Relative Existence for Recursive Utility*

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Abstract

Existence theorems for endowment economies with growth and sophisticated recursive preferences have proven difficult to come by. We offer a simple proof technique that covers many models of interest, such as the Bansal–Yaron long-run risk model, models with stochastic volatility and jumps, models with volatility of volatility and models with consumption disasters and time-varying intensities. We also prove existence for models with smooth ambiguity aversion and learning. Collectively these results cover many of the leading asset pricing models today.

Keywords: Asset pricing, long-run risk, recursive utility, relative existence.

JEL codes: G11, G12.

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1 Introduction

Recursive formulations of utility have recently taken center stage in financial economics. Such formulations have allowed for the incorporation of subtle psychological effects such as a preference for early resolution of risk (Kreps and Porteus (1978), Epstein and Zin (1989), Weil (1990)) or smooth ambiguity aversion (Klibanoff, Marinacci, and Mukerji (2009)). Combined with very persistent state processes, stochastic volatility, or jumps, these utility functions have offered promise in explaining major asset-pricing puzzles such as the equity premium (Bansal and Yaron (2004), Drechsler and Yaron (2011), Wachter (2013), Ju and Miao (2012)). Surprisingly, the question of existence for these models is not a completely settled problem. If shocks to consumption become too persistent, then agent utility can become infinite. Persistence increases the importance of the far future, which can generate divergence. A similar argument holds for jumps which can have a large influence on utility. Hence, the very features that provide explanatory power present a challenging existence problem.

We introduce a general framework to address the existence question for recursive utility and persistent consumption processes with jumps. The existence question is surprisingly delicate—very small and non-obvious changes in the consumption process can destroy existence. This provides a partial explanation of why the problem has resisted solution for so long. Our approach provides an interesting contrast to the traditional approach of contraction mappings on metric spaces in that it is resolutely economic—if convergence fails, it fails for resolutely economic reasons, rather than a lack of mathematical cleverness in choosing a metric space.

Agent utility is an ordinal notion, in that it only provides a convenient tool for summarizing agent preferences. The traditional recipe for infinite-horizon models, going back to Stokey and Lucas (1989), is to reformulate it as a contraction mapping in a metric space. This approach introduces concepts—contraction, metric—of unclear economic content. These are also cardinal notions: agent preferences are invariant to nonlinear transformations, but being a contraction mapping is not. Proving existence then becomes sensitive to both cleverness in choosing a suitable parameterization and cleverness in choosing a suitable metric. A triumphant example of this approach is Marinacci and Montrucchio (2010), which shows that a "right" choice of metric can be used to handle some growth economies.

In contrast, our approach is representation-agnostic and can be stated purely in terms of economic primitives. Rather than introduce a metric, we instead show that the existence problem has well-defined *comparative statics*—if a solution exists for a particular model, then that fact automatically implies existence for many other models. This existence has a clear economic interpretation in terms of fundamentals such as risk aversion and patience.

We show how to apply the approach to models with discrete-time affine processes (see Duffie, Pan, and Singleton (2000) and Eraker (2008)). The approach covers prominent models such as those of Bansal and Yaron (2004), Bollerslev, Tauchen, and Zhou (2009), Bollerslev, Xu, and Zhou (2015), Drechsler and Yaron (2011) and Wachter (2013). We prove, for example, existence for the model of Bansal and Yaron (2004) and show that increasing the persistence and volatility of the state processes (stochastic mean growth and stochastic variance) make the existence condition more stringent. The finding for the variance process reverses when the shocks to variance are assumed to follow a gamma instead of a normal distribution. In this case, stochastic volatility always decreases utility and hence, the existence condition becomes less demanding in a model with compared to a model without stochastic volatility. Quantitatively, the largest influence on the existence of solutions has the mean growth rate of consumption. We show that for an intertemporal elasticity of substitution (IES) larger than 1, as usually assumed in the long-run risk literature, the existence condition becomes more demanding the larger the mean growth rate and the larger the IES.[1] We provide upper bounds for the subjective discount factor that are sufficient for the existence of solutions in the model.

Furthermore, we analyse the influence of several other sources of risks on the existence of solutions. For example, volatility of volatility as proposed by Bollerslev, Tauchen, and Zhou (2009) has a significant influence on the existence condition. Persistence and volatility of the process make the existence condition more stringent and for high parameter values, solutions fail to exist. The influence of jumps in volatility (Drechsler and Yaron (2011)) on the existence of solutions depends on the nature of the jump. If jumps follow a gamma distribution, the influence on existence can either be positive or negative depending on the model parameters. For normally distributed jumps we show that as long as the mean jump size is positive, jumps always make the existence condition more stringent. Also, if volatility becomes more persistent, the influence of jumps increases exponentially making the existence condition significantly more demanding. We also consider the case of time-varying consumption disasters as in Wachter (2013). We show that, as long as the mean jump size is negative (disaster and not bonanza), the jumps have a negligible influence on the existence of solutions. Varying the persistence and variance of the intensity process does not change this conclusion. We also provide an example how the approach can be used to show existence in models with learning and smooth ambiguity aversion. We use the model of Ju and Miao (2012) and argue that solutions exist for the parameters used in their paper.

Finally, we show how the existence condition can be used when closed-form expressions are either not available or tedious to derive. In this case, the existence condition can simply be evaluated numerically as we demonstrate using the model of Drechsler and Yaron (2011) which features three state processes with multiple jumps. This approach is particularly useful for future research as it provides a quick and easy way to verify the existence of solutions in

[1]For an IES smaller 1, a solution always exist as long as consumption is assumed to be strictly positive, see Section A.3.

an asset pricing model.

Independently of this paper, Borovička and Stachurski (2019) have developed an alternate approach based on a local spectral radius theorem to show existence in models with Epstein–Zin preferences. While they focus on the case of Epstein–Zin preferences and compact domains, our approach includes other recursive preferences like smooth ambiguity aversion and non-compact state spaces. The results in Borovička and Stachurski (2019) and those in this paper do not overlap but instead nicely complement each other.

The remainder of this paper is organized as follows. In Section 2 we derive the relative existence theorem and demonstrate how it can be applied in practice. Section 3 derives the existence condition for various models and provides a detailed analysis how different sources of risks affect the existence of solutions. Section 4 concludes. The appendix contains full mathematical details and proofs.

2 Theory

In this section we prove our general theorems on the relative existence of equilibria—that is, we relate the existence of an equilibrium in a particular model specification to the existence of a solution in another model. Before we state these theorems, we first provide some details on the economic literature and establish a representation result for recursive utility. We then apply this result to derive statements on relative existence.

A reader who is mainly interested in applications may want to skip the sections on the theoretical foundation and continue immediately with Section 2.4; or (s)he may even move directly to the asset-pricing applications in Section 3.

2.1 Background

The literature has settled on a broad notion of recursive utility functions (Backus, Routledge, and Zin (2004), Ai and Bansal (2016), Strzalecki (2013)), which is sufficient to incorporate many preference effects, such as a preference over risk-timing (Kreps and Porteus (1978)) or smooth ambiguity aversion Klibanoff, Marinacci, and Mukerji (2009)). In this section we lay out the precise conditions we impose, which cover most of the recursive specifications in the literature. We also show that this class has a simple axiomatic characterization in terms of the underlying preference relation.

Most applied work in recursive utility start with what Backus, Routledge, and Zin (2004) call the "overtly practical approach": a formulation in terms of a time aggregator W_t, and a risk aggregator, I_t. The risk aggregator captures the agent's risk attitude by converting risky payoffs into certainty equivalents, while the time aggregator captures the agent's preferences over certainty equivalents. Backus, Routledge, and Zin (2004), Ai and Bansal (2016) and Strzalecki (2013) illustrate how this class captures most specifications that appear in the literature.

For a dynamic model with a single consumption good, the agent has preferences over a consumption stream, $(C_0, C_1, \ldots)$, where C_t is not known until time t. Then the agent's preferences are determined by the recursive utility specification,

$$U_t = W_t(C_t, I_t(U_{t+1})),$$

where U_t is a time t random variable that provides a summary of the agent's preferences for consumption from time t onward. The "existence problem" is then to prove that U_t is finite.

This practical approach comes with two practical difficulties. One is that this recursive relationship may fail to have any finite solutions. The other is that it can have multiple solutions. Both of these involve cardinal considerations, rather than strictly ordinal considerations. For example, since we can always transform $[-\infty, \infty]$ to $[0, 1]$, finiteness is not an intrinsic property of the preference relation. So does the existence problem matter at all? If a model assigns the maximum possible utility to a consumption stream, then usually every higher consumption stream will also be assigned that same utility, which means that the agents are satiated—they are indifferent between more and less. The significance of the existence problem is to prove that there exists a solution that rules out satiation. The consequences of multiple solutions are more ambiguous. Ultimately what matters it the ability to compare consumption bundles, so it is sufficient to pick *one* solution that correctly compares bundles.

The traditional solution to both problems is to construct a metric space and to associate a contraction operator with that metric space, such that the fixed point of that operator is a solution to the original problem; see Marinacci and Montrucchio (2010) for a general version of this approach. But again, contraction is a cardinal notion, and thus not a purely economic consideration. An arbitrary nonlinear transformation can destroy the contraction property, without changing the underlying economics.

We pursue a more direct strategy. The definition of recursive utility is unproblematic for finite horizons (we assume the agent does not receive any utility once the end horizon is reached), and only requires minimal assumptions. If we define utility for infinite horizons as the pointwise limit of finite horizons (when it exists) then again this requires minimal assumptions. To achieve uniqueness, we normalize utility such that the zero consumption sequence is assigned zero utility.

To prove that this is sufficiently general, we prove a theoretical result that serves as a converse—if a preference relation satisfies some natural properties, then it can be written as a limit of finite-horizon economies To state this theorem, we introduce a simple axiomatic approach. The axiomatic approach has a long history inspired by decision theory; see Kreps and Porteus (1978), Klibanoff, Marinacci, and Mukerji (2009), and Hayashi and Miao (2011). These axiomatizations are very complex, because to capture their full effect they must define preferences over very complex counterfactuals, such as variations in timing or second-order acts. We instead only require preferences to be defined over states that can occur in the model.

2.2 Preferences

In this section we state the precise formulation necessary to handle existence for our purposes in this paper. To make this discussion accessible to as broad an audience as possible, we have exiled all measure-theoretic details to the appendix; see Appendix A.1.

Time is discrete, $t = 0, 1, 2, \ldots$. The state space of the economy, Ω, is an infinite sequence of states, $(s_0, s_1, \ldots)$. The number of states in period t can be finite or (countably or uncountably) infinite. There is a single starting state, s_0, at time $t = 0$. Let s^t represent the history up to time t,

$$s^t = (s_0, \ldots, s^t).$$

If $s^t = (s_0, \ldots, s_t)$, then (s^t, s_{t+1}) means $(s_0, \ldots, s_t, s_{t+1})$. We impose very little structure on the nature of the s_t. The possible s_{t+1} can be constrained by s^t. (For example, technological advances in semiconductor technology can only occur in period 8 if the semiconductor was invented by period 7.) Let $\mathcal{S}_{s^t}$ represent the states that can follow history s^t. Then by definition $s_{t+1} \in \mathcal{S}_{s^t}$. Let Ω_t be the set of all permissible histories up to time t.

The set Ω serves as the underlying state space for a stochastic process. Importantly, Ω is not required to be a very large state space, as in the axiomatic approaches of Kreps and Porteus (1978), Klibanoff, Marinacci, and Mukerji (2009), or Hayashi and Miao (2011). Agents are not required to have preferences over counterfactuals such as hypothetical changes in the timing of information or second-order acts. In the "overtly practical approach", it is sufficient to model preferences over the events that can actually occur in the model.

Agents have preferences over sequences of consumption streams, $(C_0, C_1, \ldots)$, where each C_t is a time-t nonnegative random variable, $C_t = C_t(s^t)$. Consumption streams lie in some underlying domain $\mathcal{D}$ of possible consumption streams. We do not impose much structure on $\mathcal{D}$, but we do require it to permit free disposal: if $(C_0, C_1, \ldots) \in \mathcal{D}$, and if for all t and nodes s^t it holds that $C_t(s^t) \geq C'_t(s^t) \geq 0$, then $(C'_0, C'_1, \ldots) \in \mathcal{D}$.

While free disposal is a natural condition, and is a standard ingredient in general equilibrium (Debreu (1959)), it does rule out some specifications that appear in the literature. For example, for agents with CRRA utility with IES less than 1 if consumption is too low it can diverge to negative infinity. This only happens for consumption streams that are rapidly contracting, rather than consumption streams that exhibit growth, so this case is far away from the usual specifications considered. In the main body of the text we concentrate on the case that allows free disposal, but we consider the alternative in section A.3.

2.2.1 Recursive Utility

We denote the set of nonnegative real numbers by $\mathbb{R}_+ = \mathbb{R}_{>=0} = \{x \in \mathbb{R} | x \geq 0\}$ and the set of strictly positive real numbers by $\mathbb{R}_{++}$.

In this paper, we restrict attention to preferences that allow for a representation via recursive utility. An agent has utility in each state of the world, $U_t(s^t)$. The utility is then recursive with respect to functions W_t, I_t, if

$$U_t(s^t) = W_t\left(C_t(s^t), I_t\left(U_{t+1}\left((s^t, s_{t+1})\right); s^t\right); s^t\right).$$

We impose some conditions on the functions W_t, I_t.

A *time aggregator* for node s^t is an upper semicontinuous two-variable function, $W_t(x, y; s^t)$, $x, y \in \mathbb{R}_+$, with the following properties.

1. (Normalization) $W_t(0, 0; s^t) = 0$.

2. (Strict monotonicity) $W_t(x, y; s^t)$ is strictly increasing in each variable.

3. (Substitutability) For any pair x, y, there exists an x' such that $W(x, y) = W(x', 0)$.

The first condition is just a normalization: we want zero consumption for all time to be normalized to zero utility. The second condition is a standard nonsatiation assumption. The third condition is a technical condition to rule out pathological scenarios in which the agent have a very strong preference for delaying consumption. (An example that would violate this condition is $(1 - e^{-x}) + \beta(1 - e^{-y})$, with $\beta > 1$—this agent has a strong preference for consumption tomorrow, rather than today.) In most examples in the literature, W is continuous, and independent of both s^t and t. For simplicity, we suppress the state-space dependencies whenever possible, $W(x, y) = W(x(s^t), y(s^t); s^t)$.

A *risk aggregator* for node s^t is a functional $I_t(f; s^t)$ that aggregates information over states in $\mathcal{S}_{s^t}$. Let $f : \mathcal{S}_{s^t} \to \mathbb{R}_+$ denote a time-$(t+1)$ random variable. Then I_t hast the following properties.

1. (Certainty equivalence) If f is the constant function with value c, then $I_t(c; s^t) = c$.

2. (Strict monotonicity) If $f(s_{t+1}) > g(s_{t+1})$ for all s_{t+1}, then $I_t(f; s^t) > I_t(g; s^t)$.

3. (Weak continuity) If $f_1(s_{t+1}) \leq f_2(s_{t+1}) \leq f_3(s_{t+1}) \leq \cdots$ then

$$\lim_{i\to\infty} I_t\left(f_i(s_{t+1}); s^t\right) = I_t\left(\lim_{i\to\infty} f_i(s_{t+1}); s^t\right).$$

The first condition is a standard (and natural) normalization. The third condition is satisfied by any operator built out of expectations operators, because of the Monotone Convergence Theorem from real analysis. Unlike for W_t, the dependence of I_t on s^t is essential in applications. Even the simplest risk-aggregation operator, conditional expectation E_t, depends on the current state s^t. Nevertheless, we suppress again the dependence on s^t and s_{t+1} whenever possible, $I_t(f) = I_t\left(f(s_{t+1}); s_t\right)$

Dropping all explicit references to the state, we write

$$U_t = W_t(C_t, I_t(U_{t+1})). \tag{1}$$

instead of $U_t(s^t) = W_t(C_t(s^t), I_t(U_{t+1}((s^t, s_{t+1})); s^t); s^t)$. This equation may not have any solutions, but even if it does, the equation is not sufficient to uniquely identify the solution. Here is a simple example. Let $W_t(x, y) = x + \beta y$ (for $\beta < 1$), and $I_t = E_t$. Assume $0 \leq C_t \leq 1$. There is the intended solution to equation (1),

$$U_t = \sum_{\tau=t}^{\infty} \beta^{\tau-t} E_\tau(C_\tau), \tag{2}$$

but it has a continuum of other solutions. Let X_t be any martingale sequence, $X_t = E_t(X_{t+1})$. Then

$$U_t + \beta^{-t} X_t \tag{3}$$

is also a solution.

It is well known that—in special cases—these extra solutions can be ruled out. For example, if (C_t) is a Markov process and C_t depends only on the current state, s_t, we can require U_t to be a function of only s_t as well. If there is a metric space, then the contraction mapping principle picks out a unique solution; see Stokey and Lucas (1989). We do not pursue this well-known theoretical approach but instead identify a different procedure.

For a finite consumption sequence, $C_0, C_1, \ldots, C_T, 0, 0, \ldots$, there is a natural choice of solution—$U_t = 0$ for all $t > T$. For an arbitrary consumption sequence, we define the utility to be the limit of its finite truncations. For a given consumption stream, (C_t), define (C_t^T) to be the same consumption stream until time T, and then zero otherwise. Let U_t^T be the time

t utility for (C_t^T). Then we define U_t as the limit

$$U_t \equiv \lim_{T \to \infty} U_t^T, \tag{4}$$

if it exists and is finite (almost surely).

For the example above, this is the same as choosing U_t to be

$$\lim_{T \to \infty} \sum_{\tau=t}^{T} \beta^{\tau-T} E_\tau(C_\tau),$$

which converges to expression (2) and rules out solutions of the form (3).

This solution has an attractive interpretation. Imagine that we can "shut off" the economy at time t, so that the agent receives zero utility after that point. Then there are no "convergence issues". Infinite-horizon economies are idealizations of finite-horizon economies where the horizon is far away in time. Then the natural solution is the one that is the limit of finite-horizon economies. By imposing the increasing limit condition, utility is completely determined on sequences of consumption that stop at some time t.

2.2.2 Axiomatic Characterization

Interestingly, this class of "overtly practical" utility functions lends itself to a simple axiomatic characterization in terms of economic primitives. In a setting with stationarity, it is possible to work with a single preference relation, but to handle learning and dynamic ambiguity aversion requires a state-dependent preference relation. For this reason we define a state-dependent preferences $\succsim$ for each possible state in the economy.

For each state, s^t, there is a preference relation $\succsim_{s^t}$, defined over sequences of consumption streams, beginning at time t,

$$(C_\tau)_{\tau \geq t} = (C_\tau, C_{\tau+1}, \ldots),$$

satisfying some axioms.

A *recursive preference relation* is a family of preferences, $\succsim_{s^t}$, on consumption streams $(C_\tau)_{\tau \geq t}$ with the following properties.

1. (Strict monotonicity) If

$$C_t(s^t) \geq C'_t(s^t)$$

and

$$(C_\tau)_{\tau \geq t+1} \succsim_{(s^t, s_{t+1})} (C'_\tau)_{\tau \geq t+1}$$

for all $s_{t+1} \in \mathcal{S}_{s^t}$, then

$$(C_\tau)_{\tau \geq t} \succsim_{s^t} (C'_\tau)_{\tau \geq t}.$$

The same holds with all comparisons being strict.

