Rokhlin's lemma, a generalization, and combinatorial applications

Pablo Candela

joint work with Artur Avila

Combinatorial and Additive Number Theory 2016

Graz, January 2016

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1 x_1 = c_2 x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1 x_1 = c_2 x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1 x_1 = c_2 x_2$, equivalently if $(c_1 A) \cap (c_2 A) = \emptyset$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$,

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$, and $\mathcal{O}(c) = \{c^j : j \in [k]\} \leq \mathbb{Z}_p^\times$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$, and $\mathcal{O}(c) = \{c^j : j \in [k]\} \leq \mathbb{Z}_p^\times$.
Let $b_1, \dots, b_d \in \mathbb{Z}_p^\times$ such that $\mathbb{Z}_p^\times = \bigsqcup_{i\in[d]} b_i\, \mathcal{O}(c)$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$, and $\mathcal{O}(c) = \{c^j : j \in [k]\} \leq \mathbb{Z}_p^\times$.
Let $b_1, \ldots, b_d \in \mathbb{Z}_p^\times$ such that $\mathbb{Z}_p^\times = \bigsqcup_{i\in[d]} b_i \, \mathcal{O}(c)$.

Let $B = \{b_1, \ldots, b_d\}$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1 x_1 = c_2 x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1 x_1 = c_2 x_2$, equivalently if $(c_1 A) \cap (c_2 A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$, and $\mathcal{O}(c) = \{c^j : j \in [k]\} \leq \mathbb{Z}_p^\times$.
Let $b_1, \ldots, b_d \in \mathbb{Z}_p^\times$ such that $\mathbb{Z}_p^\times = \bigsqcup_{i \in [d]} b_i \, \mathcal{O}(c)$.

Let $B = \{b_1, \ldots, b_d\}$. Let $A = \bigsqcup_{j=0}^{\lfloor k/2 \rfloor - 1} c^{2j} B$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1 x_1 = c_2 x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1 x_1 = c_2 x_2$, equivalently if $(c_1 A) \cap (c_2 A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$, and $\mathcal{O}(c) = \{c^j : j \in [k]\} \leq \mathbb{Z}_p^\times$.
Let $b_1, \ldots, b_d \in \mathbb{Z}_p^\times$ such that $\mathbb{Z}_p^\times = \bigsqcup_{i \in [d]} b_i \, \mathcal{O}(c)$.

Let $B = \{b_1, \ldots, b_d\}$. Let $A = \bigsqcup_{j=0}^{\lfloor k/2 \rfloor - 1} c^{2j} B$. We have $A \cap (c\, A) = \emptyset$, so A is $(1, c)$-free.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1 x_1 = c_2 x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1 x_1 = c_2 x_2$, equivalently if $(c_1 A) \cap (c_2 A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$, and $\mathcal{O}(c) = \{c^j : j \in [k]\} \leq \mathbb{Z}_p^\times$.
Let $b_1, \ldots, b_d \in \mathbb{Z}_p^\times$ such that $\mathbb{Z}_p^\times = \bigsqcup_{i\in[d]} b_i\, \mathcal{O}(c)$.

Let $B = \{b_1, \ldots, b_d\}$. Let $A = \bigsqcup_{j=0}^{\lfloor k/2 \rfloor - 1} c^{2j}\, B$. We have $A \cap (c\, A) = \emptyset$, so A is $(1, c)$-free. Moreover, since $k \gg \log p$, we have $|A| \sim p/2$.

A question in additive combinatorics

Let G be a compact abelian group, with Haar probability μ.
Consider a linear equation $c_1x_1 = c_2x_2$ with (non-zero) coefficients $c_i \in \mathbb{Z}$.
We say a set $A \subset G$ is (c_1, c_2)-**free** if there is no $x = (x_1, x_2) \in A^2$ such that $c_1x_1 = c_2x_2$, equivalently if $(c_1A) \cap (c_2A) = \emptyset$. Consider

$$d_{(c_1,c_2)}(G) := \sup\{\mu(A) : A \subset G \text{ Borel and } (c_1, c_2)\text{-free}\}.$$

Discrete setting: focus on $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p a large prime.
Continuous setting: focus on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let us find $d_{(1,c)}(\mathbb{Z}_p)$ with $|c| > 1$.
Let $k = \min\{j \in \mathbb{N} : c^j = 1 \bmod p\}$, and $\mathcal{O}(c) = \{c^j : j \in [k]\} \leq \mathbb{Z}_p^\times$.
Let $b_1, \dots, b_d \in \mathbb{Z}_p^\times$ such that $\mathbb{Z}_p^\times = \bigsqcup_{i\in[d]} b_i\,\mathcal{O}(c)$.

