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Recovering the Original Downs Model of Spatial Party Competition

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Abstract

I argue that the popular conception of the so-called "Hotelling-Downs" model of spatial political competition overlooks two assumptions that are important for Downs' analysis: Voters may abstain if the ideologically closest party is too distant and parties are not able to choose their ideology completely freely. Omitting these assumptions, Osborne (1993) famously argued that the Hotelling-Downs model generally does not permit multi-party equilibria. I critically discuss Osborne's findings and implement a computational version of the original Downs model to show that such a model is highly conducive to multi-party equilibria, in particular when compared to a model that lacks the two additional assumptions and is inhabited by vote-maximizing parties.

Keywords: spatial voting model, agent-based model, history of economic thought

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I argue that the popular conception of the so-called "Hotelling-Downs" model of spatial political competition overlooks two assumptions that are important for Downs' analysis: Voters may abstain if the ideologically closest party is too distant and parties are not able to choose their ideology completely freely. Omitting these assumptions, Osborne (1993) famously argued that the Hotelling-Downs model generally does not permit multi-party equilibria. I critically discuss Osborne's findings and implement a computational version of the original Downs model to show that such a model is highly conducive to multi-party equilibria, in particular when compared to a model that lacks the two additional assumptions and is inhabited by vote-maximizing parties.

1 Introduction

The two most prominent works on the theory of spatial political party competition are the paper "Stability in Competition" by Hotelling (1929) and the book "An Economic Theory of Democracy" by Downs (1957). According to the canon, the so-called Hotelling-Downs model substantiates the median voter theorem in elections based on the plurality system: Assuming that two political parties or candidates compete in an election and aim to get elected, they will pursue a political program that is shared by the median voter (e.g. Osborne 1993; Osborne 1995; Brusco et al. 2012; Sengupta and Sengupta 2008; van Sloun 2023).[1]

However, this popular characterization gives a biased account of the original Downs model of spatial party competition that he describes in "An Economic Theory of Democracy" (Downs 1957a) and an accompanying paper published in the *Journal of Political Economy* (Downs 1957b). In his original model, Downs used assumptions that are much more nuanced than commonly conceived, and he showed that the median voter result does not always hold. In particular, he introduced two assumptions that are not used in the original Hotelling model, but also ignored in later work: First, that parties may not 'leap' over other parties ideologically. Second, that voters will not vote for parties which are ideologically too distant. One likely reason that these two assumptions are neglected in subsequent work is that they seem to be hard to tract analytically. The second reason is that Downs himself downplayed his additional assumptions, particularly in his book.

In this paper, I implement a computational version of Downs' original model that contains the Hotelling model as a special case, and use it to study a question that has sparked a hot debate in the last decades, namely the number and placement of parties in equilibrium. While Downs (1957a) himself suggested that a multi-peaked ideology landscape would likely be inhabited

[1] Some authors are well-aware that their characterization of the median voter result describes a Hotelling model, but not a Downs model (see, e.g., the prominent textbook "Economics of the Public Sector" by Stiglitz 2015, p. 247).

by multiple parties, later researchers such as Cox (1990) and Osborne (1993) argued that . This result was later used to justify departures from the "classic" model.

I show that a model using Downs' original assumptions is indeed, as asserted by Downs, more conducive to multi-party equilibria than a classical Hotelling model and produces arguably more realistic outcomes. On the other hand, Downs was incorrect regarding the driving force of equilibria in his model and exact positioning of parties in equilibrium.

More generally, my research highlights the potential of computational modelling to explicitly account for complex assumptions. Computational methods such as agent-based models can hence allow researchers of the history of economic thought to better understand complex theories and rigorously explore their implications.

2 Spatial Political Competition in Hotelling (1929) and Downs (1957a)

The seminal contribution by Hotelling (1929) introduced space both in a literal sense and as an analogy to economics. His idea is best illustrated by the example of two ice cream vendors on the beach who have to decide where they should locate their ice cream stand. Under the assumption that beach-goers are distributed uniformly across the beach and will eat ice cream at the closest store regardless of the distance, both vendors will settle at the center of the beach. Hotelling also (erroneously) argued that this would hold for a larger number of vendors. Even though his analysis has been proven to have flaws by later authors (see, e.g., Osborne and Pitchik 1987), he clearly inspired a whole stream of the literature by his simple model that cleverly illustrates that rational behavior on parts of consumers and competing suppliers may led to an overall inefficient outcome. While the example of ice cream vendors on a beach would represent space in a literal sense, he also argues that his model can be applied to other domains where space would rather be an analogy, including the placement of the Democrats vs. the Republicans in the US.

As of April 2024, "An Economic Theory of Democracy" (Downs 1957a) has accumulated over 41,000 citations according to Google scholar, which makes it one of the most-cited works in the field of economics. In chapter 8, he develops a model to understand how political parties will choose their political program if they are confronted with a particular shape of the voter distribution. To this end, Downs (1957a, p. 115) writes that he "borrow[s] and elaborate[s] upon an apparatus invented by Harold Hotelling."

He builds his analysis on several assumptions, although, unfortunately, he mentions some of them only in the middle of his analysis, which implies that they may escape the attention of readers. The analytically clearest version of his model is to be found in the accompanying paper "An Economic Theory of Political Action in a Democracy" (Downs 1957b).

1.) Parties want to win elections (instead of following a particular program)
2.) Voters are located on a single ideological dimension (e.g., left-right). All voters agree on the ordering of parties from one end of the scale to the other.
3.) Voters' preferences are single-peaked and monotonically down-ward sloping, meaning that, e.g., a left-wing radical will always prefer a moderate left-wing party over a moderate right-wing party.
4.) The voter distribution in a specific society is constant.
5.) Political parties may not "leap over" other parties ideologically, meaning that if party A has been on the left of B, it cannot be on the right of B in the future.

6.) Contrary to 3), voters will abstain from voting if their ideologically closest party is too far away due to alienation.

While the literature on the "Hotelling-Downs" model stays true to the first four assumptions, the nuances of the latter two assumptions have mostly been lost, even though they are "reinvented" by some recent authors. For instance, Feldman et al. (2016) argue that they "expand" the model to account for the fact people will not necessarily vote for the closest party regardless of their ideological distance. Adams (2001), on the other hand, argues that a Downsian model would imply that in a three-party system – such as the British one – parties would continuously "leapfrog" each other – which Downs also assumes cannot be the case (most probably because he was aware of the problems that would otherwise be created by the existence of three political parties.

3 Multi-party equilibria according to Downs, and in the Hotelling model

Downs argued that the location of parties in equilibrium in his model crucially depends on the voter distribution. While a unimodal distribution would produce the famous "median voter" result, multimodal distribution would be conducive to multi-party equilibria. He also argued that a polarized distribution would destroy a democratic system, as government policy would change radically between one electoral cycle to another (please note that, in this case, Downs implicitly assumed that the distribution of voters would not be fixed, but rather potentially be influenced by government policy in a negative way). However, Downs lacked the apparatus to rigorously investigate the properties of his model. This left others with the task of doing so.

