Production Theory: An Introduction

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1. Introduction

It is a characteristic feature of industrial economies that commodities are produced by means of commodities. The idea popular amongst Austrian economists (Carl Menger, Eugen von Böhm-Bawerk, Friedrich August von Hayek) that production can be conceived of as a one-way avenue of finite extension leading from the services of primary factors of production, labour and land, via a series of intermediate products, or capital goods, to consumer goods, cannot be sustained other than in singularly special cases. Production is typically circular rather than linear or unidirectional. It is another characteristic feature of these economies that production generally requires in addition to circulating capital fixed capital. While the circulating part of the capital goods advanced in production contributes entirely and exclusively to the output generated, that is, ‘disappears’ from the scene, so to speak, the fixed part of it contributes to a sequence of outputs over time, that is, after a single round of production its items are still there – older though, but still useful.

An analytical treatment of these two facts will be a major concern of the following analysis. However, it should be emphasized right at the beginning of this essay that, given the space limits to be respected, we can only offer an elementary discussion of a small subset of the problems arising in the present context. The reader interested in a more comprehensive and thorough treatment of the subject is asked to consult Kurz and Salvadori (1995). The severe limitations of the present analysis will become clear in the following.

All production, as we shall understand it in this paper, involves some dealing of man with nature. As John Stuart Mill put it, ‘man can only move matter, not create it’. The utilization of natural resources is indeed indispensable in production. There can be no doubt about this. When in certain theoretical conceptualizations this fact is not visible, then this does not mean that it is not there. It only means that the authors have for simplicity set aside the problem by assuming that natural resources are available in abundance. This amounts to assuming that their services are ‘free goods’. This is a bold assumption, not least with regard to advanced economies which are typically characterized by the scarcity of some of their lands and the depletion of some of their stocks of raw materials etc. As is well known, the treatment of exhaustible resources poses formidable problems for economic theorizing. In this short essay
we shall set aside the problem of scarce natural resources. (See, however, Kurz and Salvadori, 1995, chaps 10 and 12, for treatments of the problems of land and renewable and exhaustible resources, respectively.) Hence, in the models we shall be dealing with there will be no decreasing returns. Further, since we shall not be concerned with economic growth and the effects on labour productivity of the increase in the division of labour associated with the accumulation of capital, effects stressed by authors from Adam Smith via Allyn Young to modern growth theorists, we shall also set aside increasing returns. The assumption underlying the following discussion is therefore that throughout the economy there are constant returns to scale and no external effects.

It is also an empirical fact that production generally involves joint production, that is, the generation of more than one physically discernible output. Cases in point are the production of wool and mutton and of corn and straw. Often one or several of the joint products are ‘bads’ or ‘discommodities’ which nobody wants, but whose production is necessarily involved, given the technical options that are available, if the commodities that are wanted are to be produced. The emergence of a bad is a particularly obvious case in which the question of disposal cannot possibly be avoided. Much of economic literature assumes that disposal processes do not incur any costs whatsoever. However, from the point of view of the realism of the analysis the assumption of ‘free disposal’ is difficult to defend: most, if not all, disposal processes are in fact costly. The implication of this is that the product that is to be disposed of fetches a negative price, that is, the agent who is willing to take the product does not have to pay a price for it, but is paid a price for his willingnessness to take it. In this essay we shall set aside the intricacies of joint production proper and for the most part deal exclusively with economic systems characterized by single production. For a treatment of joint production, see Kurz and Salvadori (1995, ch. 8).

Conceiving production as a circular flow does not mean that all commodities produced in an economy play essentially the same role in the system of production under consideration. In the classical economists, in particular Adam Smith and David Ricardo, we encounter the distinction between ‘necessaries’ and ‘luxuries’. The former concept denotes essentially all commodities entering the wage basket in the support of workers. And since workers are taken to be paid at the beginning of the production period, wages form an integral part of the capital advanced in production. Moreover, since in the classical economists labour is taken to be an input needed directly or indirectly in the production of all commodities, necessaries and the means of production needed directly or indirectly in their production can be said to be indispensable in production in general and thus to ‘enter’ into all commodities, including themselves. Luxuries, like wage goods or necessaries, are pure consumption goods. However, in contradistinction to the latter the classical economists did not consider them as necessary in order to keep the
process of production going. In the terminology of the classical authors, necessaries belong to ‘productive consumption’, whereas luxuries belong to ‘unproductive consumption’. We encounter variants of the classical distinction in many later authors (see Kurz and Salvadori, 1995, chap. 13). The perhaps best known modern version of it is Piero Sraffa’s distinction between ‘basic’ and ‘non-basic’ commodities. Basics are defined as commodities that enter directly or indirectly into the production of all commodities, whereas non-basics do not (Sraffa, 1960, pp. 7-8). Since Sraffa treats wages in most of his analysis as paid at the end of the production period, the wages of labour do not belong to the capital advanced, as in the classical authors. Therefore, his criterion of a commodity entering directly or indirectly into the production of all commodities is a purely technical one. In the following we shall set aside all difficulties concerning the existence of non-basic goods by assuming that only two commodities will be produced, both of which are basics. Hence all commodities are taken to rank equally, each of them being found both among the outputs and among the inputs, and each directly or indirectly entering the production of all commodities. The reader interested in a discussion of cases with basic and non-basic products is referred to Kurz and Salvadori (1995, chs 3 and 4).

