On the Valuation and Analysis of Risky Debt: A Practical Approach Using Rating Migrations
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Abstract

This paper is concerned with the valuation and analysis of risky debt instruments with arbitrary interest and principal payments subject to default risk. For the valuation, we use a risk-neutral present value model with expected payments for risk-neutral investors and risk-free spot rates. The required risk-neutral default probabilities are derived from historically observable risk-averse migration matrices. Based on this debt valuation, we calculate various key figures for the analysis of risky debt from the point of view of risk-averse investors (e.g., promised and expected yields, yield spreads, Z-spreads, risk premia, risk-averse default probabilities, and risk-averse expected payments). Our approach is well-suited for practical applications, since the parameters required are easily available from observable data.

Keywords: risky debt, debt valuation, expected yield

JEL: G21, G31, G32
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A Practical Approach Using Rating Migrations

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April 6, 2020

Abstract
This paper is concerned with the valuation and analysis of risky debt instruments with arbitrary interest and principal payments subject to default risk. For the valuation, we use a risk-neutral present value model with expected payments for risk-neutral investors and risk-free spot rates. The required risk-neutral default probabilities are derived from historically observable risk-averse migration matrices. Based on this debt valuation, we calculate various key figures for the analysis of risky debt from the point of view of risk-averse investors (e.g., promised and expected yields, yield spreads, Z-spreads, risk premia, risk-averse default probabilities, and risk-averse expected payments). Our approach is well-suited for practical applications, since the parameters required are easily available from observable data.

* The authors are from the Institute of Finance, University of Graz, Austria. We are grateful for comments on earlier versions of the paper from seminar participants at the University of Graz.
This article provides a simple rating-based credit risk model for valuing risky debt. We present both a risk-neutral as well as a risk-adjusted approach to determine the fair price of a risky bond using historical rating transition matrices as a starting point. The model is useful for pricing non-callable corporate and government debt subject to default risk and can be used for various risk management purposes. The valuation formulas are simple, and the input parameters required for the model are easily estimated using observable data. Furthermore, to our knowledge, this is the first rating-based, reduced-form model to provide full valuation formulas for bond types other than zero-coupon bonds. In other words, our valuation framework can be applied to debt instruments with various kinds of interest and repayment modalities. The generality and practicality of our model should make it particularly attractive to practitioners.

There are numerous credit risk models that can be used for the valuation of risky debt. Previous models can be divided into two broad categories. The first class of models assumes that a stochastic process drives the value of the firm. In these structural models, the firm's debt is modeled as a contingent claim issued against the underlying assets of the firm. Default occurs when the firm value falls below a certain barrier. Structural models were first introduced by Merton (1974). Numerous extensions have been developed to include stochastic interest rates, varying default barriers, or different interest and repayment modalities (Longstaff and Schwartz, 1992, Briys and Varenne, 1997, Geske, 1977, and Fischer, Kampl, and Woeckl, 2020). Structural models are challenging to implement in practice since they require estimates for the value and volatility of the firm's assets, which are often not observable.

The second class of models evades this problem. Reduced-form models use ratings and corresponding default probabilities as a starting point to determine credit risk. They model the default event by an exogenous process, which generally does not depend explicitly on the firm’s underlying assets. In consequence, reduced-form models do not require estimates for the parameters of the firm’s underlying assets. This drasticallyfacilitates the models’ applicability in practice.

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2 We refer the reader to Niklis, Doumpos, and Zopounidis (2018) and Hao, Alam, and Carling (2010) for recent literature overviews on credit risk modeling.
The first reduced-form models used probabilities of default and recovery rates to evaluate credit risk (Fons, 1994, and Jarrow and Turnbull, 1995). Many of the earlier approaches model the default process by a doubly stochastic Poisson process with an intensity parameter $\lambda$ (Jarrow and Turnbull, 1995, Madan and Unal, 1998, and Lando, 1998). These models are, therefore, often referred to as intensity-based models. The earlier models focus explicitly on the transition to default. They do not take potential rating migrations to other rating categories into account. Jarrow, Lando, and Turnbull (1997) were the first to explicitly incorporate more detailed credit rating information into the valuation methodology. In their discrete-time model, hereinafter referred to as the JLT model, credit risk is incorporated via rating transition matrices, which are modeled using a time-homogeneous Markov chain process with default as the absorbing state. The recovery rates are assumed to be constant. Risk-neutral rating transition probabilities and probabilities of default are derived from historical, risk-averse transition matrices, and then used to value risky debt in a risk-neutral setting.

There are numerous extensions that build on the framework of the JLT model. Das and Tufano (1995) relax the assumption of a constant recovery rate and make the recovery rate in the event of default stochastic. Kijima and Komorigayashi (1998) adapt the calculation of the risk-neutral transition matrices to prevent negative transition probabilities. Recent developments include the relaxation of the assumption of time-homogeneity (Nickell, Perraudin, and Varotto, 2000, and Feng, Gourieroux, and Jasiak, 2008) as well as the extension of the model to a continuous-time setting (Fuertes and Kalotychou, 2006, Frydman and Schuermann, 2008, Kadam and Lenk, 2008). We base our model on the JLT approach because its generality and practicality make it especially attractive in practical applications. Due to its discrete-time framework and simplifying assumption regarding the recovery rates, it is easy to implement and highly intuitive. Furthermore, it can be used with all types of term structures. These advantages have motivated our choice of the JLT model as the basis for our valuation approach. However, we address a shortcoming of the model that constrains its direct applicability in practice. The JLT model and its extensions focus on the valuation of risky zero-coupon bonds. To the best of our knowledge, no model exists for the valuation of bonds with more elaborate interest and repayment structures. We provide a rating-based model that can be used for the
valuation of risky bonds with different interest and repayment modalities, making it more applicable to real-world pricing situations.

The paper is organized as follows. Section I recaps the valuation of risk-free bonds. Section II presents the model for the risk-neutral valuation of risky bonds. Section III presents the risk-adjusted valuation approach. Section IV illustrates the risk and return analysis. Section V contains a numerical example. Section VI concludes.

### I. Valuation of Risk-Free Bonds

In the simplest approach to debt valuation, it is assumed that the debt instrument under consideration is risk-free. Under the assumption of arbitrage-free and complete markets, the value of a risk-free debt instrument is determined as the present value of its promised payments to the creditor. Each promised payment is discounted using the spot rate with the same maturity as the corresponding payment. The spot rates are used as discount rates since they represent the risk-free interest rates in the market.

Let \( D_{0}^{rf} \) be the time \( t = 0 \) price of a risk-free non-terminable bond with a nominal value, \( Nom \), periodic interest payments, \( I_t \), and periodic principal repayments, \( P_t \), at \( t = 1, \ldots, T \) where \( T \) is the (residual) maturity of the bond in years. We can write this as

\[
D_{0}^{rf} = \sum_{t=1}^{T} \frac{I_t + P_t}{(1 + r_{0,t})^t}
\]

where

\[
\sum_{t=1}^{T} P_t = Nom.
\]

As can be seen from equation (1), the promised payments in the numerator are discounted using the risk-free spot rates. This is appropriate because there is no risk inherent in the payments of the bond. In reality, bonds can rarely be considered entirely risk-free. On the contrary, the riskiness of different debt instruments varies greatly depending on parameters such as issuer, seniority, or term structure. There are numerous types of risks that bonds are subject to. On the one hand, bonds are subject to market risks. These include the interest rate risk as well as the systematic spread risk. While the interest rate risk is the risk that the price of a bond is adversely
affected by fluctuations in the level of interest rates, the systematic spread risk refers to the risk of potential losses arising from changes in an interest rate or other price differential for the entire market.