Often-cited articles by Osborne (1993, 1995) explore Nash equilibria for the plurality rule (as commonly used in the USA) in political Hotelling games (i.e. omitting assumptions 5 and 6 by Downs (1957a and 1957b) mentioned above) with different setups. His results are crucially interpreted as proving that the Hotelling-Downs model "fails to admit an equilibrium in pure strategies for all distributions of voter ideal points *except for some pathological cases*" (Ronayne 2018, my emphasis). Typically, this result is then used as a justification for a departure from this model (e.g. Osborne 1996, p. 83, Ronayne 2018, Xefteris 2016). This interpretation relies on Osborne's (1993) statement that the conditions he derives for a Nash equilibrium are not given for "almost all distributions" (e.g. p. 139) and that the only counter-example he gives is the uniform distribution, but hides the fact that his conditions are indeed given for an array of very plausible ones, e.g. for an electorate distributed according to the standard normal distribution (a distribution also, e.g., discussed by Ronayne 2018 for his new model).

In the first types of games employed by Osborne (1993), there is a fixed number of potential parties (in his terminology: candidates), which have to act simultaneously. Their strategy space is given by the ability to choose either a) any position within the policy space X, i.e. all potential parties are actually running or b) any position within the policy space X OR staying out of the competition altogether. Osborne assumes that the payoff of party i is given by its plurality: $M_i = v_i - \max_{j \neq i} v_j$, where v_i denotes the number of its votes. Parties and voters are in these models perfectly informed about each other's ideologies.

Osborne (1993) argues that there is "almost no [voter] distribution" (p. 139) that yields a Nash equilibrium for this game with n >= 4 potential parties if the following assumptions hold:

1.) Parties prefer to tie for the first place rather than to lose

2.) Parties prefer to enter the competition and tie for the first place rather than to stay out.
3.) Parties prefer to stay out of the competition once rather than to lose[2].
4.) The voter distribution is single-peaked (in other words unimodal) and continuous.

Even under these very strict assumptions – which assume a priori that in any Nash equilibrium, all entrants must tie for the first place, something that clearly only happens very rarely in the real world – a Nash equilibrium is possible for the case of n=4 players who compete for, e.g., standard normally distributed voters:

Result: In a four-party equilibrium in a Hotelling game under the conditions mentioned by Osborne (1993), two parties form a left-wing camp at the first quartile of the distribution and two parties form a right-wing camp at the third quartile of the voter distribution. Each party receives exactly 25% of the votes.

Proof: If one of the left-wing parties moves to the left for an ε large enough to actually change its votes, it would instantly lose the election, since by definition only 25% of the voters are located to the left of the first quartile. The same applies vice versa to the two right-wing parties. If one of the two left-wing parties would choose to move to the right, however, it could actually increase its vote share. It could maximize its vote share by moving directly to the median. However, vote share maximization is not the target function that Osborne studies, since parties operating under the plurality rule common to the Anglo-American countries (i.e. the first-past-the-post system) could increase their vote share, but still lose their plurality. This is the case here, since the gains of the moving left-wing party would by surpassed by the gains of the left-wing party, which remains at the first quartile.

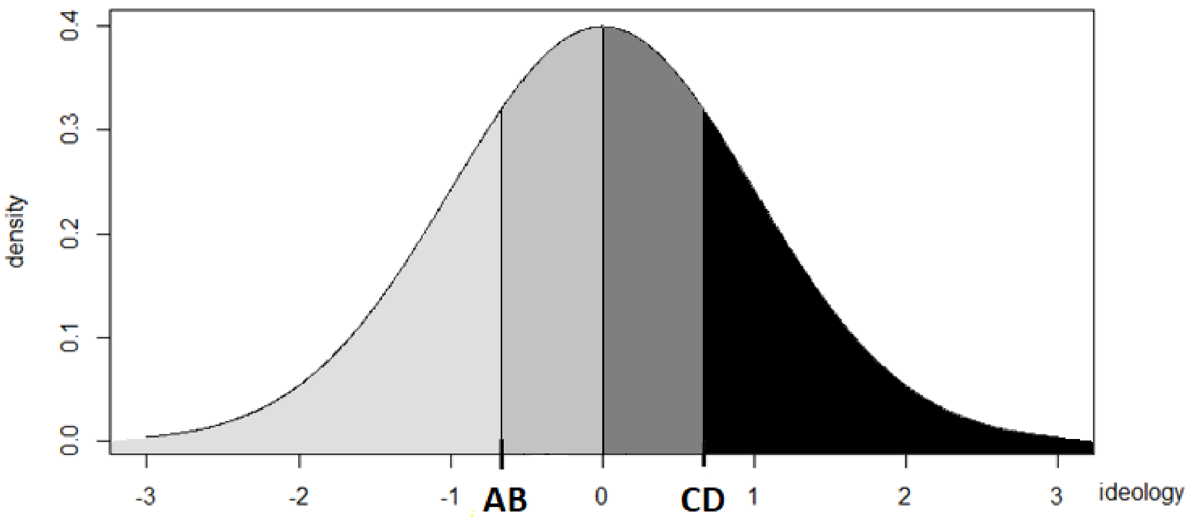


Figure 1: Symmetric Nash equilibrium for n=4 parties and standard normally distributed voters

This is illustrated by the following example: Suppose that party A moves to the median. It would increase its vote share from 25% to about 26.4%. However, party B would receive about 36.8% of the votes and thus achieve a plurality. The same result, albeit with a smaller margin, would be achieved by any move ε to the right large enough to actually change its votes to a position between the first quartile and the median. Moves beyond the median would decrease

[2] This is not equal to saying that the outside option yields a payoff of 0, because tying for the first place yields a payoff of 0 and is preferred to staying out of the competition (see assumption 2).

the vote share again (and naturally also further decrease the plurality). The same reasoning applies *mutatis mutandis* to the right-wing parties.

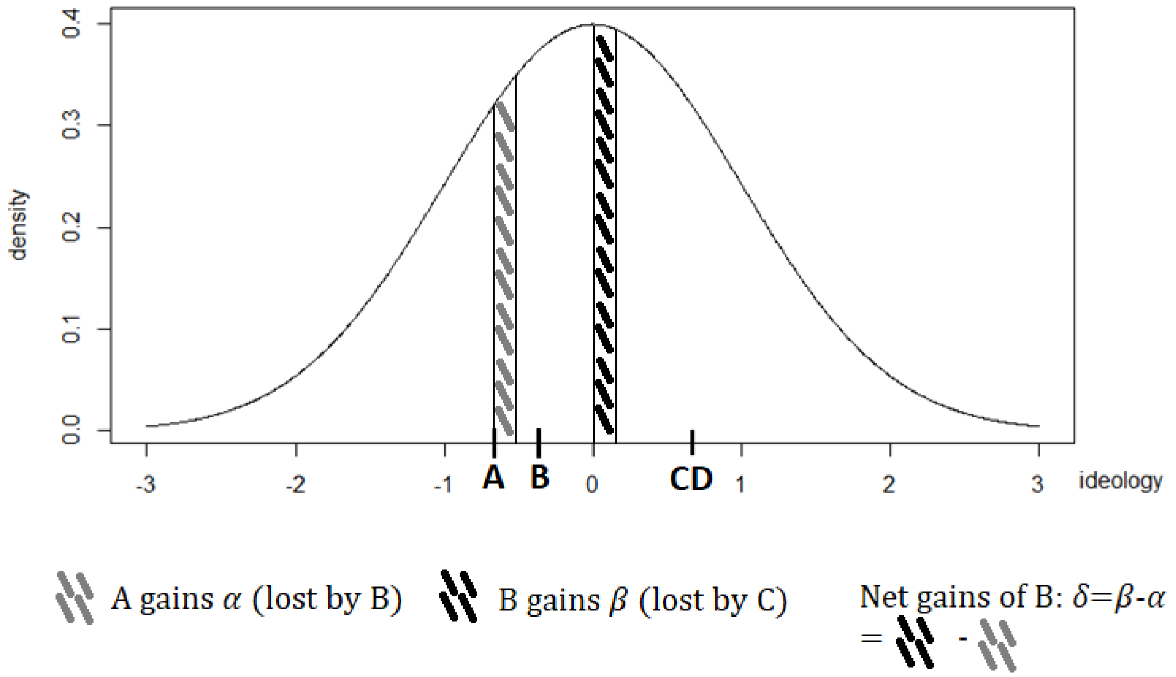


Figure 2: If B moves to the right, this party gains in votes, but loses in plurality, since its net gains (δ) will always be lower than the net gains of party B (α).