Production requires time. The assumption of ‘instantaneous’ production entertained in conventional microeconomic textbooks can only be defended as a first heuristic step in the analysis of a complex phenomenon. The different activities involved in the generation of a product typically exhibit different lengths of time. Here we shall assume for simplicity that what will be called the ‘periods of production’ are of uniform length throughout the economy. As James Mill stressed with regard to the approach chosen by the classical economists: ‘A year is assumed in political economy as the period which includes a revolving circle of production.’ Using a uniform period of production (of a possibly much shorter length) involves, of course, the introduction of some fictitious products, occasionally referred to as 'semi-finished products' or 'work in progress'.

With fixed capital there is always a problem of the choice of technique to be solved. This concerns both the choice of the pattern of utilization of a durable capital good and the choice of the economic lifetime of such a good. The utilization aspect in turn exhibits both an extensive and an intensive dimension. The former relates to the number of time units within a given time period (day, week) during which a durable capital good is actually operated, for example, whether a single-, a double-, or a treble-shift scheme is adopted; the latter relates to the intensity of operation per unit of active time (hour) of the item, for example, the speed at which a machine is run. The economic lifetime of a fixed capital good and the pattern of its operation are, of course, closely connected. Yet things are more complex because modern production processes are increasingly characterized by the joint utilization of durable means of production, a fact emphasized in contributions to the genre of industrial-technological literature, which was
an offspring of the Industrial Revolution. According to Marx modern industry was characterized by a 'system of machinery', an 'organised system of machines'. A proper discussion of fixed capital, including the empirically important case of the joint utilization of durable capital goods, however, is beyond the scope of this essay. The interested reader is once more referred to Kurz and Salvadori (1995, chs 7 and 9). Here we shall restrict ourselves to the analysis of an exceedingly simple case with only a single type of fixed capital good, which is designed to illustrate some of the issues raised by the presence of durable instruments of production.

There is another aspect that underscores the importance of fixed capital, and which should at least be mentioned. In economic systems which are subject to the principle of effective demand there is no presumption that the levels of aggregate effective demand will be such as to allow producers (possibly over a succession of booms and slumps) to utilize the productive capacity installed in the desired way. In particular, aggregate effective demand may fall short of productive capacity, with the consequence of idle plant and equipment and unemployed workers. The variability in the overall degree of utilization of plant and equipment (and, correspondingly, of the workforce) in combination with countercyclical storage activities is responsible for the remarkable elasticity of the modern industrial system, which is able to translate even widely fluctuating levels of effective demand into fluctuating levels of output and employment. However, in the following we shall set aside the problem of capital utilization in conditions of effective demand failures. We shall rather boldly assume that plant and equipment can be utilized at the normal desired degree.

We now have to specify the institutional setting to which the following analysis is taken to apply. The specification of the institutional setting has important implications for the method of analysis adopted. We shall assume that there is free competition, that is, there are no significant barriers to entry in or exit from an industry. In these conditions producers will be concerned with minimizing costs of production. The result of this concern will be a tendency toward a uniform rate of profits on capital throughout the economy. The following analysis will indeed focus attention on what may be called cost-minimizing systems of production. The prices analysed are taken to express the persistent, non-accidental and non-temporary forces governing the economy. They correspond to what the classical economists called 'normal' or 'natural' prices or 'prices of production'. The method of analysis congenial to this setting is known as the long-period method. It is indeed the application of this method that characterizes the propositions derived in this essay. As we shall see, in the context of the simplified analysis presented here normal prices depend only on two factors:

(i) the real wage rate, or, alternatively, the rate of profits; and
In the following we shall treat the level of the rate of profits as an independent variable. That is, we shall refrain from entering into a discussion of the factors affecting that level, or, in other words, from elaborating a theory of income distribution. Suffice it to emphasize that the kind of approach developed in this essay is incompatible with a determination of the rate of profits in terms of the supply of and the demand for a factor called ‘capital’, the quantity of which could be ascertained prior to and independently of the determination of the rate of profits and relative prices. (On this see Kurz and Salvadori, 1995, ch. 14). The reader will see that for the purpose of our analysis it makes no fundamental difference, whether the rate of profits or the real wage rate is taken as given from outside the system of production. We shall follow Sraffa’s lead who took the rate of profits as the independent variable.

