ruzsa, *Exact additive complements,* arXiv: 1510.00812v1 [math.NT].

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12. Finite additive complements

Two sequences A, B of non-negative integers are called *finite additive complements*, if two sequences A, B are additive complements with $|A| < +\infty$ or $|B| < +\infty$.

For example, two sequences

$$A = \{3k : k = 0, 1, 2, \ldots\}, \quad B = \{0, 1, 2\}$$

are finite additive complements.

The situation is very different from the infinite case.

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Theorem 8 *(Fang-C., 2013) For any finite additive complements A, B of non-negative integers with*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 1,$$

we have

$$A(x)B(x) - x \to +\infty \quad \textit{as} \quad x \to +\infty.$$

Ref: Jin-Hui Fang, Yong-Gao Chen, On finite additive complements, Discrete Math. 313 (2013), 595-598.

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Recently， S.Z. Kiss, E. Rozgonyi, C. Sándor made some progresses on finite additive complements, see

S.Z. Kiss, E. Rozgonyi, C. Sándor, On additive complement of a finite set, J. Number Theory 136 (2014) 195–203.

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Danzer's conjecture
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A generalization of the . . .
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Thank you!

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