Let $B = \{b_1, \dots, b_d\}$. Let $A = \bigsqcup_{j=0}^{\lfloor k/2\rfloor-1} c^{2j}\,B$. We have $A \cap (c\,A) = \emptyset$, so A is $(1, c)$-free. Moreover, since $k \gg \log p$, we have $|A| \sim p/2$.

$$\text{Hence } d_{(1,c)}(\mathbb{Z}_p) = 1/2 + o(1)_{p\to\infty}.$$

Rokhlin's lemma

We have used $B = \{b_1, \ldots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$.

Rokhlin's lemma

We have used $B = \{b_1, \ldots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Rokhlin's lemma

We have used $B = \{b_1, \ldots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Let (X, μ) be a standard probability space. An **endomorphism** on X is a measurable map $T : X \to X$ such that $\mu(T^{-1}B) = \mu(B)$ for every measurable $B \subset X$.

Rokhlin's lemma

We have used $B = \{b_1, \ldots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Let (X, μ) be a standard probability space. An **endomorphism** on X is a measurable map $T : X \to X$ such that $\mu(T^{-1}B) = \mu(B)$ for every measurable $B \subset X$. T is **aperiodic** if $\forall\, n \in \mathbb{N}$, $\mu(\{x \in X : T^n x = x\}) = 0$.

Rokhlin's lemma

We have used $B = \{b_1, \ldots, b_d\}$ as a "cross-section" for $T : x \mapsto c\, x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Let (X, μ) be a standard probability space. An **endomorphism** on X is a measurable map $T : X \to X$ such that $\mu(T^{-1}B) = \mu(B)$ for every measurable $B \subset X$. T is **aperiodic** if $\forall\, n \in \mathbb{N}$, $\mu(\{x \in X : T^n x = x\}) = 0$.

An *n*-**tower** for T is a sequence $B, T^{-1}B, ..., T^{-(n-1)}B$ of pairwise disjoint preimages of some measurable set $B \subset X$.

Rokhlin's lemma

We have used $B = \{b_1, \ldots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Let (X, μ) be a standard probability space. An **endomorphism** on X is a measurable map $T : X \to X$ such that $\mu(T^{-1}B) = \mu(B)$ for every measurable $B \subset X$. T is **aperiodic** if $\forall\, n \in \mathbb{N}$, $\mu(\{x \in X : T^n x = x\}) = 0$.

An *n*-**tower** for T is a sequence $B, T^{-1}B, ..., T^{-(n-1)}B$ of pairwise disjoint preimages of some measurable set $B \subset X$. The **measure** of such a tower is simply $\mu\big(\bigsqcup_{j=0}^{n-1} T^{-j}B\big)$.

Rokhlin's lemma

We have used $B = \{b_1, \dots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Let (X, μ) be a standard probability space. An **endomorphism** on X is a measurable map $T : X \to X$ such that $\mu(T^{-1}B) = \mu(B)$ for every measurable $B \subset X$. T is **aperiodic** if $\forall\, n \in \mathbb{N}$, $\mu(\{x \in X : T^n x = x\}) = 0$.

An *n*-**tower** for T is a sequence $B, T^{-1}B, \dots, T^{-(n-1)}B$ of pairwise disjoint preimages of some measurable set $B \subset X$. The **measure** of such a tower is simply $\mu\big(\bigsqcup_{j=0}^{n-1} T^{-j}B\big)$.

Lemma (Rokhlin, 1948)

Let $\epsilon > 0$ and let n be a positive integer. Then for every aperiodic automorphism T on a standard probability space, there exists an n-tower for T of measure at least $1 - \epsilon$.

Rokhlin's lemma

We have used $B = \{b_1, \ldots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Let (X, μ) be a standard probability space. An **endomorphism** on X is a measurable map $T : X \to X$ such that $\mu(T^{-1}B) = \mu(B)$ for every measurable $B \subset X$. T is **aperiodic** if $\forall\, n \in \mathbb{N}$, $\mu(\{x \in X : T^n x = x\}) = 0$.