On the other hand, bonds are subject to bond-specific risks, which are also referred to as unsystematic spread risks. These include the credit risk, which is the possibility of financial losses caused by changes in the credit rating or the default of the bond or its issuer, and the liquidity risk, which refers to the risk that investors might not be able to sell the debt instrument quickly and at an efficient price. Other bond-specific risks depend on the specific features of the bond under consideration. Redeemable bonds are subject to redemption date risk, for example, while foreign currency bonds are subject to currency risk.

When debt instruments are subject to risk, equation (1) needs to be adapted to account for these risks. There are two main approaches to valuing risky bonds that differ depending on whether the risk is taken into account in the payments the investor is expected to receive (i.e., risk-neutral or risk-averse expected cash flows) or in the interest rates used to discount the payments (i.e., risk-free spot rate or risk-free spot rate plus risk premium). The first approach, the risk-neutral valuation approach, is primarily used when pricing risky debt instruments. It incorporates the risk inherent in risky debt instruments in the numerator by weighting the promised payments of the bond with risk-neutral probabilities and recovery rates. The resulting expected cash flows represent a pseudo expected value or certainty equivalent. In consequence, they can be discounted using the risk-free interest rate.

The second approach is the risk-adjusted valuation approach. This approach is primarily used in practice (e.g., in project and equity/company valuation). In the risk-adjusted approach, the promised payments of the bond are weighted using the historical, or risk-averse, probabilities. The resulting risk-averse expected cash flows are then discounted using a risk-adjusted discount rate, which includes a risk premium on top of the risk-free rate. Both approaches lead to the same valuation results.

In this paper, we present a rating-based pricing approach that can be used for a wide range of risky bonds with different interest and repayment modalities. Previous rating-based reduced-form models have presented risk-neutral formulas for the valuation of risky zero-coupon bonds. To the best of our knowledge, our model is the first that can be applied to a broader range of
bonds since it accounts for different interest and repayment structures and allows the bond to default prior to maturity. As a rating-based approach, our model is well suited for practical applications. The probabilities of default must only be calculated once per rating category and not individually for each specific bond. Furthermore, unlike structural models, the approach presented in this paper does not require any firm-specific information besides the rating as well as the seniority and collateralization of the debt instrument under consideration. The model is the content of the remaining sections.

II. Risk-Neutral Valuation of Risky Bonds

First, we introduce our risk-neutral valuation approach. The discrete-time valuation model we use is based on Jarrow, Lando, and Turnbull (1997) (JLT), who present a theoretical pricing formula for risky zero-coupon bonds. We extend their valuation framework to risky bonds with more elaborate interest and repayment modalities. Additionally, unlike in the JLT model, we allow the bond to be default prior to maturity. The assumptions our model is based on follow those outlined in Jarrow, Lando, and Turnbull (1997). We assume that trading is discrete and that both risk-free and risky bonds of all maturities with different payment structures are traded in the market. Risky bonds can be grouped into different rating categories. All firms in the same rating category have the same probability of default, and the recovery rate is taken to be an exogenously given constant. The transition and default probabilities are contained in rating transition matrices. Like in the JLT model, we model the rating transition matrices using a Markov chain process. We assume that markets are complete and that no arbitrage opportunities exist. Furthermore, we assume that the bankruptcy process is uncorrelated with the risk-free spot rates. We impose this assumption to facilitate the empirical investigation. However, as Jarrow and Turnbull (1995) demonstrate, it can easily be relaxed, if necessary.

A. Valuation

Let $D_0$ be the value of a risky bond at time $t = 0$ promising to pay annual interest, $I_t$, and annual principal repayments, $P_t$, at $t = 1, \ldots, T$ where $T$ is the residual term in years. If the firm goes bankrupt, the promised payments may not be paid in full. Instead, the firm will only pay the recovery rate. The recovery rate indicates the size of the payments a creditor receives
in the event of default in % of his outstanding claims. As explained in Section I, in the risk-neutral valuation approach, the value of a risky bond is the present value of the expected cash flows of the risk-neutral investor. We can write this as

\[ D_0 = \sum_{t=1}^{T} \frac{E'_0(C_t)}{(1 + r_{0,t})^t} \]  

(2)

where \( E'_0(C_t) \) is the risk-neutral expected cash flow and \( r_{0,t} \) is the risk-free spot rate.\(^3\) Since the expected cash flows of a risk-neutral investor represent a pseudo expected value or certainty equivalent, the risk-free spot rates are used to discount the cash flows. The expected cash flows incorporate the expected risk-neutral coupon payments as well as the expected risk-neutral redemption payments such that

\[ E'_0(C_t) = \prod_{\tau=1}^{t-1} (1 - PD'_t) \cdot \left[ (1 - PD'_t) \cdot (I_t + P_t) + PD'_t \cdot RR'_t (I_t + Nom_{t-1}) \right] \]  

(3)

for \( t = 1, ..., T \). The first term in the product on the right-hand side is the cumulative pseudo survival probability of the debt instrument at the beginning of year \( t \). The term in square brackets is the conditional pseudo-expected value,

\[ E'_{t-1}(C_t|No Defaul t until t - 1) = (1 - PD'_t) \cdot (I_t + P_t) + PD'_t \cdot RR'_t (I_t + Nom_{t-1}). \]  

(4)

It represents the expected value of the cash flow at time \( t \) under the condition that no default has occurred until \( t - 1 \). Using equations (3) and (4) we can rewrite equation (2) as

\[ D_0 = \sum_{t=1}^{T} \frac{[\prod_{\tau=1}^{t-1} (1 - PD'_t)] \cdot E'_{t-1}(C_t|No Default until t - 1)}{(1 + r_{0,t})^t}. \]  

(5)

It can be seen from (3) that the expected payments of the risk-neutral investor depend on two components. The first component is the risk-neutral, or pseudo, probability of default, \( PD'_t \). This is a conditional probability of default in that it indicates the chance a default will occur at time \( t \) given that no default has occurred until \( t - 1 \). The conditional probability of default is calculated from cumulative probabilities of default using Bayes law as

\(^3\) Throughout this paper, the hyphen is used to indicate risk-neutrality.
\[ PD'_t = \text{Prob}'(\text{Default at } t|\text{No Default until } t - 1) = \frac{CPD'_{0,t} - CPD'_{0,t-1}}{1 - CPD'_{0,t-1}} \]  
where \( CPD'_{0,1} = PD'_1 \) and \( CPD'_{0,0} = 0 \). The cumulative risk-neutral probability of default, \( CDP'_{0,t} \), is the probability that default will occur at any time between 0 and \( t \). Conversely, \( 1 - CDP'_{0,t} \), is the cumulative risk-neutral survival probability until time \( t \). The third probability measure we calculate is the total probability of default, which is derived from the cumulative probabilities of default as

\[ \text{Prob}'(\text{No Default until } t - 1 \text{ and Default at } t) = CDP'_{0,t} - CDP'_{0,t-1}. \]  

As can be seen from (7), the total default probability is the probability that the debt instrument will not default until time \( t - 1 \) and that it will default at time \( t \).

