Speculative Bubbles, Heterogeneous Beliefs, and Learning *

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Abstract: Speculative bubble arises when the price of an asset exceeds every trader's valuation measured by her willingness to pay if obliged to hold the asset forever. Speculative bubble indicates speculative trade - whoever holds the asset intends to sell it at a later date. We identify a sufficient condition for speculative bubbles in a market with heterogeneous beliefs and short-sales restrictions. With Bayesian learning and heterogeneous priors, the sufficient condition is that no single prior dominates other agents' priors in the sense of monotone likelihood ratio order. We study asymptotic properties of speculative bubbles in light of merging of traders' beliefs.

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1. Introduction

If traders in asset markets have diverse (or heterogeneous) beliefs and short sales are restricted, asset prices will reflect the most optimistic beliefs. Pessimists who would want to short sell the asset, will be excluded from the market by the restriction on short selling. Harrison and Kreps (1978) pointed out that if heterogeneous beliefs are randomly changing over time so that different traders become the most optimistic at different times, asset prices may strictly exceed the most optimistic valuations because those traders anticipate to sell at a future date to new optimists. Trade becomes speculative as every trader who buys the asset intends to sell it at a future date, and hence she trades for short-term gain. Harrison and Kreps (1978) presented an example of a dynamic infinite-time market where agents are risk neutral, have heterogeneous beliefs about asset dividends, and short selling is prohibited. Because of risk neutrality, agents' valuation of the asset which in general stands for the willingness to pay if obliged to hold the asset forever, is simply the discounted expected value of dividends under individual beliefs. Agents' beliefs exhibit perpetual switching: there is no single agent who is more optimistic at all future dates and states than other agents about next period dividends of the asset. In equilibrium, the agent who has the most optimistic belief buys the asset and agents with less optimistic beliefs want to short-sell the asset but are restricted by the constraint. Asset prices persistently exceed all agents' discounted expected values of future dividends.

Heterogeneity of beliefs and short sales restrictions are generally believed to be the primary reasons for the rapid rise and fall of stock prices during the dot.com bubble of 2000-2001. Ofek and Richardson (2003) provided compelling empirical evidence that traders beliefs about newly issued internet stocks were vastly diverse and that there were stringent short sales restrictions because of lockups. Hong, Scheinkman and Xiong (2006) developed a formal analysis in a model of asset markets with heterogeneous beliefs and short sales restrictions, and demonstrated that the model can account for price changes as in the dot.com bubble. Heterogeneity of beliefs in Hong, Scheinkman and Xiong (2006)[1] model results from traders being too optimistic about information signals, that is, thinking that signals are more

[1] See also Scheinkman and Xiong (2003).

accurate than they actually are.

Asset prices in Harrison and Kreps (1978) and Hong, Scheinkman and Xiong (2006) models strictly exceed the valuation of the most optimistic agents. The difference between the price and the highest valuation is termed *speculative bubble.* Speculative bubbles should not be confused with rational bubbles as the respective definitions are based on different notions of fundamental valuation. For speculative bubble, fundamental valuation is the willingness to pay for the asset if obliged to hold it forever. For rational bubble, fundamental valuation is the discounted expected value of future dividend under the risk-neutral pricing measure (or stochastic discount factor). While rational bubbles can arise in equilibrium under rather special conditions[2], it is not so for speculative bubbles. Dynamic properties of rational and speculative bubbles are different, too. Speculative bubbles may "burst," while rational bubbles have to persist indefinitely, with positive probability.

The assumption of heterogeneity of beliefs is often met with skepticism among economists for it is at odds with the common prior doctrine. Traders' beliefs in the Harrison and Kreps (1978) example are dogmatic. They remain unchanged regardless of observed patterns of realized dividends. Heterogeneous beliefs in Scheinkman and Xiong (2003) are generated by agent's overreacting to commonly observed signals. Belief updating for the agents deviates from Bayesian updating, and this gives rise to heterogeneous conditional beliefs. Agents' updating rules remain unchanged regardless of observed dividends. Morris (1996) introduced learning in the model of speculative trade. He considered an i.i.d dividend process parametrized by a single parameter of its distribution (probability of high dividend) that is unknown to the agents. Agents have heterogeneous prior beliefs about that parameter. Morris (1996) showed that, as the agents update their beliefs over time, their posterior beliefs will exhibit switching property that leads to speculative trade as long as the prior beliefs are not ranked in the maximum likelihood ratio order. Werner (2015) showed that speculative bubbles may arise with ambiguous beliefs that are common to all traders.