There are a number of additional simplifying premises underlying the following analysis which should be mentioned. In parentheses we shall in some cases refer to contributions that transcend the limitations of this essay. The reader interested in getting a richer picture of the kind of analysis provided below is asked to consult the works cited. We shall deal with a closed economy without government (see, however, Steedman, 1979). There is no technical or organizational progress, that is, the set of technical alternatives mentioned in (ii) is given and constant (see, however, Schefold, 1976). Labour is assumed to be homogeneous, or, what amounts to the same thing, differences in quality are reduced to equivalent differences in quantity in terms of a given and constant structure of relative wage rates, the problem of the creation of skilled labour powers is set aside, and the work effort per hour is given and constant (see, however, Kurz and Salvadori, 1995, ch. 11). Wages are paid at the end of the production period.

The composition of the essay is the following. Section 2 presents the single-products model with two commodities, both of which are basic. We shall first assume that there is no choice of technique, that is, there is only a single method of production available for each commodity. In a first step we shall define the concept of viability of the economy under consideration. Next we shall determine the maximum rate of profits and then analyse the dependence of prices on the level of the rate of profits, given the technical conditions of production. Section 3 allows for a choice of technique. We shall assume that there are several methods of production available to produce each of the two commodities by means of themselves. Given the rate of profits, the question is which of the different methods will be chosen by producers. More precisely, we are interested in determining the cost-minimizing technique(s) of production. It will be shown that cost minimization involves the maximization of the dependent distributive variable, that is, the wage rate. Section 4 provides a simple model with fixed capital. It will be assumed that a
'tractor' can be produced by means of labour and 'corn' and can then be used in the production of corn. To keep things as simple as possible, it will be assumed that the tractor lasts at most for two periods and that old tractors can be disposed of at zero cost. It will be shown that there is a choice of technique problem involved concerning the length of time for which the tractor will be used (that is, one or two periods). Section 5 allows for different modes of operation of a fixed capital good; the case contemplated is single- and double-shift work. Section 6 contains some concluding considerations.

2. Two basic commodities

The two commodities will be called ‘corn’ (c) and ‘iron’ (i). Corn and iron are produced either directly or indirectly by means of corn and iron, that is, the roles of the two commodities in production are symmetrical. Table 1 summarizes the technical features of the two production processes. Accordingly, $a_{kh}$ ($h, k = c, i$) units of commodity $h$ and $l_k$ units of labour are needed to produce one unit of commodity $k$.

<table>
<thead>
<tr>
<th>material inputs</th>
<th>outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>corn process</td>
<td>corn</td>
</tr>
<tr>
<td>corn</td>
<td>$a_{cc}$</td>
</tr>
<tr>
<td>iron process</td>
<td>$a_{ic}$</td>
</tr>
</tbody>
</table>

A commodity $h$ will be said to enter directly into the production of commodity $k$, if

$$a_{kh} > 0,$$

where $h, k = c, i$. A commodity $h$ will be said to enter directly or indirectly into the production of commodity $k$, if

$$a_{kh} + a_{kc}a_{ch} + a_{ki}a_{ih} > 0.$$
\[ a_{cc} + a_{cc}^2 + a_{ci}a_{ic} > 0 \]
\[ a_{ic} + a_{ic}a_{cc} + a_{ii}a_{ic} > 0 \]
\[ a_{ci} + a_{cc}a_{ci} + a_{ci}a_{ii} > 0 \]
\[ a_{ii} + a_{ic}a_{ci} + a_{ii}^2 > 0. \]

Obviously, all the previous inequalities are satisfied if and only if
\[ a_{ci}a_{ic} > 0. \] (1)

An economy is said to be **viable** if it is able to reproduce itself. In the present context this means that there exist two real numbers \( Y_c \) and \( Y_i \), that is, activity intensities of the two processes, such that

\[ Y_c \geq Y_c a_{cc} + Y_i a_{ic} \] (2a)
\[ Y_i \geq Y_c a_{ci} + Y_i a_{ii} \] (2b)
\[ Y_c \geq 0, \ Y_i \geq 0, \ Y_c + Y_i > 0. \] (2c)

More precisely, we say that the economy is **just viable** if both weak inequalities (2a) and (2b) are satisfied as equations; and we say the economy is **able to produce a surplus** if at least one of them is satisfied as a strong inequality. Since inequality (1) holds, inequalities (2a) and (2b) imply that if at least one of the \( Y \)'s is positive, then both are. Hence, inequalities (2c) can be stated

\[ Y_c > 0, \ Y_i > 0. \] (2c')

From inequality (2a), taking account of inequalities (2c') and (1), one obtains that

\[ 1 - a_{cc} > 0 \] (4a)
\[ \frac{Y_c}{Y_i} \geq \frac{a_{ic}}{1 - a_{cc}}. \] (4b)

Similarly, from (2b),

\[ 1 - a_{ii} > 0 \] (4c)
\[ \frac{1 - a_{ii}}{a_{ci}} \geq \frac{Y_c}{Y_i}. \] (4d)
Therefore inequalities (2) are consistent if and only if (4a) and (4c) hold and there is a real number \( \frac{Y_c}{Y_i} \) such that

\[
\frac{1 - a_{ji}}{a_{ci}} \geq \frac{Y_c}{Y_i} \geq \frac{a_{ic}}{1 - a_{cc}}.
\]

Hence the economy is viable if and only if

\[
1 - a_{cc} > 0 \tag{5a}
\]

\[
(1 - a_{cc}) (1 - a_{ii}) - a_{ci}a_{ic} \geq 0. \tag{5b}
\]

Inequality (4c) is not mentioned since it is a consequence of inequalities (5). The reader will easily recognize that if the economy is just viable, then the weak inequality (5b) is satisfied as an equation, whereas if the economy is able to produce a surplus, then the weak inequality (5b) is satisfied as a strong inequality.

