ON NEUMANN PROBLEMS FOR NONLOCAL HAMILTON-JACOBI EQUATIONS WITH DOMINATING GRADIENT TERMS

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Abstract. We are concerned with the well-posedness of Neumann boundary value problems for nonlocal Hamilton-Jacobi equations related to jump processes in general smooth domains. We consider a nonlocal diffusive term of censored type of order strictly less than 1 and Hamiltonians both in coercive form and in noncoercive Bellman form, whose growth in the gradient make them the leading term in the equation. We prove a comparison principle for bounded sub-and supersolutions in the context of viscosity solutions with generalized boundary conditions, and consequently by Perron's method we get the existence and uniqueness of continuous solutions. We give some applications in the evolutive setting, proving the large time behaviour of the associated evolutive problem under suitable assumptions on the data.

1. Introduction

The aim of this work is the analysis of the well-posedness of Neumann boundary value problems for partial-integro differential equations (PIDEs in short) of Hamilton-Jacobi type, where the nonlocal terms are singular integrals related to the infinitesimal generator of discontinuous jump processes. To be more specific, we consider the following

$$\begin{cases} u(x) - \mathcal{I}[u](x) + H(x, Du) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $H : \bar{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function, $\Omega \subset \mathbb{R}^N$ is an open (smooth enough) domain and $\mathcal{I}[u]$ is an *integro-differential operator of censored type and of order strictly less than* 1 (see (1.2) for the definition).

In the probabilistic approach to PDEs, Neumann boundary conditions are associated to stochastic processes being reflected on the boundary. The underlying idea is to force the stochastic process to remain inside the domain of the equation. Classically, this is obtained essentially by a reflection on the boundary (see the method developed by Lions and Sznitman [32] in the continuous setting). A key result in the classical setting is that, for a PDE with Neumann boundary conditions, there is a unique underlying reflection process and any consistent approximation will converge to it (see [32] and Barles, Lions [12]).

When dealing with discontinuous jumping processes, the underlying idea is the same but the situation is different. This is essentially due to the fact that the jump process may exit the domain without having first hit the boundary. The consequence is that Neumann boundary conditions can be obtained in many ways,

This work was partially supported by the ERC advanced grant 668998 (OCLOC) under the EU's H2020 research programme.

depending on the kind of reflection we impose on the outside jumps. Moreover, the choice of a reflection on the boundary changes the equation inside the domain.

The starting point of our work is the paper [6] where Barles, Chasseigne, Georgeline and Jakobsen studied problems as in (1.1) in the case of linear equations (that is, without the Hamiltonian H) and when the domain Ω is the halfspace. In [6] different models of reflection are presented, among which two types of reflections are particularly relevant for possible extensions in a more general setting. The first is the *normal projection*, close to the approach of Lions-Sznitman in [32], where outside jumps are immediately projected to the boundary by killing their normal component. This model has been thoroughly investigated in the paper [8] for fully non-linear equations set in general domains.

The second, the *censored* model, is the one we consider in our paper. In this case, any outside jump of the underlying process is cancelled (censored) and the process is restarted (resurrected) at the origin of that jump.

In particular, in the present work, we consider the boundary value problem (1.1) where $\mathcal{I}[u]$ is an *integro-differential operator of censored type and of order stricly less than* 1 defined as

$$\mathcal{I}[u](x) = \lim_{\delta \to 0^+} \int_{\substack{|z| > \delta, \\ x + j(x,z) \in \bar{\Omega}}} [u(x + j(x,z)) - u(x)] d\mu_x(z), \tag{1.2}$$

where μ_x is a singular nonnegative Radon measure representing the intensity of the jumps from x to $x + z$ and satisfying the following integrability condition

$$\int |z| \wedge 1 d\mu_x(z) < +\infty,$$

and $j(x,z)$ is a jump function (see assumptions (M), (J0), (J1) in the following section for details). A meaningful example is

$$d\mu_x(z) \sim \frac{g(x,z)}{|z|^{N+\sigma}} dz, \quad \sigma \in (0,1), \quad j(x,z) = f(x)z, \tag{1.3}$$

where g, f are bounded and Lipschitz functions. Note that $\mathcal{I}$ has to be interpreted as a principal value (P.V.) integral. We remark that the domain of integration is restricted to the z such that $x + j(x,z) \in \bar{\Omega}$, avoiding thus any outside jump. Note also that, as a consequence of the fact that censored type processes are not allowed to jump outiside $\bar{\Omega}$, we don't need any condititidions on Ω^c in the boundary value problem (1.1).

We follow the PIDE analytical approach developed in [6], in the sense that we directly work with the infinitesimal generator and not yet with the processes themselves. For more details and probabilistic references on censored processes, we refer to e.g. [10], [24], [30], [28] and to the introduction of [6]. We just mention that the underlying processes in this paper are related to the censored stable processes of Bogdan [10] and the reflected σ-stable process of Guan and Ma [28].

We stress that the boundary value problem (1.1) is interpreted in the sense of viscosity solutions, meaning that the Neumann boundary condition could also be not attained (and in this case the equation holds up to the boundary).

In the case of linear PIDES, as considered in [6], the kind of singularity of μ influences the nature of the boundary value problem (1.1), in the sense that the Neumann boundary condition is attained only if the measure is singular enough. In particular in [6] it is shown that, when the singularity is of order stricly less than 1 as in (1.3), the equation holds up to the boundary and the process never reaches the boundary. On the other hand, when the singularity of the measure is strong, i.e. when μ is of the type (1.3) with $\sigma \in [1, 2)$, the situation is far more complicated, mainly due to the "ugly" dependence in x of the operator in (1.2) and to the interplay between the singularity of the measure and the geometry of the boundary. In [6] this difficulty is tackled by considering solutions which are in some sense Hölder continuous up to the boundary and the comparison principle is established only in this class. Though the result could not be optimal, it is consistent with the "natural" Neumann boundary condition for the reflected σ-stable process (proved by Guann and Ma [28] through the variational formulation and Green type formulas) which in the case of the halfspace reads

$$\lim_{t\to 0} t^{2-\sigma} \frac{\partial u}{\partial x_N}(x + te_N) = 0. \tag{1.4}$$

This allows the normal derivative to growth less than $|x_N|^{\sigma-2}$ and then suggests that it is appropriate to look for solutions which are β-Hölder continuous, with $\beta > \sigma - 1$, as assumed in [6]. We remark that the previous argument suggests also that, on the contrary, in the case $\sigma < 1$ there is no need to assume any further regularity.

The situation is different when dealing with nonlinear equations as (1.1), which we consider in this paper. Indeed, the presence of the Hamiltonian term H in (1.1) entails further difficulties even in the case of measures of order strictly less than 1 (e.g. as in (1.3)). This is due to the fact that the nonlinear term H could force the process to hit the boundary and, consequently, the Neumann boundary condition to be attained.

In order to deal with this difficulty, we consider a class of Hamiltonians with a gradient growth stronger than the diffusive term in the nonlocal operator (1.2). The first example are Hamiltonians H with *superfractional coercive* growth in the gradient variable, namely

$$H(x, p) = a(x)|p|^m - f(x), \tag{1.5}$$

where $m > \sigma$, $\sigma \in (0, 1)$, $a, f : \bar{\Omega} \mapsto \mathbb{R}$ are bounded and continuous functions and $a(x) \geq a_0 > 0$ for some fixed constant a_0. We remark that the positivity of a and the condition $m > \sigma$ make the first-order term the leading term in the equation. We also observe that we have no other additional restriction to m (in particular, we can deal with Hamiltonians as in (1.5) with $m < 1$), allowing the study of Hamiltonians which are concave in Du.

The second main example are Hamiltonians H of *Bellman type*, which arises in the study of Hamilton-Jacobi equations associated to optimal exit time problems, such as

$$H(x, p) = \sup_{\alpha \in \mathcal{A}} \{-b(x, \alpha) \cdot p - l(x, \alpha)\}, \tag{1.6}$$

where $\mathcal{A}$ is a compact metric space (the control space) and b, l are continuous and bounded functions (we refer the reader to [3] and [23] for some connections between this type of equations and control problems). Note that the diffusive term of $\mathcal{I}$ defined in (1.2) is of weaker order than the first-order term when we assume $\sigma < 1$. We also observe that, as in [15] and [35], the well-posedness of (1.1) with Hamiltonians as in (1.6) is based on a careful study of the effects of the drift b at each point of $\partial\Omega \times (0, +\infty)$.

The main result of our paper is the comparison princile between bounded sub and super-viscosity solutions to (1.1), see Theorem 3.1. We remark that the proof of this result is not standard even in the case $\sigma < 1$ in the halfspace. The difficulties are mainly due to the fact that operators as in (1.2) behave badly in x. The main idea which is behind the proof is to localize the argument on points which have the same distance from the boundary and this is carried out through the use of a non-standard non regular test function. The main assumption which allows us to localize on equidistant points is the superfractional growth of the Hamiltonian term, see in particular the proof of Lemma 5.1 and Lemma 5.6 (more precisely, Lemma 5.2 and Lemma 5.7) for Bellman and coercive Hamiltonians respectively.
After the localization procedure, the rest of the proof in the case of the halfspace is simple, whereas in the case of general domains, further technical difficulties arise form the way the x-depending set of integration of $\mathcal{I}$ interferes with the geometry of the boundary. To face these extra technical difficulties, we rectify the boundary relying on the smoothness of Ω. This is done in Lemma 4.1 which is a key result used in the proof of Theorem 3.1, which we prove before Theorem 3.1 in Section 4.
The first main application of our result is the proof of existence and uniqueness for (1.1), by standard Perron's method (Corollary 3.2).
Finally, in Section 6, we present some applications of our results to the evolutive setting. In particular, we prove the well-posedness of the Cauchy problem associated to (1.1) and we study two different kind of asymptotic behaviour under suitable assumptions on the data. We refer to Section 6 for precise assumptions, statement of the results and proofs.

1.1. **Organization of the paper.** In Section 2 we state the assumptions on the nonlocal operator and the Hamiltonian and we give the definition of solution to problem (1.1). In Section 3 we state the main results, that is the uniqueness and existence for problem (1.1) for Hamiltonian either coercive or of Bellman type (Theorem 3.1 and Corollary 3.2). In Section 4 we prove Lemma 4.1 and in Section 5 we prove Theorem 3.1. In Section 6 we treat the associated evolutive problem, studying uniqueness, existence and asymptotic behaviour of the associated evolutive problem for large time. Finally, in the Appendix we prove some lemmas used in the proof of Theorem 3.1.

2. Assumptions and definition of solutions

We consider $\Omega \subset \mathbb{R}^N$ such that

(O) $$\Omega \text{ is of class } W^{2,\infty}.$$

This means that for any $\hat{s} \in \partial\Omega$ there exists $r = r(\hat{s})$ and a $W^{2,\infty}$-diffeomorphism $\psi : B_r(\hat{s}) \mapsto \mathbb{R}^N$ satisfying $\psi_n(s) = d(s)$ for any $s \in B_r(\hat{s})$, where d is the signed distance from the boundary of Ω.

Remark 1. By assumption (O), there exists a neighbourhood of the boundary of Ω where the distance from the boundary d is smooth. Unless otherwise specified, throughtout the paper we denote by d a function which coincides with the signed distance from the boundary of $\partial\Omega$ in this neighbourhood and is bounded in all the domain. We denote by $n(x)$ the exterior unit normal vector to $\partial\Omega$ and we write $n(x) = -Dd(x)$ in the neighbourhood of the boundary where d is smooth.