We thus have found a Nash equilibrium, which also holds for the other type of game studied by Osborne, where candidates are plurality-maximizers and enter the competition in any case (i.e. when there is no outside option).

More generally, we can find a symmetric four-party equilibrium for a continuous unimodal voter distribution, where two left-wing parties settle at the first quartile and two right-wing parties at the third quartile, if these two necessary conditions are fulfilled:

1.) The distance between the first quartile and the median must equal the distance between the third quartile and the median. A sufficient condition for this to be fulfilled is that the distribution of voters is symmetric.
2.) The density at the median does not exceed twice the density at the first and third quartile.

Proof for 1: If the distance between the first quartile and the median was shorter than the distance between the median and the third quartile, the median voter (and any voter to the left of it) would fully support the left-wing camp, which means that it would gain more than half of the votes. But this cannot be an equilibrium, since the two left-wing parties would gain more than 50% of their votes from their right-wing voters, just like the two right-wing parties would gain more than 50% of their votes from their right-wing voters. All parties would thus have an incentive to enter the election with a more right-wing platform. The new situation, in which all parties are again voted by *their* median voter (just like in the symmetric equilibrium case), however, also cannot be an equilibrium, since one of the right-wing parties could move towards the left again to increase its vote share (this is always true if we assume unimodality of the voter distribution) and thus reduce the plurality of the left-wing parties. A more left-wing platform of the right-wing parties cannot result in an equilibrium because again, the two parties would not be voted by their median voter.

Proof for 2: If the density at the median was more than twice as high as the density at the first (third) quartile, any left-wing (right-wing) party could move slightly to the right (left) and gain not only in vote share, but also in plurality, because the net gains of the moving party in votes δ would be higher than the net gains of the remaining party α, as can be seen in fig. 2. Only the densities at the median and the quartiles are important, because the relative difference between the densities is highest for the interval between them due to the assumption of unimodality.

Please note that proving that a Nash equilibrium exists for the standard normal distribution does not contradict the original Osborne (1993) article. He was careful enough to use phrases like "almost no distribution" (p. 139) and "almost every distribution" (p. 141) and it is indeed true that most unimodal distributions violate the first necessary condition. Later interpretations of his paper often missed this careful phrasing, however, since they often do not state the very restrictive assumptions and attribute the results to any type of Hotelling(-Downs) game (see e.g. Osborne 1996, Xefteris 2016 and Ronayne 2018).

This result is also not robust against entries: A fifth party could (for the standard normal distribution) enter at the median and gain 26.4% of the votes, thus achieving a plurality and it is true that we cannot find a Nash equilibrium for five or more potential parties for the standard normal distribution. But what about a multimodal case?

In a simple symmetric bimodal case shown in figure 3, we can find equilibria for 2, 4 and 8 parties. More importantly, the equilibrium involving 4 parties is also robust against entries, which means that any further potential entrant cannot enter at any point to immediately achieve a plurality[3]. This means that if there are 8 potential parties, there are two types of equilibria: in one, 4 parties enter the competition and 4 stay out, in the other type all 8 are entering. The equilibria found for 2 and 4 parties also hold for a case, in which parties do not aim to maximize their plurality, but their (share of) votes.

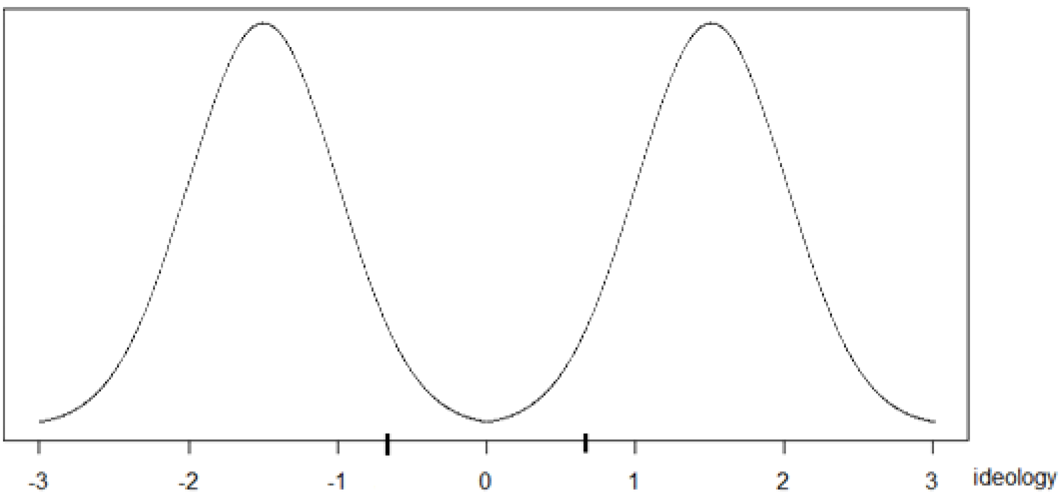


Figure 3: Bimodal distribution. For n=2 players, both players will settle at the median. For n=4 players, a left-wing and a right-wing camp will emerge, with two parties at the first quartile

[3] An entrant could enter slightly to the left or right of each mode to gain slightly less than ¼ of the votes, which would be less than the vote share achieved by at least two other parties, which is exactly ¼.

(the top of the first mode) and two parties at the third quartile (at the top of the second mode). For n=8 players, four camps emerge with two parties each at quantiles 1/8, 3/8, 5/8, 7/8.

This result is not limited to an even number of parties. If there was a distribution with three modes, we can imagine a mode in the middle with only half the density of the left- and the right-wing modes. This distribution also has a Nash equilibrium for five parties (two parties at the top of the left-wing mode, two at the right-wing mode and one at the center), which is robust against entries and is also a Nash equilibrium for vote (share) maximizing parties.

We can thus generalize our result by stating that under plurality rule, there may be one, two or four parties per mode of the voter distribution in any Nash equilibrium of the “classical” Hotelling model of political competition and are able to confirm the result of Downs (1957) that distributions with multiple nodes favor multi-party equilibria, as well as the result of Eaton and Lipsey (1975) that these equilibria exhibit “local clustering” of two. Local clustering is imperative at left-most and the right-most position occupied (a result already emphasized by Osborne), because the left-most (right-most) party could otherwise move to the right (left) to gain votes.

So far, we only considered symmetric equilibria (i.e. equilibria, in which all parties tie for the win), because those are the only equilibria possible in the presence of plurality maximizing parties, which have an outside option that would be preferred to losing. If we assume that such an outside option is not available (or that its payoff is negative), asymmetric equilibria are possible.