2. (Future substitutability) Future consumption can be traded for current consumption. For any sequence $(C_\tau)_{\tau \geq t}$, there exists a time-t random variable, $\tilde{C}_t$ such that for all

$$(C_\tau)_{\tau \geq T} \sim_{s^t} (\tilde{C}_t, 0, \ldots)$$

for all s^t. We call $\tilde{C}_t$ the present consumption equivalent.

3. (Risk substitutability) Risky consumption can be traded for riskless consumption. For any sequence

$$(C_t(s^t), C_{t+1}(s^{t+1}), 0, 0, \ldots)$$

we can replace $C_{t+1}(s^{t+1})$ (which is risky from the point of view of time t) with $\bar{C}_{t+1}(s^t)$ (which is riskless from the point of view of time t), such that

$$(C_t(s^t), C_{t+1}(s^{t+1}), 0, 0, \ldots) \sim_{s^t} (C_t(s^t), \bar{C}_{t+1}(s^t), 0, 0, \ldots)$$

for all s^t. We call $\bar{C}_{t+1}$ the riskless consumption equivalent.

4. (Increasing limits) For each i,

$$(C^i_\tau)_{\tau \geq t}$$

is a consumption stream such that $C^i_t(s^t) \leq C^{i+1}_t(s^t)$ for all i and s^t. Suppose that the pointwise limit is well-defined and

$$C_t(s^t) = \lim_{i \to \infty} C^i_t(s^t) \in \mathcal{D}.$$

Then for any consumption stream

$$(C'_\tau)_{\tau \geq t}$$

if

$$(C^i_\tau)_{\tau \geq t} \succsim_{s^t} (C'_\tau)_{\tau \geq t}$$

for all i then

$$(C_\tau)_{\tau \geq t} \succsim_{s^t} (C'_\tau)_{\tau \geq t}.$$

The strict monotonicity axiom builds in conditional weak independence (Johnsen and Donaldson (1985)), i.e. consequentialism Machina (1989). The substitutability axioms ensure that the agents experience both a risk trade-off and an intertemporal trade-off. The increasing limits assumption is again a weak continuity assumption.

We can associate recursive utility functions with recursive preference relations.

Theorem 1 (Utility Representation). *Let $\succsim$ be a recursive preference relation. Then $\succsim$ can be represented by a utility function. There exists a sequence of time aggregators W_t and risk aggregators I_t, such that*

$$U_t = W_t(C_t, I_t U_{t+1})$$

with U_t being the finite limit in equation (4)*.*

Conversely, suppose that we have a sequence of time aggregators W_t and risk aggregators I_t. Then there exists a unique solution U_t, such that U_t assigns zero to zero consumption and U_t preserves increasing limits. This U_t defines a recursive preference relation.

The recursive utility U_t is not unique, but the future substitutability axiom enables us to pick a standard one, where the utility of a consumption stream is its present consumption equivalent. In other words, U_t for $(C_\tau)_{\tau \geq t}$ satisfies

$$(U_t, 0, \ldots) \sim_{s^t} (C_\tau)_{\tau \geq t}. \tag{5}$$

In the course of proving Theorem 1 we will show that there exists W_t and I_t that make U_t a recursive utility; see Appendix 1.

2.3 Comparative Results

We explicitly developed the framework of the previous section for the purpose of stating a comparative existence result. Suppose Theorem 1 holds for an agent with recursive preferences on a set of consumption streams, $\mathcal{D}$. We now want to state sufficient conditions on the preferences of another agent, such that a well-defined utility representation exists for this agent's preferences as well.

We first introduce some terminology. Consider two agents with recursive preferences defined over the same domain $\mathcal{D}$ of permissible consumption streams. For consumption C_{t+1} agent i has a riskless equivalent

$$(\ldots, C_{t+1}, 0, 0, \ldots) \sim (\ldots, \bar{C}^i_{t+1}, 0, 0, \ldots),$$

for $i = 1, 2$. We say that agent 1 has a *higher effective risk aversion* than agent 2, if $\bar{C}^1_{t+1} \leq \bar{C}^2_{t+1}$ for all permissible C_{t+1} and with strict inequality for at least one permissible value. If the riskless equivalents are always equal, we say the agents have an *identical level of effective risk aversion*.

Let C_{t+1} be consumption at time $t+1$, but the value is known at time t. By assumption, each agent has a time-t present consumption equivalent,

$$(\ldots, C_t, C_{t+1}, 0, \ldots) \sim (\ldots, \bar{C}_t^i, 0, \ldots).$$

We say that agent 1 has *higher effective impatience* than agent 2 if $\bar{C}_t^1 \leq \bar{C}_t^2$ for all permissible C_{t+1} and with strict inequality for at least one permissible value. We say the agents have *identical levels of effective patience* if the condition always holds with equality.

Theorem 2 (Comparative Existence)**.** *Suppose agents 1 and 2 have recursive preferences over consumption streams in a domain $\mathcal{D}$. Suppose further that a well-defined utility function exists for agent 2. Then a well-defined utility function also exists for agent 1, if one of the following sufficient conditions holds.*

1. *Both agents have identical levels of effective patience, and agent 1 has higher effective risk aversion than agent 2.*

2. *Both agents have identical levels of effective risk aversion, and agent 1 has a higher level of effective impatience.*

Suppose that the agent i has utility given explicitly by W_t^i and I_t^i. A special case of the first case above is when $W_t^1 = W_t^2$ and $I_t^1 U_{t+1} \leq I_t^2 U_{t+1}$ for all U_{t+1}. A special case of the second case is when $I_t^1 U_{t+1} = I_t^2 U_{t+1}$ and $W_t^1(C_t, x_{t+1}) \leq W_t^2(C_2, x_{t+1})$.

For time aggregators, we focus on one special case.

Corollary 1. *Let*

$$W^i(c, x) = u(c) + \delta_i x.$$

Then if $\delta_1 \leq \delta_2$ then agent 1 has higher effective impatience.

We now consider risk aggregators. For the sake of notational simplicity we leave out parentheses for function application, so $fx = f(x)$ and $fgx = f(g(x))$. Suppose the model is specified in terms of a conditional distribution of the next state s_{t+1} which is a function of the current state, s^t, μ_{s^t}. Under Kreps-Porteus-Epstein–Zin utility, the risk aggregator is of the form

$$f^{-1}E(fU_{t+1}|s^t) \tag{6}$$

for some f.

Klibanoff, Marinacci, and Mukerji (2009) show that under appropriate conditions, an agent with smooth ambiguity aversion will use Bayesian updating. In that case, the agent does not know the true parameter, but instead θ is modelled as having a conditional distribution that depends on s^t.

The risk aggregator is

$$E(gE(g^{-1}U_{t+1}|s^t,\theta)|s^t). \tag{7}$$

Hayashi and Miao (2011) combine both effects into one, as follows:

$$f^{-1}E(fg^{-1}E(gU_{t+1}|s^t,\theta)|s^t). \tag{8}$$

Equation (6) is the special case where g is the identity, while equation (7) is when f is the identity. When both are the identity, the risk aggregator is just an expectation.

All three specifications are built out of operators of the form

$$\mathcal{M}_{f,\mu}(x) = f^{-1}E_\mu f(x)$$

where μ is a probability measure and f is an increasing function. These are all in our class of risk aggregators because of the Monotone Convergence Theorem for integration.

There is already a well-developed theory for comparing $\mathcal{M}_{f,\mu}$, in the form of the Arrow–Pratt theory of risk aversion (Pratt (1964)). While stated in terms of risk aversion, it captures properties of general integral operators. The main comparison result, in our notation, is as follows.

Theorem 3. *Let g be an increasing function. If g is convex, then*

$$M^{g\circ f}(x) \geq M^f(x)$$

while if g is concave

$$M^{g\circ f}(x) \leq M^f(x).$$

The converse is also true: $M^h(x) \geq M^f(x)$ if and only if $h = g\circ f$ for some convex g, and similarly for concave.

We call h *more convex* than f if $h = g \circ f$ for convex g, and *more concave* if g is concave. If f and h are twice-differentiable, then this result can be stated in terms of either the absolute or relative Arrow-Pratt coefficients. Recall that the absolute Arrow–Pratt risk aversion level of f at x is $-f''(x)/f'(x)$, while the level of relative risk aversion is $-xf''(x)/f'(x)$. If either level for h is less than or equal to the one for f for all x, then h is more concave than f. In particular, for powers, where the relative risk-aversion is a constant, the comparison is particularly simple.

Corollary 2. *Let $f = x^\alpha$ and $g = x^{\alpha'}$, where $\alpha' \geq \alpha > 0$. Then $M^f \leq M^g$.*

In other words, any operator based on f features greater effective risk aversion compared to that based on g.

This provides a particularly simple comparison with discounted expected utility. We state the result for Hayashi–Miao preferences, since it embraces both smooth ambiguity aversion and preferences for timing resolution.

Theorem 4. *Consider two agents who are endowed with the same consumption stream. Both agents have identical time aggregators,*

$$W(c, x) = u(c) + \delta x,$$

and Hayashi–Miao risk aggregators, (8)*. Suppose agent 2 is neutral towards both timing resolution and ambiguity (so f and g are both the identity). Then the following statements hold.*

1. *If agent 1 prefers early resolution of risk and is smooth-ambiguity averse, then existence for agent 2 implies existence for agent 1.*

2. *If agent 1 prefers later resolution of risk, and is smooth-ambiguity seeking, then existence for agent 1 implies existence for agent 2.*

Consider for example smooth ambiguity aversion preferences as in Ju and Miao (2012),

$$U_t = (1-\delta)C_t^{1-\frac{1}{\psi}} + \delta \left[E\left(\left(E(U_{t+1}^{\frac{1-\gamma}{1-\frac{1}{\psi}}} |s^t, \theta) \right)^{\frac{1-\eta}{1-\gamma}} |s^t \right) \right]^{\frac{1-\frac{1}{\psi}}{1-\eta}} \tag{9}$$

where the two Hayashi and Miao (2011) risk aggregators in (8) are given by

$$\begin{aligned} f(x) &= x^{\frac{1-\eta}{1-\frac{1}{\psi}}}, \quad \eta > 0, \neq 1, \quad \psi > 1 \\ g(x) &= x^{\frac{1-\gamma}{1-\frac{1}{\psi}}}, \quad \gamma > 0, \neq 1, \quad \psi > 1. \end{aligned}$$

Here, ψ denotes the intertemporal elasticity of substitution. The investor has a preference for the early resolution of risks if and only if $\gamma > \frac{1}{\psi}$. Furthermore the agent displays ambiguity aversion if and only if $\eta > \gamma$. For $\eta = \gamma$ the agent is ambiguity neutral and the preferences reduce to recursive preferences as in Epstein and Zin (1989) and Weil (1989):

$$U_t = (1-\delta)C_t^{1-\frac{1}{\psi}} + \delta \left[E\left(\left(E(U_{t+1}^{\frac{1-\gamma}{1-\frac{1}{\psi}}} |s^t, \theta) \right) |s^t \right) \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}. \tag{10}$$

For $\gamma = \frac{1}{\psi} = \eta$ the investor is neutral towards both timing resolution and ambiguity so the investor has standard CRRA preferences:

$$U_t = (1-\delta)C_t^{1-\frac{1}{\psi}} + \delta \left[E\left(\left(E(U_{t+1}|s^t, \theta) \right) |s^t \right) \right]. \tag{11}$$

Hence, Theorem 4 can be used to compare to a standard agent with CRRA utility. In the following we illustrate how the theorem can be used to prove existence for Epstein–Zin and smooth ambiguity aversion preferences. We first consider the case of standard Epstein–Zin (EZ) preferences ($\eta = \gamma$) and hence the agents are ambiguity neutral.

Corollary 3. *Consider two agents who are endowed with the same consumption stream and preferences as in* (9)*. Denote by $\psi^i, \gamma^i, \delta^i$ the preference parameters of agent i. Suppose that $\delta^1 = \delta^2$, $\psi^1 = \psi^2 > 1$ and that agent 2 is neutral towards timing resolution, that is $\gamma^2 = \frac{1}{\psi^2}$. Then*

1. *If agent 1 prefers early resolution of risk or is neutral towards timing resolution, that is $\gamma^1 \geq \frac{1}{\psi^1}$, then existence for agent 2 implies existence for agent 1.*

2. *If agent 1 prefers later resolution of risk or is neutral towards timing resolution, that is $\gamma^1 \leq \frac{1}{\psi^1}$, then existence for agent 1 implies existence for agent 2.*

Hence, the CRRA case can be used as a reference case to show existence in models with EZ preferences and a preference for the early resolution of risks. Next consider the case with smooth ambiguity aversion.

Corollary 4. *Consider two agents who are endowed with the same consumption stream and preferences as in* (9)*. Denote by $\psi^i, \gamma^i, \eta^i, \delta^i$ the preference parameters of agent i. Suppose that $\delta^1 = \delta^2$, $\psi^1 = \psi^2 > 1$ and that agent 2 is neutral towards both timing resolution and ambiguity, that is $\gamma^2 = \eta^2 = \frac{1}{\psi^2}$. Then*

1. *If agent 1 prefers early resolution of risk (or is neutral towards timing resolution) and is smooth-ambiguity averse (or ambiguity neutral), that is $\eta^1 \geq \gamma^1 \geq \frac{1}{\psi^1}$, then existence for agent 2 implies existence for agent 1.*

2. *If agent 1 prefers later resolution of risk (or is neutral towards timing resolution) and is smooth-ambiguity seeking (or ambiguity neutral), that is $\eta^1 \leq \gamma^1 \leq \frac{1}{\psi^1}$, then existence for agent 1 implies existence for agent 2.*

So again, the CRRA case serves as a reference case to prove existence for models with preference for the early resolution of risks and ambiguity aversion. In the following section we show how to prove existence for CRRA preferences for a broad class of models.

We briefly mention the relationship of our results to the existing literature. The existence result in Marinacci and Montrucchio (2010) is not directly comparable to ours (they use a metric rather than the increasing limits condition), but their argument relies on equation (1) having a fixed point for a constant lower bound and upper bound. For the special case of Epstein–Zin utility Hansen and Scheinkman (2012) use a CRRA comparison argument as part of their larger strategy of proving existence for Markov processes by examining eigenfunctions of operator equations.

2.4 Absolute Existence with CRRA Utility

Theorem 2 is a relative existence result. Simply put, it provides sufficient conditions under which the existence of an equilibrium in a particular model implies also the existence of an equilibrium in another model. Clearly, for this theorem to be of any use to us, we need a set of benchmark models for which we can prove the existence of an equilibrium by other means. Theorem 4 points to discounted expected utility models as benchmarks, but existence for expected utility is not automatic when growth is involved.

In this section we identify a simple class of models that allow for closed-form solutions, namely models with CRRA preferences when the underlying consumption process is a discrete-time affine process. In practice, most specifications of exogenous processes fall in the class of discrete-time affine processes; see Eraker (2008) for an introduction. Prominent models with such processes include models with long-run risks and stochastic volatility as in Bansal and Yaron (2004) and Schorfheide, Song, and Yaron (2018), models with stochastic volatility and jumps as in Drechsler and Yaron (2011), models with volatility-of-volatility as in Bollerslev, Tauchen, and Zhou (2009), Tauchen (2011), and Bollerslev, Xu, and Zhou (2015), and models with time-varying consumption disasters as in Wachter (2013). In the following we show how to prove existence for such model with CRRA preferences. Lifetime utility for CRRA preferences at time t is given by

$$V_t = (1-\delta)\frac{C_t^\alpha}{\alpha} + \delta E_t\left(V_{t+1}\right) \tag{12}$$

with $\alpha = 1 - \frac{1}{\psi}$, where $0 < \psi \neq 1$ is the intertemporal elasticity of substitution. Dividing both sides by $(1-\delta)C_t^\alpha/\alpha$ and defining

$$v_t \equiv \frac{V_t}{(1-\delta)C_t^\alpha/\alpha},$$

we obtain a reformulation of the CRRA recursion for growth economies,

$$v_t = 1 + \delta E_t\left(e^{\alpha \Delta c_{t+1}} v_{t+1}\right), \tag{13}$$

with $\Delta c_{t+1} = \log\frac{C_{t+1}}{C_t}$. In the following we derive a sufficient condition for a finite solution to (13) when log consumption growth, Δc_{t+1}, follows a discrete-time affine process.

A *discrete-time affine process* is a vector stochastic process, $(X_t^1, \ldots, X_t^n)'$, such that the moment-generating function (MGF) is exponential-linear in the X_t^i; specifically,

$$E_t\left(e^{k_1 X_{t+1}^1 + \cdots k_n X_{t+1}^n}\right) = \exp\left\{h_0(k_1,\ldots,k_n) + h_1(k_1,\ldots,k_n)X_t^1 + \cdots + h_n(k_1,\ldots k_n)X_t^n\right\} \tag{14}$$

for fixed constants, k_i, and functions, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots n$. We slightly rewrite the equation above to obtain a compact notation and make it applicable to the CRRA setup. Let $X_t^0 \equiv 1$ and for the vector $k = (k_0, k_1, \ldots, k_n)$ define $h(v)$ to be the vector function term

$$h(k) = (k_0 + h_0(k_1,\ldots,k_n), h_1(k_1,\ldots,k_n), \ldots, h_n(k_1,\ldots,k_n)).$$

Let $X_t = (X_t^0, X_t^1, \ldots, X_t^n)'$. We then obtain $E_t(e^{k\cdot X_{t+1}}) = \exp\{h(k)\cdot X_t\}$. Using the same vector notation as in the compact notation for the MGF, we can rewrite the CRRA recursion (13) as

$$v_t = 1 + E_t\left(e^{\tau\cdot X_{t+1}} v_{t+1}\right), \tag{15}$$

where the state variable Δc_{t+1} is the element X_{t+1}^1 and $\tau = [\ln(\delta), \alpha, 0, \ldots]$. Existence of a solution to (15) now requires the existence of a finite solution, v_t, for any t. We conjecture that the solution for v_t in the recursive equation (15) has the form

$$v_t = \sum_{i=0}^{\infty} e^{A_i\cdot X_t} \tag{16}$$

with a sequence of coefficient vectors $(A_i)_{i\in\mathbb{N}}$. Substituting the conjectured expression for v_t into equation (15) we obtain the condition

$$\begin{aligned}
e^{A_0\cdot X_t} + e^{A_1\cdot X_t} + e^{A_2\cdot X_t} + \cdots &= 1 + E_t\left(e^{\tau\cdot X_{t+1}}\left(e^{A_0\cdot X_{t+1}} + e^{A_1\cdot X_{t+1}} + \cdots\right)\right)\\
&= 1 + E_t\left(e^{(\tau+A_0)\cdot X_{t+1}}\right) + E_t\left(e^{(\tau+A_1)\cdot X_{t+1}}\right) + \cdots\\
&= 1 + e^{h(\tau+A_0)\cdot X_t} + e^{h(\tau+A_1)\cdot X_t} + \cdots .
\end{aligned}$$

We can solve this equation by equating exponents one at a time, so

$$\begin{aligned}
A_0 &= 0\\
A_1 &= h(A_0 + \tau)\\
A_2 &= h(A_1 + \tau)\\
&\vdots
\end{aligned}$$

This approach leads us to a simple recursion for the coefficients, $A_i = h(A_i + \tau)$, with the starting point $A_0 = 0$. The extension of the classical ratio test for convergence (also called the Cauchy ratio test or d'Alembert ratio test) to supremum limits states that the infinite series (15) is (absolutely) convergent if

$$\limsup_{i\to\infty} e^{(A_i - A_{i-1})\cdot X_t} < 1.$$

We thus obtain the following existence theorem.