We consider nonnegative Radon measures with density $\frac{d\mu_x}{dz}$ satisfying

(M) there exists $C_\mu, D_\mu > 0, \sigma \in (0,1)$ such that for any $x, y \in \bar{\Omega}, z \in \mathbb{R}^N$

$$\frac{d\mu_x}{dz} \leq C_\mu |z|^{-(N+\sigma)}, \quad |\frac{d\mu_x}{dz} - \frac{d\mu_y}{dz}| \leq D_\mu |x-y||z|^{-(N+\sigma)}.$$

For example, (M) is satisfied by

$$d\mu_x = g(x,z)|z|^{-(N+\sigma)}dz \quad x \in \bar{\Omega}, z \in \mathbb{R}^N, \tag{2.1}$$

where $\sigma \in (0,1)$, $g : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ is a nonnegative bounded function such that $g(\cdot, z)$ is Lipschitz uniformly with respect to z.

Concerning the jump function j we assume

(J0) for any $x \in \bar{\Omega}$ $j(x,\cdot) \in C^1(\mathbb{R}^N)$, $j(x,\cdot)$ is invertible and

$$j^{-1}(x,\cdot) \in C^1(\mathbb{R}^N), \quad |Dj^{-1}(x,\cdot)| \leq A_j;$$

(J1) there exist $\tilde{C}_j, C_j, D_j > 0$ such that for any $x, y \in \bar{\Omega}, z \in \mathbb{R}^N$, it holds

$$\tilde{C}_j|z| \leq |j(x,z)| \leq C_j|z|, \quad |j(x,z) - j(y,z)| \leq D_j|z||x-y|.$$

For example (J0), (J1) are satisfied for

$$j(x,z) = f(x)z \quad x \in \bar{\Omega}, z \in \mathbb{R}^N,$$

where $f : \mathbb{R}^N \mapsto \mathbb{R}$ is Lipschitz and bounded.

2.1. **Hamiltonian of Bellman type.** Let $\mathcal{A}$ be a compact metric space, $b : \bar{\Omega} \times \mathcal{A} \to \mathbb{R}^N$ and $f : \bar{\Omega} \times \mathcal{A} \to \mathbb{R}$ be continuous and bounded functions. We say that H is of *Bellman type* if for $x \in \bar{\Omega}, p \in \mathbb{R}^N, H(x,p)$ can be written as

$$H(x,p) = \sup_{\alpha \in \mathcal{A}} \{-b(x,\alpha) \cdot p - l(x,\alpha)\}, \tag{2.2}$$

and satisfies the assumptions below. We assume also:

(C) *Uniform continuity of the cost* l:
There exists a modulus of continuity ω_l such that

$$|l(x,\alpha) - l(y,\alpha)| \leq \omega_l(|x-y|) \quad \forall \alpha \in \mathcal{A}, \forall x, y \in \bar{\Omega};$$

(L) *Uniform Lipschitz continuity of the drift* b:

$$(\exists C > 0)\,(\forall \alpha \in \mathcal{A})\,(\forall x, y \in \bar{\Omega}) \ : \ |b(x,\alpha) - b(y,\alpha)| \leq C|x-y|.$$

We introduce the following notations

$$\Gamma_{\text{in}} := \{x \in \partial\Omega \, : \, b(x,\alpha) \cdot n(x) < 0 \ \forall \alpha \in \mathcal{A}\}, \tag{2.3}$$

$$\Gamma_{\text{out}} := \{x \in \partial\Omega \,|\, b(x,\alpha) \cdot n(x) > 0 \quad \forall \alpha \in \mathcal{A}\}, \tag{2.4}$$

$$\Gamma := \{x \in \partial\Omega \,|\, \exists \alpha_1, \alpha_2 \in \mathcal{A} \text{ s. t. } b(x,\alpha_1) \cdot n(x) < 0, b(x,\alpha_2) \cdot n(x) > 0\}. \tag{2.5}$$

Roughly speaking, Γ_{in} and Γ_{out} can be respectively understood as the set of points where the drift term pushes inside and outside Ω the trajectories.

In order to avoid two completely different drift's behaviour for arbitrarily closed points, we assume that each of these subsets is uniformly away from the others, as encoded in the following assumption (B). For example, if $\partial\Omega$ is connected, then it consists in one piece belonging to one of Γ_{in}, Γ_{out} and Γ; otherwise, we are able to deal with boundary with several components of different types, precisely each one belonging to one between Γ_{in}, Γ_{out}, Γ.

The assumptions we do on these subsets are the following

(B) $\Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma = \partial\Omega, \quad \Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma$ are unions of connected components of $\partial\Omega$.

Remark 2. Note that the strict sign in the definition of $\Gamma_{\text{in}}, \Gamma_{\text{out}}$ and Γ is fundamental, since it makes the Hamiltonian the leading order term in the equation, allowing us to control the growth of the nonlocal term, which is of order strictly less than 1.

Remark 3. In order to treat the points of Γ_{in}, we use the existence of a blow-up supersolution exploding on the boundary. We follow the same approach of [6], where the existence of a blow-up supersolution is proved for censored type operators (of order stricly less than 1) when the measure of integration satisfies specific assumptions (in particular does not depend on x and there exists at least one point where it is strictly positive). In this particular case it is shown in [6] that the integral term computed on the blow-up supersolution do not explode on the boundary. This is not true anymore when considering more general measures as we consider in (M). In order to solve this difficulty, we assume the strict sign in the behaviour of the drift term on Γ_{in}, which allows us to control the growth on the boundary of the integral term computed on this blow-up supersolution. We refer to the proof of Lemma 5.1 and in particular to Lemma 5.5 for further details.

2.2. **Coercive Hamiltonian and Examples.** We consider *superfractional* coercive Hamiltonians:

(H1) Let σ be as in (M). There exists $m > \sigma, c_0 > 0, D > 0$ such that for all $x \in \bar{\Omega}, p \in \mathbb{R}^N$
$$H(x,p) \geq c_0|p|^m - D.$$

We distinguish the case of sub or superlinear coercivity:

Sublinear coercivity: We say that H is sublinearly coercive if it satisfies (H1) for $m \leq 1$ and the following continuity condition holds:

(Ha) There exists a constant $C > 0$ and modulus of continuity ω_1 such that, for all $x, y, q, p \in \mathbb{R}^N$, we have
$$H(y,p) - H(x,q) \leq \omega_1(|x-y|)(1+|p|) + C(|p-q|).$$

Superlinear coercivity: We say that H is superlinearly coercive if:

(Hb) There exists $m > 1, A, \bar{C} > 0$ such that for all $\mu \in (0,1), x, y, p \in \mathbb{R}^N$, we have
$$H(x,p) - \mu H(x, \mu^{-1}p) \leq (1-\mu)\left(\bar{C}(1-m)|p|^m + A\right);$$

(Hc) If m is as in assumption (Hb), there exist $C > 0$ and a modulus of continuity ω_1 such that, for all $x, y, q, p \in \mathbb{R}^N$
$$H(y,p) - H(x,q) \leq \omega_1(|x-y|)(1+|p|^m \vee |q|^m) + C|p-q|(|p|^{m-1} \vee |q|^{m-1}).$$

Remark 4. *Note that condition* (*Hb*) *implies* (*H*1) *for* $m > 1$.

As it is classical in viscosity solution's theory, the comparison principle allows the application of Perron's method to conclude the existence of solutions. To this end, we introduce the following assumption, which will allows us to build sub and supersolutions:

(E) There exists $H_R > 0$ such that for any $p \in \mathbb{R}^N, |p| \leq R$
$$||H(\cdot,p)||_\infty \leq H_R.$$

As a model example for sublinearly coercive Hamiltonians, we consider

$$H(x,p) = a_1(x)|p|^m + a_2(x)|p|^l - f(x),$$

with $m \le 1, a_1 \ge a_0 > 0$ for all $x \in \Omega, l < m$ and $a_1, a_2, f : \bar{\Omega} \mapsto \mathbb{R}$ are continuous and bounded functions and a_1, a_2 are also Lipschitz continuous.
As a model example for superlinearly coercive Hamiltonian, we consider

$$H(x,p) = a_1(x)|p|^m + a_2(x)|p|^l + b(x) \cdot p - f(x),$$

with $m > 1, b$ bounded and continuous and a_1, a_2, f as before.
These Hamiltonians are coercive in p and in the case $m > 1$ we can include transport terms with a Lipschitz continuous vector field $b : \bar{\Omega} \mapsto \mathbb{R}^N$. The above assumptions are easily checkable in both cases.

2.3. **Notion of viscosity solutions.** We recall now the definition of solution to problem (1.1). We use the following notation:

$$\mathcal{I}[\phi] = \mathcal{I}_\xi[\phi] + \mathcal{I}^\xi[\phi], \tag{2.6}$$

where

$$\mathcal{I}^\xi[\phi] = \int_{\substack{|z| \ge \xi, \\ x + j(x,z) \in \bar{\Omega}}} \phi(x + j(x,z)) - \phi(x) d\mu_x(z). \tag{2.7}$$

The $\mathcal{I}^\xi$-term is well-defined for any bounded function ϕ. The $\mathcal{I}_\xi$-term is well-defined for $\phi \in C^1$ thanks to (M0).
We also denote

$$F(x, u, Du, \mathcal{I}[u]) = u(x) - \mathcal{I}[u](x) + H(x, Du).$$

Following the approach of [6], we give the definition of viscosity solution to (1.1). Let C_j be defined as in (J1).

Definition 2.1. *(i) A bounded usc function u is a viscosity subsolution to* (1.1) *if, for any test-function $\phi \in C^1(\mathbb{R}^N)$ and maximum point x of $u - \phi$ in $\bar{B}_{C_j\xi}(x) \cap \bar{\Omega}$*

$$F(x, u(x), D\phi(x), \mathcal{I}_\xi[\phi] + \mathcal{I}^\xi[u]) \le 0 \qquad x \in \Omega$$
$$\min\{F(x, u(x), D\phi(x), \mathcal{I}_\xi[\phi] + \mathcal{I}^\xi[u]), \frac{\partial \phi}{\partial n}\} \le 0 \qquad x \in \partial\Omega.$$

(ii) A bounded lsc function v is a viscosity supersolution to (1.1) *if, for any test-function $\phi \in C^1(\mathbb{R}^N)$ and minimum point x of $v - \phi$ in $\bar{B}_{C_j\xi}(x) \cap \bar{\Omega}$,*

$$F(x, v(x), D\phi(x), \mathcal{I}_\xi[\phi] + \mathcal{I}^\xi[v]) \ge 0 \qquad x \in \Omega$$
$$\max\{F(x, v(x), D\phi(x), \mathcal{I}_\xi[\phi] + \mathcal{I}^\xi[v], \frac{\partial \phi}{\partial n}\} \ge 0 \qquad x \in \partial\Omega.$$

(iii) A viscosity solution is both a sub- and a supersolution.

3. Main results

The main result of this part is the following comparison principle for the problem (1.1).

Theorem 3.1. [Comparison] *Let Ω be an open subset of $\mathbb{R}^N$ satisfying (O). Assume $(M), (J0), (J1)$. Let H be an Hamiltonian of Bellman type as in* (2.2) *satisfying (C), (L), (B) or a coercive Hamiltonian satisfying (H1), (Ha) or (H1), (Hb), (Hc). Let u be a bounded usc subsolution of* (1.1) *and v a bounded lsc supersolution of* (1.1)*. Then $u \leq v$ in $\bar{\Omega}$.*

Once the comparison holds, we use the Perron's method for integro-differential equations (see [1], [9], [33] and [20],[29] for an introduction on the method) to get as a corollary existence and uniqueness for the problem (1.1) either when H is of Bellman type either for H coercive.