Let \( p_i \) be the price of one unit of iron in terms of corn (obviously the price of corn in terms of corn equals 1). Then, if a uniform rate of profits is assumed, and if the wage rate is set equal to zero, the following equations hold:

\[
(1 + R)a_{cc} + (1 + R) a_{cij}p_i = 1 \tag{6a}
\]

\[
(1 + R)a_{ic} + (1 + R) a_{iij}p_i = p_i \tag{6b}
\]

where \( R \) is the maximum rate of profits. Further on, in order to be sensible from an economic point of view, it is required that

\[
p_i > 0, \quad 0 \leq R \leq 1. \tag{6c}
\]

Let us prove that system (6) has one and only one solution. To simplify the exposition, let us set

\[
\lambda = \frac{1}{1 + R}. \tag{7}
\]

Because of (7) system (6) can be rewritten as

\[
a_{cc} + a_{cij}p_i = \lambda \tag{8a}
\]

\[
a_{ic} + a_{iij}p_i = \lambda p_i \tag{8b}
\]

\[
p_i > 0, \quad 0 < \lambda \leq 1. \tag{9}
\]
System (8) is equivalent to the following system:

\[ z(\lambda) := \lambda^2 - (a_{cc} + a_{ii})\lambda + (a_{cc}a_{ii} - a_{ci}a_{ic}) = 0 \]  
(10a)

\[ p_i = \frac{\lambda - a_{cc}}{a_{ci}} \]  
(10b)

It is easily checked that

\[ z(a_{cc}) = z(a_{ii}) = -a_{ci}a_{ic} < 0 \]

\[ z(1) = (1 - a_{cc}) (1 - a_{ii}) - a_{ci}a_{ic} \]

Hence, if the economy is viable, then \( z(1) \geq 0 \) or, more precisely, if the economy is able to produce a surplus, \( z(1) > 0 \), whereas if it is just viable, \( z(1) = 0 \). Thus one of the two solutions to equation (10a), \( \lambda^* \), is larger than \( \max(a_{cc}, a_{ii}) \) and smaller than 1 (if the economy is able to produce a surplus) or equal to 1 (if the economy is just viable). The other solution, on the other hand, is smaller than \( \min(a_{cc}, a_{ii}) \) but not smaller than \( -\lambda^* \) since

\[ z(-\lambda^*) = 2(a_{cc} + a_{ii})\lambda^* \geq 0 \]

(\( z(-\lambda^*) = 0 \) if and only if \( a_{cc} + a_{ii} = 0 \)). Finally, if we take account of equation (10b), only the solution to equation (10a) larger than \( a_{cc} \) can be associated with a positive \( p_i \).

Thus, system (6) has one and only one solution if and only if the economy is viable. Otherwise there is no solution. \( R > 0 \) if and only if the economy is able to produce a surplus, \( R = 0 \) if and only if the economy is just viable. Figure 1 gives an example in which the economy is able to produce a surplus, \( 0 < a_{ii} < a_{cc} \) and \( a_{cc}a_{ii} < a_{ci}a_{ic} \) (if \( a_{cc}a_{ii} > a_{ci}a_{ic} \), the solution to equation (10a) associated with a negative \( p_i \) is positive).
Equations (6) determine the price of iron when the rate of profit, $r$, equals $R$. With $r < R$, the following equations hold:
where the wage rate, \( w \), and the price of iron, \( p_i \), are measured in terms of corn. For each given \( r \) such that \( 0 \leq r \leq R \), equations (11) constitute a linear system in \( p_i \) and \( w \) whose solution is

\[
\begin{align*}
\frac{w}{1+r} &= \frac{1 - (a_{cc} + a_{ii})(1+r) + (a_{cc}a_{ii} - a_{ci}a_{ic})(1+r)^2}{(1+r)a_{ci}l_i + [1 - (1+r)a_{ii}]l_c} \quad (12a) \\
\frac{p_i}{1+r} &= \frac{[1 - (1+r)a_{cc}]l_i + (1+r)a_{ic}l_c}{(1+r)a_{ci}l_i + [1 - (1+r)a_{ii}]l_c} \quad (12b)
\end{align*}
\]

Equation (12a) is known as the \textit{w-r relationship}. From equations (6a) and (6b), and taking into account inequalities (1) and (6c), we obtain

\[
1 - (1 + R)a_{cc} > 0 \\
1 - (1 + R)a_{ii} > 0.
\]

This is enough to prove that if \(-1 \leq r \leq R\), \( p_i \) as defined by equation (12b) and the denominator of the fraction in equation (12a) are positive. Moreover, since the equation (10a) has no solution greater than \( \frac{1}{1+r} \), then

\[
\left( \frac{1}{1+r} \right)^2 - (a_{cc} + a_{ii}) \left( \frac{1}{1+r} \right) + (a_{cc}a_{ii} - a_{ci}a_{ic}) > 0,
\]

for \(-1 < r < R\). Thus, \( w > 0 \) if \( 0 \leq r < R \) and \( w = 0 \) if \( r = R \).