Theorem 5. *The solution v_t to the CRRA recursion*

$$v_t = 1 + E_t\left(e^{\tau\cdot X_{t+1}} v_{t+1}\right)$$

for a discrete-time affine process X_t has the form

$$v_t = \sum_{i=0}^{\infty} e^{A_i\cdot X_t},$$

where the sequence of vectors $(A_i)_{i\in\mathbb{N}}$ is given by $A_0 = 0$ and $A_{i+1} = h(A_i + \tau)$ for some function $h : \mathbb{R}^n \to \mathbb{R}^n$. A sufficient condition for the convergence of the infinite sum is

$$\limsup_{i\to\infty} (A_{i+1} - A_i)\cdot X_t < 0.$$

To apply this general result to a concrete application, we first need to derive the function h and the sequence of vector coefficients $(A_i)_{i\in\mathbb{N}}$ for the specific model. And second, we need to check whether the sufficient condition for convergence is satisfied. We illustrate these two steps in the context of a well-known application in Section 3.1.1.

3 Applications

In the following we use Theorems 4 and 5 to derive sufficient existence conditions for different models with either Epstein–Zin or smooth ambiguity aversion preferences. In Section 3.1 we show how to apply the approach to models with gaussian and non-gaussian shocks. We begin with a simple illustrative example where consumption growth has a stochastic mean. Second we show how stochastic volatility with different kind of shocks affects the existence of solutions. Then we show how to apply the approach to models with learning and smooth ambiguity aversion. In the second part in Section 3.2, we prove existence for several models that have gained particular attention in the asset pricing literature. We consider for example the long-run risk model of Bansal and Yaron (2004), models including a volatility of volatility factor as in Bollerslev, Tauchen, and Zhou (2009), Tauchen (2011), and Bollerslev, Xu, and Zhou (2015), models with jumps to volatility as in Drechsler and Yaron (2011) and models with time varying consumption disasters as in Wachter (2013) . We not only state the sufficient existence condition for the models, but also analyze which model parameters are particularly important for the existence of solutions.

These models usually assume that shocks are gaussian which can lead to ill-defined models. For example in the model of Bansal and Yaron (2004), the stochastic variance process can become negative. Researchers in this literature have argued that quantitatively the small possibility of a negative variance has a negligible influence on asset prices (see Bansal, Kiku, and Yaron (2012) and Bansal and Shaliastovich (2013)) and as the gaussian shocks lead to tractability when linearizing the model, they have relied on the gaussian assumption. It is important to note that our approach does not require gaussian shocks as we demonstrate in Section 3.1.2. But for the practical purpose of proving existence in the models mentioned above, we stick to the original consumption dynamics. In the following examples we concentrate on the case of $\psi > 1$ as for $\psi < 1$ existence of a solution can always be achieved by the assumption that consumption is strictly positive, see Section A.3.

3.1 Existence in Models with Gaussian and Non-Gaussian Shocks and Learning

In this section we first use an illustrative example to show how Theorems 4 and 5 can be applied in practice. Then we consider a more elaborate model with stochastic volatility and non-gaussian shocks. In the last section we show how the approach can be applied to models with smooth ambiguity aversion and learning.

3.1.1 An Illustrative Example

Suppose that log consumption growth, Δc_{t+1}, has a stochastic mean as in Bansal and Yaron (2004):

$$\begin{aligned}
\Delta c_{t+1} &= \mu + x_t + \sigma\epsilon_{c,t+1} \\
x_{t+1} &= \rho x_t + \phi_x \sigma \epsilon_{x,t+1} \\
& \epsilon_{c,t+1}, \epsilon_{x,t+1} \sim i.i.d.\ N(0,1),
\end{aligned} \tag{17}$$

with $|\rho| < 1$. Define $X_t = (X_t^0, X_t^1, X_t^2)'$ with $X_t^0 = 1$, $X_t^1 = \Delta c_t$, $X_t^2 = x_t$ and $k = (k_0, k_1, k_2)$. We obtain

$$\begin{aligned}
E_t\left(e^{k\cdot X_{t+1}}\right) &= E_t\left(e^{k_0 X_{t+1}^0 + k_1 X_{t+1}^1 + k_2 X_{t+1}^2}\right) \\
&= e^{k_0 + k_1\mu + k_1 X_t^2 + 0.5k_1^2\sigma^2 + k_2\rho X_t^2 + 0.5k_2^2\phi_x^2\sigma^2}.
\end{aligned}$$

Using the notation $E_t\left(e^{k\cdot X_{t+1}}\right) = e^{h(k)\cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1,k_2), h_1(k_1,k_2), h_2(k_1,k_2)),$$

leads to

$$\begin{aligned} k_0 + h_0(k_1,k_2) &= k_0 + k_1\mu + 0.5k_1^2\sigma^2 + 0.5k_2^2\phi_x^2\sigma^2 \\ h_1(k_1,k_2) &= 0 \\ h_2(k_1,k_2) &= k_1 + k_2\rho. \end{aligned}$$

We want to compute

$$v_t = 1 + E_t(e^{\tau\cdot X_{t+1}}v_{t+1})$$

with $\tau = (\ln\delta, \alpha, 0)$. According to Theorem 5 the solution is then given by

$$v_t = \sum_{i=0}^{\infty} e^{A_i\cdot X_t}$$

with $A_0 = (0,0,0)$ and $A_i = h(A_{i-1}+\tau)$. Denote by $A_{j,i}$ the jth element of A_i. We then have

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + (A_{1,i-1}+\alpha)\mu + 0.5(A_{1,i-1}+\alpha)^2\sigma^2 + 0.5A_{2,i-1}^2\phi_x^2\sigma^2 \\ A_{1,i} &= 0 \\ A_{2,i} &= (A_{1,i-1}+\alpha) + \rho A_{2,i-1}. \end{aligned}$$

Simplifying yields

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + \alpha\mu + 0.5\alpha^2\sigma^2 + 0.5A_{2,i-1}^2\phi_x^2\sigma^2 \\ A_{2,i} &= \alpha + \rho A_{2,i-1}. \end{aligned}$$

Recall the sufficient condition for existence,

$$\lim_{i\to\infty} e^{(A_{i+1}-A_i)\cdot X_t} < 1.$$

Observe that $\lim_{i\to\infty} A_{2,i} = \frac{\alpha}{1-\rho}$ and so it must also hold that $\lim_{i\to\infty}(A_{2,i} - A_{2,i-1}) = 0$. Furthermore

$$\lim_{i\to\infty}(A_{0,i} - A_{0,i-1}) = \lim_{i\to\infty}\left(\ln\delta + \alpha\mu + 0.5\alpha^2\sigma^2 + 0.5A_{2,i-1}^2\phi_x^2\sigma^2\right).$$

By continuity, $\lim_{i\to\infty} A_{2,i-1}^2 = \frac{\alpha^2}{(1-\rho)^2}$, and so

$$\lim_{i\to\infty} (A_{0,i} - A_{0,i-1}) = \left(\ln \delta + \alpha\mu + 0.5\alpha^2\sigma^2 + 0.5\frac{\alpha^2\phi_x^2\sigma^2}{(1-\rho)^2} \right).$$

This leads us to the following proposition.

Proposition 1 (Existence for CRRA utility with long-run consumption risks)**.** *The model with log consumption growth following the process* (17) *and CRRA utility has a solution if*

$$\underbrace{\alpha\mu}_{\text{constant}} + \underbrace{\frac{1}{2}\alpha^2\sigma^2}_{\text{c-shock}} + \underbrace{\frac{1}{2}\frac{\alpha^2\sigma^2\phi_x^2}{(1-\rho)^2}}_{\text{x-shock}} < -\log(\delta) \tag{18}$$

where $\alpha = 1 - \frac{1}{\psi}$.[2]

Theorem 4 then immediately yields the following conclusion.

Corollary 5. *The model with log consumption growth following the process* (17) *and EZ utility* (10) *with* $\psi > 1$ *has a solution if the following conditions are satisfied:*

- *Condition* (18) *holds.*
- *The investor has a preference for the early resolution of risks or is neutral towards timing resolution* ($\gamma \geq \frac{1}{\psi}$).

To obtain intuition about the existence condition, note that v_t is the wealth-consumption ratio of the investor. We observe that the influence of μ on the existence condition depends on the sign of α. An increase in μ implies an improvement of investment opportunities. For $\psi > 1$ ($\alpha > 0$), the substitution effect dominates the wealth effect. That is, the investor lowers her consumption relative to wealth in response to improved investment opportunities. Hence, an increase in μ implies an increase in v_t so the existence condition becomes more stringent.

We observe that consumption volatility makes the existence condition always more stringent. An increase in volatility implies an increase in arithmetic average consumption growth. For $\psi > 1$, which implies a risk aversion smaller than 1, this effect outweighs the effect of the additional risk and hence the investor perceives an improvement in investment opportunities. As $\psi > 1$, this implies an increase in v_t and hence the existence condition becomes more stringent.

Observe that, holding all other parameters constant, a solution always exists for sufficiently small values of the discount factor $\delta > 0$. Conversely, holding all other parameters constant, a solution fails to exists for sufficiently large values of $\rho < 1$ (using the ratio test's sufficient condition for divergence).

[2] A similar result has appeared in Burnside (1998) and de Groot (2015).

3.1.2 Stochastic Volatility with Gaussian and Non-Gaussian Shocks

Next consider the case when stochastic volatility is added to the model as for example in (Bansal and Yaron (2004)). Log consumption growth Δc_{t+1} follows

$$\begin{aligned} \Delta c_{t+1} &= \mu + x_t + \sigma_t \epsilon_{c,t+1} \\ x_{t+1} &= \rho x_t + \phi_x \sigma_t \epsilon_{x,t+1} \\ \sigma^2_{t+1} &= \bar{\sigma}^2(1-\rho_\sigma) + \rho_\sigma \sigma^2_t + \phi_\sigma \epsilon_{\sigma,t+1} \\ & \epsilon_{c,t+1}, \epsilon_{x,t+1} \sim i.i.d.\ N(0,1). \end{aligned} \tag{19}$$

Bansal and Yaron (2004) assume that $\epsilon_{\sigma,t+1} \sim i.i.d.\ N(0,1)$. This implies that the variance process can become negative and hence leads to an ill-defined model. For the existence proof outlined in this paper we do not need the normal assumption. We circumvent the problem by first truncating the normal distribution of $\epsilon_{\sigma,t+1}$ to guarantee $\sigma^2_t > 0$ and second by assuming that $\epsilon_{\sigma,t+1}$ follows a gamma distribution as in Bansal and Shaliastovich (2013). For our approach to be applicable we just need that the moment-generating function of the shock is well defined which is true for both cases.[3]

The Truncated Normal Case

First consider the case of a truncated normal where ϵ^{min}_σ denotes the truncation point of the lower tail. By truncating the normal distribution we can directly analyse how the truncation point affects the existence condition and compare to the non-truncated case. Without loss of generality we assume $\phi_\sigma > 0$. To guarantee that $\sigma^2_t > 0$ for all t we impose[4]

$$\epsilon^{min}_\sigma > -\frac{\bar{\sigma}^2(1-\rho_\sigma)}{\phi_\sigma}.$$

Section B.2 contains a derivation of the following existence condition for the model with truncated normal shocks.

Proposition 2 (Existence for CRRA utility with long-run risk and stochastic volatility with truncated normal shocks)**.** *The model with log consumption growth following the process* (19) *where shocks to volatility $\epsilon_{\sigma,t+1}$ follow a truncated normal distribution with truncation param-*

[3] de Groot (2015) shows that closed form solutions for the long-run risk model with CRRA utility can be derived as long as the moment-generating function of the shocks is well defined.

[4] Note that $\sigma^2_{t+i} = \bar{\sigma}^2(1-\rho_\sigma) + \rho\bar{\sigma}^2(1-\rho_\sigma) + \rho^2\bar{\sigma}^2(1-\rho_\sigma) + \ldots + \rho^i_\sigma \sigma^2_t + \phi_\sigma \epsilon_{\sigma,t+i} + \rho_\sigma \phi_\sigma \epsilon_{\sigma,t+i-1} + \ldots$. Consider the worst case scenario that is $\epsilon_{\sigma,t} = \epsilon^{min}_\sigma\ \forall\ t$. This implies $\lim_{i\to\infty} \sigma^2_{t+i} = \bar{\sigma}^2 + \frac{\phi_\sigma}{1-\rho_\sigma}\epsilon^{min}_\sigma$. Imposing that $\sigma^2_{t+i} > 0$ yields the truncation point.

eter ϵ_σ^{min} and CRRA utility has a solution if

$$\underbrace{\alpha\mu}_{\text{constant}} + \underbrace{\frac{1}{2}\alpha^2\bar{\sigma}^2}_{\text{c-shock}} + \underbrace{\frac{1}{2}\frac{\alpha^2\bar{\sigma}^2}{(1-\rho)^2}\phi_x^2}_{\text{x-shock}} + \underbrace{\frac{1}{2}K^2}_{\sigma\text{-shock}} + \underbrace{\log\left(\frac{\Phi\left(K-\epsilon_\sigma^{min}\right)}{\Phi(-\epsilon_\sigma^{min})}\right)}_{\sigma\text{-trunc}} < -\log(\delta) \tag{20}$$

where $K = \frac{\phi_\sigma\alpha^2\left((1-\rho)^2+\phi_x^2\right)}{2(1-\rho)^2(1-\rho_\sigma)} > 0$ and $\Phi(x)$ denotes the cumulative distribution function of a standard normally distributed random variable x.

Theorem 4 then immediately yields the following conclusion.

Corollary 6. *The model with log consumption growth following the process* (19) *where shocks to volatility $\epsilon_{\sigma,t+1}$ follow a truncated normal distribution with truncation parameter ϵ_σ^{min} and EZ utility* (10) *with $\psi > 1$ has a solution if the following conditions are satisfied:*

- *Condition* (26) *holds.*
- *The investor has a preference for the early resolution of risks or is neutral towards timing resolution ($\gamma \geq \frac{1}{\psi}$).*

Note that the first three terms are identical to the example above without stochastic volatility and the last two terms arise from the stochastic volatility process (σ-shock) and from its truncation (σ-trunc). The σ-shock term is always positive and $\frac{\partial\sigma\text{-shock}}{\partial\rho_\sigma} > 0$ and $\frac{\partial\sigma\text{-shock}}{\partial\phi_\sigma} > 0$. Thus the higher the volatility and persistence of the volatility processes, the more stringent the condition for existence becomes. We analyze the quantitative influence of these effects for standard calibrations in Section 3.2.1. Furthermore note that as $K > 0$, $\Phi\left(K - \epsilon_\sigma^{min}\right) > \Phi(-\epsilon_\sigma^{min})$ and hence truncation always makes the existence condition more stringent.[5] As $\frac{\partial K}{\partial\phi_\sigma} > 0$, $\frac{\partial K}{\partial\rho_\sigma} > 0$ and $\frac{\partial\sigma\text{-trunc}}{\partial K} > 0$ it follows that also the truncation term increases with the persistence and volatility of the variance process. Furthermore note that $\frac{\partial\sigma\text{-trunc}}{\partial\epsilon_\sigma^{min}} > 0$ so the more is truncated from the normal distribution of $\epsilon_{\sigma,t+1}$, the more stringent becomes the existence condition. Hence, if there exists a solution for the model with the truncated normal distribution and truncation point ϵ_σ^{min}, the same model with a truncation point smaller than ϵ_σ^{min} always has a solution. Note that this also includes the model without truncation.

The Gamma-Distribution Case

Next, we assume that the shocks to variance $\epsilon_{\sigma,t+1}$ follow a gamma distribution, $\epsilon_{\sigma,t+1} \sim$ *i.i.d.* $\Gamma(\theta_2, \theta_1)$ where $\theta_1 > 0$ is the scale and $\theta_2 > 0$ is the shape parameter of the gamma

[5] More precisely, truncation from below makes the existence condition more stringent. Truncation from above reverses the described effects.

distribution. Section B.2 contains a derivation of the following existence condition for this case.

Proposition 3 (Existence for CRRA utility with long-run risk and stochastic volatility with gamma distributed shocks)**.** *The model with log consumption growth following the process* (19)*,* $\epsilon_{\sigma,t+1} \sim i.i.d.\ \Gamma(\theta_2, \theta_1)$ *and CRRA utility has a solution if*

$$\underbrace{\alpha\mu}_{\text{constant}} + \underbrace{\frac{1}{2}\alpha^2\bar{\sigma}^2}_{\text{c-shock}} + \underbrace{\frac{1}{2}\frac{\alpha^2\bar{\sigma}^2}{(1-\rho)^2}\phi_x^2}_{\text{x-shock}} \underbrace{- \theta_2 \log\left(1 + \frac{\alpha^2\theta_1\phi_\sigma((1-\rho)^2+\phi_x^2)}{2(1-\rho)^2(1-\rho_\sigma)}\right)}_{\sigma\text{-shock}} < -\log(\delta). \qquad (21)$$

From Theorem 4 it then follows that the same model with EZ utility and ($\gamma \geq \frac{1}{\psi}$) has a solution if (21) is satisfied. The constant, the influence of the consumption shock and the influence of the shock to x_t are again the same as in the specification without stochastic volatility. Observe that σ-shock > 0 and hence, in contrast to the case with normally distributed shocks, stochastic volatility with gamma distributed shocks always relaxes the existence condition for any model parameters. Furthermore we have that $\frac{\partial\sigma\text{-shock}}{\partial\phi_\sigma} > 0$, $\frac{\partial\sigma\text{-shock}}{\partial\rho_\sigma} > 0$, $\frac{\partial\sigma\text{-shock}}{\partial\theta_1} > 0$ and $\frac{\partial\sigma\text{-shock}}{\partial\theta_2} > 0$. So the more persistent and volatile the variance process is, the less stringent becomes the existence condition. Also increasing the scale parameter θ_1 and the shape parameter θ_2 makes the existence condition less demanding. In the asset pricing literature (see for example Bansal and Yaron (2004)) it is usually assumed that shocks to volatility follow a normal distribution. Bansal, Kiku, and Yaron (2012) and Bansal and Shaliastovich (2013) argue that the distribution assumption of the variance shocks has a negligible effect on asset price dynamics. Nevertheless it clearly poses a problem for the existence of solutions. However, from the results above we know that a model with gamma shock to the variance always has a solution if the model with normally distributed shocks has a solution. Hence, to prove existence for a model with gamma shocks, it is sufficient to show existence in the model with normal shocks.

Corollary 7. *If there exists a solution to the long-run risk model* (19) *with normally distributed shocks to volatility and CRRA preferences, there always exists a solution to the same model with stochastic volatility and gamma distributed volatility shocks* $\epsilon_{\sigma,t}$.

We can extend this argument even further. If the goal for the stochastic volatility process with gamma distributed shocks is to match the mean and volatility of the process with normally distributed shocks, it implies that $\bar{\sigma}_\Gamma^2 < \bar{\sigma}_N^2$ (see Appendix B.2.1). Here $\bar{\sigma}_\Gamma^2$ and $\bar{\sigma}_N^2$ denote the parameter $\bar{\sigma}^2$ for the processes with gamma and normal distributed shocks respectively. As an decrease in $\bar{\sigma}^2$ always makes the existence condition less demanding, this matching of moments of the processes will further relax the existence condition for the case with gamma distributed shocks compared to the case with normally distributed shocks.