This paper develops a general theory of speculative bubbles and speculative

[2]By the no-bubble theorem of Santos and Woodford (1997), see also LeRoy and Werner (2014), rational bubbles arise only with low interest rates.

trade in dynamic asset markets with short sales restrictions when agents have heterogeneous beliefs and are risk neutral. There is a single asset with arbitrary dividend process over (discrete) infinite time-horizon. Heterogeneous beliefs may arise because of overconfidence in updating beliefs upon public information, Bayesian learning with heterogeneous priors, or simply be dogmatic beliefs. We show that a condition of *valuation switching* is sufficient for speculative bubble and speculative trade. Valuation switching holds if for every event at every date there does not exist an agent whose discounted expected value of future dividends exceeds all other agents' discounted expected values from that date on forever. The condition of valuation switching is sufficient but not necessary for speculative bubbles. Interestingly, the example of Harrison and Kreps (1978) provides an illustration. One of the traders in that example is valuation dominant at every date, in every event.

Our main focus is on heterogeneous beliefs arising from updating different prior beliefs in Bayesian model of learning. We consider a general setting of priors on a parametric set of probability measures over arbitrary dividend sequences. Valuation dominance in the setting of Bayesian learning is closely related to the maximum likelihood ratio (MLR) order of priors. We show that dominance in the MLR order implies valuation dominance. For an i.i.d. binomial dividend process, valuation dominance is equivalent to MLR dominance (see Morris (1996)).

An important issue arising in settings with heterogeneous beliefs is whether or not difference in beliefs can persist in the long run as agents make observations. This is important for characterization of dynamic properties of speculative bubbles. The classical Blackwell and Dubins (1962) merging-of-opinions result states that if agents prior beliefs are absolutely continuous with respect to each other, then conditional beliefs for the future given the past converge over time. Slawski (2008) was the first to point out the relevance of merging of beliefs for the asymptotic behavior of speculative bubbles, see also Morris (1996). We show that if the true probability measure on dividends is absolutely continuous with respect to agents' beliefs, then their valuations converge to the true valuation and, moreover, asset price converges to the true valuation. This makes speculative bubble vanish in the limit. The condition of absolute continuity in infinite time is a restrictive condition. In the setting of Bayesian learning with heterogeneous priors, a weaker condition of consistency of priors with the true parameter combined with absolute continuity of priors with respect to each other is shown to be sufficient for aforementioned asymptotic properties of prices and valuations. Yet again, consistency of priors with the true parameter is not an innocuous condition and may be easily violated, for example, in infinite-dimensional parameter sets or misspecified priors. We conclude that persistent (or non-vanishing) speculative bubbles are not at all unlikely.

The paper is organized as follows. In Section 2 we present the model of dynamic asset markets with heterogeneous beliefs and short sales restrictions. We prove the main result about sufficiency of valuation switching for the existence of speculative bubbles, and discuss two examples. In Sections 3 and 4 we discuss speculative bubbles in settings with heterogeneous priors and Bayesian learning in general and with i.i.d dividends. Section 5 is about asymptotic properties of speculative bubbles in light of merging of conditional beliefs and consistency of priors.

2. Heterogeneous Beliefs and Speculative Trade.

Time is discrete with infinite horizon and begins at date 0. The set of possible states at each date is a finite set S. The product set S^∞ represents all sequences of states. For a sequence (or path) of states $(s_0, \dots, s_t, \dots)$, we use s^t the denote the partial history $(s_0, \dots, s_t)$ through date t. Partial histories are date-t events. The set S^∞ together with the σ-filed Σ of products of subsets of S is the measurable space describing the uncertainty. There is a single asset with date-t dividend x_t. Dividend x_t is a random variable on (S^∞, Σ) assumed measurable with respect to $\mathcal{F}_t$, the σ-filed of date-t events.