When there is a variation in the rate of profits \( r \), the wage rate \( w \) also varies. Let us assume that \( r \) and \( w \) move in the same direction, for example, they increase simultaneously. Then equation (11a) requires that \( p_i \) falls, but equation (11b) rewritten as

\[
\frac{1}{p_i} \left\{ (1+r)a_{ic} + wl_i \right\} = 1 - (1+r)a_{ii}
\]

requires that \( p_i \) rises. Hence we have a contradiction. Thus \( r \) and \( w \) cannot move in the same direction, that is, \( w \) is a decreasing function of the rate of profits.

3. Choice of technique
Up till now it has been assumed that there is only one way to produce each of the two commodities. In this section this assumption will be removed. In order to do so the following concepts are defined:

A method or process of production, or, for short, a process, to produce commodity h \((h = c, i)\) is defined as the triplet \((a_{hc}, a_{hi}, l_h)\). The set of all available processes is called a technology.

Assume that there exist \(u\) processes to produce corn and \(v\) processes to produce iron. These processes are referred to as

\[
(a_{hc}^{(h)}, a_{hi}^{(h)}, l_c^{(h)}) \quad h = 1, 2, ..., u
\]

\[
(a_{ic}^{(k)}, a_{ii}^{(k)}, l_i^{(k)}) \quad k = 1, 2, ..., v
\]

In addition, assume that

\[
a_{hc}^{(h)} > 0, a_{ic}^{(h)} > 0.
\]

Let \(w, r\), and \(p_i\) denote the ruling wage rate, rate of profits, and iron price, respectively. Then, processes \((a_{cc}^{(h)}, a_{ci}^{(h)}, l_c^{(h)})\) and \((a_{ic}^{(k)}, a_{ii}^{(k)}, l_i^{(k)})\) are (are not) able to pay extra profits if

\[
(1+r)a_{cc}^{(h)} + (1+r)a_{ci}^{(h)}p_i + wl_c^{(h)} < 1 \quad (\geq 1)
\]

\[
(1+r)a_{ic}^{(k)} + (1+r)a_{ii}^{(k)}p_i + wl_i^{(k)} < p_i \quad (\geq p_i),
\]

and they do (do not) incur extra costs if

\[
(1+r)a_{cc}^{(h)} + (1+r)a_{ci}^{(h)}p_i + wl_c^{(h)} > 1 \quad (\leq 1)
\]

\[
(1+r)a_{ic}^{(k)} + (1+r)a_{ii}^{(k)}p_i + wl_i^{(k)} > p_i \quad (\leq p_i),
\]

respectively. If a process is able to pay extra profits, producers would seek to adopt the new process in order to obtain the extra profits, and if they succeeded in doing so, the resulting rate of profit in the particular industry would be larger than \(r\).

In a long-period position at rate of profits \(r\) no producer can obtain a higher rate of profit by operating another process because it is a position of rest (given the data of the problem, including the level of the rate of profits). It should also be noticed that because none of the two commodities can be produced without the other also being produced, in a long-period position at least one process to produce each of the two commodities has to be operated. Hence, the pair \((w, p_i)\) is a long-period position at rate of profits \(r\), if the rate of profits \(r\), the wage rate \(w\), and the price of iron \(p_i\) are such that no process is able to pay extra profits and that there is at least
one process producing corn and at least one process producing iron that do not require extra costs. Accordingly, \((w, p_i)\) represents a long-period position at the rate of profits \(r\) if

\[
(1+r)a^{(s)}_{cc} + (1+r)a^{(s)}_{ci}p_i + w l^{(s)}_c = 1 \quad \text{some } s
\]

\[
(1+r)a^{(t)}_{ic} + (1+r)a^{(t)}_{ii}p_i + w l^{(t)}_i = p_i \quad \text{some } t
\]

\[
(1+r)a^{(h)}_{cc} + (1+r)a^{(h)}_{ci}p_i + w l^{(h)}_c \geq 1 \quad \text{each } h
\]

\[
(1+r)a^{(k)}_{ic} + (1+r)a^{(k)}_{ii}p_i + w l^{(k)}_i \geq p_i \quad \text{each } k
\]

Processes \((a^{(s)}_{cc}, a^{(s)}_{ci}, l^{(s)}_c)\) and \((a^{(t)}_{ic}, a^{(t)}_{ii}, l^{(t)}_i)\) are operated, while processes which incur extra costs are not.