The second component that influences the value of the expected cash flows is the risk-neutral, or pseudo, recovery rate, \( RR'_t \). The recovery rate indicates the size of the payments a creditor receives in the event of default in % of his outstanding claims. This fraction can depend on the seniority of the risky debt instrument, as well as the value of collateral, for instance. Recovery rates can be determined either historically or implicitly and may vary over time. Historical recovery rates are determined from past defaults and are published regularly by large credit rating agencies (see, for example, Standard & Poor’s Financial Services (2019) or Moody’s Investors Service (2011)). Risk-neutral recovery rates are calculated implicitly from market prices (see, for example, Merton, 1974, Das and Hanouna, 2009, and Schläfer and Uhrig-Homburg, 2014).\(^4\) For the risk-neutral recovery rate at time \( t \), we can write

\[ RR'_t = \frac{E'_0 (\text{Debtholder’s Share of Firm Value at } t|\text{Default at } t)}{\text{Nom}_{t-1} + I_t} \]  

Both components (i.e., probabilities of default and recovery rates) can vary for different rating classes. In rating-based approaches such as the one presented in this paper, it is, therefore, necessary to determine these parameters for each rating category individually.

Given that the relationship between cumulative and conditional survival probabilities satisfies

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\(^4\) For a detailed review of the incorporation of recovery rates in credit risk models, we refer the reader to Altman, Resti, and Sironi (2004).
\[ 1 - CPD'_{0,t} = \prod_{\tau=1}^{t} (1 - PD'_{\tau}). \]  

we can rewrite equation (5) as

\[ D_0 = \sum_{t=1}^{T} \frac{(1 - CPD_{0,t-1}') \cdot E_{t-1}' \cdot (C_t| No Default until t - 1)}{(1 + r_{0,t})^t} \]  

Under the maintained assumption of complete markets and no arbitrage opportunities, for any specific debt instrument with annual interest payments, \( I_t \), and annual principal repayments, \( P_t \), we can write the value of the risky bond as

\[ D_0 = \sum_{t=1}^{T} \frac{(1 - CPD_{0,t}') \cdot (I_t + P_t) + (1 - CPD_{0,t-1}') \cdot PD_t' \cdot RR_t' \cdot (I_t + Nom_{t-1})}{(1 + r_{0,t})^t}, \]  

or, using only cumulative probabilities of default, as

\[ D_0 = \sum_{t=1}^{T} \frac{(1 - CPD_{0,t}') \cdot (I_t + P_t) + (CPD_{0,t}' - CPD_{0,t-1}') \cdot RR_t' \cdot (I_t + Nom_{t-1})}{(1 + r_{0,t})^t}. \]

The novelty of the valuation formula in equations (11) and (12) is that they can be used to value debt instruments with different repayment modalities. Unlike many previous rating-based models, our valuation approach can be used to value a wide range of the actual bonds found in the market, since the valuation is not limited to zero-coupons bonds. Furthermore, also the determination of the risk-neutral probabilities of default and the risk-neutral recovery rates is straightforward, as we will show in the following subsection. This makes the valuation approach presented in this paper a valuable tool in practical applications.

### B. Calculation of Risk-Neutral Probabilities of Default using Zero-Coupon Bonds

In order to price risky coupon bonds using equations (11) or (12), we need to determine the rating-specific risk-neutral probabilities of default. The probabilities of default are derived from the prices of risky zero-coupon bonds with different residual maturities using a recursive bootstrapping procedure. The prices of the zero-coupon bonds can either be observed directly from the market or calculated using the observed yields or yield spreads for different rating classes. The recursive bootstrapping procedure we use in this paper is based on Jarrow, Lando,
and Turnbull (1997). Jarrow, Lando, and Turnbull (1997) derive the risk-neutral probabilities of default from their formula for the valuation of risky zero-coupon bonds which they write as

\[ D_0 = \frac{[1 - (1 - RR_T^t) \cdot CPD'_{0,t}] \cdot Nom}{(1 + r_{0,t})^t}. \]  

(13)

Rearranging equation (13) for \( CPD'_{0,t} \), the risk-neutral cumulative probability of default for a specific rating class according to the JLT model is

\[ CPD'_{0,t} = \frac{1}{1 - RR_T^t} \cdot \left[ 1 - \left( 1 + r_{0,t} \right)^t \cdot \frac{D_0}{Nom} \right]. \]  

(14)

The JLT valuation formula is based on the assumption that the zero-coupon bond can only default at maturity. We relax this assumption in order to incorporate more realistic considerations into our model. We assume that the zero-coupon bond can also default at any time before maturity. In such a case, the creditors will receive the recovery rate immediately upon default. We determine the risk-neutral probabilities of default based on equation (11). For risky zero-coupon bonds, it holds that \( I_t = 0 \) and \( P_t = 0 \). In consequence, we can write the value of a risky zero-coupon bond as

\[ D_0 = \sum_{t=1}^{T-1} \frac{(1 - CPD'_{0,t-1}) \cdot PD'_t \cdot RR'_t \cdot Nom}{(1 + r_{0,t})^t} + \frac{(1 - CPD'_{0,T-1}) \cdot [1 - (1 - RR'_T) \cdot PD'_T] \cdot Nom}{(1 + r_{0,T})^T}. \]  

(15)

We rearrange equation (15) such that

\[ PD'_t = \frac{1}{1 - RR'_t} \cdot \left\{ 1 - \frac{\left( 1 + r_{0,t} \right)^t}{1 - CPD'_{0,t-1}} \cdot \left[ \frac{D_0}{Nom} - \sum_{t=1}^{T-1} \frac{(1 - CPD'_{0,t-1}) \cdot PD'_t \cdot RR'_t}{(1 + r_{0,t})^t} \right] \right\}. \]  

(16)

The term in the curly brackets on the right-hand side of equation (16) corresponds to the expected loss at time \( t \) in % of the total claims. Equation (16) clearly illustrates that the conditional probability of default depends on the risk-neutral recovery rate, \( RR'_t \), as well as the expected loss.
Next, we show how to use the recursive procedure to bootstrap the risk-neutral conditional probabilities of default from equation (16). We start in period $t = 1$. Given that $CPD'_{0,0} = 0$, we can determine the conditional probability of default for $t = 1$ as

$$PD'_1 = \frac{1}{1 - RR'_1} \left\{ 1 - \left( 1 + r_{0,1} \right) \cdot \frac{D_0}{Nom} \right\}.$$ 

In period $t = 2$, given $PD'_1$ and $CPD'_{0,1} = PD'_1$, we get the conditional probability of default for $t = 2$ as

$$PD'_2 = \frac{1}{1 - RR'_2} \left\{ 1 - \left( 1 + r_{0,2} \right)^t \cdot \left\{ \frac{D_0}{Nom} - \frac{PD'_1 \cdot RR'_1}{(1 + r_{0,1})} \right\} \right\}.$$ 

Next, we calculate the cumulative probability of default for $t = 2$, $CPD_2$, by rearranging equation (6) such that

$$CPD'_{0,2} = PD'_2 \cdot \left( 1 - CPD'_{0,1} \right) + CPD'_{0,1}.$$ 

We can then determine the conditional probability of default for $t = 3$, $PD_3$, by inserting the parameters calculated hitherto into equation (16). The process is repeated until $t = T$ to determine the remaining conditional and cumulative risk-neutral default probabilities. The bootstrapping procedure must be carried out separately for each rating category. For example, to determine the risk-neutral probabilities for rating category A, only zero-coupon bonds with an A-rating can be used. These resulting probabilities of default can then be used to determine the fair value of a risky bond with a rating of A at $t = 0$.