Corollary 3.2. [Existence and Uniqueness] *Let Ω be an open subset of $\mathbb{R}^N$ satisfying (O). Assume $(M), (J0), (J1)$- Let H be either an Hamiltonian of Bellman type as in* (2.2) *satisfying (C), (L), (B) or a coercive Hamiltonian satisfying (H1), (Ha) or (H1), (Hb), (Hc). Assume (E).Then, there exists a unique bounded viscosity solution to problem* (1.1)*.*

4. A preliminary key lemma

We prove the following Lemma 4.1, which is a key result used in the proof of Theorem 3.1. Roughly speaking, it deals with the difficulties arising from the way the geometry of the boundary interferes with the singularity of the nonlocal terms. The scope is to estimate the nonlocal terms defined in (4.2) on points near the boundary and equidistant from it. The approach of the proof is essentially based on a rectification of the boundary, relying on its regularity.

Remark 5. In the case of domains with flat boundary, we do not need Lemma 4.1 in the proof of Theorem 3.1 since the estimation of the nonlocal terms can be carried out more easily. We refer to Remark 7, step 4 of the proof of Theorem 3.1.

Note that, if $\hat{s} \in \partial\Omega$, since Ω satisfies (O), there exists $r = r(\hat{s})$ and a $W^{2,\infty}$-diffeomorphism $\psi : B_r(\hat{s}) \mapsto \mathbb{R}^N$, satisfying

$$\psi_n(s) = d(s) \text{ for any } s \in B_r(\hat{s}), \tag{4.1}$$

where d is the signed distance from the boundary of Ω. For $s_1, s_2 \in B_{\frac{r}{2}}(\hat{s}) \cap \bar{\Omega}$, let

$$\mathcal{I}[J_{s_1}/J_{s_2}] = \int_{\substack{J_{s_1} \setminus J_{s_2},\\ |z| \leq \delta_0}} \frac{dz}{|z|^{N+\sigma-1}} \tag{4.2}$$

where $J_s = \{z \in \mathbb{R}^N \mid s + j(s,z) \in \bar{\Omega}\}$, j satisfies assumptions (J0),(J1), $\sigma \in (0,1)$ and $0 < \delta_0 < rC_j^{-1}/2$, where C_j is the constant defined in (J1).

Lemma 4.1. *Let $\mathcal{I}[J./J.]$ as in* (4.2) *and assume j satisfies $(J0), (J1)$. Let $\hat{s} \in \partial\Omega$, r given as above and s_1, s_2 satisfying*

$$d(s_1) = d(s_2), \quad s_1, s_2 \in B_{\frac{r}{2}}(\hat{s}) \cap \bar{\Omega}. \tag{4.3}$$

Then there exists a positive constant C such that

$$\mathcal{I}[J_{s_1}/J_{s_2}] \leq C|s_1 - s_2|. \tag{4.4}$$

Proof.

Step 1. *-Rectification of the boundary* We observe that since $s_1, s_2 \in B_{\frac{r}{2}}(\hat{s}) \cap \bar{\Omega}$, $\delta_0 < rC_j^{-1}/2$ and by (J1), we have for any $|z| \le \delta_0$

$$s_1 + j(s_1, z),\ s_2 + j(s_2, z) \in B_r(\hat{s}). \tag{4.5}$$

By assumption (O), we describe the domain of integration of $\mathcal{I}[J_{s_1}/J_{s_2}]$ through the diffeomorphism ψ as follows

$$\begin{aligned} s_1 + j(s_1, z) \in \bar{\Omega} &= \psi_N(s_1 + j(s_1, z)) \ge 0, \\ s_2 + j(s_2, z) \notin \bar{\Omega} &= \psi_N(s_2 + j(s_2, z)) < 0. \end{aligned}$$

We observe that by (4.1) and (4.3), we have

$$\psi_N(s_1) = \psi_N(s_2). \tag{4.6}$$

We proceed performing a change of variable in order to write the set of integration in terms of $\psi_N(s_1)$. In other words, we write

$$\psi(s_1 + j(s_1, z)) - \psi(s_1) = w, \tag{4.7}$$

that is, $j(s_1, z) = \psi^{-1}(\psi(s_1) + w) - s_1$. Then, the new set of integration can be written as follows

$$D = \{w \in \mathbb{R}^N : w_N + \psi_N(s_1) \ge 0,\ \psi_N(s_2 + j(s_2, z)) < 0,\ 0 < |w| \le \bar{C}\delta_0\}.$$

In the following step, we rewrite D in a different way.

Step 2. *-Rewriting the set D* By (4.7) and if $\psi_N(s_2 + j(s_2, z)) \le 0$, we have

$$\begin{aligned} w_N + \psi_N(s_1) &= \psi_N(s_2 + j(s_2, z)) + (\psi_N(s_1 + j(s_1, z)) - \psi_N(s_2 + j(s_2, z))) \\ &\le (\psi_N(s_1 + j(s_1, z)) - \psi_N(s_2 + j(s_2, z))). \end{aligned} \tag{4.8}$$

For convenience of notation, let for the moment

$$s(t) = ts_2 + (1-t)s_1, \quad \zeta(t) = tj(s_2, z) + (1-t)j(s_1, z). \tag{4.9}$$

Note that $s(0) + \zeta(0) = s_1 + j(s_1, z)$, $s(1) + \zeta(1) = s_2 + j(s_2, z)$. Then, since $\psi \in W^{2,\infty}$ and by (4.8) we write

$$w_N + \psi_N(s_1) \le \int_0^1 D\psi_N(s(t) + \zeta(t)) \cdot (s_1 + j(s_1, z) - (s_2 + j(s_2, z))dt = A_1 + A_2,$$

where

$$A_1 = \int_0^1 [D\psi_N(s(t) + \zeta(t)) - D\psi_N(s(t))] \cdot (s_1 + j(s_1, z) - (s_2 + j(s_2, z))dt,$$

$$A_2 = \int_0^1 D\psi_N(s(t)) \cdot (s_1 - s_2) + \int_0^1 D\psi_N(s(t)) \cdot (j(s_1, z) - j(s_2, z))dt.$$

From now on we denote by C any positive constant which may change from line to line. By definition of $\zeta(t)$

$$|\zeta(t)| = |tj(s_2, z) + (1-t)j(s_1, z)| \le 2C_j|z| \quad \text{for any } t \in [0, 1]. \tag{4.10}$$

By (J1), (O) and since $\psi \in W^{2,\infty}$

$$\tilde{C}_j|z| \le |j(s_1, z)| \le |\psi^{-1}(\psi(s_1) + w - s_1)| \le ||D\psi^{-1}||_\infty |\psi(s_1) + w - \psi(s_1)| \le C|w|. \tag{4.11}$$

Then, by (4.10), (J1) and (4.11) we get

$$A_1 \le C \int_0^1 |\zeta(t)|(|s_1 - s_2| + |j(s_1, z) - j(s_2, z)|)dt \le C|w||s_1 - s_2|.$$

Now we analyse A_2. Note that by (4.9) and (4.6)

$$\int_0^1 D\psi_N(s(t))\cdot(s_1-s_2) = \int_0^1 D\psi_N((ts_2+(1-t)s_1)\cdot(s_1-s_2) = \psi_N(s_1)-\psi_N(s_2) = 0.$$

Moreover, since $\psi \in W^{2,\infty}$, by (J1) and (4.11)

$$\int_0^1 D\psi_N(s(t)) \cdot (j(s_1, z) - j(s_2, z)))dt \leq C|w||s_1 - s_2|.$$

Then we have $A_2 \leq C|w||s_1 - s_2|$. We denote $a = \psi_N(s_1)$ and observe $a \geq 0$. By all the previous arguments, we perform the change of variable in $\mathcal{I}[J_{s_1}/J_{s_2}]$ by (J0), (J1) and using that $\psi \in W^{2,\infty}$, we get for some constant $\bar{C} > 0$

$$\mathcal{I}[J_{s_1}/J_{s_2}] \leq \bar{C} \int_{\tilde{D}} \frac{dw}{|w|^{N+\sigma-1}}, \tag{4.12}$$

where

$$D \subset \tilde{D} = \{w \in \mathbb{R}^N \,:\, -a \leq w_N \leq -a + C|s_1 - s_2||w|,\, 0 < |w| \leq \bar{C}\delta_0\}.$$

By no loss of generality and for simplicity of exposition, from now on we put $C = \bar{C} = 1$.

Step 3. -*Estimate on* $\tilde{D}$ We introduce the following notations:

$$d = (1 - |s_1 - s_2|)^{-1}, \quad \beta = (1 + |s_1 - s_2|)^{-1}. \tag{4.13}$$

Note that by the second assumption in (4.3), $|s_1-s_2| \leq r$. Without loss of generality we can suppose $r \leq \frac{1}{2}$, so that we have $|s_1 - s_2| \leq 1/2$. Then

$$2 \geq d \geq 1, \quad 1 \geq \beta \geq \frac{1}{2}. \tag{4.14}$$

Note that, if $w \in \tilde{D}$, then

$$-a \leq w_N \leq -a + |s_1 - s_2||w'| + |s_1 - s_2||w_N|. \tag{4.15}$$

We identify two cases, depending on the sign of $-a + |s_1 - s_2||w'|$ and we denote

$$D_1 = \{w' \mid -a + |s_1 - s_2||w'| \geq 0, |w'| \leq \delta_0\},$$

and

$$D_2 = \{w' \mid -a + |s_1 - s_2||w'| < 0, |w'| \leq \delta_0\}.$$

Observe that, if $w \in \tilde{D} \cap D_2$, then $-a + |s_1 - s_2||w'| < 0$ and (4.15) implies $w_N < 0$ and in particular $-a \leq w_N \leq -\beta a + \beta|s_1 - s_2||w'| < 0$. Otherwise, if $w \in \tilde{D} \cap D_1$, then $-a + |s_1 - s_2||w'| \geq 0$ and w_N can assume both negative and positive values. In particular (4.15) implies $-a \leq w_N \leq -da + d|s_1 - s_2||w'|$. Note also that $-da + d|s_1 - s_2||w'| \geq 0$.
By all the previous observations, we write

$$\int_{\tilde{D}} \frac{dw}{|w|^{N+\sigma-1}} = \int_{\tilde{D}} \frac{dw_N dw'}{(|w'|^2 + |w_N|^2)^{\frac{N+\sigma-1}{2}}} \leq \mathcal{F}_1 + \mathcal{F}_2, \tag{4.16}$$

where

$$\mathcal{F}_1 = \int_{D_1} \int_{-a}^{-da+d|s_1-s_2||w'|} \frac{dw_N dw'}{(|w'|^2 + |w_N|^2)^{\frac{N+\sigma-1}{2}}},$$

$$\mathcal{F}_2 = \int_{D_2} \int_{-a}^{-\beta a+\beta|s_1-s_2||w'|} \frac{dw_N dw'}{(|w'|^2 + |w_N|^2)^{\frac{N+\sigma-1}{2}}}.$$

For $\mathcal{F}_1$, we use that $\frac{1}{|w'|^2+|w_N|^2} \le \frac{1}{|w'|^2}$ and by Fubini's Theorem, we integrate in the N-variable and we get

$$\mathcal{F}_1 \le \int_{D_1}\int_{-a}^{-da+d|s_1-s_2||w'|} \frac{dw_N dw'}{|w'|^{N+\sigma-1}} \le \int_{D_1} \frac{-da+d|s_1-s_2||w'|+a}{|w'|^{N+\sigma-1}}dw'. \tag{4.17}$$