3.1.3 Ambiguity and Learning

In this section we show how the approach can be used to prove existence in models with smooth ambiguity aversion and learning. Comparison with discounted expected utility models, such as in Theorem 4, has an unexpected benefit in the context of models with Bayesian updating. This benefit derives from the fact that such agents are indifferent to the timing of information. For an agent with discounted expected utility, the Bayesian updating integrates out, and the model simplifies to

$$E\left(E\left(\sum_{i=0}^{\infty}\beta^t u(C_t)\middle|\,\theta)\right)\right.$$

where the final expectation is taken over the agent's time zero distribution for θ.

Suppose that for each θ, time zero utility for that model, $U_0(\theta)$, exists. It can happen that the unconditional expectation, $E(U_0(\theta))$, fails to exist. (See Geweke (2001) for an example.) If $U_0(\theta)$ is bounded above by a constant c, then $E(U_0(\theta)) \leq c$, so existence follows. One easy example is when there is a "best possible" model, θ_0, such that $U_\theta \leq U_{\theta_0}$. We apply this approach below to the model of Ju and Miao (2012).

Another application is to sharpen the existence result in Klibanoff, Marinacci, and Mukerji (2009) to show that the model solution is the unique limit of finite-horizon economies.

Theorem 6. *Suppose a consumption stream is bounded above by a constant c. For an agent with Hayashi–Miao preferences that is smooth ambiguity averse and prefers early resolution of risk, the model has a unique solution as the limit of finite-horizon economies.*

This follows immediately from the fact that

$$U(\theta) \leq \sum_{t=0}^{\infty}\beta^t u(c) = \frac{u(c)}{1-\beta}.$$

We apply the approach to prove existence in the model of Ju and Miao (2012) which features learning and smooth ambiguity aversion. Log consumption growth follows a regime-switching process:

$$\begin{aligned}\Delta c_{t+1} &= \mu_{z_t+1} + \sigma\epsilon_{t+1} \\ \epsilon_{t+1} &\sim i.i.d.\ N(0,1).\end{aligned} \tag{22}$$

z_{t+1} follows a Markov chain which takes values 1 or 2 with transition matrix (q_{ij}), where $\sum_j q_{ij} = 1, i, j = 1, 2$. We define state 1 as the good state so that $\mu_1 > \mu_2$. Ju and Miao (2012) assume that the model parameters are known to the investor but the true state is unobserved. The investor has ambiguous beliefs about the state and uses Bayes' rule to update beliefs. Let $p_t = Pr(z_{t+1} = 1|s^t)$. It follows that

$$p_{t+1} = \frac{q_{11}\Phi(\Delta c_{t+1}, \mu_1)p_t + q_{21}\Phi(\Delta c_{t+1}, \mu_2)(1-p_t)}{\Phi(\Delta c_{t+1}, \mu_1)p_t + \Phi(\Delta c_{t+1}, \mu_2)(1-p_t)} \tag{23}$$

where $\Phi(y, \mu_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\mu_i)^2}{2\sigma^2}}$ is the density of the normal distribution with mean μ_i and variance σ^2. As argued above, to show existence, it is sufficient to show existence for the best case scenario. Ju and Miao (2012) assume $\psi > 1$ so that mean consumption growth always increases utility. Hence, the best case scenario is when the economy is in state 1 with probability 1.

In this case the model simplifies to a model with a fix mean μ_1 and volatility σ. The existence condition for this model has been already derived in Section 3.1.1. Hence, the following proposition follows.

Proposition 4 (Existence for CRRA utility with regime-switching unobserved consumption growth.)**.** *The model with log consumption growth following the process* (22)*, where the investor uses bayesian updating specified in* (23) *to learn about the unobserved state* z_{t+1} *and CRRA utility has a solution if*

$$\alpha\mu_1 + \frac{1}{2}\alpha^2\sigma^2 < -\log(\delta). \tag{24}$$

Theorem 4 then immediately yields the following conclusion.

Corollary 8. *The model with log consumption growth following the process* (22) *where the investor uses bayesian updating specified in* (23) *to learn about the unobserved state* z_{t+1} *and has smooth ambiguity preferences* (9) *with* $\psi > 1$ *has a solution if the following conditions are satisfied:*

- *Condition* (24) *holds.*
- *The investor has a preference for the early resolution of risks or is neutral towards timing resolution* ($\gamma \geq \frac{1}{\psi}$)*.*
- *The investor is ambiguity averse or ambiguity neutral* ($\eta \geq \gamma$)*.*

Ju and Miao (2012) use the following benchmark parameters: $\mu_1 = 0.02251, \sigma = 0.03127, \delta = 0.975, \gamma = 2, \eta = 8.864$ and $\psi = 1.5$. Hence, the investor is both ambiguity averse and prefers early resolution of risks so that we can apply Theorem 4. The parameters imply that the left hand side of (24) is 0.0076 and $-\log(\delta) = 0.0253$. So the condition (24) holds and hence we have proved existence in the model. Ju and Miao (2012) also provide results for $\psi = 2$. In this case the left hand side (24) is 0.0114. Hence, the existence condition becomes more demanding but a solution does still exist.

3.2 Existence for Several Asset Pricing Models

In this section we derive sufficient existence conditions for different sources of risks that have been shown to be important for asset prices. Many of these models rely on approximation and are not fully specified for extreme events. For example, stochastic volatility with non-truncated normal shocks can lead to negative variance for extreme realizations. The process can be modified for these rare events to avoid negative variance. Bansal, Kiku, and Yaron (2016) consider several possibilities. If these events are rare enough, then the moment-generating function should still be well-approximated by an exponential function, so approximately we remain within the affine process class. This approximation is what allows us to apply the Campbell-Shiller log-linearization (Eraker and Shaliastovich (2008)). Under the assumption that the moment-generating function can be approximated by an exponential, we can use that approximation to apply the argument from Section 2.4.[6] The extent to which this approximation holds depends on the specific model and is outside the scope of this paper.

We conduct an extensive analysis of the risks and model parameters that are particularly important for the existence of solutions. We begin with the long-run risk model of Bansal and Yaron (2004) that includes both, a time varying mean growth rate as well as stochastic volatility. Second, we provide existence conditions for the volatility of volatility factor of Bollerslev, Tauchen, and Zhou (2009), Tauchen (2011), and Bollerslev, Xu, and Zhou (2015). Third, we analyze how jumps to volatility as in Drechsler and Yaron (2011) affect the existence condition. We consider both, gamma-distributed jumps as well as normally-distributed jumps. Fourth, we show how jumps to consumption itself affect the existence of solutions. Here we consider the discrete-time version of Wachter (2013) where the intensity of jumps is time-varying and follows a CIR process. Finally, we prove existence in the model of Drechsler and Yaron (2011) that includes three state processes to model changes in the mean growth rate of consumption, stochastic volatility and changes in mean volatility. The model features Gaussian shocks as well as multiple Poisson jumps. What all these models have in common is that they require a preference for the early resolution of risks in order to explain asset price dynamics. Hence, Theorem 4 implies that it is sufficient to prove existence for the model with CRRA preference in order to prove existence in the corresponding model with Epstein–Zin preferences. As all the following models have this property in common, we do not explicitly state it for each individual model.

3.2.1 Long-Run Risks and Stochastic Volatility (Bansal and Yaron (2004))

In this section we analyze the existence of solutions in the long-run risk model of Bansal and Yaron (2004). Here we concentrate on the case where shocks to volatility are normally

[6] In Appendix B.1 we show how to apply the approach to general affine models.

distributed as in the original model. The case with truncated normal or gamma distributed shocks to volatility to ensure positivity of the variance process has already been extensively discussed in Section 3.1.2. Closed-form solutions for the model with CRRA utility can also be found in the online appendix of de Groot (2015). Log consumption growth Δc_{t+1} follows

$$\begin{aligned}
\Delta c_{t+1} &= \mu + x_t + \sigma_t \epsilon_{c,t+1} \\
x_{t+1} &= \rho x_t + \phi_x \sigma_t \epsilon_{x,t+1} \\
\sigma_{t+1}^2 &= \bar{\sigma}^2(1-\rho_\sigma) + \rho_\sigma \sigma_t^2 + \phi_\sigma \epsilon_{\sigma,t+1} \\
&\epsilon_{c,t+1}, \epsilon_{x,t+1}, \epsilon_{\sigma,t+1} \sim i.i.d.\ N(0,1).
\end{aligned} \tag{25}$$

Section B.2 contains a derivation of the following existence condition for this process and CRRA utility.

Proposition 5 (Existence for CRRA utility with long-run risk and stochastic volatility)**.** *The model with log consumption growth following the process* (19) *and CRRA utility has a solution if*

$$\underbrace{\alpha\mu}_{\text{constant}} + \underbrace{\frac{1}{2}\alpha^2\bar{\sigma}^2}_{\text{c-shock}} + \underbrace{\frac{1}{2}\frac{\alpha^2\bar{\sigma}^2}{(1-\rho)^2}\phi_x^2}_{\text{x-shock}} + \underbrace{\frac{1}{8}\frac{\alpha^4\left((1-\rho)^2+\phi_x^2\right)^2}{(1-\rho)^4\,(1-\rho_\sigma)^2}\phi_\sigma^2}_{\sigma\text{-shock}} < -\log(\delta) \tag{26}$$

From Section 3.1.2 we know that if this model has a solution, a model with gamma distributed volatility shocks always has a solution as well. Hence, for a similar specification that ensures positivity of the variance process, a solution always exists. Expression (26) shows that the existence of solutions depends on, among other parameters, the size of the subjective discount factor δ and mean growth μ which enters only the constant term. The smaller the subjective discount factor, δ, the less demanding the existence condition becomes. As explained in Section 3.1.1, the influence of the mean growth depends on the magnitude of the IES ψ as $\alpha = 1 - \frac{1}{\psi}$. Hence, for an IES > 1, when the substitution effect dominates the income effect, the higher consumption growth μ, the more stringent becomes the existence condition.

In addition, each shock in the model, $\epsilon_{c,t+1}, \epsilon_{x,t+1}$ and ω_{t+1}, adds a new term to the existence requirement.[7] The presence of each type of shock makes the existence requirement more demanding since the three shock terms are all positive. Observe that $\frac{\partial \text{x-shock}}{\partial \rho} > 0$, $\frac{\partial \text{x-shock}}{\partial \phi_x} > 0$, $\frac{\partial \sigma\text{-shock}}{\partial \rho_\sigma} > 0$ and $\frac{\partial \sigma\text{-shock}}{\partial \phi_\sigma} > 0$.[8] Thus the higher the volatility and persistence of the

[7]Note that in this model specification stochastic volatility influences not only shocks to consumption but also shocks to long-run risk. In a more parsimonious setup, with stochastic volatility only entering the shocks to consumption, where the long-run risk factor is a standard AR(1) process the x-shock term simplifies to to $0.5\left(\frac{\alpha}{1-\rho}\right)^2\phi_x^2$ and the σ-shock term to $\frac{1}{8}\frac{\alpha^4}{(1-\rho_\sigma)^2}\phi_\sigma^2$ and so there is no interaction between the separate terms.

[8]Note that, since stochastic volatility also affects the long-run risk factor, it holds that $\frac{\partial \sigma\text{-shock}}{\partial \rho} > 0$ and $\frac{\partial \sigma\text{-shock}}{\partial \phi_x} > 0$, making the conditions for existence more stringent as ρ and ϕ_x increase.

state processes, the more stringent the condition for existence becomes. Table 1 reports the magnitudes of the four terms on the left-hand side of (26) for two different parameterizations. In particular, we use the calibration of Bansal and Yaron (2004) as well as the more recent calibration of Bansal, Kiku, and Yaron (2012) in which the influence of stochastic volatility is increased.

Table 1: Existence in the Long-Run Risk Model of Bansal and Yaron (2004)

Calibration Bansal and Yaron (2004)					
	constant	c-shock	x-shock	σ-shock	Sum
$\psi = 2$	7.5e-4	7.6e-6	3.3e-5	7.1e-9	7.9e-4
$\psi = 1.5$	5.0e-4	3.4e-6	1.5e-5	1.4e-9	5.1e-4
Calibration Bansal, Kiku, and Yaron (2012)					
	constant	c-shock	x-shock	σ-shock	Sum
$\psi = 2$	7.5e-4	6.5e-6	1.5e-5	6.7e-7	7.7e-4
$\psi = 1.5$	5.0e-4	2.9e-6	6.7e-6	1.3e-7	5.1e-4

The table displays values for the four terms in condition (26)*, which enter the sufficient existence condition for solutions in the model of Bansal and Yaron (2004) for two sets of parameter calibrations. The parameters in Bansal and Yaron (2004) are given by* $\mu = 0.0015, \bar{\sigma} = 0.0078, \rho = 0.979, \phi_x = 0.044, \rho_\sigma = 0.987, \phi_\sigma = 2.3e\text{-}6$. *The parameters in Bansal, Kiku, and Yaron (2012) are* $\mu = 0.0015, \bar{\sigma} = 0.0072, \rho = 0.975, \phi_x = 0.038, \rho_\sigma = 0.999, \phi_\sigma = 2.8e\text{-}6$.

Mean log consumption growth μ is 0.0015 in both calibrations and $\psi = 1.5$. These estimates yield a constant term of 5.0e-4. The recent estimation study of Schorfheide, Song, and Yaron (2018) reports a value of $\psi \approx 2$ so we also provide results for the higher IES case. The constant term increases in ψ and hence for $\psi = 2$, the term becomes 7.5e-4 making the existence condition more stringent. We observe that for both calibrations, the influence of the c-shock, x-shock and σ-shock terms is at least one order of magnitude smaller compared to the constant term. In the calibration of Bansal, Kiku, and Yaron (2012) the influence of stochastic volatility is increased with $\rho_\sigma = 0.999$ (instead of $\rho_\sigma = 0.987$), however, the influence on the existence condition still remains small. Of the three shock terms, the x-shock has the largest influence. So potentially, very persistent and volatile x-processes can have a significant influence on the existence of solutions. However, for the two calibrations considered here the influence is still small compared to the constant term.

The sum of the four terms is 5.1e-4 for ψ and about 7.9e-4 for $\psi = 2$. So which values for the subjective discount factor δ ensure existence for the two cases? For $\psi = 1.5$, $\delta < 0.9995$ guarantees the existence of solutions while for $\psi = 2$, a discount factor smaller then 0.9992 is needed. Bansal and Yaron (2004) use a value of $\delta = 0.998$ while Bansal, Kiku, and Yaron

(2012) use 0.9989, hence we have proven existence for both models.

Summarizing, we find that all three shocks in the model make the existence condition more stringent. Persistence of the state processes further amplifies their influence. Quantitatively we find that—using the calibrations of Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012)—the influence of the different shocks is rather small compared to the influence of mean consumption growth. Hence, the existence of a model solution is mainly driven by mean growth μ, the IES ψ and the discount factor δ. For the calibrations used in the paper the subjective discount factors are small enough to ensure that solutions exist.

3.2.2 Volatility of Volatility (Bollerslev, Tauchen, and Zhou (2009))

Next, we consider the influence of volatility of volatility as in Bollerslev, Tauchen, and Zhou (2009), Tauchen (2011), and Bollerslev, Xu, and Zhou (2015) on the existence of solutions. In both studies it is assumed that the agent has Epstein–Zin preferences (10) and a preference for the early resolution of risks. Hence, Theorem 4 implies that it is sufficient to prove existence for the corresponding model with CRRA preferences in order to prove existence for the Epstein–Zin case. We use the parsimonious model formulation of Bollerslev, Tauchen, and Zhou (2009) without the long-run risk factor to separately analyze the influence of volatility of volatility on the existence of solutions:

$$\begin{aligned} \Delta c_{t+1} &= \mu + \sigma_t \epsilon_{c,t+1} \\ \sigma_{t+1}^2 &= \bar{\sigma}^2(1-\rho_\sigma) + \rho_\sigma \sigma_t^2 + \sqrt{q_t}\epsilon_{\sigma,t+1} \\ q_{t+1} &= \mu_q(1-\rho_q) + \rho_q q_t + \phi_q \sqrt{q_t}\epsilon_{q,t+1} \\ & \epsilon_{c,t+1}, \epsilon_{\sigma,t+1}, \epsilon_{q,t+1} \sim i.i.d.\ N(0,1). \end{aligned} \tag{27}$$

Section B.3 (and the Mathematica file Existence_VoV.nb accompanying this paper) contains a derivation of the following existence condition for this process and CRRA utility.

Proposition 6 (Existence for CRRA utility with volatility of volatility)**.** *The model with log consumption growth following the process* (19) *and CRRA utility has a solution if*

$$\underbrace{\alpha\mu}_{constant} + \underbrace{\frac{\alpha^2\bar{\sigma}^2}{2}}_{c\text{-}shock} + \underbrace{\frac{\mu_q\left(1-\rho_q\right)\left(1-\rho_q-\sqrt{\left(1-\rho_q\right)^2-\frac{\alpha^4\phi_q^2}{4(1-\rho_\sigma)^2}}\right)}{\phi_q^2}}_{\sigma\text{- and } q\text{-}shock} < -\log(\delta). \tag{28}$$

Note that the influence of the σ- and q-shock cannot be separated as q_t follows a discrete time version of a continuous-time square root-type process. For this specification, the model solution can become complex, a feature that has already been mentioned in Bollerslev, Tauchen, and Zhou (2009) and is extensively discussed in Pohl, Schmedders, and Wilms

(2018). More precisely, the larger ρ_σ, ϕ_q and $|\alpha|$ and the smaller ρ_q, the more likely a complex solution becomes.

To isolate the effects of stochastic volatility and volatility of volatility on the existence of solutions, we consider the case where q_t follows a simple AR(1) process:

$$q_{t+1} = \mu_q(1-\rho_q) + \rho_q q_t + \phi_q \epsilon_{q,t+1}. \tag{29}$$

For this variation, the existence condition is given by

$$\underbrace{\alpha\mu}_{\text{constant}} + \underbrace{\frac{\alpha^2\bar{\sigma}^2}{2}}_{\text{c-shock}} + \underbrace{\frac{\alpha^4\mu_q}{8\,(1-\rho_\sigma)^{\,2}}}_{\sigma\text{-shock}} + \underbrace{\frac{\alpha^8\phi_q^2}{128\,(1-\rho_q)^{\,2}\,(1-\rho_\sigma)^{\,4}}}_{\text{q-shock}} < -\log(\delta), \tag{30}$$

see Section B.3 and the Mathematica file Existence_VoV_no_CIR_q.nb for a derivation.