### III. Risk-Adjusted Valuation of Risky Bonds

Next, we introduce our risk-adjusted valuation approach. As mentioned in Section I, the risk-neutral valuation of risky bonds is carried out based on the expected cash flows of the risk-neutral investor, $E'_0(C_t)$. For this, the risk-neutral probabilities of default and the risk-neutral recovery rates, which are used to weight the promised cash flows of the debt instruments, need to be determined. The risk-adjusted valuation of risky bonds, on the other hand, uses the expected cash flows of the risk-averse investor, $E_0(C_t)$, which are calculated by weighting the promised payments of the debt instrument using the historical probabilities of default and the historical recovery rates. However, when these historical parameters are used to weight the promised cash flows, the resulting risk-averse expected cash flows do not fully factor in the
risk inherent in the bond. In consequence, they cannot be discounted using only the risk-free spot rates. Instead, a maturity-dependent risk premium, $RP_t$, must be added to the risk-free spot rates in order to account for the risk of the debt instrument.

Let $D_0$ be the value of a risky bond at time $t = 0$. Using the risk-adjusted valuation approach, we can write

$$D_0 = \sum_{t=1}^{T} \frac{E_0(C_t)}{(1 + r_{0,t} + RP_t)^t}$$

(17)

where

$$E_0(C_t) = \prod_{t=1}^{t-1} (1 - PD_t) \cdot [(1 - PD_t)(I_t + P_t) + PD_t \cdot RR \cdot (I_t + Nom_{t-1})],$$

(18)

$PD_t$ is the risk-averse probability of default, and $RR_t$ is the risk-averse recovery rate for period $t$. Unlike the risk-neutral probabilities of default used in section II, which we must bootstrap from risky zero-coupon bonds, the risk-averse probabilities of default are derived from historical market data. Risk-averse probabilities of default are published regularly by rating agencies in so-called rating transition matrices or rating migrations (see, for example, Standard & Poor’s Financial Services (2019) or Moody’s Investors Service (2011)). A $t$-year rating transition matrix is a table listing the cumulative probabilities that an issuer stays within a specific rating category, transitions to another rating category, or defaults until the end of the $t$-year period. The rating category the debt instrument is in at the beginning of the period under consideration is indicated in the headers of the rows, while the rating category the debt instrument is in at the end of the period is indicated in the column headers.

The parameters in the numerator of equation (17) are readily available so that investors can determine the risk-averse expected cash flows. However, investors are also interested in the maturity-dependent risk premia, $RP_t$. These risk premia reflect the systematic risk of the debt instrument and are required to calculate the present value of the risk-averse expected cash flows. They can vary over time and are expressed in % p.a. There are several methods that can be used to determine risk premia. For example, we can calculate the risk premia using the Capital Asset Pricing Model (CAPM) (Treynor 1962, Sharpe 1964, Lintner 1965a, b, Mossin 1966). According to CAPM, the risk premia satisfy
where $E(r_{M,t})$ is the expected return of the market, $\beta_{D,t}$ is the sensitivity of the debt instrument’s return to the return on the market portfolio, and $r_t$ is the risk-free spot rate.

Alternatively, it is possible to derive the risk premia using the prices of risky bonds obtained from risk-neutral valuation models such as the JLT model or the approach presented in this paper.

A. Risk-Premia for Risky Zero-Coupon Bonds

For zero-coupon bonds, the approach is straightforward. Recall from Section I that, under the assumption of arbitrage-free and complete markets, both the risk-neutral and the risk-averse approach must lead to the same fair value for a risky bond. We can use this insight and equation (15) to determine the prices of the risky bonds, $D_0$, on the left side of equation (17). Additionally, we know the risk-free spot rates and can calculate the risk-averse expected cash flows using the historical default probabilities and recovery rates. The only missing parameters in equation (17) are the risk premium for each period $t$, $RP_t$.

We can use the JLT model to determine the risk premia, for example. For this, we adapt the risk-neutral valuation approach presented in Jarrow, Lando, and Turnbull (1997) to the risk-adjusted setting. We write the adapted formula for the price of a risky zero-coupon bond as

$$D_0 = \frac{\left[1 - (1 - RR_t) \cdot CPD_{0,T}\right] \cdot Nom}{(1 + r_{0,T} + RP_T)^\tau}$$

and then rearrange the equation such that

$$RP_T = \sqrt{\frac{Nom}{D_0} \cdot \left[1 - (1 - RR_T) \cdot CPD_{0,T}\right] - (1 + r_{0,T})}$$

to calculate the risk-premium of the zero-coupon bond.

As mentioned in Section II, a significant drawback of the JLT approach is that the valuation formula is based on the assumption that the zero-coupon bond can only default at maturity. In our model, we assume that the zero-coupon bond can default prior to maturity. The price of a risky zero-coupon bond based on our risk-adjusted valuation approach is given by
\[ D_0 = \sum_{t=1}^{T-1} \frac{(1 - CPD_{0,t-1}) \cdot PD_t \cdot RR_t \cdot Nom}{(1 + r_{0,t} + RP_t)^t} \]

\[ + \frac{(1 - CPD_{0,T-1}) \cdot [1 - (1 - RR_t) \cdot PD_T]}{(1 + r_{0,T} + RP_T)^T} \cdot Nom \]  

Equation (22) differs from its risk-neutral counterpart (15) in that expected payments in the numerator are calculated by weighting the promised payments with the risk-averse probabilities of default and recovery rates, \( PD_t \) and \( RR_t \), rather than the risk-neutral parameters, \( PD_0 \) and \( RR_0 \). Conversely, the denominator in the risk-adjusted valuation in (22) incorporates a risk premium, \( RP_T \), on top of the spot rate, \( r_{0,t} \), while the risk-neutral approach does not. Furthermore, we now not only have one single risk premium for the zero-coupon bond with maturity \( T \) but different risk premia for each time \( t = 1, 2, ..., T \). In consequence, we must use a recursive bootstrapping technique to calculate these time-dependent risk premia. For this, we rearrange equation (22) such that

\[ RP_t = \sqrt{ \frac{D_0 \cdot Nom - \sum_{t=1}^{T-1} \left(1 - CPD_{0,t-1}\right) \cdot PD_t \cdot RR_t}{(1 + r_{0,t} + RP_t)^t} } \]

\[ \cdot \frac{(1 - CPD_{0,t-1}) \cdot [1 - (1 - RR_t) \cdot PD_T]}{(1 + r_{0,T} + RP_T)^T} \]

For the recursive bootstrapping technique, we start in period \( t = 1 \). Given that \( CPD'_{0,0} = 0 \), we can determine the risk premium for \( t = 1 \) as

\[ RP_1 = \frac{[1 - (1 - RR_1) \cdot PD_1]}{D_0 \cdot Nom} - (1 + r_{0,1}) \]