An *n*-**tower** for T is a sequence $B, T^{-1}B, \ldots, T^{-(n-1)}B$ of pairwise disjoint preimages of some measurable set $B \subset X$. The **measure** of such a tower is simply $\mu\big(\bigsqcup_{j=0}^{n-1} T^{-j}B\big)$.

Lemma (Rokhlin, 1948)

Let $\epsilon > 0$ and let n be a positive integer. Then for every aperiodic automorphism T on a standard probability space, there exists an n-tower for T of measure at least $1 - \epsilon$.

Many applications in ergodic theory (mixing, entropy, constructions...).

Rokhlin's lemma

We have used $B = \{b_1, \dots, b_d\}$ as a "cross-section" for $T : x \mapsto c\,x$, i.e. $\mathbb{Z}_p^\times = \bigsqcup_{j=0}^{k-1} c^j B$. We want to do something similar for more general maps.

Let (X, μ) be a standard probability space. An **endomorphism** on X is a measurable map $T : X \to X$ such that $\mu(T^{-1}B) = \mu(B)$ for every measurable $B \subset X$. T is **aperiodic** if $\forall\, n \in \mathbb{N}$, $\mu(\{x \in X : T^n x = x\}) = 0$.

An n-**tower** for T is a sequence $B, T^{-1}B, \dots, T^{-(n-1)}B$ of pairwise disjoint preimages of some measurable set $B \subset X$. The **measure** of such a tower is simply $\mu\big(\bigsqcup_{j=0}^{n-1} T^{-j}B\big)$.

Lemma (Rokhlin, 1948)

Let $\epsilon > 0$ and let n be a positive integer. Then for every aperiodic automorphism T on a standard probability space, there exists an n-tower for T of measure at least $1 - \epsilon$.

Many applications in ergodic theory (mixing, entropy, constructions...).

Many generalizations, in the invertible case, to other group actions than $\mathbb{Z}$.

Towers for commuting endomorphisms

For problems such as determining $d_{(c_1,c_2)}(\mathbb{T})$ with $|c_i| > 1$, we must handle **several non-invertible** endomorphisms, i.e. the maps $\mathbb{T} \to \mathbb{T}$, $x \mapsto c_i\, x$.

Towers for commuting endomorphisms

For problems such as determining $d_{(c_1,c_2)}(\mathbb{T})$ with $|c_i| > 1$, we must handle **several non-invertible** endomorphisms, i.e. the maps $\mathbb{T} \to \mathbb{T}$, $x \mapsto c_i\, x$.

Consider a **measure-preserving action** f of $\mathbb{N}_0^d$ on X $\quad(\mathbb{N}_0 = \mathbb{Z}_{\geq 0})$ generated by commuting endomorphisms $T_1, \ldots, T_d$ on X,

Towers for commuting endomorphisms

For problems such as determining $d_{(c_1,c_2)}(\mathbb{T})$ with $|c_i| > 1$, we must handle **several non-invertible** endomorphisms, i.e. the maps $\mathbb{T} \to \mathbb{T}$, $x \mapsto c_i\, x$.

Consider a **measure-preserving action** f of $\mathbb{N}_0^d$ on X $\quad(\mathbb{N}_0 = \mathbb{Z}_{\geq 0})$
generated by commuting endomorphisms $T_1, \ldots, T_d$ on X,
thus for $n = \big(n(1), \ldots, n(d)\big) \in \mathbb{N}_0^d$ and $x \in X$ we have
$f(n,x) = f_n(x) := T_1^{n(1)} \circ \cdots \circ T_d^{n(d)}(x)$. $\quad$ (Note $f_{m+n}(x) = f_m \circ f_n(x)$.)

Towers for commuting endomorphisms

For problems such as determining $d_{(c_1,c_2)}(\mathbb{T})$ with $|c_i| > 1$, we must handle **several non-invertible** endomorphisms, i.e. the maps $\mathbb{T} \to \mathbb{T}$, $x \mapsto c_i\, x$.