We observe that the q-shock term positively depends on the persistence of volatility, ρ_σ, the volatility of volatility, ρ_q, and the volatility of the vol-of-vol factor ϕ_q. This dependence holds irrespectively of the sign of α. Hence, a persistent and volatile q_t-process makes the existence condition more stringent. The first line in Table 2 reports the magnitudes of the four terms in condition (30) for the calibration in Bollerslev, Tauchen, and Zhou (2009). In addition, to highlight the importance of the volatility of volatility channel, the table reports results for different values for the persistence ρ_q and the volatility ϕ_q of the q_t process. First, we

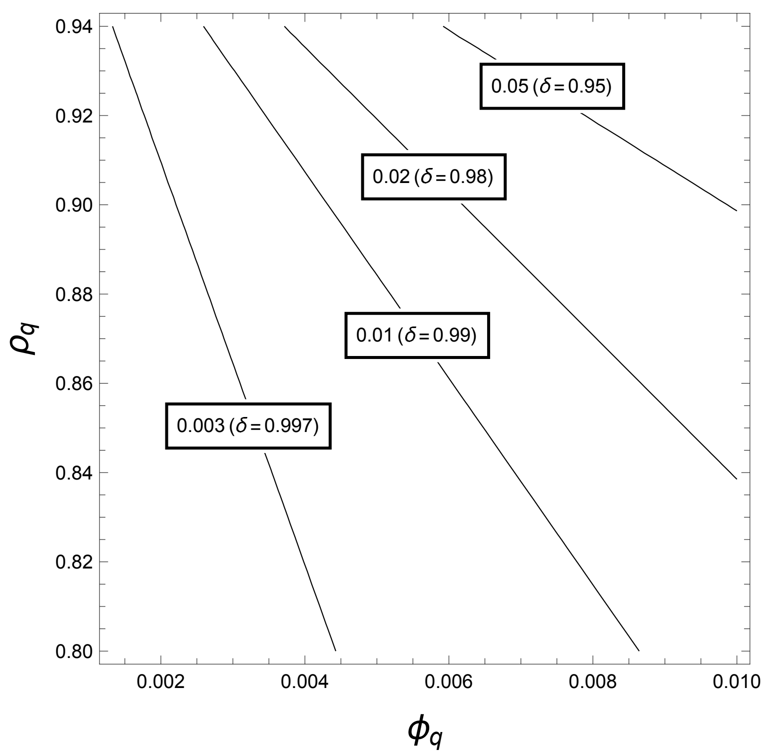
Table 2: Sufficient Condition for Existence in the Volatility-of-Volatility Model of Bollerslev, Tauchen, and Zhou (2009)

	constant	c-shock	σ-shock	q-shock	Sum
$\rho_q = 0.8, \phi_q = $ 1e-3	0.0005	3.38e-6	3.19e-6	0.00013	0.00063
$\rho_q = 0.9, \phi_q = $ 1e-3	0.0005	3.38e-6	3.19e-6	0.00051	0.00101
$\rho_q = 0.8, \phi_q = $ 5e-3	0.0005	3.38e-6	3.19e-6	0.00318	0.00368
$\rho_q = 0.9, \phi_q = $ 5e-3	0.0005	3.38e-6	3.19e-6	0.01271	0.01321

The table displays values for the four terms in condition (30)*, which enter the sufficient existence condition for solutions in the model of Bollerslev, Tauchen, and Zhou (2009). The model parameters are given by* $\psi = 1.5, \mu = 0.0015, \rho_\sigma = 0.978, \bar{\sigma}^2 = 0.0078^2$*, and* $\mu_q = 1e\text{-}6$*.*

observe that the influence of the vol-of-vol factor on existence is significantly larger compared to the influence of stochastic volatility and the influence increases rapidly with ρ_q and ϕ_q. Bollerslev, Tauchen, and Zhou (2009) use a discount factor of $\delta = 0.997$ which implies a right hand side of the existence condition of 0.003. Hence, the existence condition is satisfied for their baseline calibration. Figure 1 plots the existence condition (30) as a function of ϕ_q and ρ_q. The contour lines show the discount factor δ that is needed to ensure existence. For

Figure 1: Sufficient Condition for Existence in the Volatility-of-Volatility Model of Bollerslev, Tauchen, and Zhou (2009)



The figure plots the existence condition (30) as a function of the volatility ϕ_q and the persistence ρ_q of the vol-of-vol factor q_t for the model of Bollerslev, Tauchen, and Zhou (2009). The contour lines also show the discount factor δ that is needed to ensure existence. The model parameters are given by $\psi = 1.5, \mu = 0.0015, \rho_\sigma = 0.978, \bar{\sigma}^2 = 0.0078^2$, and $\mu_q =$ 1e-6.

moderate levels of ϕ_q and ρ_q, a discount factor of $\delta = 0.997$ is sufficient for the existence of solutions. However, for larger levels of ϕ_q and ρ_q, a significantly smaller δ is required. In sum, the volatility-of-volatility factor has a significantly larger effect on the existence of solutions compared to the standard stochastic volatility factor and existence might become an issue for highly persistent and volatile vol-of-vol processes when using monthly discount factors that are usually assumed to lie between 0.99 and 0.999.

3.2.3 Stochastic Volatility with Jumps (Drechsler and Yaron (2011))

Next we analyze the influence of jumps in volatility as in Drechsler and Yaron (2011) on the existence of solutions. As in the previous studies, Drechsler and Yaron (2011) assume that the agent has Epstein–Zin preferences (10) and a preference for the early resolution of risks. Hence, Theorem 4 implies that it is sufficient to prove existence for the corresponding model with CRRA preferences in order to prove existence for the Epstein–Zin case. We consider a special case of the model of Drechsler and Yaron (2011), which excludes the long-run risk process as well as the volatility of volatility to isolate the effects of jumps on existence (the full model of Drechsler and Yaron (2011) featuring multiple state processes and jumps with time-varying intensities is considered in Section 3.2.5). Then log consumption growth Δc_{t+1} is given by

$$\begin{aligned} \Delta c_{t+1} &= \mu + \sigma_t \epsilon_{c,t+1} \\ \sigma^2_{t+1} &= \bar{\sigma}^2(1-\rho_\sigma) + \rho_\sigma \sigma^2_t + \phi_\sigma \epsilon_{\sigma,t+1} + J_{t+1} \\ &\epsilon_{c,t+1}, \epsilon_{\sigma,t+1} \sim i.i.d.\ N(0,1), \end{aligned} \tag{31}$$

where $J_{t+1} = \sum_{j=1}^{N_{t+1}} \xi_j$, N_{t+1} is a Poisson counting process, and ξ_j is the size of the jump that occurs upon the jth increment of N_{t+1}. We assume that the jump intensity of N_{t+1} is constant and given by l_0. For the size of the jumps we consider two different distributions, a gamma and normal distribution.

Gamma-Distributed Jumps

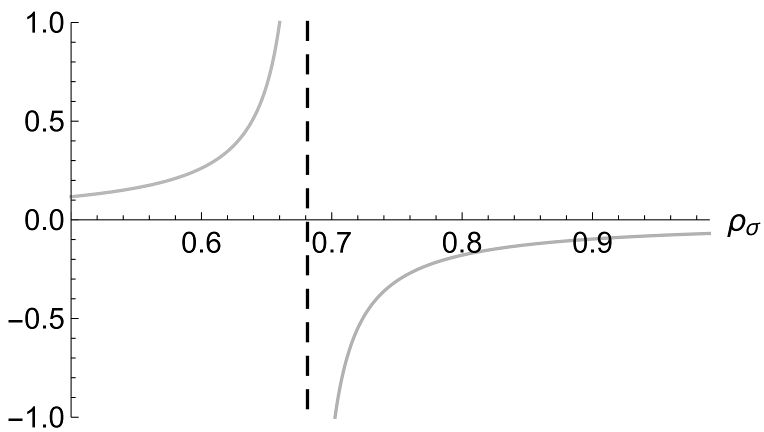
We first assume that the jumps to stochastic volatility are i.i.d. gamma-distributed as in Drechsler and Yaron (2011), $\xi_j \sim \Gamma(\nu_\sigma, \frac{\mu_\sigma}{\nu_\sigma})$. For a derivation of the following existence condition see Section B.4 (and the Mathematica file Existence_SV_with_Jumps.nb).

Proposition 7 (Existence for CRRA utility with gamma-distributed jumps)**.** *The model with log consumption growth following the process* (31) *with* $\xi_j \sim \Gamma(\nu_\sigma, \frac{\mu_\sigma}{\nu_\sigma})$ *and CRRA utility has* *a solution if*

$$\underbrace{\alpha\mu}_{constant} + \underbrace{\frac{\alpha^2\bar{\sigma}^2}{2}}_{c\text{-}shock} + \underbrace{\frac{\alpha^4\phi_\sigma^2}{8\left(1-\rho_\sigma\right)^2}}_{\sigma\text{-}shock} + \underbrace{l_0\left(\left(1-\frac{\alpha^2\mu_\sigma}{2\nu_\sigma\left(1-\rho_\sigma\right)}\right)^{-\nu_\sigma}-1\right)}_{\sigma\text{-}jump} < -\log(\delta). \tag{32}$$

As we have extensively analyzed the first three components in the previous sections, we focus on the σ-jump term, J_{t+1}, here. To demonstrate the influence of gamma-distributed jumps on existence, we consider the calibration of Drechsler and Yaron (2011) given by $\psi = 2, \mu_\sigma = 2.55$, and $\nu_\sigma = 1$. The jump intensities in Drechsler and Yaron (2011) are time varying with a mean value of 0.8/12. Therefore, we set $l_0 = 0.8/12$. As we demonstrate in the following, the sign of the jump component crucially depends on the persistence of stochastic volatility ρ_σ. In Figure 2 we plot the σ-jump term as a function of ρ_σ. We observe that for large values of persistence the term is negative and so the jumps to volatility increase the existence region. Drechsler and Yaron (2011) calibrate a value of $\rho_\sigma = 0.87$ so the jumps

Figure 2: Gamma-distributed Jumps: σ-jump Term as a Function of ρ_σ



The figure plots the σ-jump term of the existence condition (32) as a function of the persistence ρ_σ. The parameters are from Drechsler and Yaron (2011) and are given by $\psi = 2, \mu_\sigma = 2.55, l_0 = 0.8/12$, and $\nu_\sigma = 1$.

relax the existence condition. Interestingly, for smaller persistence levels, positive gamma jumps make the existence condition significantly more stringent. Furthermore, there exists a threshold value for which the σ-jump term grows unbounded and so no solution exists.

Normally Distributed Jumps

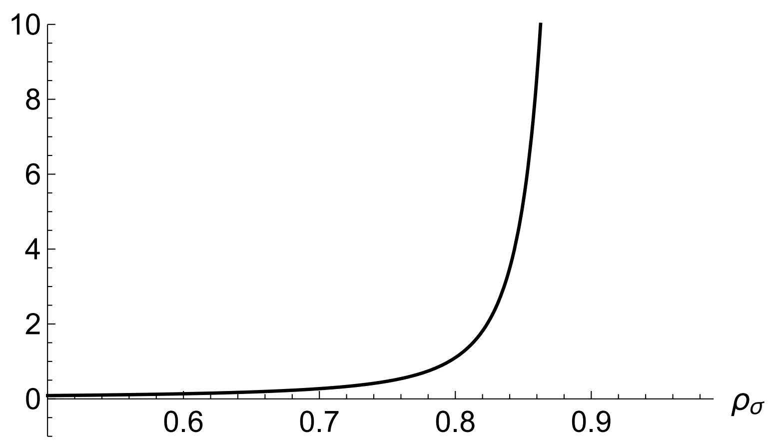
Next, we assume that the jumps to stochastic volatility are i.i.d. normal-distributed, $\xi_j \sim N(\mu_\xi, \sigma_\xi)$. For a derivation of the following existence condition see Section B.4 (and the Mathematica file Existence_SV_with_Jumps.nb).

Proposition 8 (Existence for CRRA utility with normally distributed jumps)**.** *The model with log consumption growth following the process* (31) *with* $\xi_j \sim N(\mu_\xi, \sigma_\xi)$ *and CRRA utility has a solution if*

$$\underbrace{\alpha\mu}_{constant} + \underbrace{\frac{\alpha^2\bar{\sigma}^2}{2}}_{c\text{-}shock} + \underbrace{\frac{\alpha^4\phi_\sigma^2}{8\left(1-\rho_\sigma\right)^2}}_{\sigma\text{-}shock} + \underbrace{l_0\left(\exp\left(\frac{\alpha^4\sigma_\xi^2 + 4\alpha^2\mu_\xi\left(1-\rho_\sigma\right)}{8\left(1-\rho_\sigma\right)^2}\right)-1\right)}_{\sigma\text{-}jump} < -\log(\delta). \quad (33)$$

The key difference to the case with gamma-distributed shocks is that jumps can now be both, positive or negative. We observe that $\frac{\partial\sigma\text{-jump}}{\partial\mu_\xi} > 0$ and $\frac{\partial\sigma\text{-jump}}{\partial\sigma_\xi} > 0$. So the larger the volatility of the jumps and the larger the mean of the jumps, the more stringent the existence condition becomes. Furthermore, $\lim_{\rho_\sigma\to 1}\sigma\text{-jump} = \infty$, so for highly persistent stochastic volatility processes, a solution does not exist. Figure 3 depicts this pattern. It plots the σ-jump term as a function of ρ_σ. To make the results comparable, we assume that the normally distributed

Figure 3: Normally Distributed Jumps: σ-jump Term as a function of ρ_σ



The figure plots the σ-jump term of the existence condition (33) as a function of the persistence ρ_σ. The parameters are given by $\psi = 2, \mu_\xi = 2.55, l_0 = 0.8/12$, and $\sigma_\xi = 2.55$.

jumps have the same mean and standard deviation as the gamma-distributed jumps in the previous section, hence $\mu_\xi = 2.55$ and $\sigma_\xi = 2.55$. We observe that the σ-jump is always positive and increases strongly with ρ_σ. Hence, normally-distributed jumps to stochastic volatility make the existence condition significantly more demanding.

3.2.4 Consumption Disasters with Time-Varying Disaster Intensities (Wachter (2013))

As the next application, we consider a model with consumption disasters and time-varying disaster intensities. The model is a discrete-time version of the model in Wachter (2013). As Wachter (2013) shows, the model also requires Epstein–Zin preferences (10) and a preference for the early resolution of risks to explain asset price dynamics. Hence, Theorem 4 implies that it is sufficient to prove existence for the corresponding model with CRRA preferences in order to prove existence for the Epstein–Zin case. In the model, log consumption growth Δc_{t+1} is given by:

$$\begin{aligned} \Delta c_{t+1} &= \mu + \sigma \epsilon_{t+1} + J_{t+1} \\ \lambda_{t+1} &= \bar{\lambda}(1-\rho_l) + \rho_l \lambda_t + \phi_l \sqrt{\lambda_t} \omega_{t+1} \end{aligned} \tag{34}$$

where $J_{t+1} = \sum_{j=1}^{N_{t+1}} \xi_j$ are the consumption disasters, N_{t+1} is a Poisson counting process, and ξ_j is the size of the jump that occurs upon the jth increment of N_{t+1}. As in Wachter (2013) we assume that ξ_j is an i.i.d. normal process, $\xi_j \sim N(\mu_\xi, \sigma_\xi)$, so there are both positive and negative jumps to consumption growth. The moment-generating function, M_ξ, of ξ_j is given by

$$M_\xi(u) = E_t e^{u\xi_j} = e^{u\mu_\xi + 0.5u^2\sigma_\xi^2}.$$

The intensity process for N_{t+1} follows a discrete-time version of a continuous-time square root-type process and is given by λ_t. For a derivation of the following existence condition see Section B.5 (and the Mathematica file Existence_Wachter.nb).

Proposition 9 (Existence for CRRA utility with normally distributed disaster shocks)**.** *The model with log consumption growth following the process* (34) *with* $\xi_j \sim N(\mu_\xi, \sigma_\xi)$ *and CRRA utility has a solution if*

$$\underbrace{\alpha\mu}_{constant} + \underbrace{0.5\alpha^2\sigma^2}_{c\text{-}shock} + \underbrace{\frac{\bar{\lambda}\,(1-\rho_l)\left(1-\rho_l-\sqrt{(1-\rho_l)^2 - 2\,(M_\xi(\alpha)-1)\,\phi_l^2}\right)}{\phi_l^2}}_{c\text{-}jump} < -\log(\delta). \tag{35}$$

To obtain some intuition, first assume that disaster intensities are constant, $(\phi_l, \rho_l = 0)$. By the rule of l'hopital we then have that the c-jump term is given by $\bar{\lambda}(M_\xi(\alpha) - 1)$. So we observe that volatility of the jump σ_ξ always makes the existence condition more stringent. The influence of the mean jump size μ_ξ depends on the size of α. For $\psi > 1$ $(\alpha > 0)$ the substitution effect dominates the wealth affect. Hence agents utility increases with consumption growth so the existence condition becomes more stringent, the larger μ_ξ (see Section 3.1.1).

Next we turn to the case with time varying disaster intensities. We first analyze the square-root term in (35) to determine whether the solution is real. The square-root term is real if and only if

$$(1-\rho_l)^2 - 2\left(e^{\alpha\mu_\xi + 0.5\alpha^2\sigma_\xi^2} - 1\right)\phi_l^2 \geq 0.$$

This term is the difference of two non-negative expressions. We observe that volatility of jumps σ_ξ always increases the second expression and thus the possibility of a complex solution irrespective of the sign of α. If $\alpha > 0$, that is for $\psi > 1$, large negative jumps ($\mu_\xi < 0$) reduce the probability of a complex solution. For $\alpha < 0$, that is for $\psi < 1$, the opposite holds true.