In period \( t = 2 \), for the risk premium we can write

\[ RP_2 = \sqrt{ \frac{D_0 \cdot Nom - PD_1 \cdot RR_1}{(1 + r_{0,1} + RP_1)^{t-1}} } \cdot \frac{(1 - CPD_{0,1}) \cdot [1 - (1 - RR_2) \cdot PD_2]}{(1 + r_{0,2} + RP_1)^{T-1}} - (1 + r_{0,2}) \]

This process is repeated until \( t = T \). Again, the risk premia must be calculated separately for each rating category.
B. Risk-Premia for Risky Coupon Bonds

When determining the risk premia of risky debt instruments other than risky zero-coupon bonds, investors must first determine the expected future prices of the corresponding risky debt instrument. Let \( E_0(D_t^+|\text{No Default until } t) \) be today’s (i.e., at time \( t = 0 \)) expected price of a debt instrument at time \( t^+ \) (i.e., directly after the interest and principal payments are paid at time \( t \)). As with the current fair price for a debt instrument, we get the same expected future fair price irrespective of whether we use the risk-neutral or the risk-averse valuation approach. In the risk-neutral valuation approach, the expected price of a bond is given by

\[
E_0(D_t^+|\text{No Default until } t) = \frac{(1 - PD'_{t+1}) \cdot I_t + P_t + E'_0(D_{(t+1)}^+|\text{No Default until } t + 1)}{1 + E_0(r_{t,t+1})} + \frac{PD'_{t+1} \cdot RR^t_{t+1} \cdot (I_t + Nom_{t-1})}{1 + E_0(r_{t,t+1})}
\]

where

\[
E'_0(D_{T+}|\text{No Default until } T) = 0
\]

and \( E_0(r_{t,t+1}) \) is the expected future spot rate from \( t \) to \( t + 1 \). Any desired risk-free term structure model can be used to determine the expected future spot rates. Numerous estimation techniques have been developed to determine future risk-free spot interest rates.\(^5\) In this paper, we derive the expected future spot rates based on the assumption that the Pure Expectations Hypothesis holds. The Pure Expectations Hypothesis postulates that the current term structure fully incorporates all information on the future development of the interest rates. Using the Spot-Forward-Relation, the future rates can be derived as the geometric mean of the current interest rates. We can write this as

\[
E_0(r_{t,t+1}) = \frac{(1 + r_{0,t+1})^{t+1}}{(1 + r_{0,t})^t} - 1.
\]

We determine the expected future prices backward, starting in period \( t = T - 1 \). Once we have determined all expected future bond prices, we can calculate the risk-premia of risky

\(^5\) We refer the reader to Marangio, Massimo, and Ramponi (2002) for an overview of the spot rate estimation literature.
coupon bonds. For this, we write the value at of a risky bond at time $t = 0$ in the risk-adjusted valuation setting as

$$D_0 = \sum_{t=1}^{T-1} \left(1 - CPD_{0,t-1}\right) \cdot \left((1 - PD) \cdot (I_t + P_t) + PD \cdot RR_t \cdot (I_t + Nom_{t-1})\right)\left(1 + r_{0,t} + RP_t\right)^t$$

$$+ \left(1 - CPD_{0,T-1}\right) \cdot \left((1 - PD) \cdot [I_t + P_t + E_0(D_t|\text{No Default until } t)] + PD \cdot RR_t \cdot (I_t + Nom_{t-1})\right)\left(1 + r_{0,T} + RP_t\right)^T$$

where

$$E_0(D_T|\text{No Default until } T) = 0.$$ We then rearrange equation (26) such that

$$RP_t = \sqrt{D_0 - \sum_{t=1}^{T-1} \left(1 - CPD_{0,t-1}\right) \cdot \left((1 - PD) \cdot (I_t + P_t + E_0(D_t|\text{No Default until } t)] + PD \cdot RR_t \cdot (I_t + Nom_{t-1})\right)\left(1 + r_{0,t} + RP_t\right)^t} - (1 + r_{0,t}).$$

The risk premia can be derived using a bootstrapping method starting in period $t = 1$ up to period $t = T$.

### IV. Risk and Return Analysis

An important application of the model presented in this paper is in the area of risk management, where it can be used to determine various risk and return parameters. For example, it can be used to compute the following three commonly used statistics: yield to maturity, yield spread, and Z-spread. When determining the yield of a risky bond, investors focus predominantly on the promised yield to maturity, $y_T$, which is the yield of the risky debt instrument in the case that no default occurs. It is calculated based on the promised payments of the bond by solving

$$D_0 = \sum_{t=1}^{T} \frac{I_t + P_t}{(1 + y_T)^t}$$
for \( y_T \). Since it does not take a potential default of the debt instrument into account, the promised yield to maturity is the upper bound on the actual yield of a debt instrument. In consequence, it may be significantly too high to be used in risk management considerations. It may, therefore, be advisable to additionally determine the expected, or implied yield to maturity, \( E_0(y_T) \). The expected yield is derived based on the expected risk-averse cash flows of the risky bond by solving

\[
D_0 = \sum_{t=1}^{T} \frac{E_0(C_t)}{(1 + E_0(y_T))^t}
\]

for \( E_0(y_T) \) with

\[
E_0(C_t) = (1 - CPD_{0,t-1}) \cdot [(1 - PD_t) \cdot (I_t + P_t) + PD_t \cdot RR_t \cdot (I_t + Nom_{t-1})].
\]

The expected yield takes the risk-averse probabilities of default and recovery rates into account and is, therefore, a more realistic estimate for the actual yield of a risky bond.

Based on the yields, it is possible to determine the yield spread. The yield spread measures the difference between the yield of a risky debt instrument and the yield of an otherwise identical risk-free debt instrument. Investors can again differentiate between the promised and expected yield spread. The promised yield spread, \( YS_T \), is the difference between the promised yield to maturity and the yield of a risk-free bond of identical maturity.

\[
YS_T = y_T - y_T^{rf}
\]

The expected yield spread, \( E_0(YS_T) \), on the other hand, is derived using the expected yield to maturity as

\[
E_0(YS_T) = E_0(y_T) - y_T^{rf}
\]

Finally, it may also be of interest to investors to determine the zero-volatility spread, or Z-spread, of a risky debt instrument. The Z-spread is a constant credit spread that is added to each point on the term structure when discounting the cash flows of the bond, which ensures that the cash flows of the debt instrument equal its current fair price. In other words, it corresponds to the parallel shift of the term structure that is required to make the present value of the debt instrument’s cash flows equal to its market price. The promised Z-spread, \( ZS_T \), is calculated based on the promised payments of the bond. It is derived by solving
\[ D_0 = \sum_{t=1}^{T} \frac{I_t + P_t}{(1 + r_{0,t} + ZS_T)^t} \]  
for $ZS_T$. The expected Z-spread, $E_0(ZS_T)$, is calculated based on the risk-averse expected payments from  
\[ D_0 = \sum_{t=1}^{T} \frac{E_0(C_t)}{(1 + r_{0,t} + E_0(ZS_T))^t}. \]  

When determining the yield, the yield spread, or the Z-spread, it is important to determine the fair price of the risky debt on the left-hand side of all equations (28-33) as accurately as possible. Recall that the valuation model presented in this paper – unlike the models developed before – takes into account the unique interest and repayment structure of the debt instrument to be valued and allows the debt instrument to default before maturity.