Consider a **measure-preserving action** f of $\mathbb{N}_0^d$ on X $\quad(\mathbb{N}_0 = \mathbb{Z}_{\geq 0})$
generated by commuting endomorphisms $T_1, \dots, T_d$ on X,
thus for $n = \big(n(1), \dots, n(d)\big) \in \mathbb{N}_0^d$ and $x \in X$ we have
$f(n, x) = f_n(x) := T_1^{n(1)} \circ \cdots \circ T_d^{n(d)}(x)$. $\quad$ (Note $f_{m+n}(x) = f_m \circ f_n(x)$.)

Call f a **free action** if $\forall\, k \neq \ell$ in $\mathbb{N}_0^d$, $\quad \mu(\{x \in X : f_k(x) = f_\ell(x)\}) = 0$.

Towers for commuting endomorphisms

For problems such as determining $d_{(c_1,c_2)}(\mathbb{T})$ with $|c_i| > 1$, we must handle **several non-invertible** endomorphisms, i.e. the maps $\mathbb{T} \to \mathbb{T}$, $x \mapsto c_i\, x$.

Consider a **measure-preserving action** f of $\mathbb{N}_0^d$ on X $\quad (\mathbb{N}_0 = \mathbb{Z}_{\geq 0})$ generated by commuting endomorphisms $T_1, \ldots, T_d$ on X, thus for $n = \big(n(1), \ldots, n(d)\big) \in \mathbb{N}_0^d$ and $x \in X$ we have $f(n,x) = f_n(x) := T_1^{n(1)} \circ \cdots \circ T_d^{n(d)}(x)$. $\quad$ (Note $f_{m+n}(x) = f_m \circ f_n(x)$.)

Call f a **free action** if $\forall\, k \neq \ell$ in $\mathbb{N}_0^d$, $\quad \mu(\{x \in X : f_k(x) = f_\ell(x)\}) = 0$.

For $k, \ell \in \mathbb{Z}^d$, write $k < \ell$ (resp. $k \leq \ell$) if for every $j \in [d]$ we have $k(j) < \ell(j)$ (resp. $k(j) \leq \ell(j)$).

Towers for commuting endomorphisms

For problems such as determining $d_{(c_1,c_2)}(\mathbb{T})$ with $|c_i| > 1$, we must handle **several non-invertible** endomorphisms, i.e. the maps $\mathbb{T} \to \mathbb{T}$, $x \mapsto c_i\, x$.

Consider a **measure-preserving action** f of $\mathbb{N}_0^d$ on X $\quad (\mathbb{N}_0 = \mathbb{Z}_{\geq 0})$ generated by commuting endomorphisms $T_1, \ldots, T_d$ on X, thus for $n = \big(n(1), \ldots, n(d)\big) \in \mathbb{N}_0^d$ and $x \in X$ we have
$f(n,x) = f_n(x) := T_1^{n(1)} \circ \cdots \circ T_d^{n(d)}(x)$. $\quad$ (Note $f_{m+n}(x) = f_m \circ f_n(x)$.)

Call f a **free action** if $\forall\, k \neq \ell$ in $\mathbb{N}_0^d$, $\quad \mu(\{x \in X : f_k(x) = f_\ell(x)\}) = 0$.

For $k, \ell \in \mathbb{Z}^d$, write $k < \ell$ (resp. $k \leq \ell$) if for every $j \in [d]$ we have $k(j) < \ell(j)$ (resp. $k(j) \leq \ell(j)$).

For $N \in \mathbb{N}^d$ and $B \subset X$, write $B_{(N)} = (f_k^{-1}(B))_{0 \leq k < N}$. If the preimages are pairwise disjoint, we say that $B_{(N)}$ is an N-**tower for** f with **base** B.

Towers for commuting endomorphisms

For problems such as determining $d_{(c_1,c_2)}(\mathbb{T})$ with $|c_i| > 1$, we must handle **several non-invertible** endomorphisms, i.e. the maps $\mathbb{T} \to \mathbb{T}$, $x \mapsto c_i\, x$.

Consider a **measure-preserving action** f of $\mathbb{N}_0^d$ on X $\quad(\mathbb{N}_0 = \mathbb{Z}_{\geq 0})$ generated by commuting endomorphisms $T_1, \ldots, T_d$ on X, thus for $n = \big(n(1), \ldots, n(d)\big) \in \mathbb{N}_0^d$ and $x \in X$ we have $f(n,x) = f_n(x) := T_1^{n(1)} \circ \cdots \circ T_d^{n(d)}(x)$. $\quad$ (Note $f_{m+n}(x) = f_m \circ f_n(x)$.)