Let us now check whether the above system allows for solutions. In order to do this, define a technique as a set of two processes consisting of one process producing corn and the other iron. A technique is said to be cost-minimizing at a rate of profits \(r^*\) if at the corresponding wage rate and iron price no known process is able to pay extra profits.

Figure 2 is useful to prove that for a given rate of profits \(r = r^*\) the Propositions stated below hold. A corn process can be plotted in the \((p_i, w)\) plane as a straight line such as FA in Figure 2 (since \(r = r^*\)). (Processes for which \(1 < (1+r^*)a_{cc}\) can be left out of consideration.) Similarly, each iron process can be plotted as a straight line such as EA in Figure 2. Notice that the decreasing straight line cuts the vertical axis at a positive value, whereas the increasing straight line cuts the vertical axis at a negative value.
Figure 2

**Proposition 1:** If a process $\alpha$ is able to pay extra profits at the prices of technique $\beta$, then there exists a technique $\gamma$ which can pay a wage rate larger than that paid by technique $\beta$.

**Proof:** The wage rate and the price of iron associated with technique $\beta$ are represented in Figure 2 by point A. Hence the relation between $w$ and $p_i$ relative to process $\alpha$ at $r = r^*$ intersects the line AD by hypothesis. If this relation is a decreasing line (that is, process $\alpha$ produces corn), it will intersect line AE at a point above and to the right of A. If, on the contrary, this relation is an increasing line (that is, process $\alpha$ produces iron), it will intersect line AF at a point above and to the left of A.

Q.E.D.

Proposition 1 is sufficient to sustain that each technique which at the given rate of profits $r^*$ is able to pay the highest possible wage rate is cost-minimizing at that rate of profits. The following Proposition 2 precludes the case in which a technique which is not able to pay the highest possible wage rate can be a technique which minimizes costs.

**Proposition 2.** If technique $\alpha$ is able to pay a larger wage rate than technique $\beta$, then there is a process in technique $\alpha$ which is able to pay extra profits at the wage rate and iron price of technique $\beta$.

**Proof:** The wage rate and the iron price of technique $\beta$ are represented in Figure 2 by point A. Hence, the wage rate and the iron price of technique $\alpha$ are represented by a point which is either located in quadrant CAD or in quadrant BAD. In the first case, the process which produces corn if technique $\alpha$ is used will pay extra profits at prices relative to technique $\beta$. In the second case, the process which produces iron if technique $\alpha$ is used will pay extra profits at prices relative to technique $\beta$.

Q.E.D.

**Proposition 3.** If both technique $\alpha$ and technique $\beta$ are cost-minimizing at the rate of profits $r^*$, then the iron price corresponding to $r = r^*$ is the same for both techniques.

**Proof:** The wage rate and the iron price of technique $\beta$ are represented in Figure 2 by point A. Since both techniques $\alpha$ and $\beta$ pay the same wage rate, the wage rate and the iron price of technique $\alpha$ are given either by point A, or by a point located on AC (excluding point A), or by a point located on AB (excluding point A). In the second case, the process which produces corn if technique $\alpha$ is used will pay extra profits at prices of technique $\beta$. In the third case, the process which produces iron if technique $\alpha$ is used will pay extra profits at prices of technique $\beta$.
It follows that the wage rate and the iron price of technique $\alpha$ are of necessity given by point A. Q.E.D.

While Propositions 1-3 are important in themselves, they can also be utilized to prove the following

**Theorem 1:**

(a) If there is a technique which has a positive $p_i$ and a positive wage rate $w$ for $r = r^*$, then there is a cost-minimizing technique at the rate of profits $r^*$.

(b) A technique which yields a positive price $p_i^*$ and a nonnegative wage rate $w^*$ for $r = r^*$ minimizes costs at the rate of profits $r^*$ if and only if no other technique allows a wage rate higher than $w^*$ for $r = r^*$.

(c) If there is more than one technique which minimizes costs at a rate of profits $r^*$, then these techniques yield the same wage rate and the same $p_i^*$ at $r = r^*$.

**Proof:** The "only if" part of statement (b) is a direct consequence of Proposition 2. The "if" part is a consequence of Proposition 1. Statement (c) is equivalent to Proposition 3. Statement (a) is a direct consequence of statement (b) since the number of processes is finite. Q.E.D.
Thus, if the $w$-$r$ relationships relative to all techniques available are drawn in the same diagram (see Figure 3), the outer envelope represents the wage-profit frontier for the whole technology. The points on the wage-profit frontier at which two techniques are cost-minimizing are called switch points. If a technique is cost-minimizing at two disconnected ranges of the rate of profits and not so in between these ranges, we say that there is a reswitching of technique.