Next, assuming the existence of a real solution, we analyze the influence of the jump-term on the existence condition. First note that (assuming $\bar{\lambda} > 0$ and $\rho_l > 0$)

$$\frac{\partial \text{c-jump}}{\partial M_\xi(\alpha)} = \frac{\bar{\lambda}\,(1-\rho_l)}{\sqrt{(1-\rho_l)^2 - 2(M_\xi(\alpha)-1)\phi_l^2}} > 0.$$

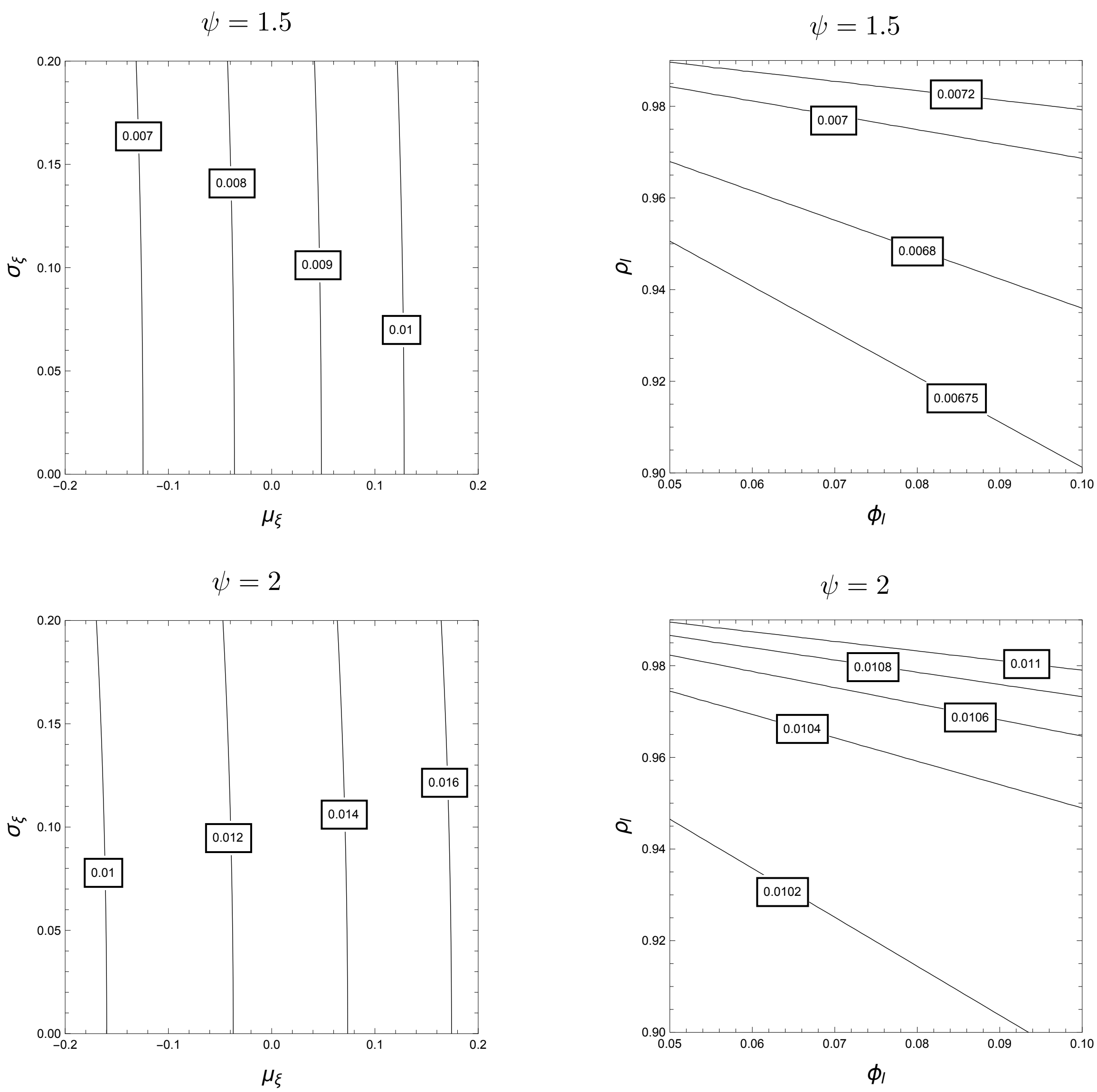
Hence, the conclusions for the constant disaster intensity case also hold for the case with time-varying disasters. The larger $M_\xi(\alpha)$, the more stringent becomes the existence condition. If $\alpha > 0$ ($\psi > 1$), the larger the mean jump size μ_ξ, the larger the c-jump term. Furthermore, irrespective of α, volatility of jumps, σ_ξ, always makes the existence condition more demanding.

To quantitatively analyze the influence of time-varying consumption disasters, we consider the calibration provided by Wachter (2013). The parameters are given by $\mu = 0.0252, \bar{\lambda} = 0.0355, \sigma = 0.02, \rho_l = e^{-0.08} = 0.923, \phi_l = 0.067, \mu_\xi = -0.15$, and $\sigma_\xi = 0.1$. Note first that we use the discrete-time equivalent of the parameters for the continuous-time model provided in Wachter (2013) and second, that—in contrast to the previous models—the parameters are calibrated to an annual frequency. The rate of time preference in Wachter (2013) is 0.012, which corresponds to $\delta = 0.988$. Figure 4 shows values for the left-hand side of the existence condition (35) for different values of $\mu_\xi, \sigma_\xi, \rho_l, \phi_l$ and ψ (Wachter (2013) assumes a unit IES but Tsai and Wachter (2018) argue that an IES > 1 can significantly improve the fit of the model. Hence, we provide results for $\psi = 1.5$ and $\psi = 2$.). We observe that the existence condition hardly changes with σ_ξ. Hence, the volatility of disasters has a negligible influence on existence. This is true for both, an IES of 1.5 and 2. In line with the results above, the left-hand side of the existence condition increases in the mean jump size, μ_ξ. For $\delta = 0.988$ the value of the left-hand side must be smaller than 0.12. We observe that the condition is only violated if μ_ξ approaches zero from below and $\psi = 2$. Hence, for negative mean jumps to consumption growth, as usually assumed in disaster models, a model solution is more likely to exist. Furthermore, while both, the persistence ρ_l and the volatility ϕ_l of the intensity process λ_t are affecting the existence condition, their quantitative impact is rather small and for the benchmark calibration the existence condition remains well below 0.012.

3.2.5 Existence for General Affine Processes with Gaussian and Poisson Jump Shocks (Drechsler and Yaron (2011))

In the previous sections we have demonstrated how the existence Theorem 5 can be applied to analytically analyze the existence of solutions for different models. In the following we

Figure 4: Existence with Consumption Disasters: Values for the Left-hand Side of (35)



The figure plots the existence condition (35) for different values of $\mu_\xi, \sigma_\xi, \rho_l, \phi_l$ and ψ for the consumption disaster model of Wachter (2013). The model parameters are given by $\mu = 0.0252, \bar{\lambda} = 0.0355, \sigma = 0.02, \rho_l = 0.923, \phi_l = 0.067, \mu_\xi = -0.15$, and $\sigma_\xi = 0.1$.

use the methodology described in Section B.1 to numerically proof existence in a model for which analytical solutions are either not available or are tedious to derive. The supplementary material includes a simple Matlab script that can be used to prove existence for any model that falls in the affine model class as in equation (41). For this, we need to specify the model dynamics in terms of $X_t, \mu, F, G_t, l_0, l_1, h, H^i$ and $M_{\xi^i}(k)$, see Section B.1. As an example we use the specific model used in Drechsler and Yaron (2011) which features a stochastic mean growth rate of consumption, stochastic volatility with a time varying mean and jumps to the mean growth rate of consumption as well as to volatility. The model dynamics are given by

$$X_{t+1} = \mu + FX_t + G_t z_{t+1} + J_{t+1} \tag{36}$$

with

$$X_t = \begin{bmatrix} 1 \\ \Delta c_t \\ x_t \\ \bar{\sigma}_t^2 \\ \sigma_t^2 \end{bmatrix}, \mu = \begin{bmatrix} 1 \\ \mu_c \\ 0 \\ \mu_{\bar{\sigma}} \\ \mu_\sigma \end{bmatrix}, F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_x & 0 & 0 \\ 0 & 0 & 0 & \rho_{\bar{\sigma}} & 0 \\ 0 & 0 & 0 & (1-\tilde{\rho}_\sigma) & \rho_\sigma \end{bmatrix} h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_c^2(1-\omega_c)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{\bar{\sigma}}^2(1-\omega_{\bar{\sigma}})^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$l_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, l_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{1,x} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{1,\sigma} \end{bmatrix} H_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_c^2\omega_c^2 & 0 & 0 & 0 \\ 0 & 0 & \phi_x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_\sigma^2 \end{bmatrix},$$

Drechsler and Yaron (2011) assume that $\bar{J}_{t+1}^i = \sum_{j=1}^{N_{t+1}^i} \xi_j^i$ where N_{t+1}^i is the standard Poisson counting process. The jumps in x_t follow the negative of a demeaned gamma distribution $\xi^x \sim -\Gamma(\nu_{\xi_x}, \frac{\mu_{\xi_x}}{\nu_{\xi_x}}) + \mu_{\xi_x}$ and the jumps in σ_t^2 follow a gamma distribution $\xi_\sigma \sim \Gamma(\nu_{\xi_\sigma}, \frac{\mu_{\xi_\sigma}}{\nu_{\xi_\sigma}})$. The corresponding moment-generating functions of J_{t+1}^i are then given by

$$M_{J^x}(k) = e^{\lambda_t^x \left(\left(1+\frac{\mu_{\xi_x}}{\nu_{\xi_x}} k\right)^{-\nu_{\xi_x}} e^{k\mu_{\xi_x}} - 1 \right)}$$

and

$$M_{J^\sigma}(k) = e^{\lambda_t^\sigma \left(\left(1-\frac{\mu_{\xi_\sigma}}{\nu_{\xi_\sigma}} k\right)^{-\nu_{\xi_\sigma}} - 1 \right)}.$$

Drechsler and Yaron (2011) rewrite model dynamics in terms of compensated jump shocks so that the model is easier to calibrate. They define $\tilde{J}_{t+1} = J_{t+1} - E_t(J_{t+1})$ such that

$$X_{t+1} = \tilde{\mu} + \tilde{F} X_t + G_t z_{t+1} + \tilde{J}_{t+1} \tag{37}$$

where

$$\tilde{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_x & 0 & 0 \\ 0 & 0 & 0 & \rho_{\bar{\sigma}} & 0 \\ 0 & 0 & 0 & (1-\tilde{\rho}_\sigma) & \tilde{\rho}_\sigma \end{bmatrix},$$

$\tilde{\mu} = (I - \tilde{F})E(X_t)$ and I is the identity matrix. The advantage of the compensated notation is that the model can be calibrated in terms of the true persistence parameter $\tilde{\rho}_\sigma$ and the unconditional means $E(X_t)$. The parameters used in Drechsler and Yaron (2011) are given by $\psi = 2, \delta = 0.999, E(\Delta c_t) = 0.0016, E(\bar{\sigma}_t^2) = 1, E(\sigma_t^2) = 1, \rho_x = 0.976, \rho_{\bar{\sigma}} = 0.985, \tilde{\rho}_\sigma = 0.87, \phi_c = 0.0066, \omega_c = 0.5, \phi_x = 0.032\phi_c, \phi_{\bar{\sigma}} = 0.1, \phi_\sigma = 0.35, l_{1,x} = 0.8/12, , l_{1,\sigma} = 0.8/12, \mu_{\xi_x} = 3.645\phi_x, \nu_{\xi_x} = 1, \mu_{\xi_\sigma} = 2.55, \nu_{\xi_\sigma} = 1$. Furthermore they use $\gamma = 9.5$ so the agent prefers the early resolution of risks. Hence, existence for CRRA preferences is a sufficient condition for the existence with Epstein–Zin preferences as in the original model. Using the specification for the model dynamics together with the moment-generating functions, we can now compute the h_i functions (44) to numerically compute the limit of the A_i coefficients. Using the Matlab script with an iteration limit for i of 1,000,000, we obtain that

$$\lim_{i\to\infty} e^{(A_{i+1}-A_i)\cdot X_t} = 0.99074 < 1.$$

Further increasing i to 2,000,000 does not change the limit up to the 12th digit. Hence, we have proven existence for the model of Drechsler and Yaron (2011).

4 Conclusion

The paper presents a simple approach to prove existence and uniqueness for asset pricing models with recursive preferences. The leading models in the literature, such as Bansal and Yaron (2004), Wachter (2013) and Ju and Miao (2012), feature very persistent processes and increasingly complex preferences. We show that both problems can be handled simultaneously by comparing to tractable benchmark models.

We demonstrate that existence can be shown comparatively. The existence for one model automatically implies existence for many related models, and this implication has a direct economic interpretation. For example, increasing the amount of smooth ambiguity aversion, or the strength of preference for early resolution of risk, preserves existence. For uniqueness, we show that the models have a unique solution as a limit of finite-horizon economies.

The methodology can be applied to a wide range of models covering for example the models of Bansal and Yaron (2004), Bollerslev, Tauchen, and Zhou (2009), Bollerslev, Xu, and Zhou (2015), Drechsler and Yaron (2011) and Wachter (2013). We provide a detailed analysis on how the different sources of risks used in these studies affect the existence of a solution.

For the state processes used in the studies, increasing the persistence and variance of the states usually makes the existence condition more demanding. Hence, the model features that provide explanatory power for asset pricing dynamics also intensify the existence problem. Furthermore we show that for very persistent parameters, solutions fail to exist.

The influence of jumps on the existence of solutions highly depends on the nature of the jumps. For example for jumps to stochastic volatility as in Drechsler and Yaron (2011) there is a large difference between when jumps follow a normal distribution rather than a gamma distribution. Normally distributed jumps with a positive mean always make the existence condition more demanding and the influence increases exponentially with the persistence of the volatility process. For gamma distributed jumps, the influence on existence can either be positive or negative, depending on the calibration of the state process. Jumps to consumption itself as in Wachter (2013) have a negligible influence on the existence of solutions as long as they are negative on average (disasters and not bonanzas).

We also show how to apply the approach to models with learning and smooth ambiguity aversion. We sharpen the existence and uniqueness results from Klibanoff, Marinacci, and Mukerji (2009). We also take the model of Ju and Miao (2012) and show that solutions exist in the calibrations used in the paper. Finally, we show how the existence theorem can be applied in cases where closed-form solutions do not exist or are tedious to derive. This is of particular practical use as researchers can simply use the accompanying Matlab script to show existence for a particular model.

Appendix

A Proofs for Section 2

A.1 Measure-Theoretic Details

The set Ω of sequences $(s_0, s_1, \ldots)$ is the underlying state space. Let $\mathcal{F}$ be a σ-algebra on Ω and $\mathbb{P}$ be a probability distribution.

A standard approach in the literature is to build up $\mathbb{P}$ out of conditional distributions on Ω_t by using extension theorems such as the Kolmogorov extension theorem or the Ionescu-Tulcea extension theorem. Our approach does not depend on the exact details of how $\mathbb{P}$ was constructed, so we instead assume that $\mathbb{P}$ is given to us, and we construct a filtration compatible with the state space representation.

We define a filtration $\mathcal{F}_i$ on $i = 0, 1, \ldots$, such that $\mathcal{F}_i \subset \mathcal{F}_{i+1} \subset \mathcal{F}$. Intuitively, $\mathcal{F}_t$ consists of all events that have happened as of time t.

For an arbitrary sequence, $(s_0, s_1, \ldots)$, let $s|_t$ represent the history up to time t,

$$s|_t = (s_0, \ldots, s_t).$$

$\mathcal{F}_t$ is defined as follows. An event $E \in \mathcal{F}_t$ if whenever $s \in E$ and $s'|t = s|t$, then $s' \in E$. In other words, it is exactly the events such that you can determine whether s is a member of E entirely by looking only up to time t.

A stochastic process X_t is a sequence of random variables. The process is adapted to $\mathcal{F}_t$ if X_t is $\mathcal{F}_t$-measurable. In Section 2.2, we assume that a time t random variable can be written as a function of s^t. For our choice of filtration, this is without loss of generality:

Lemma 1. *Every $\mathcal{F}_t$-measurable function $f : \Omega \to \mathbb{R}$ can be written as $g(s|_t)$ for a function $g : \Omega_t \to \mathbb{R}$.*

Proof. We prove that $s|_t = s'|_t$ then $f(s) = f(s')$. Let $f(s) = r$. For any $r \in \mathbb{R}$, $E = f^{-1}(r)$ is an element of $\mathcal{F}_t$. By assumption, if $s'|t = s|t$, then $s' \in E$, so $f(s') = r$.

For s^t, we define $g(s^t) = f(s)$ for some s such that $s|t = s^t$. By the previous paragraph this is independent of choice of such s. $\square$

So for our purposes a time t random variable is a function $g : \Omega_t \to \mathbb{R}$ such that $g(s|t)$ is $\mathcal{F}_t$-measurable.

We require that W_t be a jointly measurable function of its arguments. For a fixed f, $I_t(f)(s)$ is measurable as a function of s.

A.2 Proofs

Proof of Theorem 1. It's sufficient to show one recursive utility specification exists for a recursive preference relation, so we show that the present consumption equivalent is one such. So let U_t for $(C_\tau)_{\tau \geq t}$ be such that

$$(U_t, 0, \ldots) \sim_{s^t} (C_\tau)_{\tau \geq t}.$$

Let x and y be fixed deterministic consumption levels. Define $W_t(x, y; s^t)$ to be the present consumption equivalent

$$(x, y, 0, \ldots) \sim_{s^t} (W(x, y; s^t), 0, 0, \ldots).$$

Let $s_{t+1} \in \mathcal{S}_{s^t}$. For a function f of s_{t+1}, let $C^f_{t+1}((s^t, s_{t+1})) = f(s_{t+1})$, and zero for all other time $t+1$ states. Then define $I_t(f; s^t)$ to be the riskless equivalent,

$$(0, C^f_{t+1}, 0, \ldots) \sim_{s^t} (0, I_t(f; s_t)).$$

The substitutability and increasing limits preference properties directly translate to the equivalent properties for W_t and I_t. It remains to show U_t satisfies equation 1.

By assumption,

$$(U_{t+1}, 0,) \sim_{s^{t+1}} (C_\tau)_{\tau \geq t+1}.$$

By the strong monotonicity axiom, this implies that

$$(U_t, 0, 0, \ldots) \sim_{s^t} (C_t, U_{t+1}, 0, \ldots).$$

Fix s^t. Let $x = C(s^t)$, and for $s_{t+1} \in \mathcal{S}_{s^t}$ let $f(s_{t+1}) = U_{t+1}((s^t, s_{t+1})$. Since $C_t(s^t) = x$ and $C^f_{t+1}((s^t, s_{t+1})) = U_{t+1}((s^t, s_{t+1}))$ for all $s_{t+1} \in \mathcal{S}_{s^t}$, it follows from strong montonicity that

$$(x, C^f_{t+1}, 0, \ldots) \sim_{s^t} (C_t, U_{t+1}, 0, \ldots).$$

Then $U_t = W_t(C_t, I_t(U_{t+1}))$, as expected.

Conversely, suppose we are given W_t and I_t. Suppose we truncate a consumption stream at time T. We define utility over the truncated sequence, U^T_t, as follows. If $t > T$, $U^T_t = 0$. Otherwise, define U^T_t by backwards induction:

$$\begin{aligned} U^T_T &= W(C_T, 0) \\ U^T_t &= W(C_t, I_t(U^T_{t+1})). \end{aligned}$$

For arbitrary sequence, define

$$U_t = \lim_{T\to\infty} U_t^T.$$

By the continuity properties of I_t and W_t,

$$\begin{aligned}
W(C_t, U_{t+1}) &= W(C_t, I_t(\lim_{T\to\infty} U_{t+1}^T)) \\
&= W(C_t, \lim_{T\to\infty} I_t(U_{t+1}^T)) \\
&= \lim_{T\to\infty} W(C_t, I_t(U_{t+1}^T)) \\
&= \lim_{T\to\infty} W(C_t, U_{t+1}^T) \\
&= \lim_{T\to\infty} U_t^T = U_t.
\end{aligned}$$

So U_t satisfies increasing limits. □

We call the utility construction in the previous theorem the *recursive certainty equivalent* utility.

Our existence results come from the following simple lemma. Let H_t be a sequence of operators that turns a time $t+1$ random variable into a time t random variable. (This will usually be an operator of the form $H_t x = W_t(C_t, I_t x)$, for some choice of C_t.)

We assume H_t satisfies the strict monotonicity and weak continuity properties for a risk aggregator:

- If $U_{t+1} \leq V_{t+1}$ a.s., then $I_t U_{t+1} \leq I_t V_{t+1}$, and if $U_{t+1} < V_{t+1}$ a.s. then $I_t U_{t+1} < I_t V_{t+1}$ a.s.

- If U_{t+1}^i is a sequence of time $t+1$ random variables such that $U^i \leq U^{i+1}$ a.s., and if $\lim_{i\to\infty} U_t^i$ exists, then
$$\lim_{t\to\infty} H_t U_{t+1}^i = H_t \lim_{t\to\infty} U_{t+1}^i.$$

We consider sequences of operators that satisfy conditions 2 and 3 to be a risk aggregator. Then we can prove the following.