This is not true for a continuous unimodal distribution: As already argued before, the only possible equilibria in this case feature one, two or four parties. One party can by definition not form an asymmetric equilibrium. Two parties will always converge at the median voter. Four parties could only form an equilibrium, if two parties settle at the first quartile and two parties at the third quartile and their location is at the same time the location of their respective median voters. An asymmetric outcome is only possible, if the distance between the median voter of the left-wing camp and the total median voter does not equal the distance between the median voter of the right-wing camp and the median voter. But this violates our first necessary condition for an equilibrium. It is, however, possible for a discrete unimodal distribution (see fig. 4):

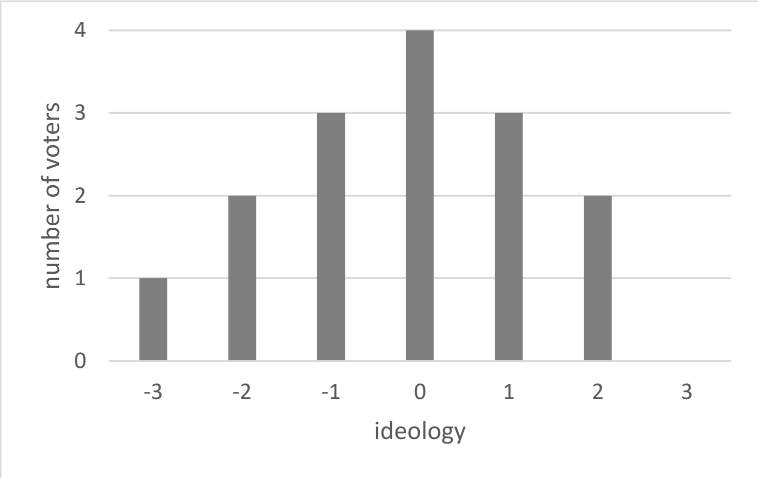


Figure 4: An asymmetric discrete voter distribution, which yields to an asymmetric 4-party-equilibrium. Two parties settle at the ideology -1 and are voted by 4, whereas two parties settle at the ideology 1 and get 3.5 votes each.

The difference between a discrete and a continuous distribution is the condition that the distance between the first quartile and the median equals the distance between third quartile and the median is met more easily in the former case. The example in fig. 4 is set on purpose to be easy and retraceable, but it is not limited to a small policy space. If the ideological space would range from -2000 to +2000 and one voter was located at -2000, two at -1999 and so on, two at +1999, but none at +2000 we would also find an asymmetric equilibrium, where the two left-wing parties would win.

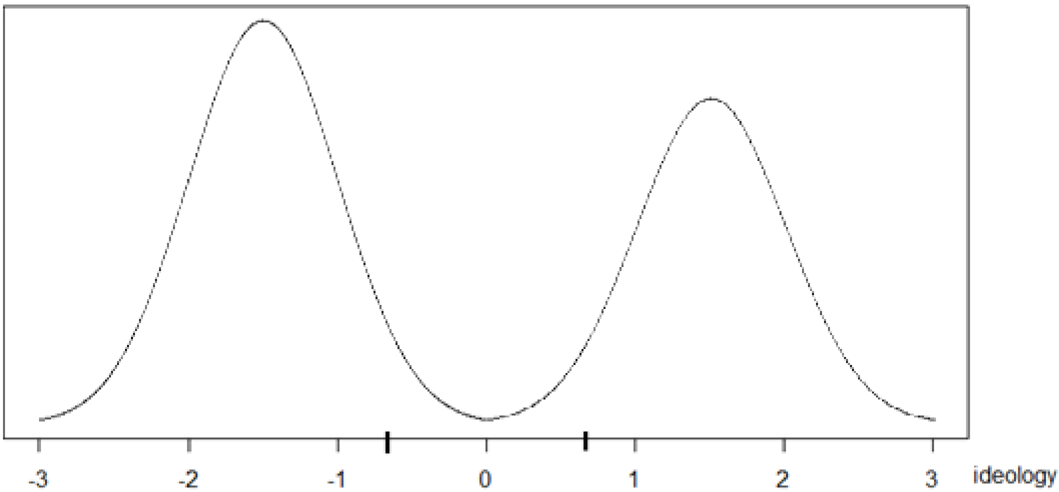
We can thus rewrite our conditions for the case of discrete voter distributions in the following way:

We can find a Nash equilibrium involving 4 parties competing for voters, which are distributed according to a discrete unimodal distribution, where two left-wing parties settle at the first quartile and two right-wing parties at the third quartile if these two necessary conditions hold:

1.) The distance between the first quartile and the median equals the distance between the third quartile and the median.
2.) The number of voters at the first/third quartile does not exceed twice the number of the median.

We will further find this equilibrium to be symmetric, if the number of voters located to the left of the median equals the number of voters located to the right of the median and asymmetric (i.e. one of the two camps win) otherwise.

Matters are different for the multi-modal case. Think of an asymmetric bimodal distribution:



Result: In an asymmetric bimodal voter distribution, a four-party equilibrium is possible, where two parties locate at the top of the left mode and two at the top of the right mode, if the following necessary condition is fulfilled:

1.) Neither of the two camps achieves less than 2/3 of the combined vote share of the other camp.

Proof: Assume that the right-wing camp only achieves 2/3 of the votes of the left-wing camp, which means that each left-wing party achieves 3/10 of the votes and each right-wing party 2/10 of the votes. A right-wing party cannot increase its vote share or plurality by moving anywhere to the right, because only 2/10 of the votes are located to the right (which is already their equilibrium vote share). It also cannot increase its vote share or plurality by moving only slightly to the left, because it would lose votes at the top of the right-wing mode to win them at the bottom of the left-wing mode, where the density is lower. It can finally increase its vote share by moving drastically to the left, entering just slightly to the left or right of the two left-wing parties to end up with 3/10 of the votes. But this move would increase the vote share of the remaining right-wing party to 4/10 of the votes, leaving the moving party with an unchanged plurality of -1/10[4]. If the right-wing camp could achieve more than 2/3 of the vote at the top of the right-wing mode, such a move would decrease the plurality of the moving right-wing party. If it was lower than 2/3, it would increase its plurality (it thus cannot be an equilibrium).

While an equilibrium involving 8 parties is thus a very special case in a bimodal voter distribution, an equilibrium an equilibrium involving 4 parties is a not-so-special case.

This exercise (which could be repeated for distributions involving more than two modes and thus an increased number of parties) tells us that the strategic behavior of political parties does not rule out a multi-party equilibrium under plurality rule. The real reason why multi-

[4] Note that if parties are “complete plurality maximizers” (e.g., Cox 1987), which means that they first maximize their plurality against the best competitor, then the difference between their votes and the second-best competitor and so on, the special case, where one camp achieves exactly 2/3 of the votes of the other camp is not an equilibrium. While a move would leave the difference against the best competitor unchanged, the difference against the second- and third-best competitors would be increased. Complete plurality maximization would, however, not change the results if both camps achieve strictly more than 2/3 of the votes of the other camp.

party systems are rare under plurality rule is therefore more likely to be found in the strategic behavior of voters (as already pointed out by Duverger 1963). Think of the bimodal distribution presented in Fig. 1. For n=4 we find a symmetric equilibrium, which is robust against entries. But it is not robust against strategic voting: Any voter of any left-wing party could switch to voting for the other left-wing party and cause an outright victory for the left-wing camp (the voters are indifferent between the two left-wing parties anyway). The same applies to the right-wing voters. But that reduces our problem to the two-party-problem again. But if we assume that both parties settle exactly at the median voter, a third party could enter at the median + ε and gain a vote share which is slightly less than 50%, but more than that of the other two parties, which would split a vote share of slightly more than 50% (i.e. slightly more than 25% for each party) and strategic voting would not change that. A stable party system under plurality rule can thus only exist, if we introduce additional assumptions: either the existence of barriers to entry or a non-centrist solution given by another model.