4. Fixed capital

Up till now it has been assumed that each process produces one and only one commodity. Let us now introduce a model that allows for joint production, but only in a special way. We shall assume that a fixed capital good, say a tractor, can be produced by means of another commodity, say corn. Corn can in turn be produced by means of itself and the tractor. The crucial idea now is that when a new tractor is used to produce corn, it produces at the same time a one year old tractor as a joint product. For simplicity we shall assume that the maximum technical lifetime of the tractor is two years. The equivalent of Table 1 for the new model is Table 2.
There are now three processes to produce corn instead of just one. Process (1) uses the new tractor as an input and produces an old tractor as a by-product. Process (2) uses an old tractor and produces no joint product, given our assumptions about the technical lifetime of the tractor. (The word "tractor" must not lead the reader to think that at the end of the tractor's lifetime there is some scrap to be disposed of; yet, in the case in which there happens to be some scrap to be disposed of, an appropriate additional assumption would have to be introduced). Process (4) is a consequence of the fact that we assume free disposal with regard to the one year old tractor. The fact that a one year old tractor can be disposed of is important because it allows us to determine the economic lifetime of the tractor as opposed to its technical lifetime. (The assumption that this disposal is free rather than costly is entertained only for the sake of simplicity.) If, in fact, it were not economically convenient to employ the tractor for a second year, it would be jettisoned. This means that instead of process (1) process (4) will be used. (In this case we are effectively back in a model with only circulating capital goods – just as the one analysed in the preceding two sections.) Which of the two alternatives will be adopted is, of course, a choice of technique problem, analogous at the one already investigated.

We shall assume that old tractors are not consumables. If they were, then, in fact, process (1) should be operated, even if cost minimization would necessitate producers to select process (4) instead of (1), in order to produce old tractors for the consumers. Finally, it will not have escaped the reader's attention that in this simple model we have set aside the possibility of the joint utilization of tractors, that is, no process is using both old and new tractors.

When we dealt with single production we distinguished between two different problems. However, towards the end of Section 3, the results of the analyses of the two problems were
put together in terms of the outer envelope of the different w-r relationships, or wage frontier. The first problem studied was the dependence of the price of iron in terms of corn and the real wage rate (also in terms of corn) with regard to a single technique for all feasible levels of the rate of profits (see Section 2). The second problem concerned the choice of technique for a given rate of profits \( r = r^* \) (see Section 3). Having understood the basic logic underlying this procedure, the reader will not mind, not least in the interest of brevity, if with respect to fixed capital we shall skip the first step and immediately turn to the second one. Hence it will be assumed that there are altogether \( u \) processes of the kind of process (1) and the same number of processes of the kind of process (4), and there are \( v \) and \( z \) processes of the kind of processes (2) and (3), respectively.

In order to show that the same argument developed in the previous section applies, we first need to show that for \( r = r^* \) (\( p_{nt} \) is the price of a new tractor in terms of corn)

(i) each of the \( z \) processes producing new tractors can be represented as an increasing straight line in the \((p_{nt}, w)\) plane;

(ii) each of the \( u \) processes producing corn with new tractors without producing old tractors can be represented as a decreasing straight line in the \((p_{nt}, w)\) plane;

(iii) each of the \( u \times v \) pair of processes of the kind of processes (1) and (2) can be represented as a decreasing straight line in the \((p_{nt}, w)\) plane.

Statements (i) and (ii) are obvious, because the reference is to single-products processes like those we encountered in Section 3. As regards statement (iii), the two processes mentioned determine the following equations, given the condition that each of them pays the rate of profits \( r^* \):

\[
(1+r^*)a_1 + (1+r^*)p_{nt} + w_l_1 = b_1 + p_{ot} \quad \text{(13a)}
\]

\[
(1+r^*)a_2 + (1+r^*)p_{ot} + w_l_2 = b_2 \quad \text{(13b)}
\]

where \( p_{ot} \) is the price of an old tractor in terms of corn. By multiplying both sides of equation (13a) by \((1+r^*)\) and adding the resulting equation and equation (13b), the following equation is obtained:

\[
(1+r^*)^2a_1 + (1+r^*)^2p_{nt} + (1+r^*)w_l_1 + (1+r^*)a_2 + w_l_2 = (1+r^*)b_1 + b_2 \quad \text{(14)}
\]
that is
\[ w = \frac{(1+r^*)(b_1 - (1+r^*)a_1) + [b_2 - (1+r^*)a_2] - (1+r^*)^2p_{nt}}{(1+r^*)(l_1 + l_2)} \]

and since we are interested only in the cases in which
\[ (1+r^*)(b_1 - (1+r^*)a_1) + [b_2 - (1+r^*)a_2] > 0 \]

statement (iii) is proved. But there is something more. If we look at equation (14), it resembles the equation of a single-product process producing corn, where \((1+r^*)b_1 + b_2\) is the output of corn, \((1+r^*)a_1 + a_2\) is the input of corn, \((1+r^*)\) is the input of a new tractor, and \((1+r^*)l_1 + l_2\) is the labour input.¹ In the following we shall refer to this fictitious process as a 'core process' to produce corn; of course, there are \(u \times v\) of them. Similarly to the definitions given in the previous section we say that a core process is (is not) able to pay extra profits at the rate of profit \(r^*\) if
\[ (1+r^*)^2a_1 + (1+r^*)^2p_{nt} + (1+r^*)w_1 + (1+r^*)a_2 + w_2 < (1+r^*)b_1 + b_2 \quad (\geq (1+r^*)b_1 + b_2) \]

and it does (does not) incur extra costs at the rate of profit \(r^*\) if
\[ (1+r^*)^2a_1 + (1+r^*)^2p_{nt} + (1+r^*)w_1 + (1+r^*)a_2 + w_2 > (1+r^*)b_1 + b_2 \quad (\leq (1+r^*)b_1 + b_2) \]