There are I agents. Each agent i is risk-neutral and discounts future consumption by discount factor β, common to all agents. Agent's i beliefs are represented by a probability measure P^i on (S^∞, Σ). Agent's i utility function of consumption plan $c = \{c_t\}_{t=0}^\infty$ adapted to $\mathcal{F}_t$ is

$$\sum_{t=0}^{\infty} \beta^t E^i[c_t], \tag{1}$$

where E^i denotes the expectation under probability measure P^i. Endowments e_t^i

are measurable w.r. to $\mathcal{F}_t$, positive, and bounded. Initial holdings of the asset are $\hat{h}_0^i \geq 0$. The supply of the asset $\hat{h}^0 = \sum_i \hat{h}_i^0$ is strictly positive.

The agent faces the following budget and portfolio constraints

$$c(0) + p(0)h(0) \leq e^i(0) + p(0)\hat{h}_0^i, \tag{2}$$

$$c(s^t) + p(s^t)h(s^t) \leq e^i(s^t) + [p(s^t) + x(s^t)]h(s_-^t) \quad \forall s^t, \tag{3}$$

$$h(s^t) \geq 0, \quad \forall s^t \tag{4}$$

Condition (4) is the short-sales constraint.

An equilibrium consists of prices p and consumption-portfolio allocation $\{c^i, h^i\}$ such that plans (c^i, h^i) are optimal and markets clear. Market clearing is

$$\sum_i c_t^i = \bar{e}_t^i + \hat{h}_0 x_t, \text{ and } \sum_i h_t^i = \hat{h}_0,$$

for every t.

Because of the short-sales constraint, equilibrium asset price p_t at date t satisfies the relationship

$$p_t(s^t) = \max_i \beta E^i[p_{t+1} + x_{t+1}|s^t]. \tag{5}$$

The agent (or agents) whose one-period-ahead conditional belief $P^i(\cdot|s^t)$ is the maximizing one on the right-hand side of (5) holds the asset in s^t while the other agents whose conditonal beliefs give lower expectation have zero holding. We call the agent whose beliefs is the maximizing one the optimist (about next-period price plus dividend) at s^t.

Market belief $\hat{P}(\cdot|s^t)$ is the maximizing probability in (5), i.e., the optimist's belief. Let $\hat{P}$ be the probability measure on S^∞ derived from one-period-ahead probabilities $\hat{P}(\cdot|s^t)$.[3] It follows that $\hat{P}$ is a risk-neutral pricing measure (or state-price process) for p. Since the asset is in strictly positive supply and the discounted present value of the aggregate endowment $\sum_{t=0}^\infty \beta^t E_{\hat{P}}[\bar{e}_t]$ is finite, the no-bubble theorem (see Theorem 3.3 in Santos and Woodford (1997), or Theorem 32.1 in LeRoy and Werner (2014)) implies that equilibrium price of the asset is equal to the infinite sum of discounted expected dividends under the market belief. That

[3] The existence of probability measure $\hat{P}$ on S^∞ follows from the Kolmogorov Extension Theorem, see Halmos (1974), Sec. 38.

is,

$$p_t(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\hat{P}}[x_\tau | s^t], \tag{6}$$

for every s^t. The fundamental value of the asset under agent's i belief is the discounted sum of expected dividends conditional on event s^t, that is,

$$V^i(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E^i[x_\tau | s^t]. \tag{7}$$

Because of risk-neutral utilities, agents' fundamental values represent their willingness to pay for the asset if obliged to hold it forever. It follows from (5) that

$$p_t(s^t) \geq V_t^i(s_t), \tag{8}$$

for every i, every s^t. The following lemma will be often used in the analysis to follow.

Lemma 1: *If $p_t(s^t) > V_t^i(s^t)$ for agent i in some event s^t, then $p_\tau(s^\tau) > V_\tau^i(s^\tau)$ for every predecessor event s^τ of s^t, where $\tau < t$.*

PROOF: We first prove that $p_{t-1}(s^{t-1}) > V_{t-1}^i(s^{t-1})$ for the immediate predecessor of s^t. From (5) we have

$$p_{t-1}(s^{t-1}) \geq \beta E_{t-1}^i[p_t + x_t | s^{t-1}] > \beta E^i[V_t^i + x_t | s^{t-1}] = V_{t-1}^i(s^{t-1}), \tag{9}$$

where we used (8) and (for strict inequality) the assumption that $p_t(s^t) > V_t^i(s_t)$. The proof for non-immediate predecessor events is an iteration of the argument in (9). $\square$.