By the first of (4.13) and (4.14) and since $da \ge 0$, we have $-da+d|s_1-s_2||w'|+a = -da|s_1-s_2|+d|s_1-s_2||w'| \le 2|s_1-s_2||w'|$. Therefore

$$\mathcal{F}_1 \le d|s_1-s_2|\int_{D_1}\frac{dw'}{|w'|^{N+\sigma-2}}. \tag{4.18}$$

From now on we denote by C any positive constant which may change from line to line. Note that, since $w' \in \mathbb{R}^{N-1}$ and $\sigma < 1$, we have

$$\int_{D_1}\frac{dw'}{|w'|^{N+\sigma-2}} \le C. \tag{4.19}$$

Then by the previous observations, we get

$$\mathcal{F}_1 \le C|s_1-s_2|. \tag{4.20}$$

Now we analyse $\mathcal{F}_2$. For simplicity of notations, we denote

$$\zeta(w') = \int_{-a}^{-\beta a+\beta|s_1-s_2||w'|}\frac{dw_N}{(|w'|^2+w_N^2)^{\frac{N+\sigma-1}{2}}}$$

and then, by Fubini's Theorem, we have

$$\mathcal{F}_2 = \int_{D_2}\zeta(w')dw'. \tag{4.21}$$

We split the domain as follows

$$\int_{D_2}\zeta(w')dw = \int_{D_2\cap\{a\le|w'|\}}\zeta(w')dw' + \int_{D_2\cap\{a>|w'|\}}\zeta(w')dw'. \tag{4.22}$$

We estimate the first term by

$$\int_{D_2\cap\{a\le|w'|\}}\zeta(w')dw' \le \int_{D_2\cap\{a\le|w'|\}}\frac{-\beta a+\beta|s_1-s_2||w'|+a}{|w'|^{N+\sigma-1}}dw' \le C|s_1-s_2|, \tag{4.23}$$

where in the first inequality we used that $-\beta a+\beta|s_1-s_2||w'|+a \le 2|w'||s_1-s_2|$, since $\beta \le 1$ and $a \le |w'|$, and in the second inequality we used (4.19).

Take now the second term in (4.22). Note that, if $a > |w'|$, by (4.13) and (4.14), we have $-\beta a+\beta|s_1-s_2||w'| \le -\beta a d^{-1} \le -a4^{-1} \le 0$. By all the previous observations, since the function $w_N \mapsto \frac{1}{(|w'|+w_N^2)^{\frac{N+\sigma-1}{2}}}$ is increasing on the negative halfline, we have

$$\zeta(w') \le \frac{|s_1-s_2|(a+|w'|)}{(|w'|^2+4^{-2}a^2)^{\frac{N+\sigma-1}{2}}} \le 2^{N+\sigma-1}|s_1-s_2|\frac{a+|w'|}{(|w'|^2+a^2)^{\frac{N+\sigma-1}{2}}}. \tag{4.24}$$

Then

$$\int_{D_2\cap\{|w'|\le a\}}\frac{a+|w'|}{(|w'|^2+a^2)^{\frac{N+\sigma-1}{2}}} \le 2a\int_{D_2}\frac{dw'}{|(w',a)|^{N+\sigma-2}} \le C\int_{D_2}\frac{dw'}{|w'|^{N+\sigma-2}} \le C \tag{4.25}$$

and coupling (4.24) and (4.25), we get

$$\int_{D_2\cap\{a\geq|w'|\}} \zeta(w')dw' \leq C|s_1 - s_2|. \tag{4.26}$$

Then coupling (4.21), (4.22), (4.23) and (4.26), we obtain

$$\mathcal{F}_2 \leq C|s_1 - s_2| \tag{4.27}$$

and we conclude the proof by coupling (4.12), (4.16), (4.20) and (4.27). □

5. Proof of the comparison principle

We prove Theorem 3.1 and we split the proof into two parts, depending whether H is of Bellman type or coercive.

5.1. **Hamiltonians of Bellman type.** The proof of Theorem 3.1 follows mainly by the following lemma, which we prove first. At the end of the proof of Lemma 5.1, we will prove Theorem 3.1.

Lemma 5.1. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying (O). Let $\mathcal{I}$ as in* (1.2) *and assume μ satisfies (M), j satisfies $(J0),(J1)$. Let H be an Hamiltonian of Bellman type as in* (2.2) *satisfying (C), (L), (B) and let u, v be respectively bounded sub and supersolutions to* (1.1)*. Then the function*

$$\omega(x) := u(x) - v(x)$$

satisfies, in the viscosity sense, the equation

$$\begin{cases} \omega - \mathcal{I}[\omega](x) - B|D\omega| \leq 0 & \text{in } \Omega, \\ \frac{\partial\omega}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where B is a positive constant depending on the data.

Proof. Let $x_0 \in \bar{\Omega}$ and $\phi \in C^1(\mathbb{R}^N)$ such that $\omega - \phi$ has a strict maximum point at x_0. We observe that if $x_0 \in \Omega$ the proof is rather standard, since in this case the maximum points (x, y) of $u - v - \phi$ converge as $\varepsilon \to 0$ to (x_0, x_0) and hence they are bounded away from the boundary for ε small enough. This last property implies that we can directly use the equations and then proceed as in the following case.

Let $\Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma$ be defined respectively in (2.3), (2.4) and (2.5) and recall they satisfy (B). We suppose $x_0 \in \partial\Omega$ and we split the proof depending if

(a) $x_0 \in \Gamma_{\text{in}}$;
(b) $x_0 \in \Gamma_{\text{out}}$;
(c) $x_0 \in \Gamma$.

In case (a) we use the existence of the blow-up supersolution which explodes at the boundary and allows us to keep the maximum points far from the boundary. Since the proof in this case is inspired by a similar approach used in [6], we give the details at the end of the proof of (b) and (c) in Remark 9. Now we treat case (b) and (c). Since the proofs are similar, we treat them at the same time.
We suppose that

$$\frac{\partial\phi}{\partial n}(x_0) > 0, \tag{5.2}$$

and prove that the F-viscosity inequality of Definition (2.1) hold for ω.

Step 1. *Localising on equidistant points (that is, $d(x)=d(y)$)* Let $\varepsilon>0$ and d be a function as in Remark 1. We double the variable by introducing the function for $\varepsilon,\delta>0$

$$\tilde{\phi}(x,y)=\phi((x+y)/2)+\varepsilon^{-1}\chi_\varepsilon(|x-y|)+K\varepsilon^{-1}\chi_\delta(|d(x)-d(y)|), \tag{5.3}$$

where $\chi_\varepsilon:\mathbb{R}\to\mathbb{R}$ (and similarly χ_δ) is defined as follows

$$\chi_\varepsilon(r)=\sqrt{r^2+\varepsilon^4}\quad r\in\mathbb{R} \tag{5.4}$$

and $K>0$ is a constant large enough such that

$$K>(2+C_2)\gamma^{-1}, \tag{5.5}$$

where $\gamma, C_2>0$ depend on x_0 and are precisely defined in Lemma 6.9 in the Appendix (for $\hat{s}=x_0$). Let

$$\Phi(x,y)=u(x)-v(y)-\tilde{\phi}(x,y) \tag{5.6}$$

and denote by $(\bar{x},\bar{y})$ the maximum point of Φ in $\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\times\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}$. We observe that $(\bar{x},\bar{y})$ depends now also on δ and we omit the dependence.
Now consider

$$\Psi=u(x)-v(y)-\psi(x,y), \tag{5.7}$$

where

$$\psi(x,y)=\phi((x+y)/2)+\varepsilon^{-1}\chi_\varepsilon(|x-y|)+K\varepsilon^{-1}|d(x)-d(y)|, \tag{5.8}$$

where d is the signed distance from the boundary (see Remark 1), χ_ε is defined as in (5.4) and K is as in (5.5). Note that the test function in (5.8) is not differentiable on the points such that $d(x)=d(y)$. By upper-continuity, Ψ in (5.7) attains its maximum over

$$A:=\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\times\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}$$

at a point (x,y). By classical arguments in viscosity solution theory, we get as $\varepsilon\to 0$

$$x,y\to x_0,\quad \varepsilon^{-1}\chi_\varepsilon(|x-y|)\to 0,\quad \varepsilon^{-1}|d(x)-d(y)|\to 0 \tag{5.9}$$

and $u(x)-v(y)-\tilde{\psi}(x,y)\to u(x_0)-v(x_0)-\phi(x_0)$.
We prove the following key lemma.

Lemma 5.2. *Under the above notations, we have*

(i) $\bar{x}\to x,\ \bar{y}\to y,\ u(\bar{x})\to u(x),\ v(\bar{y})\to v(y)$ *as* $\delta\to 0$;
(ii) $d(x)=d(y)$*;*

Proof. Note that (i) follows by classical argument in viscosity solution theory. We remark that the proof of (ii) is slightly different in case (b) and case (c). We argue by contradiction and we suppose that $d(x)\neq d(y)$. First we prove that the F-viscosity inequalities for u and v of Definition (2.1) hold. Suppose that $x\in\partial\Omega$, then $d(x)=0$ and $d(y)\neq 0$. We denote

$$\hat{p}=\frac{x-y}{|x-y|} \tag{5.10}$$

and we write

$$\frac{\partial\psi}{\partial n}(\cdot,y)(x)=\frac{1}{2}\frac{\partial\phi}{\partial n}((x+y)/2)+\varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p}\cdot n(x)+K\varepsilon^{-1}. \tag{5.11}$$

Note that

$$0 \le \chi'_\varepsilon(|x-y|) \le 1. \tag{5.12}$$

Note that by (5.9), we can suppose that x, y are close to the boundary, by taking ε small enough. By the Taylor's formula for the distance function, we have for ε small enough

$$n(x)\cdot(x-y) + \frac{1}{2}(x-y)^T D^2 d(x)(x-y) + o(|x-y|^2) = d(y) \ge 0$$

and then

$$n(x)\cdot(x-y) \ge -||D^2 d||_\infty |x-y|^2/2 + o(|x-y|^2). \tag{5.13}$$

By (5.10), (5.12), (5.13) and (5.9), we have

$$\varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p}\cdot n(x) \ge o_\varepsilon(1). \tag{5.14}$$

Note that, from (5.2), for ε small enough we have also

$$\frac{\partial \phi}{\partial n}((x+y)/2) > \frac{1}{2}\frac{\partial \phi}{\partial n}(x_0) > 0. \tag{5.15}$$

By (5.11), (5.15), (5.14) and since $K \ge 0$, we conclude for ε small enough

$$\frac{\partial \psi}{\partial n}(\cdot, y)(x) \ge \frac{1}{4}\frac{\partial \phi}{\partial n}(x_0) + o_\varepsilon(1) + K\varepsilon^{-1} > 0. \tag{5.16}$$

Then, since u is a viscosity subsolution and the function $u(\cdot) - v(y) - \psi(\cdot, y)$ has a local maximum at x, the F-viscosity inequality of Definition (2.1)(i) holds. A similar argument can be carried out for v. From now on, we treat separately *Case* (*b*) ($x_0 \in \Gamma_{\text{out}}$) and *Case* (*c*) ($x_0 \in \Gamma$).