¹ It will not escape the reader's attention that if the tractor exhibits the same efficiency throughout its life, that is, \(a_1 = a_2 := a\), \(l_1 = l_2 := l\), and \(b_1 = b_2 := b\), then equation (14) can be written as
\[ (1+r^*)a + \frac{r^*(1+r^*)^2}{(1+r^*)^2 - 1} p_{nt} + wl = b. \]

The reader will also notice that the second term involves nothing but the well-known annuity formula giving the annual charge of a fixed capital good lasting \(n\) years,
\[ \frac{r(1+r)^n}{(1+r)^n - 1}, \]

in the special case in which \(n = 2\).
It is easily checked that if a process of type (1) or (2) is able to pay extra profits, then there is a core process which is paying extra profits, and vice versa. This is enough to recognize that Propositions 1-3 of Section 3 hold also in the present context. Only two remarks about the proofs are needed: first, anytime the word "process" occurs the reader has to read "process or core process"; second, with respect to Proposition 3 the proof refers to the price of new tractors only. If at least one of $\alpha$ and $\beta$ is a technique which does not include a core process, then no change is required. If both techniques include a core process, then the proof provided in Section 3 is incomplete since we need to prove that also the price of old tractors is uniquely determined. But this is certainly the case, in fact, if the price of old tractors related to technique $\alpha$ were to be smaller than the one related to technique $\beta$. Then, since both the wage rate and the price of new tractors are proved to be the same in the two techniques, the process of kind (2) in technique $\alpha$ would be able to pay extra profits at the prices of technique $\beta$, and the process of kind (1) in technique $\beta$ would be able to pay extra profits at the prices of technique $\alpha$, that is, a contradiction would arise. Once Propositions 1-3 are proved, Theorem 1 of Section 3 is also proved.

5. Capital utilization

In the preceding section we have assumed that the pattern of utilization of fixed capital is given. This concerns both an extensive and an intensive dimension: the number of hours during which the durable instrument of production is operated per period, say per day, and the speed at which it is operated. In this section we shall allow for different modes of utilization, which poses just another kind of choice of technique problem. The example discussed will be shift work. More precisely, we shall assume that a machine can be operated in a single- and in a double-shift system.

Suppose that a pin manufacturer has the choice of operating a machine under a single (day) shift or under a double (day and night) shift. Suppose for simplicity that under the night shift the amount of direct labour and the quantities of the means of production used up per unit of output are the same as under the day shift. Hence, under the double-shift system the same yearly output could be produced by working half of the machinery twice as long each day as under the single-shift system. Assume in addition that the machine's lifetime lasts two years under the single- and one year under the double-shift system, respectively. The technical alternatives from which the producer can choose are summarized in Table 3. Process (1) relates to the single-shift system when the new machine is utilized, whereas process (2) relates to the single-shift system when the one year old machine is utilized. Process (3) refers to the double-shift system.
Process (4) relates to the single shift when the economic life of the machine is truncated. (The process to produce the new machine is not in the table.)

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<td>(1) pin processes (single shift)</td>
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<td>(2)</td>
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<td>(3) pin process (double shift)</td>
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<td>(4) pin process (truncated)</td>
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Assume that for the work performed at night not only the basic hourly wage rate $w$ but also a premium $\alpha > 0$ per hour will have to be paid, so that the wage rate per hour during the night shift amounts to $w(1 + \alpha)$. Obviously, the producer can compare the cheapness of the three alternatives for each rate of profits and the corresponding prices and basic wage rate. Under the conditions specified, the question of whether it would be profitable to schedule work regularly both at day and night instead of only at day is easily decided. By adopting the double-shift system the producer could economize on his machinery by one half per unit of output. On the other hand he would incur a larger wages bill. Hence, whether the double-shift system proves superior depends on the wage premium and the rate of profits, and can be decided in a way that is analogous to the problem of the choice of technique investigated in the previous section.

6. Concluding remarks

The paper has analysed in a general framework of the analysis, using, however, only simple models, some of the outstanding features of modern industrial economies, that is, commodities are produced by means of commodities and fixed capital goods play an important part in the production process. It has been shown that the framework elaborated allows a discussion of the intricate problem of the choice of technique and a consistent determination of the dependent
variables under consideration: one of the distributive variables (the rate of profits or, alternatively, the real wage rate) and relative prices. It has also been shown that problems such as different patterns of utilization of plant and equipment can easily be analysed in the present framework. The paper is designed to provide the basis for an analysis that takes production seriously.

References