Call f a **free action** if $\forall\, k \neq \ell$ in $\mathbb{N}_0^d$, $\quad \mu(\{x \in X : f_k(x) = f_\ell(x)\}) = 0$.

For $k, \ell \in \mathbb{Z}^d$, write $k < \ell$ (resp. $k \leq \ell$) if for every $j \in [d]$ we have $k(j) < \ell(j)$ (resp. $k(j) \leq \ell(j)$).

For $N \in \mathbb{N}^d$ and $B \subset X$, write $B_{(N)} = (f_k^{-1}(B))_{0 \leq k < N}$. If the preimages are pairwise disjoint, we say that $B_{(N)}$ is an N-**tower for** f with **base** B.

Theorem (Towers for $\mathbb{N}_0^d$ actions – Avila & C, 2015)

Let $\epsilon > 0$ and let $N \in \mathbb{N}^d$. Then for every free measure-preserving action f of $\mathbb{N}_0^d$ on a standard probability space, there exists an N-tower for f of measure at least $1 - \epsilon$.

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$.

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$. Then $f_n(x) = c_1^{n(1)} c_2^{n(2)} x$ defines a free measure-preserving action of $\mathbb{N}_0^2$ on $\mathbb{T}$.

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$. Then $f_n(x) = c_1^{n(1)} c_2^{n(2)} x$ defines a free measure-preserving action of $\mathbb{N}_0^2$ on $\mathbb{T}$. Fix any $\delta > 0$, and apply the towers theorem with $\epsilon = \delta/2$ and $N = (t, 2)$, with $t > 1/\delta$.

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$. Then $f_n(x) = c_1^{n(1)} c_2^{n(2)} x$ defines a free measure-preserving action of $\mathbb{N}_0^2$ on $\mathbb{T}$. Fix any $\delta > 0$, and apply the towers theorem with $\epsilon = \delta/2$ and $N = (t, 2)$, with $t > 1/\delta$. Let B be the resulting base of an N-tower, with $\mu(B_{(N)}) \geq 1 - \delta/2$.

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$. Then $f_n(x) = c_1^{n(1)} c_2^{n(2)} x$ defines a free measure-preserving action of $\mathbb{N}_0^2$ on $\mathbb{T}$. Fix any $\delta > 0$, and apply the towers theorem with $\epsilon = \delta/2$ and $N = (t, 2)$, with $t > 1/\delta$. Let B be the resulting base of an N-tower, with $\mu(B_{(N)}) \geq 1 - \delta/2$. Now let

$$A = \bigsqcup_{j=1}^{t-1} (c_1^j\ c_2)^{-1}(B).$$

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$. Then $f_n(x) = c_1^{n(1)} c_2^{n(2)} x$ defines a free measure-preserving action of $\mathbb{N}_0^2$ on $\mathbb{T}$. Fix any $\delta > 0$, and apply the towers theorem with $\epsilon = \delta/2$ and $N = (t, 2)$, with $t > 1/\delta$. Let B be the resulting base of an N-tower, with $\mu(B_{(N)}) \geq 1 - \delta/2$. Now let

$$A = \bigsqcup_{j=1}^{t-1} (c_1^j\, c_2)^{-1}(B).$$

We have $c_1 A \cap c_2 A = \emptyset$.

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$. Then $f_n(x) = c_1^{n(1)} c_2^{n(2)} x$ defines a free measure-preserving action of $\mathbb{N}_0^2$ on $\mathbb{T}$. Fix any $\delta > 0$, and apply the towers theorem with $\epsilon = \delta/2$ and $N = (t, 2)$, with $t > 1/\delta$. Let B be the resulting base of an N-tower, with $\mu(B_{(N)}) \geq 1 - \delta/2$. Now let

$$A = \bigsqcup_{j=1}^{t-1} (c_1^j\, c_2)^{-1}(B).$$

We have $c_1 A \cap c_2 A = \emptyset$. Also, $\mu(A) \geq (1-\delta/2)/2 - 1/(2t) \geq 1/2 - \delta$.