V. Numerical Example

This section lays out our formulas for the rating-based valuation of risky debt via one step-by-step example. First, we illustrate the calculation of risk-neutral probabilities of default. Using these risk-neutral default rates, we then show how to value risky bonds using this article’s risk-neutral and risk-averse valuation frameworks. The example is based on two fictitious coupon bonds. Each bond has a nominal value of 100, a fixed coupon of 4 % p.a., and a residual term of three years. The two bonds are identical except for their ratings. While the first bond belongs to rating category A at time $t = 0$, the second bond has an initial rating of B. In total, there are three credit rating categories. The highest rating category is denoted by “A”, the second-highest rating category is “B”, and the third rating category is “D”, which stands for default. The parameters of the risky bonds are summarized in Table I.
Table I
Parameters of the Risky Bonds
This table reports the parameters of the two risky coupon bonds, which are used in the three numerical examples. The bonds are identical except for their initial ratings. While the first bond has an initial rating of A, the second bond is initially rated in category B of our three-part rating scale (A, B, and D).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>$T$</td>
<td>3</td>
</tr>
<tr>
<td>Nominal value</td>
<td>$Nom_0$</td>
<td>100.00</td>
</tr>
<tr>
<td>Nominal interest rate</td>
<td>$i_{nom}$</td>
<td>4.00%</td>
</tr>
<tr>
<td>Rating</td>
<td>A or B</td>
<td></td>
</tr>
</tbody>
</table>

In addition to these parameters, we require information on the rating migrations the two bonds can potentially undergo. Such information is summarized in rating transition matrices, which are published regularly by credit rating agencies (see, for example, Standard & Poor’s Financial Services (2019) or Moody’s Investors Service (2011)). The estimates in such rating transition matrices are obtained from historical observations of credit rating changes. In other words, the credit ratings of a fixed group of firms are observed at the beginning and at the end of a particular time period and then summarized in the rating transition matrix. Since only the very beginning and the end of the period are compared, rating transition matrices do not include the exact timing of any transitions within the period. This means that rating migrations are based on the assumption that every firm has made either exactly one transition or has not transitioned at all throughout the specified period.

In our example, we use a fictitious one-year rating transition matrix, which contains the cumulative probabilities from $t = 0$ until $t = 1$. To facilitate our calculations, we assume that the one-year transition probabilities follow a time-homogeneous Markov process. This means that the probabilities for a transition to another state are the same for each one-year period in $t = [0,T]$. The matrix contains fictitious risk-averse transition and default probabilities for bonds with ratings A, B, or D. The one-year rating transition matrix is shown in Table II. Examining the first row, the probability of staying in the highest credit rating category A over a period of one year is 0.9. The transition rate from the highest rating category to the second-
highest rating category, B, is 0.06, and the probability that a firm in the highest rating category will default before the end of the one-year period is 0.04. This rate of default does not take into account the possibility that the firm is first downgraded (occurring with a probability of 0.06) and then subsequently defaults (occurring with a rate of 0.1 in rating category B) within the one-year period. As can be seen from the last row in the rating transition matrix, for simplicity of estimation, we assume that bankruptcy (state D) is an absorbing state. This means that once a bond defaults, it can no longer be upgraded again to the higher rating categories A and B.

Table II
One-Year Rating Transition Matrix
This table reports the one-year rating transition matrix used in the numerical example. The transition matrix is based on a three-part rating scale with the ratings A, B, and D. A corresponds to the highest rating, B corresponds to the second-highest rating, and D corresponds to default.

<table>
<thead>
<tr>
<th>1-year</th>
<th>A</th>
<th>B</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.9000</td>
<td>0.0600</td>
<td>0.0400</td>
</tr>
<tr>
<td>B</td>
<td>0.1000</td>
<td>0.8000</td>
<td>0.1000</td>
</tr>
<tr>
<td>D</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Since the two bonds used in our examples both have a maturity of three years, we are also interested in the two- and three-year rating transition matrices. Table III illustrates the multi-period transition matrices, which contain the cumulative probabilities from $t = 0$ until $t = 2$ and $t = 3$, respectively. Since the one-year transition matrix is time-homogenous, we calculate the two-year transition matrix by multiplying the one-year matrix by itself. The three-year rating matrix is derived by multiplying the resulting two-year matrix with the initial one-year rating transition matrix. All transition probabilities follow a first-order Markov process. Therefore, each transition probability only depends on the previous period.
Table III
Two-Year and Three-Year Rating Transition Matrices
This table reports the two- and three-year rating transition matrices derived from the one-year rating transition matrix in Table II. The two- and three-year transition matrices are derived by multiplying the one-year rating transition matrix one and two times by itself, respectively. The transition matrix is based on a three-part rating scale with the ratings A, B, and D. A corresponds to the highest rating, B corresponds to the second-highest rating, and D corresponds to default.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>two-year</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>0.8160</td>
<td>0.1020</td>
<td>0.0820</td>
</tr>
<tr>
<td>B</td>
<td>0.1700</td>
<td>0.6460</td>
<td>0.1840</td>
</tr>
<tr>
<td>D</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>three-year</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>0.7446</td>
<td>0.1306</td>
<td>0.1248</td>
</tr>
<tr>
<td>B</td>
<td>0.2176</td>
<td>0.5270</td>
<td>0.2554</td>
</tr>
<tr>
<td>D</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

To be able to value the bonds specified in Table I, we need additional input parameters. These input parameters are summarized in Table IV. First, we need the risk-free spot rates, $r_{0,t}$, for $t = 1, 2, 3$. In this numerical example, we use a fictitious normal term structure. Second, we require the prices of risky zero-coupon bonds for the rating categories A and B in order to calculate the risk-neutral probabilities of default for the two rating classes. In our examples, we determine the risky zero-coupon prices using the yield of risky zero-coupon bonds with residual maturities of 1, 2, and 3 years based on (28). Table IV contains exemplary yields for ratings A and B. In practice, the market prices of the zero-coupon bonds can be used directly, if available. Otherwise, the yields of zero-coupon bonds with different ratings observable in the market should be used.

Third, we need the risk-neutral and risk-averse recovery rates. For both rating categories A and B, we assume constant risk-neutral and the risk-averse recovery rates of 0.55 and 0.75, respectively. This assumption is imposed to simplify the estimation. It is equivalent to the
assumption that both bonds have the same seniority. In practice, the recovery rates of bonds with the same seniority but different ratings may differ.

Table IV
Input Parameters for the Valuation

This table reports the additional input parameters used for the valuation of the risky coupon bonds described in Table I. The recovery rates are independent of the bonds’ rating.

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-free Spot Rates $r_{0,t}$</td>
<td>1.00%</td>
<td>1.50%</td>
<td>2.00%</td>
</tr>
<tr>
<td>Yield of Risky Zero-Coupon Bond (Rating A) $y_A$</td>
<td>2.50%</td>
<td>3.50%</td>
<td>5.00%</td>
</tr>
<tr>
<td>Yield of Risky Zero-Coupon Bond (Rating B) $y_B$</td>
<td>4.00%</td>
<td>5.00%</td>
<td>6.50%</td>
</tr>
<tr>
<td>Risk-Neutral Recovery Rate $RR_t$</td>
<td>0.55</td>
<td>0.55</td>
<td>0.55</td>
</tr>
<tr>
<td>Risk-Averse Recovery Rate $RR_t$</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
</tbody>
</table>

A. Risk-Neutral Probabilities of Default

This section demonstrates how we calculate the risk-neutral probabilities of default for rating categories A and B using the yields of the risky zero-coupon bonds specified in Table III. We calculate the risk-neutral probabilities of default based both on our valuation formulas (FKW) as well as using the approach presented in Jarrow, Lando, and Turnbull (1997) (JLT). The results are shown in Table V.
Table V
Risk-Neutral Probabilities of Default

This table reports the risk-neutral probabilities of default derived using the approach presented in this paper (FKW) as well as based on the valuation approach proposed by Jarrow, Lando, and Turnbull (1997) (JLT). For each model, the table shows the cumulative, total, and conditional probabilities of default derived for each period for bonds with rating A as well as for bonds with rating B.