Case (*b*) In this case $x_0 \in \Gamma_{\text{out}}$, where Γ_{out} is defined in (2.4). Suppose $d(x) > d(y)$. Then, for $1 > \xi' > 0$, by Definition (2.1)(i) and by (5.16), we have

$$u(x) - \mathcal{I}_{\xi'}[\psi(\cdot, y)](x) - \mathcal{I}^{\xi'}[u](x) + H(x, D[\psi(\cdot, y)](x)) \le 0. \tag{5.17}$$

Note that

$$D[\psi(\cdot, y)](x) = \varepsilon^{-1}(\chi'_\varepsilon(|x-y|)\hat{p} - Kn(x)) + q,$$

where $\hat{p}$ is defined in (5.10) and

$$q = D\phi((x+y)/2)/2. \tag{5.18}$$

We apply Lemma 6.9, (6.16) with $\hat{s} = x_0$, $p = \varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p} + q$ and $\lambda = \varepsilon^{-1}K$ and by the definition (5.5) of K, we get for ε small

$$\begin{aligned} H(x, D[\psi(\cdot, y)](x)) &\ge \varepsilon^{-1}\gamma K - C_2\left|\varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p} + q\right| - C_2 \\ &\ge \varepsilon^{-1}(\gamma K - C_2) - C \\ &\ge \varepsilon^{-1} - C, \end{aligned} \tag{5.19}$$

where by C, here and in the following, we denote any positive constant independent of ε which may change from line to line. To estimate the nonlocal terms we use the following lemma, which we prove in the Appendix.

Lemma 5.3. *Let $\mathcal{I}^\xi$,$\mathcal{I}_\xi$ be as in (2.7), (2.6) and assume the first of (M) and (J1). Under the above notations, for any $\xi > 0$, there exist a positive constants C_1 independents of ε such that*

(*i*) $-\mathcal{I}_\xi[\psi(\cdot, y)](x) - \mathcal{I}^\xi[u](x) \ge -\varepsilon^{-1}C_1\xi^{1-\sigma} - C_1\xi^{-\sigma}$;
(*ii*) $-\mathcal{I}_\xi[\psi(x, \cdot)](y) - \mathcal{I}^\xi[v](y) \le \varepsilon^{-1}C_1\xi^{1-\sigma} + C_1\xi^{-\sigma}$.

Then, by (5.19), by Lemma 5.3 (i) with $\xi' = \varepsilon$ and by the boundedness of u, we write (5.17) as follows

$$-\varepsilon^{-\sigma} + \varepsilon^{-1} \leq C,$$

and we reach a contradiction for ε small enough, since C is independent of ε and $\sigma < 1$.

Now suppose $d(x) < d(y)$. In this case we use the following F-viscosity inequality for the supersolution v for $2 > \xi' > 0$

$$v(y) - \mathcal{I}_{\xi'}[-\psi(x,\cdot)](y) - \mathcal{I}^{\xi'}[v](y) + H(y, -D[\psi(x,\cdot)](y)) \geq 0. \tag{5.20}$$

We have

$$D[-\psi(\cdot,y)](x) = \varepsilon^{-1}(\chi'_\varepsilon(|x-y|)\hat{p} + Kn(y)) - q,$$

where $\hat{p}$ is defined in (5.10) and q is defined in (5.18). Then , for ε small enough, we apply Lemma 6.9, (6.17) with $\hat{s} = x_0$, $p = \varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p} - q$ and $\lambda = \varepsilon^{-1}K$ and by (5.5) we get

$$\begin{aligned} H(y, -D[\psi(x,\cdot)](y)) &\leq -\varepsilon^{-1}\gamma K + C_2 \left|\varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p} - q\right| \\ &\leq -\varepsilon^{-1}(\gamma K + C_2) + C \\ &\leq -\varepsilon^{-1} - C. \end{aligned} \tag{5.21}$$

We proceed as in the previous case, we apply Lemma 5.3 (ii) with $\xi = \varepsilon$ and by (5.21) and the boundedness of v , we get

$$\varepsilon^{-\sigma} - \varepsilon^{-1} \geq C$$

and we reach a contradiction for ε small enough as above.

Case (c) In this case $x_0 \in \Gamma$, where Γ is defined in (2.5). If $d(x) > d(y)$ the proof is the same. If $d(x) < d(y)$ we write again equation (5.17) and since

$$D[\psi(\cdot,y)](x) = \varepsilon^{-1}(\chi'_\varepsilon(|x-y|)\hat{p} + Kn(x)) + q,$$

where $\hat{p}$ is defined in (5.10), we apply Lemma 6.9 (6.19) with $\hat{s} = x_0$, $p = \varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p} + q$ and $\lambda = \varepsilon^{-1}K$, for ε enough small, and we conclude as above. □

Step 2. *Writing the viscosity inequalities* By 5.9 and Lemma 5.2 (i), from now on we consider δ, ε small enough so that

$$\bar{x}, \bar{y}, x, y \in B_{C_j}(x_0) \cap \bar{\Omega}. \tag{5.22}$$

Now we prove that the F-viscosity inequalities for u and v hold. We take $\bar{x} \in \partial\Omega$ and we show that the boundary conditions do not hold, so the F-viscosity inequalities hold as in Definition (2.1). We proceed exactly as in Step 1, Lemma 5.2, so we omit the details. We recall that for all $\delta > 0$

$$0 \leq \chi'_\delta(|x-y|) \leq 1 \quad \text{ for all } x, y \in \bar{\Omega}, \tag{5.23}$$

and we note only that since $d(\bar{x}) = 0$, we have for ε, δ small enough

$$\frac{\partial\tilde{\phi}}{\partial n}(\cdot,\bar{y})(\bar{x}) \geq \frac{1}{4}\frac{\partial\phi}{\partial n}(x_0) + K\varepsilon^{-1}\chi'_\delta(d(\bar{y})) + o_{\delta,\varepsilon}(1) > 0,$$

where $o_{\delta,\varepsilon}(1)$ means that $\lim_{\delta\to 0} o_{\delta,\varepsilon}(1) = o_\varepsilon(1)$. Then for $1 > \xi' > 0$, we have

$$\begin{aligned} u(\bar{x}) - v(\bar{y}) &\leq H(\bar{y}, -D[\tilde{\phi}(\bar{x},\cdot)](\bar{y})) - H(\bar{x}, D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})) \\ &+ \mathcal{I}^{\xi'}[u](\bar{x}) - \mathcal{I}^{\xi'}[v](\bar{y}) + \mathcal{I}_{\xi'}[\tilde{\phi}(\cdot,y)](\bar{x}) - \mathcal{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot)](\bar{y}). \end{aligned} \tag{5.24}$$

Since $\tilde\phi \in C^1$, by (J1) and the first of (M), we have

$$\mathcal{I}_{\xi'}[\tilde\phi(\cdot,\bar y)](\bar x) \le C_j ||D\tilde\phi||_{L^\infty(\bar B(0,C_j\xi'))} \int_{\mathbb{R}^n} 1_{|z|\le\xi'}|z|d\mu_x(z) = o_{\xi'}(1). \tag{5.25}$$

where C_j is as in (J1) and $o_{\xi'}(1)$ is independent of δ. The same holds for $-\mathcal{I}_{\xi'}[-\tilde\phi(\bar x,\cdot)](\bar y)$. Note that

$$\left|D[\tilde\phi(\cdot,\bar y)](\bar x) - D[-\tilde\phi(\bar x,\cdot)](\bar y)\right| = \varepsilon^{-1}\left|K\chi_\delta'(|d(\bar x)-d(\bar y)|)\tilde p\,(n(\bar y)-n(\bar x))\right| + |D\phi((\bar x+\bar y)/2)|, \tag{5.26}$$

where

$$\tilde p = \frac{d(\bar x)-d(\bar y)}{|d(\bar x)-d(\bar y)|}. \tag{5.27}$$

For δ, ε small enough, we suppose that $\bar x, \bar y$ belong to the neighbourhood of the boundary where the distance is smooth. By (5.23) and the smoothness of the distance function we have

$$\begin{aligned} \left|D[\tilde\phi(\cdot,\bar y)](\bar x) - D[-\tilde\phi(\bar x,\cdot)](\bar y)\right| &\le \varepsilon^{-1}K|n(\bar y)-n(\bar x)| + |D\phi((\bar x+\bar y)/2)| \\ &\le \varepsilon^{-1}K|\bar x-\bar y| + |D\phi((\bar x+\bar y)/2)|. \end{aligned} \tag{5.28}$$

By the definition of H and (5.28), we have

$$H(\bar y, -D[\tilde\phi(\bar x,\cdot)](\bar y)) - H(\bar y, D[\tilde\phi(\cdot,\bar y)](\bar x)) \le B\left(|D\phi((\bar x+\bar y)/2)| + K\varepsilon^{-1}|\bar x-\bar y|\right), \tag{5.29}$$

where $B = \sup_{x\in\bar\Omega,\alpha\in\mathcal{A}} b(x,\alpha)$. Moreover by $(C), (L)$, we have

$$H(\bar y, D[\tilde\phi(\cdot,\bar y)](\bar x) - H(\bar x, D[\tilde\phi(\cdot,\bar y)](\bar x)) \le B|\bar x-\bar y||D[\tilde\phi(\cdot,\bar y)](\bar x)| + \omega_l(|\bar x-\bar y|)$$

and since

$$|D[\tilde\phi(\cdot,\bar y)](\bar x)| \le K\varepsilon^{-1} + \varepsilon^{-1}|\bar x-\bar y| + 2^{-1}||D\phi||_{L^\infty(B_{2C_j}(x_0))}, \tag{5.30}$$

we get

$$H(\bar y, D[\tilde\phi(\cdot,\bar y)](\bar x) - H(\bar x, D[\tilde\phi(\cdot,\bar y)](\bar x)) \le C\left(\varepsilon^{-1}|\bar x-\bar y| + |\bar x-\bar y|\right) + \omega_l(|\bar x-\bar y|), \tag{5.31}$$

where $C>0$ is a constant depending on B, K and $||D\phi||_{L^\infty(B_{2C_j}(x_0))}$. By coupling (5.29) and (5.31) and by (5.9) and (iii) of Lemma 5.2, we get

$$H(\bar y, -D[\tilde\phi(\bar x,\cdot)](\bar y)) - H(\bar x, D[\tilde\phi(\cdot,\bar y)](\bar x) \le B|D\phi((\bar x+\bar y)/2)| + o_{\delta,\varepsilon}(1), \tag{5.32}$$

where $o_{\delta,\varepsilon}(1)$ means $\lim_{\delta\to 0} o_{\delta,\varepsilon}(1) = o_\varepsilon(1)$. Plugging (5.32) and (5.25) into (5.24), we get

$$u(\bar x) - v(\bar y) \le B|D\phi((\bar x+\bar y)/2)| + \mathcal{I}^{\xi'}[u](\bar x) - \mathcal{I}^{\xi'}[v](\bar y) + o_{\delta,\varepsilon}(1) + o_{\xi'}(1). \tag{5.33}$$

Step 3. *Sending* $\delta \to 0$ We want to send first $\delta \to 0$ in (5.33) and we observe that the nonlocal terms are uniformly bounded in δ. Consider $\mathcal{I}^{\xi'}[u](\bar x)$, observing that the same argument works similarly for $\mathcal{I}^{\xi'}[v](\bar y)$. Note that by (5.22) and (J1), if $|z|<1$, then $\bar x + j(\bar x,z) \in B_{2C_j}(x_0)$. Since $(\bar x,\bar y)$ is a maximum point on $\bar B_{2C_j}(x_0)\cap\bar\Omega \times \bar B_{2C_j}(x_0)\cap\bar\Omega$ of Φ defined in (5.6), we have for δ,ε small

$$\begin{aligned} u(\bar x + j(\bar x,z)) - u(\bar x) &= u(\bar x + j(\bar x,z)) - v(\bar y) - (u(\bar x) - v(\bar y)) \\ &\le \tilde\phi(\bar x + j(\bar x,z),\bar y) - \tilde\phi(\bar x,\bar y). \end{aligned} \tag{5.34}$$

Note that χ_δ is Lipschitz with Lipschitz constant independent of δ thanks to (5.23). Then, by the definition of $\tilde{\phi}$, since $\chi_\varepsilon, \chi_\delta, \phi$ are Lipschitz and by (J1), we have

$$u(\bar{x}+j(\bar{x},z)) - u(\bar{x}) \le C\varepsilon^{-1}|z| + C|z|, \tag{5.35}$$

which, by the first of (M), gives the uniform boundedness in δ of $\mathcal{I}^{\xi'}[u](\bar{x})$ when $|z|<1$. When $|z|\ge 1$, the claim simply follows by the boundedness of u and the first of (M).
Then, we send $\delta \to 0$ in (5.33) and we apply Fatou's Lemma. By the semicontinuity and boundedness of u and v and Lemma 5.2 (i), we get

$$u(x) - v(y) \le B|D\phi((x+y)/2)| + \mathcal{I}^{\xi'}[u](x) - \mathcal{I}^{\xi'}[v](y) + o_\varepsilon(1) + o_{\xi'}(1). \tag{5.36}$$

Note that now, thanks to Lemma 5.2 (ii), we have that $d(x)=d(y)$.