Lemma 2 (Relative Existence Lemma)**.** *Let H_t, H_t' be two sequences of operators that satisfy the monotonicity and weak continuity properties of being an aggregator. Suppose that for any random variable z_{t+1}, $0 \leq H_t z_{t+1} \leq H_t' z_{t+1}$. If H_t' has a sequence of solutions,*

$$x_t' = H_t' x_{t+1}',$$

where $x_t' > 0$. Then H_t has a sequence of solutions,

$$x_t = H_t x_{t+1},$$

such that $0 \leq x_t \leq x'_t$.

Proof. Define $x_t^T, x_t'^T$, $t \leq T$ by recursion. $x_T^T = 0, x_T'^T = 0$, and $x_t^T = H_t x_{t+1}^T, x_t'^T = H'_t x_{t+1}'^T$.

We first show

$$x_t^T \leq x_t^{T+1}. \tag{38}$$

We induct on the difference $i = T - t$. $x_T^T = 0 \leq H_T 0 = x_T^{T+1}$, so the base case of $i = 0$ holds. Suppose it holds for $i - 1$. Then $x_{t+1}^T \leq x_{t+1}^{T+1}$. By monotonicity,

$$x_t^T = H_t x_{t+1}^T \leq H_t x_{t+1}^{T+1} = x_t^{T+1},$$

and the result follows.

We next show that

$$x_t^T \leq x_t'^T \tag{39}$$

Again we induct on $i = T - t$. $0 = x_T^T = x_T^T$, so $i = 0$ holds. If it holds for $i-1$, then $x_{t+1}^T \leq x_{t+1}'^T$. By monotonicity, $x_t^T = H_t x_{t+1}^T \leq H_t x_{t+1}'^T$. By assumption, $H_t x_{t+1}'^T \leq H'_t x_{t+1}^T = x_t'^T$, and the result follows.

Finally, we show that

$$x_t'^T \leq x'_t \tag{40}$$

$x_T'^T = 0$, so the base case holds. If $x_{t+1}'^T \leq x'_{t+1}$, then $x_t'^T = H_t x_{t+1}'^T \leq H_t x'_{t+1} = x'_t$.

Thus x_t^T is an increasing sequence, and $x_t^T \leq x_t'^T \leq x'_t$. Let

$$x_t = \lim x_t^T.$$

This is an increasing sequence, so $\lim x_t^T = \sup x_t^T \leq x'_t$ and the limit exists and is finite. □

Corollary 9. *1. Let W_t, W'_t be two time aggregators such that $W_t \leq W'_t$, and I_t, I'_t two risk aggregators such that $I_t \leq I'_t$. If W'_t and I'_t has a solution, $U'_t \geq 0$, such that*

$$U'_t = W(C_t, I'_{t+1} U'_{t+1}),$$

then

$$U_t = W(C_t, I_{t+1} U_{t+1}),$$

also exists.

2. Let $C_t \leq C'_t$. Then if U'_t exists such that

$$U'_t = W(C'_t, I_{t+1} U'_{t+1}),$$

then U_t exists such that

$$U_t = W(C_t, I_{t+1}U_{t+1}),$$

Proof. For the first one, let $H_t x = W_t(C_t, I_t x)$ and $H'_t x = W'_t(C_t, I'_t x)$. If $I_t x \leq I'_t x$, then since W_t is increasing in its first argument $W_t(C_t, I_t x) \leq W_t(C_t, I'_t x)$. Thus

$$W(C_t, I_t x) \leq W(C_t, I'_t x) \leq W'(C_t, I'_t x),$$

and the result follows from the Relative Existence Lemma.

For the second one, let $H_t x = W_t(C_t, I_t x)$ and $H'_t x = W_t(C'_t, I_t x)$. W_t is monotonic in the first argument, so $H_t x \leq H'_t x$, and the result follows from the Relative Existence Lemma. □

Proof of Theorem 2. Let W_t, I_t be the recursive certainty equivalent operators for agent 1, and W'_t, I'_t be the same for agent 2.

If agent 1 has higher effective risk aversion, then $I_t x \leq I'_t x$. If agent 1 has higher effective impatience, then $W_t(C_t, x) \leq W'_t(C_t x)$. Thus both results follow from part 1 of corollary 9. □

Proof of Theorem 1. For $(C_t, C_{t+1}, \ldots)$, where C_{t+1} is known at time t, the utility of each agent is

$$u(C_t) + \delta_i u(C_{t+1}).$$

The certainty equivalent for each agent is

$$u^{-1}\left(u(C_t) + \delta_i u(C_{t+1})\right).$$

We have $u(C_t) + \delta_1 u(C_{t+1}) \leq u(c) + \delta_2 u(C_{t+1})$, and u^{-1} is monotonic, so agent 1 has higher effective impatience.

□

Proof of Theorem 3. This is theorem 1 in Pratt (1964). □

Proof of Corollary 2. This follows immediately from the previous theorem. □

Proof of Theorem 4. This is just an application of the comparison theorem for powers, Corollary 2. Increasing a preference for early resolution and increasing smooth ambiguity aversion both lower the exponent relative to the CRRA baseline. Likewise a preference for late resolution or smooth-ambiguity seeking raises the exponent relative to that baseline. □

Proof of Corollary 3. This is a special case of Theorem 4. □

Proof of Corollary 4. This is also special case of Theorem 4. □

A.3 Relaxing the Free Disposal Assumption

In Section 2 we consider the consequences of dropping the free disposal assumption. The assumption is avoidable, at the price of additional bookkeeping. While natural, the free disposal assumption is violated for constant EIS less than one, since utility is $-\infty$ for consumption streams very near zero.

Since the main focus of research is growth economies, the simplest solution is to assume that the constant EIS condition does not apply for very low consumption, and that free disposal does hold. But we also consider some alternatives.

The existence results require that the domain of consumption streams, $\mathcal{C}$, contain a "worst case" consumption stream, which in the main body of the paper is constant zero consumption. The theoretical results work by approximating the solution from below starting with the worst case. The arguments do not require any particular form of worst case, so an arbitrary consumption stream with well-defined preferences would work. One particularly simple solution is to assume that consumption is bounded below at a small positive number ϵ. The worst case is simply ϵ every period. Since utility for CRRA preferences with EIS smaller than one is bounded above (by zero), this makes the existence question for this case entirely trivial.

Alternatively, a more symmetric approach is to consider the mirror image problem, and try to identify rates of economic contraction that prevent utility from diverging to $-\infty$. The condition of preserving increasing limits is replaced with the condition of preserving decreasing limits. Comparisons then must be used to bound utility from below, rather than from above. For example, for agents with late resolution of uncertainty, CRRA utility can be used to provde a bound below. This type of model is far from what is standard in the current asset pricing literature, however.

B Proofs for Section 3

In this section we present the derivations of the sufficient existence conditions that we stated for various log growth consumption processes from the asset-pricing literature and CRRA utility. All proofs follow the general approach illustrated in Section 3.1.1. We apply the general statement of Theorem 5 to the concrete applications. For this purpose, we need to derive the function h and the sequence of vector coefficients $(A_i)_{i\in\mathbb{N}}$ for the specific model. This sequence is the solution to a recursive system of equations. A sufficient existence condition then follows from the limit condition of the theorem. (We refer to Mathematica files that actually solve for the existence conditions.) In Section B.1 we show how Theorem 5 can be applied to general processes in the affine class. The subsequent sections contain derivations for specific models.

B.1 General Example for Affine Processes with Gaussian and Poisson Jump Shocks

In the following we show how to apply Theorem 5 to models with general processes. For this purpose we consider the affine framework used in Eraker (2008) and Drechsler and Yaron (2011). The framework uses a VAR specification with both Gaussian shocks and Poisson jump shocks to model consumption dynamics (Eraker and Shaliastovich (2008) use a similar framework in continuous time):

$$X_{t+1} = \mu + FX_t + G_t z_{t+1} + J_{t+1} \tag{41}$$

where $X_t \in \mathbb{R}^n$ denotes the vector of state processes, $z_{t+1} \sim N(0, I)$ is a vector of Gaussian shocks and J_{t+1} is the vector of compound poisson jumps.[9] Hence, the ith component of J_{t+1} is given by $J^i_{t+1} = \sum_{j=1}^{N^i_{t+1}} \xi^i_j$, where N^i_{t+1} is a Poisson counting process and ξ^i_j is the size of the jump that occurs upon the jth increment of N^i_{t+1}. Following Drechsler and Yaron (2011), we let the N^i_{t+1} be independent of each other conditional on time-t information and assume that the ξ^i_j are i.i.d. The intensity process of the N^i_{t+1} can be time-varying and is given by the ith component of $\lambda_t = l_0 + l_1 X_t$ with $l_0 \in \mathbb{R}^n$ and $l_1 \in \mathbb{R}^{n\times n}$. To ensure that the dynamics fall in the affine class (Duffie, Pan, and Singleton (2000)) we impose $G_t G_t' = h + \sum_i H^i X^i_t$ with $h \in \mathbb{R}^n$ and $H^i \in \mathbb{R}^{n\times n}$. For the application of Theorem 5 we need the MGF of X_t:

$$M_X(k) = E_t(e^{k\cdot X_{t+1}}) = e^{k\mu + kFX_t} E_t(e^{kG_t z_{t+1}}) E_t(e^{kJ_{t+1}}) \tag{42}$$

where we use the independence of z_{t+1} and J_{t+1}. Drechsler and Yaron (2011, Appendix A.1) show how to derive $M_X(k)$. In the following we provide a brief summary of their derivation.

As z_{t+1} are Gaussian, we have $E_t(e^{kG_t z_{t+1}}) = e^{\frac{1}{2}kG_t G_t' k'} = e^{\frac{1}{2}khk' + \frac{1}{2}\sum_i kH^i k' X^i_t}$. Furthermore note that $E_t(e^{k^i J^i_{t+1}}) = e^{\lambda^i_t (M_{\xi^i}(k^i) - 1)}$ where $M_{\xi^i}(k^i)$ is the MGF of the jump size ξ^i and $\Psi^i_t(k^i) = \lambda^i_t(M_{\xi^i}(k^i) - 1)$ is the cumulant generating function of J^i_{t+1}. We denote by $M_\xi(k) = [M_{\xi^1}(k), M_{\xi^2}(k), \ldots]$ the vector of the MGFs of the jump sizes. We then obtain $E_t(e^{kJ_{t+1}}) = e^{(M_\xi(k)-1)\lambda_t} = e^{(M_\xi(k)-1)l_0 + (M_\xi(k)-1)l_1 X_t}$. Then $M_X(k)$ is given by

$$M_X(k) = e^{k\mu + kFX_t + \frac{1}{2}khk' + \frac{1}{2}\sum_i kH^i k' X^i_t + (M_\xi(k)-1)l_0 + (M_\xi(k)-1)l_1 X_t}. \tag{43}$$

[9] Note that the first element of X_t is 1 as defined above and the dynamics driving consumption growth follow thereafter.

Using $M_X(k)$, we can now determine the vector function $h(k)$:

$$\begin{aligned} k_0 + h_0(k) &= k\mu + \frac{1}{2}khk' + (M_\xi(k) - 1)l_0 \\ h_i(k) &= (kF)^i + \frac{1}{2}kH^ik' + ((M_\xi(k) - 1)l_1)^i \quad \forall \quad i \geq 1, \end{aligned} \tag{44}$$

where $(y)^i$ denotes the ith element of the vector y.

Having the vector function $h(k)$, we can compute $A_{i+1} = h(A_i + \tau)$ with $A_0 = 0$. The calculations can be done in closed-form for many model specifications that have appeared in the literature in recent years as we demonstrate in Section 3. For complex models, the derivations can become quite tedious. So instead—to prove existence in a specific parameterized application—the expressions can simply be evaluated numerically. In Section 3.2.5 we use this approach to show existence in the model of Drechsler and Yaron (2011) that includes 3 state processes, Gaussian shocks, and multiple Poisson jumps.

B.2 Long-Run Risks and Stochastic Volatility (Bansal and Yaron (2004))

We derive the sufficient existence condition for the model with the long-run risk process of Bansal and Yaron (2004) with different kind of shocks to stochastic volatility $\epsilon_{\sigma,t+1}$, see Proposition 5 and Proposition 3 .

Proof. Set $X_t = (X_t^0, X_t^1, X_t^2, X_t^3)'$ with $X_t^0 = 1$, $X_t^1 = \Delta c_t$, $X_t^2 = x_t, X_t^3 = \sigma_t^2$ and $k = (k_0, k_1, k_2, k_3)$. We obtain

$$\begin{aligned} E_t e^{k \cdot X_{t+1}} &= E_t e^{k_0 X_{t+1}^0 + k_1 X_{t+1}^1 + k_2 X_{t+1}^2 + k_3 X_{t+1}^3} \\ &= e^{k_0 + k_1\mu + k_1 X_t^2 + 0.5k_1^2 X_t^3 + k_2\rho X_t^2 + 0.5k_2^2\phi_x^2 X_t^3 + k_3 \cdot \bar{\sigma}^2(1-\rho_\sigma) + k_3\rho_\sigma X_t^3 + \log(M_{\epsilon_\sigma}(\phi_\sigma k_3))} \end{aligned}$$

where $M_{\epsilon_\sigma}(k)$ denotes the moment-generating function of $\epsilon_{\sigma,t+1}$. Using the notation $E_t(e^{k \cdot X_{t+1}}) = e^{h(k) \cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1, k_2, k_3), h_1(k_1, k_2, k_3), h_2(k_1, k_2, k_3), h_3(k_1, k_2, k_3)),$$

leads to

$$\begin{aligned} k_0 + h_0(k_1, k_2, k_3) &= k_0 + k_1\mu + k_3\bar{\sigma}^2(1 - \rho_\sigma) + \log(M_{\epsilon_\sigma}(\phi_\sigma k_3)) \\ h_1(k_1, k_2, k_3) &= 0 \\ h_2(k_1, k_2, k_3) &= k_1 + k_2\rho \\ h_3(k_1, k_2, k_3) &= 0.5k_1^2 + 0.5k_2^2\phi_x^2 + k_3\rho_\sigma. \end{aligned}$$

We need to compute $v_t = 1 + E_t(e^{\tau \cdot X_{t+1}} v_{t+1})$ with $\tau = (\ln \delta, \alpha, 0, 0)$ and $\alpha = 1 - \frac{1}{\psi}$. According to Theorem 5 the solution is then given by

$$v_t = \sum_{i=0}^{\infty} e^{A_i \cdot X_t}$$

with $A_0 = (0, 0, 0, 0)$ and $A_i = h(A_{i-1} + \tau)$. Denote by $A_{j,i}$ the jth element of A_i. We then have

$$\begin{aligned}
A_{0,i} &= A_{0,i-1} + \ln \delta + (A_{1,i-1} + \alpha)\mu + A_{3,i-1}\bar{\sigma}^2(1 - \rho_\sigma) + \log(M_{\epsilon_\sigma}(\phi_\sigma A_{3,i-1})) \\
A_{1,i} &= 0 \\
A_{2,i} &= (A_{1,i-1} + \alpha) + \rho A_{2,i-1} \\
A_{3,i} &= 0.5(A_{1,i-1} + \alpha)^2 + \rho_\sigma A_{3,i-1} + 0.5(A_{2,i-1})^2 \phi_x^2.
\end{aligned}$$

Simplifying yields

$$\begin{aligned}
A_{0,i} &= A_{0,i-1} + \ln \delta + \alpha\mu + A_{3,i-1}\bar{\sigma}^2(1 - \rho_\sigma) + \log(M_{\epsilon_\sigma}(\phi_\sigma A_{3,i-1})) \\
A_{2,i} &= \alpha + \rho A_{2,i-1} \\
A_{3,i} &= 0.5\alpha^2 + \rho_\sigma A_{3,i-1} + 0.5(A_{2,i-1})^2 \phi_x^2.
\end{aligned}$$

Recall the sufficient condition for existence from Theorem 5,

$$\lim_{i \to \infty} e^{(A_{i+1} - A_i) \cdot X_t} < 1.$$

Observe that $\lim_{i \to \infty} A_{2,i} = \frac{\alpha}{1-\rho}$ and so it must also hold that $\lim_{i \to \infty} (A_{2,i} - A_{2,i-1}) = 0$. Similarly,

$$\lim_{i \to \infty} A_{3,i} = \frac{1}{1 - \rho_\sigma} \left(\frac{1}{2}\alpha^2 + \frac{1}{2} \frac{\alpha^2}{(1 - \rho)^2} \phi_x^2 \right)$$

and so $\lim_{i \to \infty} (A_{3,i} - A_{3,i-1}) = 0$. To compute the limit of $A_{0,i} - A_{0,i-1}$ we need to specify $M_{\epsilon_\sigma}(k)$. In the case where $\epsilon_{\sigma,t+1}$ is standard normally distributed we have $M_{\epsilon_\sigma}(k) = e^{\frac{k^2}{2}}$ and hence we obtain

$$\lim_{i \to \infty} (A_{0,i} - A_{0,i-1}) = \ln \delta + \alpha\mu + \lim_{i \to \infty} \left(A_{3,i-1}\bar{\sigma}^2(1 - \rho_\sigma) + 0.5(A_{3,i-1})^2 \phi_\sigma^2 \right).$$

Substituting $\lim_{i \to \infty} A_{3,i}$ into the last expression leads to condition (26) in Proposition 5.

Now consider the case where $\epsilon_{\sigma,t+1}$ is truncated at some value ϵ_σ^{min}. The moment-generating function of $\epsilon_{\sigma,t+1}$ then becomes

$$M_{\epsilon_\sigma^{trunc}}(k) = e^{\frac{k^2}{2}} \frac{\Phi(k - \epsilon_\sigma^{min})}{\Phi(-\epsilon_\sigma^{min})}.$$

Note that for $\epsilon_\sigma^{min} \to \infty$ we obtain the standard normal MGF. Hence, we obtain

$$\lim_{i\to\infty} (A_{0,i} - A_{0,i-1}) = \ln\delta + \alpha\mu + \lim_{i\to\infty} \left(A_{3,i-1}\bar{\sigma}^2(1-\rho_\sigma) + 0.5(A_{3,i-1})^2\phi_\sigma^2 + \log\left(\frac{\Phi(\phi_\sigma A_{3,i-1} - \epsilon_\sigma^{min})}{\Phi(-\epsilon_\sigma^{min})} \right) \right).$$

Substituting $\lim_{i\to\infty} A_{3,i}$ into the last expression leads to condition (20) in Proposition 2.

If $\epsilon_{\sigma,t+1} \sim i.i.d.\ \Gamma(\theta_2, \theta_1)$, the moment-generating function is given by

$$M_{\epsilon^\Gamma}(k) = e^{-\theta_2 \log(1-\theta_1 k)}$$

Hence, we obtain

$$\lim_{i\to\infty} (A_{0,i} - A_{0,i-1}) = \ln\delta + \alpha\mu + \lim_{i\to\infty} \left(A_{3,i-1}\bar{\sigma}^2(1-\rho_\sigma) - \theta_2 \log\left(1 - \theta_1\phi_\sigma A_{3,i-1}\right) \right).$$

Substituting $\lim_{i\to\infty} A_{3,i}$ into the last expression leads to condition (21) in Proposition 3.

□

The derivations of A_i and the sufficient existence condition can also be found in the Mathematica files Existence_BY.nb and Existence_BY_Gamma.nb, which accompanies this paper.