<table>
<thead>
<tr>
<th>Model</th>
<th>FKW</th>
<th>JLT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Rating A</td>
<td>Cumulative $PD^c_t$</td>
<td>3.25%</td>
</tr>
<tr>
<td></td>
<td>Total $PD^t_t$</td>
<td>3.25%</td>
</tr>
<tr>
<td></td>
<td>Conditional $PD^c_t$</td>
<td>3.25%</td>
</tr>
<tr>
<td>Rating B</td>
<td>Cumulative $PD^c_t$</td>
<td>6.41%</td>
</tr>
<tr>
<td></td>
<td>Total $PD^t_t$</td>
<td>6.41%</td>
</tr>
<tr>
<td></td>
<td>Conditional $PD^c_t$</td>
<td>6.41%</td>
</tr>
</tbody>
</table>

Recall that in this paper, we make the assumption that the risky debt instrument can default at any point before maturity. Jarrow, Lando, and Turnbull (1997), on the other hand, assume that default can only occur at maturity. As can be seen from Table V, these different assumptions have an impact on the probabilities of default derived using the two models. For rating category A, all probabilities of default calculated using our approach are strictly greater than the default probabilities derived using the JLT model. This can also be seen from Figure 1, which illustrates the cumulative risk-neutral probabilities of default for rating category A derived using both models. The grey line indicates the probabilities derived based on the JLT model, while the dotted black line illustrates the probabilities we calculate based on our model. The JLT model underestimates the risk of default. When investors use the JLT model to price bonds, they may underestimate the risk that the bond will default and may, therefore, determine a price that is too high.
Figure 1. Comparison of the cumulative risk-neutral probabilities of default for rating category A derived using the approach presented in this paper and the Jarrow, Lando, and Turnbull (1997) approach. Values contained in Table V.

B. Bond Valuation

In this section, we value the two risky bonds utilizing the risk-neutral and risk-averse procedures described in this paper. We value both bonds three times, assuming a different repayment agreement (lump-sum, constant principal, annuity) in each round. For comparison purposes, we also determine the price of the two bonds assuming they are risk-free based using equation (1) for all three repayment cases. Tables VI and VII only contain one risk-free value for both bonds. This is the case because the bonds are identical when we disregard the ratings of the two bonds and assume that both are risk-free.

In our first valuation round, we assume that both bonds feature lump-sum repayment. Table VI summarizes the valuation results. The value of the risk-free bond is 105.88. To calculate the value of the risky bonds, we first determine the expected cash flows. We calculate the risk-neutral and risk-averse expected cash flows using equations (3) and (18), respectively. Next, we determine the value of the two risky bonds using our risk-neutral valuation framework in
equation (11). The value of the risky bond with rating A is 97.22, while the price of the B-rated bond is 93.11. The price of the A-rated bond is higher because the probabilities of default are lower for this rating category. Finally, we are also interested in the risk-premia required in the risk-averse valuation approach. For this, we first calculate the expected bond prices based on equation (24) using the expected future spot rates derived from equation (25). We then calculate the risk-premia using our bootstrapping technique based on equation (28).

Table VI
Valuation of Risky Lump-Sum Bonds (Ratings A & B)
This table reports the valuation results for the two bonds described in Table I. Both bonds are assumed to feature lump-sum repayment. $E_0'(C_t)$ and $E_0(C_t)$ are the risk-neutral and risk-averse expected cash flows, respectively. $E_0(D_{t+1}|\text{no Default at } t)$ is today’s (i.e., at time $t = 0$) expected price of a debt instrument at time $t^+$ (i.e., directly after the interest and principal payments are paid at time $t$).

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-free</td>
<td>Value of risk-free debt $D_{t+1}^{rf}$</td>
<td>105.88</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rating A</td>
<td>Value of risky debt $D_0$</td>
<td>97.22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk Premium $RP_t$</td>
<td>0.53%</td>
<td>0.99%</td>
<td>2.06%</td>
<td></td>
</tr>
<tr>
<td>$E_0'(C_t)$</td>
<td>5.73</td>
<td>6.71</td>
<td>90.24</td>
<td></td>
</tr>
<tr>
<td>$E_0(C_t)$</td>
<td>6.96</td>
<td>6.95</td>
<td>94.36</td>
<td></td>
</tr>
<tr>
<td>$E_0(D_{t+1}</td>
<td>\text{no Default at } t)$</td>
<td>95.57</td>
<td>95.83</td>
<td>0.00</td>
</tr>
<tr>
<td>Rating B</td>
<td>Value of risky debt $D_0$</td>
<td>93.11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk Premium $RP_t$</td>
<td>0.72%</td>
<td>1.30%</td>
<td>2.56%</td>
<td></td>
</tr>
<tr>
<td>$E_0'(C_t)$</td>
<td>7.41</td>
<td>8.17</td>
<td>82.61</td>
<td></td>
</tr>
<tr>
<td>$E_0(C_t)$</td>
<td>11.40</td>
<td>9.82</td>
<td>83.01</td>
<td></td>
</tr>
<tr>
<td>$E_0(D_{t+1}</td>
<td>\text{no Default at } t)$</td>
<td>92.57</td>
<td>94.05</td>
<td>0.00</td>
</tr>
</tbody>
</table>

As can be seen from Table VI, the expected cash flows at $t = 1$ and $t = 2$ are higher for rating category B than for rating category A. This is due to the fact that the probabilities of default for category B are higher than for category A. In consequence, for rating B, more weight is put on the case that the bond will default and that the recovery rate will be paid to the creditor.
compared to rating A, making the expected cash flows higher. The risk premia for both rating A and rating B are positive. As mentioned in Section III, in the risk-adjusted approach, the expected cash flows do not fully factor in the risk inherent in the bond. These risk premia must be added to the risk-free spot rates when discounting the risk-averse cash flows in order to account for the risk of the debt instrument.

In our second and third valuation round, we repeat the valuation but change the underlying assumption on the repayment modality of the two loans to constant principal and annuity repayment, respectively. The value of the risk-free and risky debt is again determined using equations (1) and (11), respectively. We then perform a simple risk and return analysis for the three scenarios. The results are contained in Table VII.
Table VII
Valuation of Risky Bonds with Different Repayment Agreements (Rating A)
This table reports the valuation results and various risk and return parameters for the two bonds described in Table I under three different scenarios. In the first valuation round, both bonds are assumed to feature lump-sum repayment. In the second round, both bonds feature constant principal repayment, and in the third round, the bonds have annuity repayment.