Step 4. *Estimate of the nonlocal terms* We prove the following lemma.

Lemma 5.4. *Under the above notations, we have*

$$\mathcal{I}^{\xi'}[u](x) - \mathcal{I}^{\xi'}[v](y) \le C\varepsilon^{-1}|x-y| + \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1) + o_{\xi'}(1), \tag{5.37}$$

where $C>0$ is independent of all the parameters.

Remark 6. *In the proof of Lemma 5.4, we deeply rely on the assumption $\sigma \in (0,1)$.*

Proof. For simplicity of exposition, we first conclude the proof when the measure μ in the nonlocal terms has no dependence on x, i.e. $\mu_x \equiv \mu$. We refer to Remark 8 for details in the case of x-dependence. We write

$$\mathcal{I}^{\xi'}[u](x) - \mathcal{I}^{\xi'}[v](y) = \mathcal{I}^{\xi'}[J_x/J_y] + \mathcal{I}^{\xi'}[J_y/J_x] + \mathcal{T}^{\xi'}[J_x\cap J_y], \tag{5.38}$$

where $J_x = \{z\in\mathbb{R}^n \mid x + j(x,z) \in \bar{\Omega}\}$ and

$$\mathcal{I}^{\xi'}[J_x/J_y] = \int_{\substack{J_x/J_y,\\ |z|\ge\xi'}} u(x+j(x,z)) - u(x) d\mu(z), \tag{5.39}$$

$$\mathcal{I}^{\xi'}[J_y/J_x] = \int_{\substack{J_y/J_x,\\ |z|\ge\xi'}} v(y) - v(y+j(y,z)) d\mu(z),$$

$$\mathcal{T}^{\xi'}[J_x\cap J_y] = \int_{\substack{J_x\cap J_y,\\ |z|\ge\xi'}} [u(x+j(x,z)) - u(x) - (v(y+j(y,z)) - v(y))] d\mu(z). \tag{5.40}$$

Consider $\mathcal{T}^{\xi'}[J_x\cap J_y]$. Recall that $(\bar{x},\bar{y})$ satisfy for any $x',y' \in \bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\times\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}$

$$u(\bar{x}) - v(\bar{y}) - \tilde{\phi}(\bar{x},\bar{y}) \ge u(x') - v(y') - \tilde{\phi}(x',y'). \tag{5.41}$$

Letting $\delta\to 0$ in (5.41), by (i) of Lemma 5.2, the definition of $\tilde{\phi}$ and the semicontinuity of u, v, we get for any $x',y' \in \bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\times\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}$

$$\begin{aligned} u(x') - u(x) - (v(y') - v(y)) \le{}& \varepsilon^{-1}\chi_\varepsilon(|x'-y'|) - \varepsilon^{-1}\chi_\varepsilon(|x-y|) \\ &+ \phi((x'+y'/2) - \phi((x+y)/2). \end{aligned} \tag{5.42}$$

If $|z| < 1$, then by (5.22) and (J1), $x + j(x,z), y + j(y,z) \in B_{2C_j}(x_0)$. Then we write (5.42) for $x' = x + j(x,z), y' = y + j(y,z)$ and we have

$$\begin{aligned} u(x+j(x,z)) - u(x) \quad &- \quad (v(y+j(y,z)) - v(y)) \\ &\leq \quad \varepsilon^{-1}\chi_\varepsilon(|x+j(x,z)-y-j(y,z)|) - \varepsilon^{-1}\chi_\varepsilon(|x-y|) \\ &+ \quad \phi((x+j(x,z)+y+j(y,z))/2) - \phi((x+y)/2). \end{aligned}$$

Note that by the Lipschitz continuity of χ_ε, (J1) and (5.9), we have

$$\varepsilon^{-1}\chi_\varepsilon(|x+j(x,z)-y-j(y,z)|) - \varepsilon^{-1}\chi_\varepsilon(|x-y|) \leq D_j|z|\varepsilon^{-1}|x-y| = |z|o_\varepsilon(1), \tag{5.43}$$

where D_j is defined in (J1) and then

$$\begin{aligned} &u(x+j(x,z)) - u(x) - (v(y+j(y,z)) - v(y)) \\ &\qquad \leq \phi((x+j(x,z)+y+j(y,z))/2) - \phi((x+y)/2) + |z|o_\varepsilon(1). \end{aligned} \tag{5.44}$$

Then for $0 < \xi' < \xi < 1$, by (5.44) and the first of (M), we get

$$\mathcal{T}^{\xi'}[J_x \cap J_y] \leq \mathcal{P}_\xi - \mathcal{P}_{\xi'} + \mathcal{K}^\xi + o_\varepsilon(1), \tag{5.45}$$

where $o_\varepsilon(1)$ is independent of ξ' and

$$\mathcal{K}^\xi = \int_{\substack{J_x \cap J_y, \\ |z| \geq \xi}} u(x+j(x,z)) - u(x) - (v(y+j(y,z)) - v(y))d\mu(z), \tag{5.46}$$

$$\mathcal{P}_{\xi'} = \int_{\substack{J_x \cap J_y, \\ |z| \leq \xi'}} \phi((x+j(x,z)+y+j(y,z))/2) - \phi((x+y)/2)d\mu(z), \tag{5.47}$$

$$\mathcal{P}_{\xi} = \int_{\substack{J_x \cap J_y, \\ |z| \leq \xi}} \phi((x+j(x,z)+y+j(y,z))/2) - \phi((x+y)/2)d\mu(z). \tag{5.48}$$

Since ϕ is Lipschitz, by (J1) and the first of (M), we have

$$\mathcal{P}_{\xi'} = o_{\xi'}(1). \tag{5.49}$$

Now we consider the term $\mathcal{I}^{\xi'}[J_x/J_y]$, defined in (5.39), observing that the same argument works similarly for $\mathcal{I}^{\xi'}[J_y/J_x]$. Take $0 < \delta_0 < 1$ enough small (note that δ_0 will be defined more precisely at the end of the proof of Lemma 5.37). We split the domain of integration in $\{z : |z| \geq \delta_0\}$ and $\{z : \xi' \leq |z| \leq \delta_0\}$. We write

$$\mathcal{I}^{\xi'}[J_x/J_y] = \mathcal{I}^{\xi'}[B^c_{\delta_0}] + \mathcal{I}^{\xi'}[B_{\delta_0}], \tag{5.50}$$

where

$$\mathcal{I}^{\xi'}[B^c_{\delta_0}] = \int_{\substack{J_x/J_y, \\ |z| \geq \delta_0}} u(x+j(x,z)) - u(x)d\mu(z),$$

$$\mathcal{I}^{\xi'}[B_{\delta_0}] = \int_{\substack{J_x/J_y, \\ \xi' \leq |z| < \delta_0}} u(x+j(x,z)) - u(x)d\mu(z).$$

By the boundedness of u, we have

$$\mathcal{I}^{\xi'}[B^c_{\delta_0}] \leq 2C||u||_\infty \int_{|z|\geq\delta_0} 1_{J_x/J_y} d\mu(z),$$

and since $|J_x/J_y| \to 0$ as $\varepsilon \to 0$, by the first of (M) and the Dominated Convergence theorem, we get

$$\mathcal{I}^{\xi'}[B^c_{\delta_0}] \le o_\varepsilon(1), \tag{5.51}$$

where $o_\varepsilon(1)$ is independent of ξ'. For $\mathcal{I}^{\xi'}[B_{\delta_0}]$ we use again the maximum point inequality (5.42) with $x' = x + j(x,z), y' = y$ and since $\phi \in C^1$ and by (J1), the first of (M), we get

$$\mathcal{I}^{\xi'}[B_{\delta_0}] \le C\varepsilon^{-1} \int_{\substack{J_x/J_y, \\ \xi' \le |z| \le \delta_0}} \frac{dz}{|z|^{N+\sigma-1}}, \tag{5.52}$$

where we remark that $C > 0$ is independent of all the parameters. We couple (5.38), (5.45), (5.49), (5.50), (5.51) and (5.52) with (5.38) and we get
(5.53)

$$\mathcal{I}^{\xi'}[u](x) - \mathcal{I}^{\xi'}[v](y) \le C\varepsilon^{-1}\mathcal{I}^{\xi'}[J_x/J_y] + C\varepsilon^{-1}\mathcal{I}^{\xi'}[J_y/J_x] + \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1) + o_{\xi'}(1),$$

where for all $x, y \in \mathbb{R}^N$, we denote

$$\mathcal{I}^{\xi'}[J_x/J_y] := \int_{\substack{J_x/J_y, \\ \xi' \le |z| \le \delta_0}} \frac{dz}{|z|^{N+\sigma-1}}. \tag{5.54}$$

Now we estimate the term in (5.54) by Lemma 4.1. Let $r := r(x_0)$, where $r(x_0)$ is defined in assumption (O) for $\hat{s} = x_0$. Take $rC_j^{-1}/2 > \delta_0$. Note that, by (5.22), (x, y) satisfy (4.3) for $\hat{s} = x_0$ and $r = r(x_0)$. Then we apply Lemma 4.1 by taking $\{s_1, s_2\} = \{x, y\}, \hat{s} = x_0$ in order to estimate $\mathcal{I}^{\xi'}[J_x/J_y], \mathcal{I}^{\xi'}[J_y/J_x]$ defined in (5.54) and we get for all $\xi' > 0$

$$\mathcal{I}^{\xi'}[J_x/J_y] \le C|x-y|, \quad \mathcal{I}^{\xi'}[J_y/J_x] \le C|x-y|. \tag{5.55}$$

Then the claim of the lemma follows by plugging (5.55) into (5.53). □

Note that Lemma 4.1 is not necessary when dealing with domains with flat boundary. In the following remark we consider the case when Ω is the halfspace and we show how the estimate of the nonlocal terms can be carried out more easily without Lemma 4.1.