We say that there is *speculative bubble* in event s^t, if

$$p_t(s^t) > max_i V_t^i(s^t). \tag{10}$$

If (10) holds, then the optimist who buys the asset at s^t pays the price exceeding her valuation of the asset if she were to hold the asset forever. This means, of course, that she intends to sell the asset at a later date. Thus, speculative bubble indicates speculative trade. It follows from Lemma 1 that if there is speculative bubble in event s^t at date t, then there is speculative bubble at every date $\tau < t$, in each predecessor event. Thus speculative bubble has to originate at date 0, or more generally at the time of initial offering, but it can cease to exist (or burst) at a later date, or be permanent.

Agent i is (weakly) *valuation dominant* in event s^t if

$$V^i(s^\tau) \geq max_j V^j(s^\tau), \tag{11}$$

for every event s^τ which is a successor of s^t. If there is no valuation dominant agent in event s^t, then we say that agents' beliefs exhibit *valuation switching* at s^t. There is *perpetual valuation switching* from s^t on if beliefs exhibit valuation switching in every successor of s^t.

The main result of this section shows that valuation switching is sufficient for the existence of speculative bubble.

Theorem 1: *If agents' beliefs exhibit valuation switching in event s^t, then in equilibrium there is speculative bubble in s^t.*

PROOF: Suppose by contradiction that $p_t(s^t) = V_t^i(s^t)$ for some agent i. It follows from Lemma 1 that $p_\tau(s^\tau) = V_\tau^i(s^\tau)$ for every successor event s^τ. Since agent i is not valuation dominant, there exists j and a successor event s^τ such that $V_\tau^j(s^\tau) > V_\tau^i(s^\tau) = p_\tau(s^\tau)$. But this contradicts (8). $\square$.

If there is perpetual valuation switching from s^t on, then, by Theorem 1, there is permanent speculative bubble in every successor event of s^t. The condition of valuation switching is sufficient for speculative bubble but it is not necessary. This is illustrated by the following example.

Example 1, Harrison and Kreps (1978): The dividend process x_t is a Markov chain taking two values 0 and 1 for every $t \geq 1$. There are two agents whose beliefs are described by transition matrices Q_1 and Q_2 given by

$$Q_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \tag{12}$$

Note that agent 1 is more optimistic than agent 2 about next-period high dividend when current dividend is 1 and vice versa when the current dividend is 0. Discount factor is $\beta = 0.75$.

Fundamental values of the asset depend only on the current dividend and can found to be equal to

$$V^1(0) = \frac{4}{3}, \qquad V^1(1) = \frac{11}{9}, \tag{13}$$

$$V^2(0) = \frac{16}{11}, \qquad V^2(1) = \frac{21}{11}. \tag{14}$$

Thus agent 2 is valuation dominant.

In equilibrium, the agent who is more optimistic about next period dividend is the optimist (about price plus dividend) and holds the asset. Equilibrium prices can be found from equation (5). We have

$$p(0) = \beta[\frac{1}{2}p(0) + \frac{1}{2}(p(1) + 1)] \tag{15}$$

$$p(1) = \beta[\frac{1}{4}p(0) + \frac{3}{4}(p(1) + 1)] \tag{16}$$

It follows that

$$p(0) = \frac{24}{13}, \qquad p(1) = \frac{27}{13}. \tag{17}$$

One can easily verify that the right-hand sides of equations (16) and (16) are the respective maximal values among the two agents. We have

$$p(0) > \max_i V^i(0) \text{ and } \ p(1) > \max_i V^i(1),$$

implying that there is speculative bubble. □

We conclude this section with another example in which Theorem 1 is used to demonstrate that there is speculative bubble in equilibrium.

Example 2, Overreaction to News.