<table>
<thead>
<tr>
<th>Repayment Form</th>
<th>Lump-Sum</th>
<th>Constant Principal</th>
<th>Annuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-free</td>
<td>Value of risk-free debt $D_{0}^{rf}$</td>
<td>105.88</td>
<td>104.57</td>
</tr>
<tr>
<td>Rating A</td>
<td>Value of risky debt $D_{0}$</td>
<td>97.22</td>
<td>99.87</td>
</tr>
<tr>
<td></td>
<td>Promised Yield to Maturity $y_{T}$</td>
<td>5.02%</td>
<td>4.07%</td>
</tr>
<tr>
<td></td>
<td>Expected Yield to Maturity $E_{0}(y_{T})$</td>
<td>3.92%</td>
<td>2.97%</td>
</tr>
<tr>
<td></td>
<td>Promised Yield Spread $YS_{T}$</td>
<td>3.06%</td>
<td>2.42%</td>
</tr>
<tr>
<td></td>
<td>Expected Yield Spread $E_{0}(YS_{T})$</td>
<td>1.95%</td>
<td>1.32%</td>
</tr>
<tr>
<td></td>
<td>Promised Z-Spread $ZS_{T}$</td>
<td>3.05%</td>
<td>2.40%</td>
</tr>
<tr>
<td></td>
<td>Expected Z-Spread $E_{0}(ZS_{T})$</td>
<td>1.96%</td>
<td>1.35%</td>
</tr>
<tr>
<td>Rating B</td>
<td>Value of risky debt $D_{0}$</td>
<td>93.11</td>
<td>97.05</td>
</tr>
<tr>
<td></td>
<td>Promised Yield to Maturity $y_{T}$</td>
<td>6.61%</td>
<td>5.61%</td>
</tr>
<tr>
<td></td>
<td>Expected Yield to Maturity $E_{0}(y_{T})$</td>
<td>4.30%</td>
<td>3.16%</td>
</tr>
<tr>
<td></td>
<td>Promised Yield Spread $YS_{T}$</td>
<td>4.64%</td>
<td>3.93%</td>
</tr>
<tr>
<td></td>
<td>Expected Yield Spread $E_{0}(YS_{T})$</td>
<td>2.34%</td>
<td>1.51%</td>
</tr>
<tr>
<td></td>
<td>Promised Z-Spread $ZS_{T}$</td>
<td>4.63%</td>
<td>3.96%</td>
</tr>
<tr>
<td></td>
<td>Expected Z-Spread $E_{0}(ZS_{T})$</td>
<td>2.39%</td>
<td>1.56%</td>
</tr>
</tbody>
</table>

The value of both the risky-free and the risky debt varies depending on the repayment modality of the bond. While the value of risk-free debt is highest for lump-sum repayment (105.88), this repayment modality leads to the lowest debt value for both the bond with rating A (97.22) and the bond with rating B (93.11) when the risk is taken into account. The yields to maturity of the two bonds, on the other hand, are highest for lump-sum repayment. Since the promised yields for rating B are generally higher than the respective yields for rating A, also
the yield and Z-spreads are higher for rating B than for rating A. An important insight from our model for risk management purposes can be derived from the comparison of the promised yield and Z-spreads with their expected counterparts. When we take the risk of the two bonds into account, the yield and Z-spreads are drastically reduced. The expected spreads of the B-rated bond are approximately 48-62% lower than the promised spreads. For the A-rated bond, the yields are lower by approximately 37-44%. This highlights the fact that investors may significantly overestimate the performance of a risky bond when focusing solely on the promised payments. The promised yields, yield spreads, and Z-spreads are merely the upper limit for their risky counterparts.

VI. Conclusion

This article presents a model for the valuation of risky debt, which uses historical rating transition matrices. The model is based on the seminal paper by Jarrow, Lando, and Turnbull (1997), and includes both a risk-neutral as well as a risk-adjusted approach to determine the fair price of a risky bond. The model is well-suited for practical applications since it provides simple valuation formulas that can be applied to debt instruments with various kinds of interest and repayment modalities. Furthermore, the input parameters required for the model are easily estimated using observable data. An illustrative example is provided to highlight the easiness of application.
REFERENCES


Fuertes and Kalotychou, 2006


Standard & Poor’s Financial Services, 2019, *2018 Annual Global Corporate Default and Rating Transition Study*.

APPENDIX

The risk-averse valuation approach presented in this paper represents a direct analogy to the structural approach presented by Fischer, Kampl, and Woeckl (2019). The authors extend the classic structural valuation model by Merton (1974) using the multivariate normal distribution. The analogy is best depicted when we replace the discrete cumulative risk-neutral survival probability until time $t$, $(1 - CPD_{0,t})$, and the discrete total probability of default, $(CPD'_{0,t} - CPD'_{0,t-1})$, in equation (12) by their continuous equivalents,

$$1 - CPD'_{0,t} = N_t(d_1^t, ..., d_2^t; \rho_t)$$

$$CPD'_{0,t} - CPD'_{0,t-1} = N_{t-1}(d_1^t, ..., d_2^{t-1}; \rho_{t-1}) - N_t(d_1^t, ..., d_2^t; \rho_t),$$

and transition from discrete discounting by continuous discounting. Equation (12) can then be transformed into the closed-form solution derived by Fischer, Kampl, and Woeckl (2020) as

$$D_0 = \sum_{t=1}^{\tau} (I_t + P_t) \cdot e^{-r \cdot t} \cdot N_t(d_1^t, ..., d_2^t; \rho_t) + V_0 \cdot [1 - N_T(d_1^T, ..., d_2^T; \rho_T)]$$

where

$$RR^t = E_0\left(\frac{\text{Debtholder’s Share of Firm Value at } t}{\text{Value of Claims of Debtholder at } t}\left|\text{Default at } t\right.\right)$$

$$= E_0\left(\frac{V_t}{I_t + N_{om_{t-1}}} \mid V_t \leq V^*_t\right)$$

$$= \frac{V_0 \cdot e^{-r \cdot t} \cdot N_{t-1}(d_1^{t-1}, ..., d_2^{t-1}; \rho_{t-1}) - N_t(d_1^t, ..., d_2^t; \rho_t)}{I_t + N_{om_{t-1}}} \cdot \frac{N_{t-1}(d_1^{t-1}, ..., d_2^{t-1}; \rho_{t-1}) - N_t(d_1^t, ..., d_2^t; \rho_t)}{ho_t}$$

and

$$\rho_t = \left\{ \begin{array}{ll}
1 & \text{if } \tau_1 = \tau_2, \tau_1 = 1, ..., \tau, \tau_2 = 1, ..., \tau \\
\sqrt{\tau_1 / \tau_2} & \text{if } \tau_1 < \tau_2, \tau_1 = 1, ..., \tau, \tau_2 = 1, ..., \tau \\
0 & \text{else.}
\end{array} \right.$$

$V_t$ is the value of the firm (i.e., the total assets), $N(\cdot)$ denotes the standard normal cumulative distribution function, $\rho_t$ is the correlation matrix, and $r$ is the risk-free interest rate. Like in the classic Merton (1974) model, Fischer, Kampl, and Woeckl (2020) assume that the risk-free interest rate is constant. The valuation framework presented in Fischer, Kampl, and Woeckl (2020) is the theoretical counterpart to the practical valuation framework we present in this...
paper. Like the model presented here, it can be applied to debt instruments with any kind of interest and repayment modalities. However, the structural approach is far more challenging to implement in practice since many of the required parameters cannot be determined easily.