Remark 7. Take $\Omega := \{(x_1, \cdots, x_N = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$. For simplicity, we suppose that $j(x,z) = z$ if $x + z \in \bar{\Omega}$. Note that (i),(ii) of Lemma 5.2 read

$$\bar{x}_N - \bar{y}_N \to 0 \quad \text{as } \delta \to 0. \tag{5.56}$$

Consider the nonlocal terms in (5.24) and restrict ourselves to a subsequence such that $\bar{x}_N \ge \bar{y}_N$ (if $\bar{x}_N \le \bar{y}_N$ the argument is similar). Then we can write

$$\begin{aligned} \mathcal{I}^{\xi'}[u](\bar{x}) - \mathcal{I}^{\xi'}[v](\bar{y}) &= \int_{\substack{-\bar{x}_N \le z_N < -\bar{y}_N, \\ |z| \ge \xi'}} [u(\bar{x}+z) - u(\bar{x})] d\mu_{\bar{x}}(z) \\ &+ \int_{\substack{-\bar{y}_N \le z_N, \\ |z| \ge \xi'}} [u(\bar{x}+z) - v(\bar{y}+z) - (u(\bar{x}) - v(\bar{y}))] d\mu_{\bar{x}}(z) \\ &:= \mathcal{I}^{\xi'}[J_{\bar{x}}/J_{\bar{y}}] + \mathcal{T}^{\xi'}[J_{\bar{x}} \cap J_{\bar{y}}], \end{aligned}$$

where in the last line we used the same notations as in the previous step, see in particular (5.39), (5.40). The term $\mathcal{T}^{\xi'}[J_{\bar{x}} \cap J_{\bar{y}}]$ is treated exactly as in the non flat case (see the previous step). On the contrary, note that in this case the estimate of the term $\mathcal{I}^{\xi'}[J_{\bar{x}}/J_{\bar{y}}]$ is easier, since by (5.56) $|J_{\bar{x}}/J_{\bar{y}}| \to 0$ as $\delta \to 0$ and then by the Dominated Convergence Theorem, we have $\mathcal{I}^{\xi'}[J_{\bar{x}}/J_{\bar{y}}] \to 0$ as $\delta \to 0$.

Step 5. *-Sending the other parameters to their limits* We couple (5.36) with (5.37) and we get

$$u(x) - v(y) - B|D\phi((x+y)/2)| \leq C\varepsilon^{-1}|x-y| + \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1) + o_{\xi'}(1),$$

where $C > 0$ is independent of the parameters. Then, we first send $\xi' \to 0$ by the Dominated Convergence Theorem and we get

$$u(x) - v(y) - B|D\phi((x+y)/2)| \leq C\varepsilon^{-1}|x-y| + \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1), \tag{5.57}$$

where C is a constant independent of ξ. Moreover, since ϕ is C^1, by the first of (M), the Dominated Convergence Theorem and since $x, y \to x_0$ as $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon \to 0} \mathcal{P}_\xi \leq \mathcal{I}_\xi[\phi](x_0)$$

and by the boundedness and semicontinuity of u, v and applying Fatou's lemma for each $\xi > 0$ fixed, we have

$$\limsup_{\varepsilon \to 0} \mathcal{K}^\xi \leq \mathcal{I}^\xi[\omega(\cdot, t_0)](x_0),$$

and, by the previous estimates, we conclude by sending $\varepsilon \to 0$ in (5.57). □

Remark 8. We give some details of the analysis of the nonlocal terms in step 4 when the measure μ depends on x. We write (5.38) with

$$\mathcal{I}^{\xi'}[J_x/J_y] = \int_{\substack{J_x/J_y, \\ |z| \geq \xi'}} u(x + j(x,z)) - u(x) d\mu_x(z),$$

$$\mathcal{I}^{\xi'}[J_y/J_x] = \int_{\substack{J_y/J_x, \\ |z| \geq \xi'}} v(y) - v(y + j(y,z)) d\mu_y(z),$$

$$\mathcal{T}^{\xi'}[J_x \cap J_y] = \int_{\substack{J_x \cap J_y, \\ |z| \geq \xi'}} [u(x+j(x,z)) - u(x)] d\mu_x(z) - [(v(y+j(y,z)) - v(y))] d\mu_y(z).$$

For $\mathcal{I}^{\xi'}[J_x/J_y]$ and $\mathcal{I}^{\xi'}[J_y/J_x]$ we proceed as above (Step 4), noting that the x-dependence plays no role by the first of (M). For the $\mathcal{T}$-term, we write

$$\mathcal{T}^{\xi'}[J_x \cap J_y] = \mathcal{T}_1^{\xi'}[J_x \cap J_y] + \mathcal{T}_2^{\xi'}[J_x \cap J_y],$$

where

$$\mathcal{T}_1^{\xi'}[J_x \cap J_y] = \int_{\substack{J_x \cap J_y, \\ |z| \geq \xi'}} u(x + j(x,z)) - u(x) - (v(y + j(y,z)) - v(y)) d\mu_y(z),$$

$$\mathcal{T}_2^{\xi'}[J_x \cap J_y] = \int_{\substack{J_x \cap J_y, \\ |z| \geq \xi'}} [(u(x + j(x,z)) - u(x))](d\mu_x(z) - d\mu_y(z)).$$

For $\mathcal{T}_1^{\xi'}[J_x \cap J_y]$, we proceed as above (in Step 3, for $\mathcal{T}^{\xi'}[J_x \cap J_y]$ defined in (5.40)) and we prove (5.45). Now consider $\mathcal{T}_2^{\xi'}[J_x \cap J_y]$. Take $0 < \xi' < \xi < 1$ and denote

$$\mathcal{T}_2^{\xi'}[J_x \cap J_y] = \mathcal{T}_2^{\xi'}[B_\xi] + \mathcal{T}_2^{\xi'}[B_\xi^c],$$

where

$$\mathcal{T}_2^{\xi'}[B_\xi] = \int_{\substack{J_x \cap J_y, \\ \xi \geq |z| \geq \xi'}} [(u(x+j(x,z)) - u(x))](d\mu_x(z) - d\mu_y(z)),$$

$$\mathcal{T}_2^{\xi'}[B_\xi^c] = \int_{\substack{J_x \cap J_y, \\ |z| > \xi}} [(u(x+j(x,z)) - u(x))](d\mu_x(z) - d\mu_y(z)).$$

For $\mathcal{T}_2^{\xi'}[B_\xi]$ we use the maximum point inequality (5.42) and we write for $|z| \leq \xi$

$$u(x+j(x,z)) - u(x) \leq \varepsilon^{-1}\chi_\varepsilon(|x+j(x,z)-y|) - \varepsilon^{-1}\chi_\varepsilon(|x-y|) + \phi((x+j(x,z)+y)/2) - \phi(x+y)/2. \quad (5.58)$$

Then by the lipschitz continuity of χ_ε and ϕ, (J1), (M) and (5.9) we get

$$\mathcal{T}_2^{\xi'}[B_\xi] \leq C \int_{\substack{J_x \cap J_y, \\ \xi \geq |z| \geq \xi'}} (\varepsilon^{-1}|z| + |z|)(d\mu_x(z) - d\mu_y(z)) \leq o_\varepsilon(1), \quad (5.59)$$

where we observe $o_\varepsilon(1)$ is independent of ξ' and from now on may change from line to line in the following. For $\mathcal{T}_2^{\xi'}[B_\xi^c]$, by the boundedness of u, (M), (5.9), we get

$$\mathcal{T}_2^{\xi'}[B_\xi^c] \leq 2||u||_\infty \int_{\substack{J_x \cap J_y, \\ |z| > \xi}} (d\mu_x(z) - d\mu_y(z)) \leq o_\varepsilon(1). \quad (5.60)$$

Then, by (5.59) and (5.60), we get

$$\mathcal{T}_2^{\xi'}[J_x \cap J_y] \leq o_\varepsilon(1), \quad (5.61)$$

where $o_\varepsilon(1)$ is independent of ξ'. From now on the proof is the same as above.

Remark 9. We give the details of the proof of Lemma 5.1 in case (a), when $x_0 \in \Gamma_{\text{in}}$ is a strict maximum point of $\omega - \phi = u - v - \phi$, for $\phi \in C^1(\mathbb{R}^N)$.

The strategy of the proof relies on the existence of a blow-up supersolution exploding on the boundary, which allows us to keep the maximum points away from the boundary. The existence of such a supersolution is stated in the following lemma, whose proof is given in the Appendix.

Lemma 5.5. *For any $\bar{x} \in \Gamma_{in}$, there exists $r = r(\bar{x}) > 0$ and a positive function $U_r \in C^2(B_r(\bar{x}) \cap \Omega)$ satisfying for any ξ small enough (with respect to r, that is, $\xi < C_j^{-1}\frac{r}{2}$)*

(i) $-b(x,\alpha) \cdot DU_r - \mathcal{I}_\xi[U_r](x) \geq 0$ *in* $B_{\frac{r}{2}}(\bar{x}) \cap \Omega, \quad \forall \alpha \in \mathcal{A}$;

(ii) $U_r(x) \geq \frac{1}{\omega_r(d(x))}$ *in $B_r(\bar{x}) \cap \Omega$, for some function ω_r which is nonnegative, continuous, stricly increasing in a neighbourhood of* 0 *and satisfies* $\omega_r(0) = 0$.

Proof of case (a). Let $r = r(x_0)$ be defined in Lemma 5.5 for $\bar{x} = x_0$. We localize the argument in a ball of radius r around x_0 and we use the existence of the blow-up function U_r defined in Lemma 5.5 for $\bar{x} = x_0$. Let $\varepsilon > 0$. We double the variable and we consider (x,y) maximum point on $\bar{B}_{\frac{r}{2}}(x_0) \cap \bar{\Omega} \times \bar{B}_{\frac{r}{2}}(x_0) \cap \bar{\Omega}$ of the function $\Phi(x,y) = u(x) - v(y) - \tilde{\phi}(x,y)$, where

$$\tilde{\phi}(x,y) = \phi\left(\frac{x+y}{2}\right) + \frac{|x-y|^2}{\varepsilon^2} + k[U_r(x) + U_r(y)].$$

Note that, by (ii) of Lemma 5.5, we have that $(x, y) \in \bar{B}_{\frac{r}{2}}(x_0) \cap \Omega \times \bar{B}_{\frac{r}{2}}(x_0) \cap \Omega$; moreover, again by (ii) of Lemma 5.5, we have for k small enough

$$d(x), d(y) \geq \omega_r^{-1}\left(\frac{k}{2L}\right) =: \bar{\delta}, \tag{5.62}$$

where $L = ||u||_{L^\infty(\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega})} + ||v||_{L^\infty(\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega})} + ||\phi||_{L^\infty(\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega})} + 1$. Note that the existence of the blow-up function plays its mayor role here to get (5.62). This estimate tells us, roughly speaking, that the maximum points are away from the boundary. For fixed k, a standard argument shows that

$$\frac{|x-y|^2}{\varepsilon^2} \to 0 \text{ as } \varepsilon \to 0. \tag{5.63}$$

By the previous estimate on x, y and extracting subsequences if necessary, we can assume, without loss of generality, that as $\varepsilon, k \to 0$

$$x, y \to x_0, \quad u(x) - v(y) - \tilde{\phi}(x, y) \to u(x_0) - v(x_0) - \phi(x_0). \tag{5.64}$$