Hotelling-Models under Proportional Rule

The situation is different, however, for political parties competing in a proportional representation (PR) system. In a system with perfectly proportional representation, the share of seats in parliament equals in theory the share of popular votes. If we assume (as we did before for plurality rule) that parties want to maximize their political power, we can therefore assume that parties would try to maximize their vote share. If we assume that there are no abstentions, this is equal to vote maximization.

For a unimodal distribution, our equilibrium where the left-wing and the right-wing camp settle at the first and third quartile respectively collapses, since every party would increase its vote share by moving toward the median. For our bimodal distribution, the equilibrium with four parties holds.[5]

Hermsen and Verbeek (1992) explored the existence of equilibria for vote-maximizing parties in a multi-party system using an otherwise unchanged Hotelling model. Like under plurality rule, the left-most and the right-most position are in equilibrium always occupied by exactly two parties, since the left-most (right-most) party could otherwise move to the right (left) to gain votes.[6] Parties located between the left- and the right-most parties may share their ideology in equilibrium with at most one other party. If three or more parties shared the same ideology in a continuous voter distribution, one party could move slightly to the right (or left) to win 50% of the total votes shared previously by three (or more) parties and thus increase its vote share.

What about strategic voting? In a perfectly representative system, there does not seem to be an incentive to vote strategically. If two parties share the same ideology, a voter is indifferent between voting for either, because changing their vote will not change the outcome of the election. This is an important difference to strategic voting under plurality rule explored before. In practice, however, there is always a hurdle for a party to enter parliament, the lowest possible hurdle being that the number of seats are limited. In addition to that, however, most PR systems also feature a certain percentage hurdle that parties need to

[5] It was already observed by Eaton and Lipsey (1975) that the number of profit-maximizing firms for a spatial location game cannot exceed twice the number of modes of the distribution.

[6] This also explains, why there cannot be any equilibrium for n=3 parties. It is impossible to have exactly two left-most and two right-most parties if the total number of parties is 3.

surpass in order to enter parliament. Two left-wing parties that split votes may sometimes not surpass this hurdle, where a unified one does. More generally, strategic voting in a PR system does not preclude that ideologies are shared by two or more parties, as long as every party collects enough votes to pass the hurdle. It may even be beneficial for a voter to have two parties in their ideologic proximity, which share the same ideology, since the competition between the two parties would serve to keep them in line (provided that they do not collude, which may be the case in a coalition government).

4 A computational implementation of the Original Downs Model

I now move on to study the existence and character of equilibria in a novel implementation of the "original" Downs model. I choose to formalize the original Downs model by creating a computational model inhabited by heterogeneous interacting agents, i.e. a so-called agent-based model. Setting up a computational model calls for an exact specification of the implementation, which is as follows:

1.) There is a discrete number of voters N^{voters} located in a discrete ideological space ranging from x^{min} (the left-most ideology) to x^{max} (the right-most ideology). This distribution is fixed over time and known to the parties.
2.) All voters have single-peaked preferences.
3.) Voters vote truthfully for the ideologically closest political party, but only if it lies within the maximum voting distance r^{max}. If there is a tie, their vote is split. If no party is located within r^{max}, voters abstain. Voters are perfectly informed about the ideologies chosen by the parties.
4.) The model is discrete-time. Parties can change their ideology in each time step and do so sequentially. I test two specifications: In one, similar to Downs (1957a), parties are not able to freely change their ideology but can only do so incrementally, i.e., they may change their ideology by -1 (one step to the left) or +1 (one step to the right) up to x^{min} or x^{max} respectively. In the other specification, each party can choose any ideology (similar to the model studied by Osborne 1993).
5.) Existing parties only care about either maximizing their vote share (i.e., their number of votes divided by the total number of votes) or about maximizing their plurality (i.e., the difference between their vote share and the strongest other party) and thus choose the move that achieves this goal.[7] If they are indifferent between moving and keeping their ideological stance, they do not move. They do not perform any backward induction to analyze whether their move is best in the long term.

Attentive readers will notice that the assumption that parties can only incrementally change their ideology is slightly different from Downs' original assumption that parties may not "leap beyond" their competitors. Downs argues that he generalizes the original Hotelling model in a way that it allows for multi-party competition due to this specific assumption. He explains that for the case of three parties, the original model lacks an equilibrium, as the two parties to the left and to the right of the median party would try to approach it and thus steal away votes from the median party. The median party would then react by leaping over either the

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left-wing or the right-wing party to escape being squeezed out. The new median party would then behave in the same way.

This argument is, however, not very compelling. Downs is not explicit about the result of two parties adopting the same ideology. Either this is not possible. In this case, the two-party case would not have only one, as Downs argues, but an infinite amount of Nash equilibria (if we assume a continuum of ideologies, as Downs does).[8] If it is, however, possible that two parties share the same ideology, we could further assume that the votes are evenly split between those parties. In this case, Downs result for the two-party case is restored, but the three-party case again ends up in disequilibrium. A median party caught between a left-wing and a right-wing party would simply adopt the ideological stance of the strongest competing party, who would then again have an incentive to differentiate itself and so on.

The assumption that parties may only change their ideological stance incrementally does not exhibit these problematic implications of Downs' assumption, but serves the same purpose, i.e. that parties must seem reliable to the electorate and thus are not allowed to radically change their ideology frequently. I also validated this assumption using the Chapel Hill Expert Survey trend file (Polk et al. 2017 and Bakker et al. 2015) that captures (shifts of) the ideological stance of all major and many minor political parties in the EU as perceived by experts. Although the shifts in countries that experienced greater instability like Spain can be more pronounced than in very stable countries like Germany, dramatic shifts are extremely rare (see Fig. 5 for the German case).