B.2.1 The Stochastic Volatility Process with Normal and Gamma Shocks

In Section B.2 a stochastic volatility process with normally distributed shocks

$$\sigma_{N,t+1}^2 = \bar{\sigma}_N^2(1-\rho_\sigma) + \rho_\sigma\sigma_{N,t}^2 + \phi_\sigma\epsilon_{\sigma,t+1}, \quad \epsilon_{\sigma,t+1} \sim i.i.d.\ N(0,1).$$

and a process with gamma distributed shocks

$$\sigma_{\Gamma,t+1}^2 = \bar{\sigma}_\Gamma^2(1-\rho_\sigma) + \rho_\sigma\sigma_{\Gamma,t}^2 + \phi_\sigma\epsilon_{\sigma,t+1}^\Gamma, \quad \epsilon_{\sigma,t+1}^\Gamma \sim i.i.d.\ \Gamma(\theta_2, \theta_1).$$

where $\theta_1 > 0$ is the scale and $\theta_2 > 0$ is the shape parameter of the gamma distribution are considered. Note that the shape parameter θ_2 determines the skewness and excess kurtosis of the shocks and $E(\epsilon^\Gamma) = \theta_1\theta_2$ and $Var(\epsilon^\Gamma) = \theta_2\theta_1^2$.

Assume the goal is to calibrate $\bar{\sigma}_\Gamma$, θ_1 and θ_2 to match the mean and variance of the two volatility processes. We have one degree of freedom and the parameter θ_2 can be freely calibrated to match the skewness and excess kurtosis of the variance. We then have

$$E(\sigma_{N,t}^2) = \bar{\sigma}_N^2$$

and

$$E(\sigma_{\Gamma,t}^2) = \bar{\sigma}_\Gamma^2 + \frac{\phi_\sigma}{1-\rho_\sigma}\theta_1\theta_2$$

So it follows that

$$\bar{\sigma}_\Gamma^2 = \bar{\sigma}_N^2 - \frac{\phi_\sigma}{1-\rho_\sigma}\theta_1\theta_2.$$

So we observe that the parameter $\bar{\sigma}_\Gamma^2$ of the process with gamma distributed shocks must be smaller than the parameter $\bar{\sigma}_N^2$ of the process with normally distributed shocks in order for the two processes to have the same mean. Furthermore

$$Var(\sigma_{N,t}^2) = \frac{\phi_\sigma^2}{1-\rho_\sigma^2}$$

and

$$Var(\sigma_{\Gamma,t}^2) = \frac{\phi_\sigma^2}{1-\rho_\sigma^2}\theta_2\theta_1^2.$$

Hence it follows that

$$\theta_1 = \frac{1}{\sqrt{\theta_2}}$$

in order to match the variance of the two processes.

B.3 Volatility of Volatility (Bollerslev, Tauchen, and Zhou (2009))

We derive the sufficient existence condition for the model with the volatility-of-volatility process of Bollerslev, Tauchen, and Zhou (2009), see Proposition 6.

Proof. Set $X_t = (X_t^0, X_t^1, X_t^2, X_t^3)'$ with $X_t^0 = 1$, $X_t^1 = \Delta c_t$, $X_t^2 = \sigma_t^2, X_t^3 = q_t$ and $k = (k_0, k_1, k_2, k_3)$. We obtain

$$\begin{aligned} E_t e^{k\cdot X_{t+1}} &= E_t e^{k_0 X_{t+1}^0 + k_1 X_{t+1}^1 + k_2 X_{t+1}^2} \\ &= e^{k_0 + k_1\mu + 0.5k_1^2 X_t^2 + k_2\bar{\sigma}^2(1-\rho_\sigma) + k_2\rho_\sigma X_t^2 + 0.5k_2^2 X_t^3 + k_3\mu_q(1-\rho_q) + k_3\rho_q X_t^3 + 0.5.\phi_q^2 k_3^2 X_t^3} \end{aligned}$$

Using the notation $E_t(e^{k\cdot X_{t+1}}) = e^{h(k)\cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1,k_2,k_3), h_1(k_1,k_2,k_3), h_2(k_1,k_2,k_3), h_3(k_1,k_2,k_3)),$$

leads to

$$
\begin{aligned}
k_0 + h_0(k_1, k_2, k_3) &= k_0 + k_1\mu + k_2\bar{\sigma}^2(1-\rho_\sigma) + k_3\mu_q(1-\rho_q) \\
h_1(k_1, k_2, k_3) &= 0 \\
h_2(k_1, k_2, k_3) &= 0.5k_1^2 + k_2\rho_\sigma \\
h_3(k_1, k_2, k_3) &= 0.5k_2^2 + k_3\rho_q + 0.5\phi_q^2 k_3^2.
\end{aligned}
$$

We need to compute $v_t = 1 + E_t(e^{\tau \cdot X_{t+1}} v_{t+1})$ with $\tau = (\ln\delta, \alpha, 0, 0)$ and $\alpha = 1 - \frac{1}{\psi}$. According to Theorem 5 the solution is then given by

$$
v_t = \sum_{i=0}^{\infty} e^{A_i \cdot X_t}
$$

with $A_0 = (0,0,0,0)$ and $A_i = h(A_{i-1} + \tau)$. Denote by $A_{j,i}$ the jth element of A_i. We then have

$$
\begin{aligned}
A_{0,i} &= A_{0,i-1} + \ln\delta + (A_{1,i-1} + \alpha)\mu + A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q) \\
A_{1,i} &= 0 \\
A_{2,i} &= 0.5(A_{1,i-1} + \alpha)^2 + \rho_\sigma A_{2,i-1} \\
A_{3,i} &= 0.5(A_{2,i-1})^2 + \rho_q A_{3,i-1} + 0.5(A_{3,i-1})^2\phi_q^2.
\end{aligned}
$$

Simplifying yields

$$
\begin{aligned}
A_{0,i} &= A_{0,i-1} + \ln\delta + \alpha\mu + A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q) \\
A_{2,i} &= 0.5\alpha^2 + \rho_\sigma A_{2,i-1} \\
A_{3,i} &= 0.5(A_{2,i-1})^2 + \rho_q A_{3,i-1} + 0.5(A_{3,i-1})^2\phi_q^2.
\end{aligned}
$$

Recall the sufficient condition of Theorem 5,

$$
\lim_{i\to\infty} e^{(A_{i+1} - A_i)\cdot X_t} < 1.
$$

Observe that $\lim_{i\to\infty} A_{2,i} = \frac{\alpha^2}{2(1-\rho_\sigma)}$ and so it must also hold that $\lim_{i\to\infty}(A_{2,i} - A_{2,i-1}) = 0$. The fixed-point equation based on the recursion for $A_{3,i}$ is a quadratic equation and so has (generically) two solutions. However, starting from $(A_{2,0}, A_{3,0}) = (0,0)$, the recursion can reach only one of the two fixed points. Since the infinite recursion has a well-defined finite limit for $\phi_q \to 0$, we can identify the correct limit from the two solutions of the quadratic equations, since one of them does not remain finite for $\phi_q \to 0$. The limit is then

$$\lim_{i\to\infty} A_{3,i} = \frac{1-\rho_q - \sqrt{(1-\rho_q)^2 - \frac{\alpha^4\phi_q^2}{4(1-\rho_\sigma)^2}}}{\phi_q^2}.$$

Finally, we obtain

$$\lim_{i\to\infty}(A_{0,i} - A_{0,i-1}) = \ln\delta + \alpha\mu + \lim_{i\to\infty}\left(A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q)\right).$$

Substituting $\lim_{i\to\infty} A_{2,i}$ and $\lim_{i\to\infty} A_{3,i}$ into the last expression leads to condition (28). □

We also derive the sufficient existence condition (30) for the case of q_t following the standard AR(1) process (29).

Proof. If q_t follows the process (29), we obtain

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + \alpha\mu + A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q) + 0.5(A_{3,i-1})^2\phi_q^2 \\ A_{2,i} &= 0.5\alpha^2 + \rho_\sigma A_{2,i-1} \\ A_{3,i} &= 0.5(A_{2,i-1})^2 + \rho_q A_{3,i-1}. \end{aligned}$$

Observe that $\lim_{i\to\infty} A_{2,i} = \frac{\alpha^2}{2(1-\rho_\sigma)}$ and so it must also hold that $\lim_{i\to\infty}(A_{2,i} - A_{2,i-1}) = 0$. Similarly,

$$\lim_{i\to\infty} A_{3,i} = \frac{\alpha^4}{8(1-\rho_\sigma)^2(1-\rho_q)}$$

and so $\lim_{i\to\infty}(A_{3,i} - A_{3,i-1}) = 0$. Finally, we obtain

$$\lim_{i\to\infty}(A_{0,i} - A_{0,i-1}) = \ln\delta + \alpha\mu + \lim_{i\to\infty}\left(A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q) + 0.5(A_{3,i-1})^2\phi_q^2\right).$$

Substituting $\lim_{i\to\infty} A_{2,i}$ and $\lim_{i\to\infty} A_{3,i}$ into the last expression leads to condition (30). □

The derivations of A_i and the sufficient existence conditions can also be found in the Mathematica file Existence_VoV_no_CIR_q.nb, which accompanies this paper.

B.4 Stochastic Volatility with Jumps (Drechsler and Yaron (2011))

We derive the sufficient existence condition for the model with gamma-distributed jumps of Drechsler and Yaron (2011), see Proposition 7.

Proof. For the jump process J_{t+1} it holds that

$$E_t e^{uJ_{t+1}} = e^{l_0(M_\xi(u)-1)}$$

with the moment-generating function

$$M_\xi(u) = E_t e^{u\xi} = (1 - \frac{\mu_\sigma}{\nu_\sigma} u)^{-\nu_\sigma}.$$

Set $X_t = (X_t^0, X_t^1, X_t^2)'$ with $X_t^0 = 1$, $X_t^1 = \Delta c_t, X_t^2 = \sigma_t^2$ and $k = (k_0, k_1, k_2)$. We obtain

$$\begin{aligned} E_t e^{k \cdot X_{t+1}} &= E_t e^{k_0 X_{t+1}^0 + k_1 X_{t+1}^1 + k_2 X_{t+1}^2} \\ &= e^{k_0 + k_1 \mu + 0.5 k_1^2 X_t^2 + k_2 \bar{\sigma}^2 (1-\rho_\sigma) + k_2 \rho_\sigma X_t^2 + 0.5 k_2^2 \phi_\sigma^2 + l_0((1 - \frac{\mu_\sigma}{\nu_\sigma} k_2)^{-\nu_\sigma} - 1)}. \end{aligned}$$

Using the notation $E_t(e^{k \cdot X_{t+1}}) = e^{h(k) \cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1, k_2), h_1(k_1, k_2), h_2(k_1, k_2))$$

leads to

$$\begin{aligned} k_0 + h_0(k_1, k_2) &= k_0 + k_1 \mu + k_2 \bar{\sigma}^2 (1 - \rho_\sigma) + 0.5 k_2^2 \phi_\sigma^2 + l_0((1 - \frac{\mu_\sigma}{\nu_\sigma} k_2)^{-\nu_\sigma} - 1) \\ h_1(k_1, k_2) &= 0 \\ h_2(k_1, k_2) &= 0.5 k_1^2 + k_2 \rho_\sigma. \end{aligned}$$

We need to compute $v_t = 1 + E_t(e^{\tau \cdot X_{t+1}} v_{t+1})$ with $\tau = (\ln \delta, \alpha, 0)$ and $\alpha = 1 - \frac{1}{\psi}$. According to Theorem 5 the solution is then given by

$$v_t = \sum_{i=0}^{\infty} e^{A_i \cdot X_t}$$

with $A_0 = (0, 0, 0)$ and $A_i = h(A_{i-1} + \tau)$. Denote by $A_{j,i}$ the jth element of A_i. We then have

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln \delta + (A_{1,i-1} + \alpha)\mu + A_{2,i-1} \bar{\sigma}^2 (1 - \rho_\sigma) + 0.5 (A_{2,i-1})^2 \phi_\sigma^2 \\ &\quad + l_0 \left(\left(1 - \frac{\mu_\sigma}{\nu_\sigma} A_{2,i-1} \right)^{-\nu_\sigma} - 1 \right) \\ A_{1,i} &= 0 \\ A_{2,i} &= 0.5 (A_{1,i-1} + \alpha)^2 + \rho_\sigma A_{2,i-1}. \end{aligned}$$

Recall the sufficient condition of Theorem 5,

$$\lim_{i \to \infty} e^{(A_{i+1} - A_i) \cdot X_t} < 1.$$

Observe that $\lim_{i \to \infty} A_{2,i} = \frac{\alpha^2}{2(1-\rho_\sigma)}$ and so it must also hold that $\lim_{i \to \infty} (A_{2,i} - A_{2,i-1}) = 0$.

We obtain

$$\lim_{i\to\infty}(A_{0,i}-A_{0,i-1}) = \ln\delta+\alpha\mu+\lim_{i\to\infty}\Bigg\{A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma)+0.5(A_{2,i-1})^2\phi_\sigma^2+ \\ + l_0\left(\left(1-\frac{\mu_\sigma}{\nu_\sigma}A_{2,i-1}\right)^{-\nu_\sigma}-1\right)\Bigg\}.$$

Substituting $\lim_{i\to\infty} A_{2,i}$ into the last expression leads to condition (32) in Proposition 7. $\square$

We also derive the sufficient existence condition (33) for normally distributed jumps in Proposition 8.

Proof. When the jumps to stochastic volatility are i.i.d. normally distributed, $\xi_j \sim N(\mu_\xi, \sigma_\xi)$, the moment-generating function of ξ_j becomes

$$M_\xi(u) = e^{u\mu_\xi+0.5u^2\sigma_\xi^2}$$

and we obtain

$$\begin{aligned} A_{0,i} &= A_{0,i-1}+\ln\delta+\alpha\mu+A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma)+0.5(A_{2,i-1})^2\phi_\sigma^2 \\ &\quad + l_0(e^{A_{2,i-1}\mu_\xi+0.5(A_{2,i-1})^2\sigma_\xi^2}-1) \\ A_{2,i} &= 0.5\alpha^2+\rho_\sigma A_{2,i-1}. \end{aligned}$$

Substituting $\lim_{i\to\infty} A_{2,i} = \frac{\alpha^2}{2(1-\rho_\sigma)}$ into the expression for $\lim_{i\to\infty}(A_{0,i}-A_{0,i-1})$ leads to condition (33) in Proposition 8. $\square$

The derivations of A_i and the sufficient existence conditions can be found in the Mathematica file Existence_SV_with_Jumps.nb.

B.5 Consumption Disasters with Time-varying Disaster Intensities (Wachter (2013))

We derive the sufficient existence condition for the model with consumption disasters and time-varying disaster intensities of Wachter (2013), see Proposition 9.

Proof. For the jump process J_{t+1} to consumption growth it holds that

$$E_t e^{uJ_{t+1}} = e^{\lambda_t(M_\xi(u)-1)}$$

with

$$M_\xi(u) = E_t e^{u\xi_j} = e^{u\mu_\xi+0.5u^2\sigma_\xi^2}.$$

Set $X_t = (X_t^0, X_t^1, X_t^2)'$ with $X_t^0 = 1$, $X_t^1 = \Delta c_t$, $X_t^2 = \lambda_t$ and $k = (k_0, k_1, k_2)$. We obtain

$$\begin{aligned} E_t e^{k \cdot X_{t+1}} &= E_t e^{k_0 X_{t+1}^0 + k_1 X_{t+1}^1 + k_2 X_{t+1}^2} \\ &= e^{k_0 + k_1 \mu + 0.5 k_1^2 \sigma^2 + X_t^2 (e^{k_1 \mu_\xi + 0.5 k_1^2 \sigma_\xi^2} - 1) + k_2 \bar{\lambda}(1-\rho_l) + k_2 \rho_l X_t^2 + 0.5 k_2^2 \phi_l^2 X_t^2}. \end{aligned}$$

Using the notation $E_t(e^{k \cdot X_{t+1}}) = e^{h(k) \cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1, k_2), h_1(k_1, k_2), h_2(k_1, k_2))$$

leads to

$$\begin{aligned} k_0 + h_0(k_1, k_2) &= k_0 + k_1 \mu + 0.5 k_1^2 \sigma^2 + k_2 \bar{\lambda}(1 - \rho_l) \\ h_1(k_1, k_2) &= 0 \\ h_2(k_1, k_2) &= (e^{k_1 \mu_\xi + 0.5 k_1^2 \sigma_\xi^2} - 1) + k_2 \rho_l + 0.5 k_2^2 \phi_l^2. \end{aligned}$$

We need to compute $v_t = 1 + E_t(e^{\tau \cdot X_{t+1}} v_{t+1})$ with $\tau = (\ln \delta, \alpha, 0)$ and $\alpha = 1 - \frac{1}{\psi}$. According to Theorem 5 the solution is then given by

$$v_t = \sum_{i=0}^{\infty} e^{A_i \cdot X_t}$$

with $A_0 = (0, 0, 0)$ and $A_i = h(A_{i-1} + \tau)$. Denote by $A_{j,i}$ the jth element of A_i.

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln \delta + (A_{1,i-1} + \alpha)\mu + 0.5(A_{1,i-1} + \alpha)^2 \sigma^2 + A_{2,i-1} \bar{\lambda}(1 - \rho_l) \\ A_{1,i} &= 0 \\ A_{2,i} &= (e^{(A_{1,i-1} + \alpha)\mu_\xi + 0.5(A_{1,i-1} + \alpha)^2 \sigma_\xi^2} - 1) + A_{2,i-1} \rho_l + 0.5 (A_{2,i-1})^2 \phi_l^2. \end{aligned}$$

Simplifying yields

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln \delta + \alpha \mu + 0.5 \alpha^2 \sigma^2 + A_{2,i-1} \bar{\lambda}(1 - \rho_l) \\ A_{2,i} &= (e^{\alpha \mu_\xi + 0.5 \alpha^2 \sigma_\xi^2} - 1) + A_{2,i-1} \rho_l + 0.5 (A_{2,i-1})^2 \phi_l^2. \end{aligned}$$

Recall the sufficient condition of Theorem 5,

$$\lim_{i \to \infty} e^{(A_{i+1} - A_i) \cdot X_t} < 1.$$

The fixed-point equation based on the recursion for $A_{2,i}$ is a quadratic equation and so has (generically) two solutions. However, starting from $A_{2,0} = 0$, the recursion can reach only one of the two fixed points. Since the infinite recursion has a well-defined finite limit for $\phi_l \to 0$, we can identify the correct limit from the two solutions of the quadratic equations, since one of them does not remain finite for $\phi_l \to 0$. The limit is then

$$\lim_{i\to\infty} A_{2,i} = \frac{1-\rho_l - \sqrt{(1-\rho_l)^2 - 2\left(e^{\alpha\mu_\xi + 0.5\alpha^2\sigma_\xi^2} - 1\right)\phi_l^2}}{\phi_l^2}$$

and so it must also hold that $\lim_{i\to\infty}(A_{2,i} - A_{2,i-1}) = 0$. We obtain

$$\lim_{i\to\infty}(A_{0,i} - A_{0,i-1}) = \ln\delta + \alpha\mu + 0.5\alpha^2\sigma^2 + \bar{\lambda}(1-\rho_l)\lim_{i\to\infty} A_{2,i-1}$$

Substituting $\lim_{i\to\infty} A_{2,i}$ into the last expression leads to condition (34) in Proposition 9. □

The derivations of A_i and the sufficient existence conditions can be found in the Mathematica file Existence_Wachter.nb.

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