Towers for $\mathbb{N}_0^d$ actions – Applications

Fiz-Pontiveros had observed that an earlier special case of the theorem for a single map $x \mapsto c\,x$ implied that $d_{(1,c)}(\mathbb{T}) = 1/2$. (2012)
Applying the theorem for two such maps easily yields the following result.

Proposition

For every distinct integers c_1, c_2, we have $d_{(c_1,c_2)}(\mathbb{T}) = 1/2$.

Proof.

W.L.O.G. c_1, c_2 are coprime and $|c_i| > 1$. Then $f_n(x) = c_1^{n(1)} c_2^{n(2)} x$ defines a free measure-preserving action of $\mathbb{N}_0^2$ on $\mathbb{T}$. Fix any $\delta > 0$, and apply the towers theorem with $\epsilon = \delta/2$ and $N = (t, 2)$, with $t > 1/\delta$. Let B be the resulting base of an N-tower, with $\mu(B_{(N)}) \geq 1 - \delta/2$. Now let

$$A = \bigsqcup_{j=1}^{t-1} (c_1^j\, c_2)^{-1}(B).$$

We have $c_1 A \cap c_2 A = \emptyset$. Also, $\mu(A) \geq (1-\delta/2)/2 - 1/(2t) \geq 1/2 - \delta$.

Hence $d_{(c_1,c_2)}(\mathbb{T}) \geq 1/2 - \delta$. □

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.
The problem: determine $d_{\mathcal{F}}(\mathbb{T}) = \sup\{\mu(A) : A \subset \mathbb{T} \text{ is } \mathcal{F}\text{-free}\}$.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.
The problem: determine $d_{\mathcal{F}}(\mathbb{T}) = \sup\{\mu(A) : A \subset \mathbb{T} \text{ is } \mathcal{F}\text{-free}\}$.

Using the towers theorem, we show that $d_{\mathcal{F}}(\mathbb{T})$ is equal to a quantity d_Γ defined as follows.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0,..,t-1\} \times \{0,1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.
The problem: determine $d_{\mathcal{F}}(\mathbb{T}) = \sup\{\mu(A) : A \subset \mathbb{T} \text{ is } \mathcal{F}\text{-free}\}$.

Using the towers theorem, we show that $d_{\mathcal{F}}(\mathbb{T})$ is equal to a quantity d_Γ defined as follows. For $N \in \mathbb{N}^d$, let $\pi(N)$ denote the product $N(1) \cdots N(d)$.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.
The problem: determine $d_{\mathcal{F}}(\mathbb{T}) = \sup\{\mu(A) : A \subset \mathbb{T} \text{ is } \mathcal{F}\text{-free}\}$.

Using the towers theorem, we show that $d_{\mathcal{F}}(\mathbb{T})$ is equal to a quantity d_Γ defined as follows. For $N \in \mathbb{N}^d$, let $\pi(N)$ denote the product $N(1) \cdots N(d)$. We say a set $S \subset \mathbb{Z}^d$ is Γ-**admissible** if for every edge $i\,j$ of Γ we have $(S - e_i) \cap (S - e_j) = \emptyset$.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.
The problem: determine $d_{\mathcal{F}}(\mathbb{T}) = \sup\{\mu(A) : A \subset \mathbb{T} \text{ is } \mathcal{F}\text{-free}\}$.

Using the towers theorem, we show that $d_{\mathcal{F}}(\mathbb{T})$ is equal to a quantity d_Γ defined as follows. For $N \in \mathbb{N}^d$, let $\pi(N)$ denote the product $N(1) \cdots N(d)$. We say a set $S \subset \mathbb{Z}^d$ is Γ-**admissible** if for every edge $i\,j$ of Γ we have $(S - e_i) \cap (S - e_j) = \emptyset$. For each $N \in \mathbb{N}^d$, let

$$M_\Gamma(N) = \max\{\, |S|/\pi(N) \ : \ S \subset \textstyle\prod_{i=1}^{d} [0, N(i)) \ \text{ is } \Gamma\text{-admissible}\}.$$

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.
The problem: determine $d_{\mathcal{F}}(\mathbb{T}) = \sup\{\mu(A) : A \subset \mathbb{T} \text{ is } \mathcal{F}\text{-free}\}$.