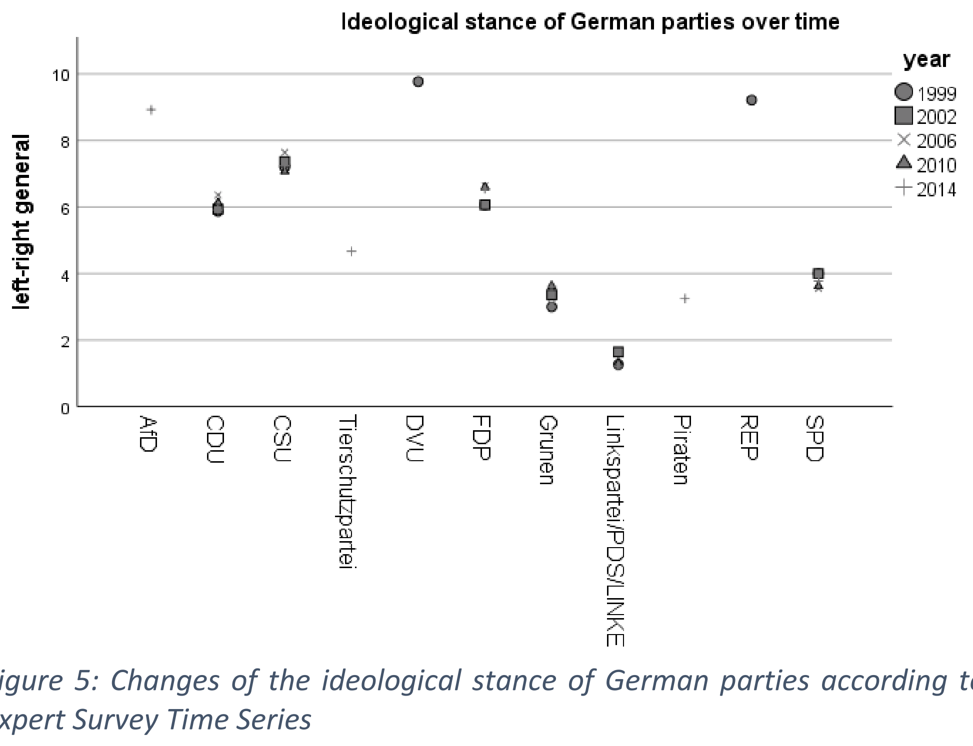

Figure 5: Changes of the ideological stance of German parties according to the Chapel Hill Expert Survey Time Series

[8] As soon as one party is right (or left) to the median voter with the competing party even further to the right (left), the party would be invincible in any election and thus have no incentive to shift their ideological position. The other party could only come very close without ever winning a plurality. This is at least true for the uniform distribution. For other distributions, it is also true for forward-looking parties.

Parties in equilibrium

Depending on the voter distribution, the abstention parameter, number and initial location of the parties, multiple equilibria are possible for this model.

I first present some analytical results and illustrative examples for the existence of equilibria for 2, 3 or >=4 parties which are only able to move incrementally. I then move on to simulations to understand how likely it is that multi-party systems will end up in an equilibrium in this model and compare it to the "classical Hotelling model" a la Osborne (1993).

N=2 parties

Consider first the two-party case. If either of the two new assumptions does not hold, two vote share or plurality maximizing parties will always (sooner or later) converge towards the median voter and settle there for an equilibrium. If both assumptions hold, but the voter distribution is unimodal, both parties will converge as well, but depending on the distribution not necessarily at the median voter.[9] Those equilibria are, however, not robust against entry: In almost all cases, a new political party could enter slightly to the right or to the left of the two parties and win an outright majority.[10] If the voter distribution is multimodal, however, equilibria in which both parties adopt distinct political platforms are possible, provided that the maximum voting range is small enough. The most illustrative case is a polarized distribution (i.e., a bimodal one).

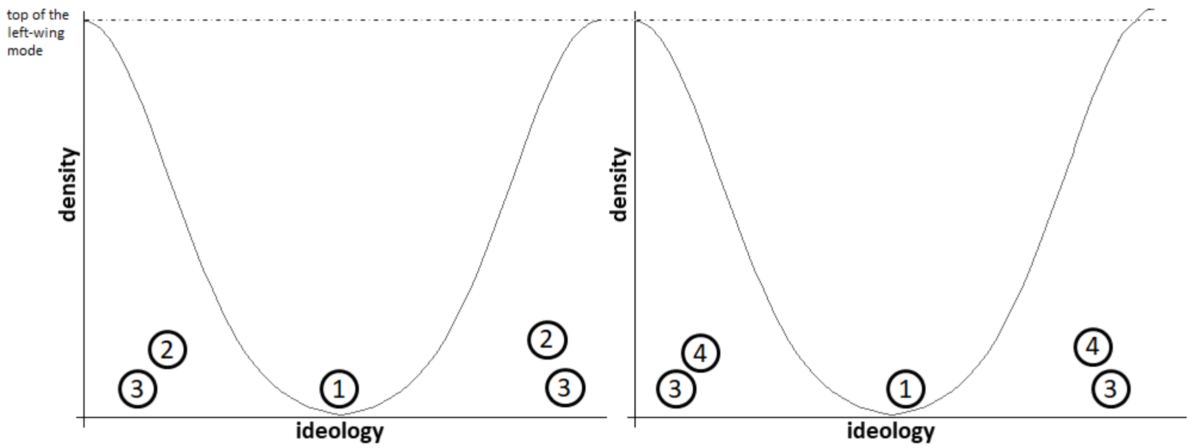

Figure 6: A symmetric (left) and an asymmetric (right) polarized voter distribution and possible locations of parties A and B in equilibrium for n=2 total parties. (1): The location of A and B for a "classical" Hotelling model a la Osborne (1993). (2): A or B for the model of Downs (1957). (3): Additional A and B for the new model. (4): Additional A or B for the new model. Depending on the parameter setting, the new model also contains equilibria (1) and (2).

If the maximum voting range is very large (or absent), this distribution would yield a (very unconvincing) equilibrium, where both parties settle at the political center, which is populated by almost nobody as it is the case for the "classical" Hotelling model. If the maximum voting

[9] This is true if, e.g., the only mode is the very left or very right position. In such a situation it would be beneficial to move from the median voter towards the mode, though not all the way. In equilibrium the two parties gain as much votes from their left-wing voters as they get from their right-wing voters.
[10] In the other cases, such as when all voters share the same ideology, a new entrant could simply enter with the same ideology as the two incumbents and tie for the win.

range is small enough, however, it features three equilibria, two of which are still unconvincing: either both parties settle at the left-most ideology, or both parties settle at the right-most ideology, or one party at the left-most and one at the right-most ideology. All unconvincing equilibria are prone to entry, but the divergent equilibrium is robust against it. The model by Downs (1957a) has the advantage that it precludes the unconvincing equilibria. For the case of an asymmetric polarized voter distribution, however, an equilibrium fails to exist for his model, because the losing party could adopt the ideology of the winner, which is not possible, if movement is restricted to the ideological neighborhood.

N=3 parties

If the maximum voting distance is small enough, equilibria are now even possible for three parties. Consider as a simple example a uniform distribution with a normalized continuous ideological space ranging from 0 to 1. If the maximum range allows a party to attract voters within 1/6 of the ideological space to the left and 1/6 to the right, a symmetric equilibrium is possible, where one party settles at 1/6, the second at 3/6 and the third at 5/6.

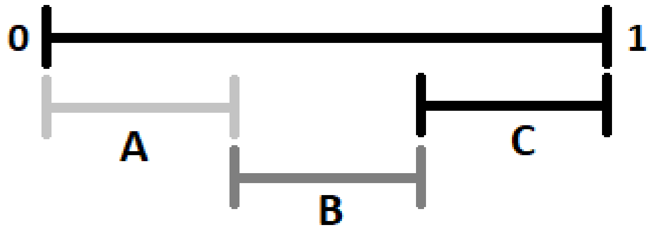

Figure 7: Symmetric equilibrium for three parties in a uniform distribution

If the maximum range is smaller, infinitely many symmetric equilibria are possible. All the aforementioned equilibria are also possible, if parties are not restricted to incremental change. They thus also can be considered to be Nash equilibria for a game in which they choose their platforms simultaneously. If movement is restricted and parties are vote share-maximizing and not plurality-maximizing, a smaller maximum voting range can also lead to asymmetric equilibria.