Suppose that dividends x_t are an i.i.d. sequence taking two values H or L such that $L < H$. There are I agents. At each date $t \geq 1$, agents observe realization of a public signal y_t. Signals are independent and with identical distribution conditional on next-period dividend. That is, conditional on x_{t+1} signals $(y_1, \ldots, y_t)$ are i.i.d. Signals can take arbitrary positive or negative values. The distribution of y_τ conditional on x_{t+1} for $\tau \leq t$ is two-sided exponential with the following densities that can be different across agents:

$$f_i(y|H) = \begin{cases} k_i a_i^y & \text{if } \ y \geq 0, \\ k_i b_i^{-y} & \text{if } \ y < 0, \end{cases} \tag{18}$$

and

$$f_i(y|L) = \begin{cases} k_i b_i^y & \text{if } \ y \geq 0, \\ k_i a_i^{-y} & \text{if } \ y < 0, \end{cases} \tag{19}$$

where $0 < a_i < 1$ and $0 < b_i < 1$, and $k_i = 1/[\frac{1}{\ln a_i} + \frac{1}{\ln b_i}]$.

The prior probability of high dividend is 1/2. The posterior probability of date-$t+1$ dividend equal to H after observing history of signals $y^t = (y_1, \ldots, y_t)$ is

$$\pi_i(H|y^t) = \frac{1}{1 + (\frac{b_i}{a_i})^m} \tag{20}$$

where we used Bayes rule and where $m = y_1 + \ldots y_t$. Thus the posterior probability depends only on the cumulative value m of past and current signals. We assume that $\frac{b_i}{a_i} < 1$ for every i, so that greater cumulative signals are considered favorable to high dividend. The ratio $\frac{b_i}{a_i}$ can be interpreted as the *strength of reaction to the signal*. The higher the ratio, the higher are the probabilities assigned to high dividend for positive cumulative values of the signal and to low dividends for negative values of the cumulative signal.

Agent's i fundamental valuation of the asset at date t depends only on the cumulative signal and is given by

$$V^i(m) = \frac{\beta}{1-\beta}[L(1 - \pi_i(H|m) + H\pi_i(H|m)] \tag{21}$$

for every m. If agents differ in the strength of reaction to the signal, that is, if there are i and j such that

$$\frac{b_i}{a_i} \neq \frac{b_j}{a_j} \tag{22}$$

then there is perpetual valuation switching. Indeed, if $\frac{b_i}{a_i} > \frac{b_j}{a_j}$ so that agent i over-react to the signal relative to j, then $\pi_i(H|m) > \pi_j(H|m)$ (and therefore $V^i(m) > V^j(m)$) for large positive value of m, and $\pi_i(H|m) < \pi_j(H|m)$ for large negative m. Theorem 1 implies that there is speculative bubble. $\square$

3. Speculative Trade and Bayesian Learning.

Bayesian learning in the setting of Section 2 is described as follows: There is a family of probability measures P_θ on (S^∞, Σ) parametrized by θ in the set of parameters Θ. The set Θ can be finite or infinite. There is σ-filed $\mathcal{G}$ of subsets of

Θ and the mapping $\theta \to P_\theta(A)$ is measurable for every $A \in \Sigma$. An agent, who does not know the true probability measure on (S^∞, Σ), has a prior belief μ on $(\Theta, \mathcal{G})$. The prior μ induces a joint distribution of states and parameters Π_μ defined by

$$\Pi_\mu(A \times B) = \int_A P_\theta(B)\mu(d\theta),$$

for $A \in \mathcal{G}$ and $B \in \Sigma$. Conditional probability on $\mathcal{G} \times \Sigma$ upon observing s^t is $\Pi_\mu(\cdot|s^t)$ and it induces the posterior belief on Θ denoted by $\mu_t(\cdot|s^t)$ and conditional probability of the future given the past on Σ denoted by $P_\mu(\cdot|s^t)$. For example, if μ is a Dirac point-mass measure at some θ, then $\mu_t = \mu$ for every t and $P_\mu(\cdot|s^t) = P_\theta(\cdot|s^t)$. This is "dogmatic" belief, as in Example 1, that is unaffected by learning.