Using the towers theorem, we show that $d_{\mathcal{F}}(\mathbb{T})$ is equal to a quantity d_Γ defined as follows. For $N \in \mathbb{N}^d$, let $\pi(N)$ denote the product $N(1) \cdots N(d)$. We say a set $S \subset \mathbb{Z}^d$ is Γ-**admissible** if for every edge $i\,j$ of Γ we have $(S - e_i) \cap (S - e_j) = \emptyset$. For each $N \in \mathbb{N}^d$, let

$$M_\Gamma(N) = \max\{\ |S|/\pi(N)\ :\ S \subset \textstyle\prod_{i=1}^d [0, N(i))\ \text{ is } \Gamma\text{-admissible}\}.$$

Fact: for every sequence of $N_j \in \mathbb{N}^d$ with $\min_i N_j(i) \to \infty$, the sequence $M_\Gamma(N_j)$ converges to the same limit d_Γ.

In the last proof, the N-tower was used to reduce the problem of finding a large (c_1, c_2)-free set in $\mathbb{T}$ to finding a large $S \subset \{0, .., t-1\} \times \{0, 1\} \subset \mathbb{Z}^2$ such that $(S - e_1) \cap (S - e_2) = \emptyset$. We can generalize this argument.

Large subsets of $\mathbb{T}$ avoiding several 2-variable equations

Let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers. Let Γ be a graph on $[d]$, and let $\mathcal{F}$ be the family of equations $c_i x_1 = c_j x_2$ where $i\,j$ is an edge in Γ. We say that a measurable set $A \subset \mathbb{T}$ is $\mathcal{F}$-**free** if there are no solutions $(x_1, x_2) \in A^2$ to any of the equations in $\mathcal{F}$.
The problem: determine $d_{\mathcal{F}}(\mathbb{T}) = \sup\{\mu(A) : A \subset \mathbb{T} \text{ is } \mathcal{F}\text{-free}\}$.

Using the towers theorem, we show that $d_{\mathcal{F}}(\mathbb{T})$ is equal to a quantity d_Γ defined as follows. For $N \in \mathbb{N}^d$, let $\pi(N)$ denote the product $N(1) \cdots N(d)$. We say a set $S \subset \mathbb{Z}^d$ is Γ-**admissible** if for every edge $i\,j$ of Γ we have $(S - e_i) \cap (S - e_j) = \emptyset$. For each $N \in \mathbb{N}^d$, let

$$M_\Gamma(N) = \max\{\ |S|/\pi(N)\ :\ S \subset \textstyle\prod_{i=1}^d [0, N(i))\ \text{ is } \Gamma\text{-admissible}\}.$$

Fact: for every sequence of $N_j \in \mathbb{N}^d$ with $\min_i N_j(i) \to \infty$, the sequence $M_\Gamma(N_j)$ converges to the same limit d_Γ. We show that $d_{\mathcal{F}}(\mathbb{T}) = d_\Gamma$.

We give estimates for d_Γ in terms of known graph parameters. Let us end with a special case in which we can tell the exact value.

We give estimates for d_Γ in terms of known graph parameters. Let us end with a special case in which we can tell the exact value.

Proposition (Avoiding a bipartite family of equations)

Let $c_1, c_2, \ldots, c_d \in \mathbb{Z} \setminus \{0\}$ be multiplicatively independent, let Γ be a non-empty bipartite graph on $[d]$, and let $\mathcal{F}$ be the corresponding family of 2-variable equations.

We give estimates for d_Γ in terms of known graph parameters. Let us end with a special case in which we can tell the exact value.

Proposition (Avoiding a bipartite family of equations)

Let $c_1, c_2, \ldots, c_d \in \mathbb{Z} \setminus \{0\}$ be multiplicatively independent, let Γ be a non-empty bipartite graph on $[d]$, and let $\mathcal{F}$ be the corresponding family of 2-variable equations. Then $d_{\mathcal{F}}(\mathbb{T}) = 1/2$.

We give estimates for d_Γ in terms of known graph parameters. Let us end with a special case in which we can tell the exact value.

Proposition (Avoiding a bipartite family of equations)

Let $c_1, c_2, \ldots, c_d \in \mathbb{Z} \setminus \{0\}$ be multiplicatively independent, let Γ be a non-empty bipartite graph on $[d]$, and let $\mathcal{F}$ be the corresponding family of 2-variable equations. Then $d_{\mathcal{F}}(\mathbb{T}) = 1/2$.

Thank you !