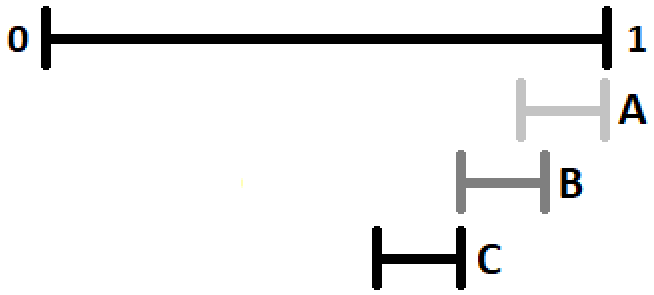

Figure 8: Asymmetric equilibrium for vote-share maximizing parties with a low maximum vote range and restricted movement. C wins the election, neither A nor B can increase their vote share by moving only slightly to the left or right.

If the range is larger however, unrestricted movement cannot result in an equilibrium, since the left and the right-wing parties would approach the center again and the center would counter this move by moving left of the left-wing party or right of the right-wing party. For the case of restricted movement, however, the center-ground party cannot immediately "jump beyond" a competitor. We therefore find an equilibrium, in which (for the uniform distribution case) the left- and the right-wing parties will be victorious and "squeeze out" the center. The larger the maximum voting range gets, the closer will the left- and the right-wing party come to the center. If the maximum voting range is large enough, they will settle close enough to the center such that the center party is able to adopt the ideology of one of the two other parties, resulting in disequilibrium again.

In a unimodal distribution, where the mode is located sufficiently far away from the edge of the political spectrum, such equilibria are impossible, since either the left or the right-wing party could improve their vote share by moving toward the mode, until the center party can overcome this party once again. In a bimodal distribution, an equilibrium is possible if two parties are located at the left (or right) and the other one at the right (or left) mode. If all parties are initially located at one mode, reaching an equilibrium might be impossible, if the maximum voting range is too low and there is thus no incentive for a party to converge to the other end of the political spectrum. A distribution with three or more modes is finally very suitable for a 3-party equilibrium, where each party is located at the top of one node (provided that the modes are not "too different").

N>=4 parties

If the maximum voting range is smaller than or equal to 1/2N, we will *mutatis mutandis* find symmetric and asymmetric equilibria for the uniform distribution case as in the case with 3 parties. If the maximum voting range is large enough and movement is unrestricted, we are back in a case where the maximum voting range does not matter at all. [11] If movement is restricted, however, the results are hard to tract analytically. Multi-party systems typically feature more than three parties (and sometimes a lot more than that). To better understand the model dynamics in these cases, I hence turn to simulations.

Simulation setup

As is common in the literature on agent-based models (see, e.g., Dosi et al. 2010), I investigate the properties of the model using an array of simulations, known as Monte Carlo method. This approach helps to rigorously investigate the model properties in spite of the fact that each single simulation run can be influenced by stochastic factors.

I run Monte Carlo simulations with 3000 different voter distributions: 1000 are drawn from a uniform distribution, where each voters' ideology is between x^{min} (-10) or x^{max} (+10). This process effectively creates a multi-modal distribution. The remaining 2000 are drawn from normal distributions with a mean ideology of 0 and a standard deviation of 4 (for 1000 voter distributions) – which amounts to a unimodal voter distribution – and a standard deviation of 12 (for the remaining 1000 distributions). If a voter would be assigned to an ideology lower than the minimum of -10, I set it to -10. The same procedure applies to values larger than 10. Hence, the last configuration typically produces a voter distribution with three large modes (left, center, right). All distributions are created with a pseudo-random generator commanded by a random-seed, which allows us to a) study each configuration with exactly the same voter distributions and b) to replicate the results. Example voter distributions which are created by the three algorithms are illustrated in Fig. 9.

[11] Two parties on the left and two parties on the right will in equilibrium always share the same ideologies respectively.

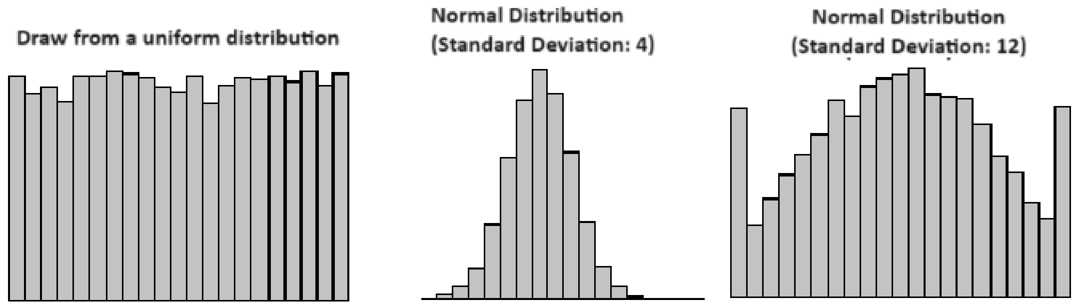

Figure 9: Distributions created by the various voter distribution algorithms using the random seed 1.

Each voter distribution is then used to simulate the model in various parameter setups:

Number of parties	4-8
Maximum voting distance	5/10/21
Optimizing behavior	Vote maximization / plurality maximization
Available strategies	Incremental change / all ideologies

I simulate each parameter combination, resulting in 3000*5*3*2*2=180,000 combinations for 500 different simulation runs. In each run, the parties are placed randomly on the political spectrum and are able to take action according to their available strategies for 100 periods. I then record whether the parties should reach an equilibrium (i.e., no party wants to change their strategy anymore) in one of these 500 runs.

Figure 10 shows the simulation results for parties which engage in vote maximization. We can see that a model which stays true to Downs (1957a) and (1957b), i.e. a reasonably low maximum voting range, and the assumption that parties generally are not able to freely change their ideology from one period to the next, is much more conducive to the existence of equilibria than the classical Hotelling model, where parties can choose any ideology they want to.