Returning to the model of asset trading of Section 2, let agent's i prior belief be μ^i on $(\Theta, \mathcal{G})$. We use E^i to denote the expectation under probability measure P_{μ^i} and $E^i[\cdot|s^t]$ (or simply E^i_t) for conditional expectation under conditional probability $P_{\mu^i}(\cdot|s^t)$. As pointed out by Morris (1996), the condition of valuation dominance is related to the monotone likelihood ratio order of priors. Suppose that $\Theta \subset R$ and that each prior μ^i has density function f^i on Θ. Recall that prior μ^i dominates μ^j in the monotone likelihood ratio (MLR) order if

$$\frac{f^i(\theta')}{f^i(\theta)} \geq \frac{f^j(\theta')}{f^j(\theta)} \qquad \text{for every } \theta' \geq \theta. \tag{23}$$

Proposition 2: *Suppose that $V_{\theta'}(s^t) \geq V_\theta(s^t)$ for every $\theta' \geq \theta$ and every $s^t \in \mathcal{S}^\infty$. If agent's i prior μ^i MLR-dominates every other agents' prior, then agent i is valuation dominant from s^0 on.*

Proof: It is well-known that if μ^i MLR-dominates μ^j, then μ^i dominates μ^j in the sense of first-order stochastic dominance. Since $V_\theta(s^0)$ is increasing in θ, it follows that

$$V^i(s^0) = \int V_\theta(s^0) f^i(\theta) d\theta \geq \int V_\theta(s^0) f^j(\theta) d\theta = V^j(s^0) \tag{24}$$

for every j. Further, if μ^i MLR-dominates μ^j, then the posterior $\mu^i(\cdot|s^t)$ MLR-dominates the posterior $\mu^j(\cdot|s^t)$ for every s^t. As in (24), this implies $V^i(s^t) \geq V^j(s^t)$ for every j. $\square$

4. Speculative Trade and Learning with I.I.D. Dividends.

In this section we consider an important special cases of the model of Section 3: i.i.d. dividends.

Suppose that there is a family of probability measures π_θ on the state space $\mathcal{S}$. Let P_θ be the product measure π_θ^∞ making random variables $\{x_t\}$ independent with common distribution π_θ on $\mathcal{S}$. Here, the mean of the posterior distribution $\mu_t^i(\cdot|s^t)$ on Θ is the Bayes estimate of the unknown true parameter. Let $E_\theta[x]$ denote the expected value of the dividend under π_θ.

Let $E^i[x_{t+1}|s^t]$ be the expected value of next-period dividend under the probability $P_{\mu^i}(\cdot|s^t)$. Then

$$V^i(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E^i[x_{t+1}|s^t] = E^i[x_{t+1}|s^t]\frac{\beta}{1-\beta}, \tag{25}$$

for every s^t. Note that $E^i[x_{t+1}|s^t] = \int E_\theta[x] d\mu_t^i(\theta|s^t)$. It follows that agent i is valuation dominant in event s^t if and only if her conditional expectation of the next period dividend (weakly) exceeds every other agent's conditional expectation of the dividend in every successor event s^τ of s^t. Otherwise, there is valuation switching in s^t. If each prior μ^i has density function f^i on Θ and $\Theta \subset R$, then we have the following corollary of Proposition 2:

Corollary 3: *In the model with i.i.d. dividends, if agent's i prior μ^i MLR-dominates every other agents' prior and $E_\theta[x]$ is a non-decreasing function of θ, then agent i is valuation dominant from s^0 on.*

Example 4, Morris(1996): Suppose that x_t can take two values, 0 or 1, and θ is the probability of dividend 1 where $\theta \in [0,1] = \Theta$. Agents' priors on $[0,1]$ have density functions f^i. The expected value of the dividend is equal to the probability of dividend taking value one. That probability conditional on s^t depends only on the number of successes, i.e., dividends equal to one from date 0 through t. Denote that probability by $\nu^i(t,k)$, for k successes in t periods. We have

$$\nu^i(t,k) = \frac{\int_0^1 \theta^{k+1}(1-\theta)^{t-k} f^i(\theta) d\theta}{\int_0^1 \theta^{k}(1-\theta)^{t-k} f^i(\theta) d\theta} \tag{26}$$

If μ^j the uniform prior on $[0,1]$ with $f^j(\theta) \equiv 1$, then

$$\nu^j(t,k) = \frac{(k+1)}{(t+2)}. \tag{27}$$

If μ^i is the Jeffrey's prior (see Morris(1996) with $f^i(\theta) = \frac{1}{\sqrt{\theta}(1-\theta}$, then

$$\nu^i(t,k) = \frac{(k+1/2)}{(t+1)}. \tag{28}$$

These two popular priors under ignorance give rise to perpetual valuation switching and, by Theorem 1, to permanent speculative bubbles.