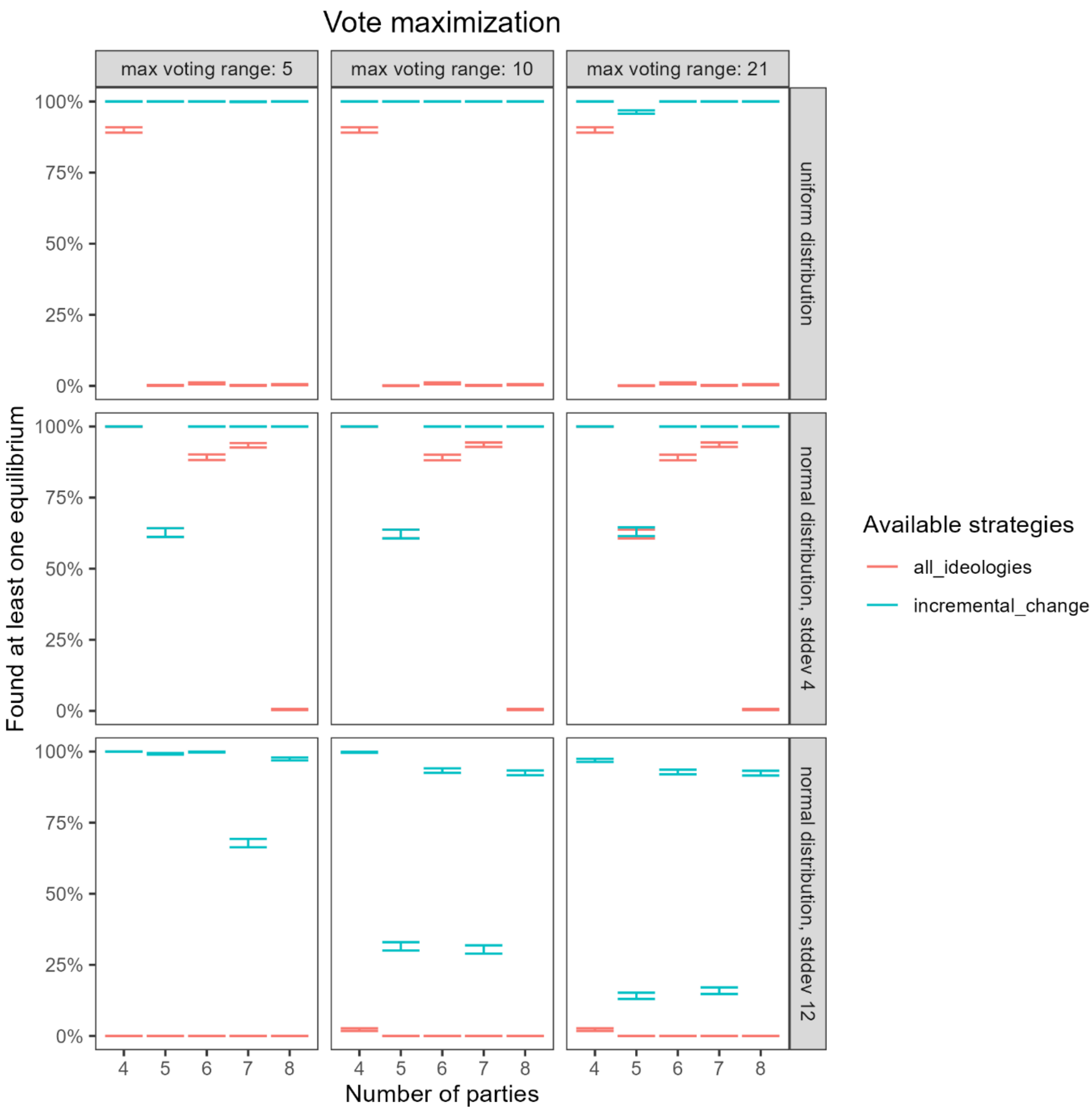

Figure 10: Share of parameter combinations where an equilibrium was found (vote maximization).

Figure 11 then shows the results for plurality-maximizing parties. Here, we can see the biggest differences between a situation in which parties can only incrementally change their ideology and a situation in which parties can choose any ideology at will for odd numbers of parties (i.e., 5 and 7). Furthermore, the maximum voting range seems to play a large role for odd numbers of parties.

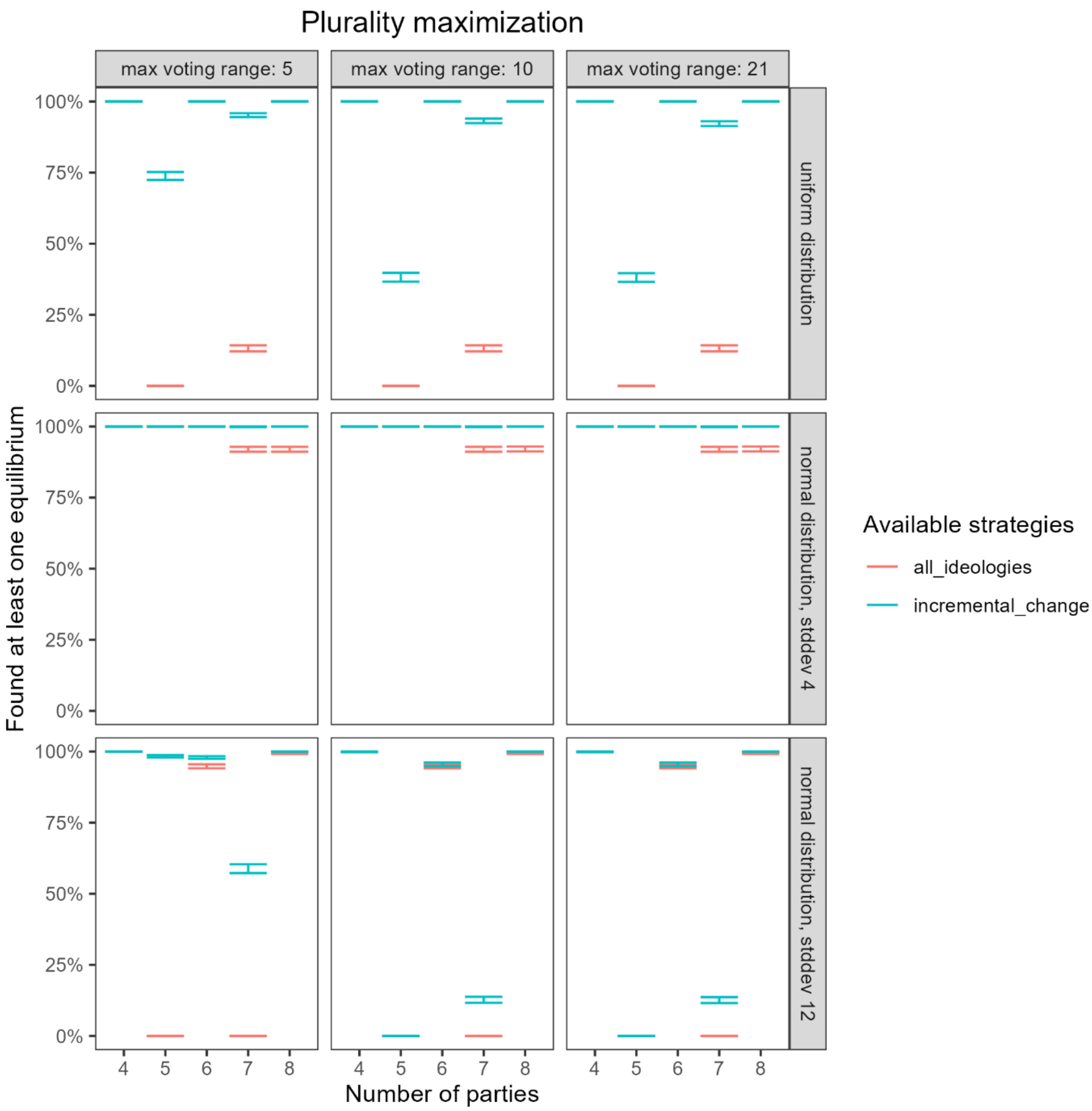

Figure 11: Share of parameter combinations where an equilibrium was found (plurality maximization).

In the final version of the paper, I will provide a better characterization of the resulting equilibria.

5 Conclusion

In this paper, I argued that the popular characterization of the so-called "Hotelling-Downs model" overlooks important nuances in the assumptions made by Downs (1957a). This misunderstanding caused subsequent authors to assume that the Downsian approach would be incapable of dealing with multi-party systems.

After critically discussing this line of literature, focusing in particular on the famous contribution by Osborne (1993), I draw on computational modelling to create a so-called agent-based model. This flexible method allows me to better implement two crucial assumptions by Downs (1957a) which are hard to tract analytically: Namely, that i) voters do not vote for parties which are ideologically "too distant", even if they are the closest, and ii) parties cannot freely determine their ideology, but are constrained by the ideology they pursued in the previous period.

I showed that adding these two assumptions drastically increases the share of voter distributions for which I find at least one equilibrium. This is particularly true for vote-maximizing parties, which is highly important as multi-party equilibria are improbable in an electoral system operating under the plurality rule due to strategic voting.

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