More generally, if prior μ^i has beta distribution with parameters α_i and β_i, then

$$\nu^i(t,k) = \frac{(k+\alpha_i)}{(t+\alpha_i+\beta_i)}. \tag{29}$$

(see Morris (1996)). If μ^j has beta distribution as well, with α_j and β_j, then μ^j valuation dominates μ^i for every (t,k) if and only if $\alpha_j \geq \alpha_i$ and $\beta_j \leq \beta_i$. Otherwise, there is perpetual valuation switching between μ^i and μ^j. Note that prior μ^j dominates μ^i in the MLR-order if and only if $\alpha_j \geq \alpha_i$ and $\beta_j \leq \beta_i$. Thus, MLR-order dominance and valuation dominance are equivalent within the class of beta priors and $0-1$ dividends. $\square$

5. Merging of Beliefs and Speculative Bubbles.

In this section we discuss asymptotic properties of speculative bubbles. Slawski (2009) pointed out the relevance of the Blackwell and Dubins (1962) merging-of-opinions result for the asymptotics of bubbles. If conditional beliefs merge in the sense of becoming close to each other in variational norm, then fundamental values converge to a common limiting value. Blackwell and Dubins theorem says that conditional beliefs merge if initial beliefs are absolutely continuous.

As in Section 2, suppose that the beliefs of agent i are represented by a probability measure P^i on (S^∞, Σ). Further, let P^0 be the true probability measure on (S^∞, Σ). Blackwell and Dubins theorem says that if P^0 is absolutely continuous with respect to P^i, then

$$\lim_{t\to\infty} \{\sup_{A\in\Sigma} |P^i(A|s^t) - P^0(A|s^t)|\} = 0, \quad P^0 - a.e. \tag{30}$$

Condition (30) is called merging of conditional beliefs.[4] If the merging condition holds, then $\lim_t [V_t^i(s^t) - V_t^0(s^t)] = 0$ P^0-a.e where V^0 is the fundamental value of the asset under the true measure P^0. Absolute continuity of P^0 with respect to P^i says that $P^0(A) > 0$ for every $A \in \Sigma$ such that $P^i(A) > 0$. It is a strong condition. It does not follow from a rather innocuous condition that date-t marginal P_t^0 is absolutely continuous with respect to P_t^i for all t. For example, if P^0 and P^i are infinite products of measures on S as in the case of iid true distribution and iid beliefs, then P^0 is absolutely continuous with respect to P^i only if they are identical. The same holds for stationary Markov beliefs. The beliefs in Example 4 are, of course, not absolutely continuous with respect to each other.

By the same argument, if the true measure P^0 is absolutely continuous with respect to the market belief $\hat{P}$, then then equilibrium asset price p converges to the true fundamental value P^0-a.e. We shall prove next that P^0 is absolutely continuous with respect to $\hat{P}$ if P^0 is absolutely continuous with respect to every agent's belief P^i. We apply a criterion for absolutely continuity of measures on the product space (S^∞, Σ) due to Darwich (2009), which is a simplified version of the main result of a seminal paper by Kabanov, Liptser and Shiryaev (1985).

Probability measure P^0 is absolutely continuous with respect to another measure Q on (S^∞, Σ) if

$$\sum_{t=0}^{\infty} E_Q[(1 - \frac{Q(s^{t+1}|s^t)}{P^0(s^{t+1}|s^t)})^2 | s^t] < \infty, \quad P^0 - a.e., \tag{31}$$

where the ratio of conditional probabilities is set to zero if the denominator is zero.

Recall from Section 3 that the market belief $\hat{P}$ is formed by selecting at each s^t the one-period-ahead probability $P^i(\cdot|s^t)$ which maximizes (5). If the sum in (31) is finite for $Q = P^i$ for each i, then the sum for $Q = \hat{P}$ must be finite, as well. It follows that P^0 is absolutely continuous with respect to the market belief $\hat{P}$. We summarize our discussion in the following theorem

Theorem 2: *Suppose that P^0 is absolutely continuous with respect to P^i for every i. Then*

$$\lim_t [V_t^i(s^t) - V_t^0(s^t)] = 0, \quad P^0 - a.e. \tag{32}$$

[4] Absolute continuity of probability measures on the product space (S^∞, Σ) is not only sufficient but also necessary for merging of conditional beliefs for any pair of measures whose date-t marginals are absolutely continuous for every t.

Moreover P^0 is absolutely continuous with respect to the market belief $\hat{P}$ and

$$\lim_t [p_t(s^t) - V_t^0(s^t)] = 0, \quad P^0 - a.e. \tag{33}$$

Consequently, the speculative bubble vanishes in the limit P^0-almost surely.

The analysis of asymptotic properties of speculative bubbles is somewhat different when beliefs arise from Bayesian learning with heterogeneous priors. As in Section 3, let Θ be the set of parameters with a σ-filed of subsets $\mathcal{G}$. Prior belief of agent i is measure μ^i on $(\Theta, \mathcal{G})$. Let θ_0 be the true parameter so that the true probability distribution on states is $P^0 = P_\theta$ for $\theta = \theta_0$. If Θ is a finite set, then the condition $\mu^i(\theta_0) > 0$ guarantees that the Dirac point-mass measure at θ_0 is absolutely continuous with respect to μ_i. This in turn implies that P^0 is absolutely continuous with respect to P_μ, and by the Blackwell-Dubins Theorem, that conditionals $P^0(\cdot|s^t)$ and $P_{\mu^i}(\cdot|s^t)$ merge $P^0 - a.e.$ If Θ is an infinite set, then the condition $\mu^i(\theta_0) > 0$ may be unnatural. In Example 4, there is no θ in the support of any of the priors that has strictly positive measure. Consistency of prior belief μ^i and θ_0 becomes an important issue.

Recall that prior μ_i is consistent at θ_0 if the posterior belief μ_t^i converges weakly to the Dirac measure at θ_0 in the weak-star topology that is

$$\lim_{t\to\infty} \int_\Theta g d\mu_t^i(\cdot|s^t) = g(\theta_0), \qquad P^0 - a.e. \tag{34}$$

for every continuous and bounded function g on Θ. We have

Proposition 3: *Suppose that $E_\theta(x_t)$ is continuous in θ for every t. If every prior μ^i is consistent at θ_0 and μ^i are absolutely continuous with respect to each other, then the hypotheses (32) and (33) of Theorem 2 hold and the speculative bubble vanishes in the limit P^0-almost surely.*

Proof: If μ^i is consistent at θ_0, then μ_t^i converges weakly to the Dirac measure at θ_0. This implies that $\lim_t[V_t^i(s^t) - V_t^0(s^t)] = 0$ P^0-a.e. Furthermore, if μ^i are absolutely continuous with respect to each other, then P_{μ^i} are absolutely continuous with respect to each other and, by the same argument as in Theorem 4, P_{μ^i} is absolutely continuous with respect to the market belief $\hat{P}$. This implies that $\lim_t[p_t(s^t) - V_t^0(s^t)] = 0$. $\square$.

Conditions for consistency of prior μ^i with the true parameter θ_0 depend on whether the dividend process $\{x_t\}$ is i.i.d. or not, and whether the parameter set Θ is finite or infinite dimensional. If $\{x_t\}$ is i.i.d. and Θ is finite dimensional such as a probability simplex on a finite set of values that x_t may take, then μ_i is consistent at θ_0 if and only if θ_0 lies in the support of μ^i, that is, $\theta_0 \in \text{supp } \mu^i$, see Friedman (1962). Support of μ^i is the smallest closed set of full measure. This result does not extend to infinite-dimensional parameter sets (see Diaconis and Friedman (1965)). A recent account of conditions for consistency for non-i.i.d. processes can be found in Shalizi (2009). Those conditions include some restrictions on temporal dependence of the process.

Slawski (2009) provides an example in which the true parameter lies outside of the common support of priors and there is persistent speculative bubble. If the true parameter lies outside of the support of a prior, the prior is called misspecified.

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