Let C_j be as in (J1) and $C_j^{-1}\frac{r}{4} > \xi' > 0$. Thanks to (5.64), we can take ε, k small enough so that $x, y \in B_{\frac{r}{4}}(x_0) \cap \Omega$. We proceed as in Step 2 in the above proof, we write the viscosity inequalities (5.24) and, using that $\tilde{\phi} \in C^1$, (J1) and the first of (M), we get

$$\begin{aligned} u(x) - v(y) &\leq H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot,y)](x)) \\ &+ \mathcal{I}^{\xi'}[u](x) - \mathcal{I}^{\xi'}[v](y) + o_{\xi'}(1), \end{aligned} \tag{5.65}$$

where $o_{\xi'}(1)$ is independent of δ. First we analyse the term $\mathcal{I}^{\xi'}[u](x) - \mathcal{I}^{\xi'}[v](y)$. For simplicity of exposition, we conclude the proof in the case the measure μ in the nonlocal terms has no dependence on x, i.e. $\mu_x \equiv \mu$. The result can be easily extended in the case of x-dependence analogously as already shown in Remark 8 for case (b) and (c). We use the same notation of Lemma 5.1, see (5.39), (5.40) and we write

$$\mathcal{I}^{\xi'}[u](x) - \mathcal{I}^{\xi'}[v](y) = \mathcal{I}^{\xi'}[J_x/J_y] + \mathcal{I}^{\xi'}[J_y/J_x] + \mathcal{T}^{\xi'}[J_x \cap J_y]. \tag{5.66}$$

As in the proof of Lemma 5.1, the term $\mathcal{T}^{\xi'}[J_x \cap J_y]$ can be estimated as follows for $0 < \xi' < \xi < C_j^{-1}\frac{r}{4}$

$$\begin{aligned} \mathcal{T}^{\xi'}[J_x \cap J_y] &\leq k\mathcal{I}_\xi[U_r](x) + k\mathcal{I}_\xi[U_r](y) - k\mathcal{I}_{\xi'}[U_r](x) - k\mathcal{I}_{\xi'}[U_r](y) \\ &+ \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1) + o_{\xi'}(1), \end{aligned}$$

where we use the notations (5.46), (5.48) of Lemma 5.1. Note that $o_\varepsilon(1)$ is independent of ξ'. Since U_r is Lipschitz, by (J1) and the first of (M), we have $\mathcal{I}_{\xi'}[U_r](x), \mathcal{I}_{\xi'}[U_r](y) \leq o_{\xi'}(1)$, and then

$$\mathcal{T}^{\xi'}[J_x \cap J_y] \leq k\mathcal{I}_\xi[U_r](x) + k\mathcal{I}_\xi[U_r](y) + \mathcal{P}_\xi + \mathcal{K}^\xi + o_{\xi'}(1) + o_\varepsilon(1), \tag{5.67}$$

where $o_\varepsilon(1)$ is independent of ξ'. Now we estimate the terms $\mathcal{I}^{\xi'}[J_x/J_y]$ and $\mathcal{I}^{\xi'}[J_y/J_x]$. Thanks to (5.62), in this case the estimate is easier than in the cases (b) and (c) treated above. Take for example $\mathcal{I}^{\xi'}[J_x/J_y]$ (the argument being analogous for $\mathcal{I}^{\xi'}[J_y/J_x]$) and note that by (5.62) the integral is independent of ξ' as soon as $\xi' < \bar{\delta}$ where $\bar{\delta}$ is defined in (5.62). Then by the boundedness of u, we have

$$\mathcal{I}^{\xi'}[J_x/J_y] \leq 2||u||_\infty \int_{|z|\geq\bar{\delta}} 1_{J_x/J_y} d\mu(z)$$

and since $|J_x/J_y| \to 0$ as $\varepsilon \to 0$, by the first of (M), the Dominated Convergence theorem, we get

$$\mathcal{I}^{\xi'}[J_x/J_y] \le o_\varepsilon(1), \tag{5.68}$$

where $o_\varepsilon(1)$ is independent of ξ'. Then plugging (5.68) and (5.67) into (5.66) and then coupling it with (5.65), we get for $C_j^{-1}\frac{r}{4} > \xi > \xi' > 0$

$$\begin{aligned} u(x) - v(y) \quad &\le \quad H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot, y)](x) \\ &+ \quad k\mathcal{I}_\xi[U_r](x) + k\mathcal{I}_\xi[U_r](y) + \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1) + o_{\xi'}(1), \end{aligned} \tag{5.69}$$

where $o_\varepsilon(1)$ is independent of ξ'. Now, by (i) of Lemma 5.5, we estimate the integrals terms of the left hand side of (5.69) together with the first order terms involving U_r in $H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot, y)](x)$ and we get

$$\begin{aligned} H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot, y)](x) + k\mathcal{I}_\xi[U_r](x) + k\mathcal{I}_\xi[U_r](y) \\ \le B|D\phi((x+y)/2)| + o_\varepsilon(1). \end{aligned} \tag{5.70}$$

Then, plugging (5.70) into (5.69), we get

$$u(x) - v(y) \le B|D\phi((x+y)/2)| + \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1) + o_{\xi'}(1), \tag{5.71}$$

where $o_\varepsilon(1)$ is independent of ξ'. The rest of the proof is the same as in the previous cases, by sending first $\xi' \to 0$, then $\varepsilon \to 0$. For the details we refer to the end of the proof of Lemma 5.1. □

Now we prove Theorem 3.1 for H of Bellman type.

Proof of Theorem 3.1. By contradiction, we suppose that $M = \sup_\Omega\{u - v\} > 0$. Denote $\omega(x) = u(x) - v(x)$ and for $\nu > 0$, consider $\Phi(x) = \omega(x) - \psi(R^{-1}|x|) + \nu d(x)$, where ψ is a smooth function such that

$$\psi(s) = \begin{cases} 0 & \text{for } 0 \le s < \frac{1}{2}, \\ \text{increasing} & \text{for } \frac{1}{2} \le s < 1, \\ ||u||_\infty + ||v||_\infty + 1 & \text{for } s \ge 1. \end{cases} \tag{5.72}$$

and d is the signed distance from the boundary (see Remark 1). Note that $\sup \Phi \to M$ as $R \to \infty$ and $\nu \to 0$. Since $\Phi \le -1/2$ for $|x|$ large and ν small enough and $M > 0$, the function Φ achieves its positive maximum $\sup \Phi > \frac{M}{2}$ at a point x for R big and ν small enough. We give the details in the case where all maximum points x are located on the boundary. We have

$$\omega(x) = M + o_{R,\nu}(1), \tag{5.73}$$

where with $o_{R,\nu}(1)$ we mean that $o_{R,\nu}(1) \to 0$ if $R \to \infty, \nu \to 0$.

We use $\phi(\cdot) := \psi(R^{-1}|\cdot|) - \nu d(\cdot)$ as a test function at x. Note that, if $x \in \partial\Omega$ and for $\nu > R^{-1}||\psi'||_{L^\infty}$, we have $\frac{\partial \phi}{\partial n} \ge -R^{-1}||\psi'||_{L^\infty} + \nu > 0$. Then, by Lemma 5.1, we get

$$\omega(x) - \mathcal{I}[\phi](x) - B(|D\phi(x)|) \le 0 \quad \text{in} \quad \Omega.$$

By Lemma 6.10 (see the Appendix), $\mathcal{I}[\phi](\cdot), |D\phi(\cdot)| \le o_{\nu,R}(1)$ and by (5.73), we get

$$M + o_{\nu,R}(1) \le o_{\nu,R}(1),$$

and by letting $R \to \infty$, $\nu \to 0$, we get a contradiction since $M > 0$. □

5.2. **Coercive Hamiltonians.** We proceed analogously as for Hamiltonian of Bellman type and we prove Lemma 5.6. Once proved Lemma 5.6, the proof of Theorem 3.1 for H coercive follows by standard arguments as already showed for Hamiltonian of Bellman type. We sketch first the proof of Theorem 3.1 and then we prove Lemma 5.6.

Proof of Theorem 3.1. We just observe that we proceed again by contradiction, supposing that $M = \sup_{\Omega}\{u - v\} > 0$. We fix $0 < \mu < 1$ and define for $x \in \Omega$ $\omega_\mu(x) = \mu u(x) - v(x)$. We proceed as in the Bellman case and we use Lemma 5.6 to get

$$M + o_{\nu,R,\mu}(1) - o_{\nu,R}(1) \leq CA(1-\mu).$$

Then, by letting $R \to \infty$, $\nu \to 0$ and finally $\mu \to 1$, we get a contradiction since $M > 0$ and we conclude the proof. □

Lemma 5.6. *Let $\mathcal{I}$ as in* (1.2) *and assume* (M), $(J0), (J1)$. *Let H be a coercive Hamiltonian satisfying (H1), (Ha) or (H1), (Hb), (Hc) and let u, v be respectively bounded sub and supersolutions to* (1.1). *Let $\mu \in (0,1)$ if H is superlinearly coercive, $\mu = 1$ is H is sublinearly coercive. Then the function $\omega(x) := \mu u(x) - v(x)$ satisfies, in the viscosity sense, the equation*

$$(5.74) \qquad \begin{cases} \omega - \mathcal{I}[\omega](x) - C_{m,\mu}|D\omega|^m \leq A(1-\mu) & \text{in } \Omega, \\ \frac{\partial \omega}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A, C_{m,\mu}$ are positive constants which depend on the data. Precisely, if $\mu = 1$, $C_{m,\mu} = \tilde{C}$ where $\tilde{C}$ is defined in (Hb) and, if $\mu \in (0,1)$, $C_{m,\mu} = \bar{C}^{1-m}C^m 2^{m-1}m^{-m}(1-\mu)^{1-m}$

Proof. We give the details when H has superlinear form i.e. when $m > 1 > \sigma$, since the proof in the sublinear case is similar with easier computations. Since the proof is similar to that of Lemma 5.1, we focus only on the main differences.

We start by noting that if u is a subsolution of (1.1), then $\bar{u} = \mu u$ is a viscosity subsolution to

$$(5.75) \qquad \bar{u} - \mathcal{I}[\bar{u}](x) + \mu H(x, \mu^{-1}D\bar{u}) \leq 0, \quad \text{in } \Omega.$$

Let $x_0 \in \bar{\Omega}$ and ϕ a smooth function such that $\omega - \phi$ has a strict maximum point. We suppose that $x_0 \in \partial\Omega$, since the other case being similar and even simpler.

Step 1. *-Localising on equidistant points (i.e. $d(x) = d(y)$)* We double the variable and consider the function

$$(5.76) \qquad \Phi(x,y) := \bar{u}(x) - v(y) - \tilde{\phi}(x,y),$$

where $\tilde{\phi}$ is as in (5.3) with $K = 2$. Let $(\bar{x}, \bar{y})$ be a maximum point of Φ over the set $A := \bar{B}_{2C_j}(x_0) \cap \bar{\Omega} \times \bar{B}_{2C_j}(x_0) \cap \bar{\Omega}$. Let $\Psi = \bar{u}(x) - v(y) - \psi(x,y)$, where ψ is defined as in (5.8) with $K = 2$ and let (x, y) be a point of maximum of Ψ over A. By classical argument, we have as $\varepsilon \to 0$

$$(5.77) \qquad x, y \to x_0, \quad \varepsilon^{-1}\chi_\varepsilon(|x-y|) \to 0, \quad \varepsilon^{-1}|d(x) - d(y)| \to 0.$$

We have the following lemma.

Lemma 5.7. *Under the above notations, we have*

(i) $\bar{x} \to x$, $\bar{y} \to y$, $u(\bar{x}) \to u(x)$, $v(\bar{y}) \to v(y)$ *as* $\delta \to 0$,

(ii) $d(x) = d(y)$.

Proof. The proof is very similar to that of Lemma 5.2. We give a sketch of the proof of (ii). We suppose that $\frac{\partial\phi}{\partial n}(x_0) > 0$, then for ε small enough $\frac{\partial\psi}{\partial n}((x+y)/2) > \frac{1}{2}\frac{\partial\phi}{\partial n}(x_0) > 0$. We assume that $d(x) > d(y)$. Then for $0 < \xi' < 2$ and $0 < \mu < 1$, we have

$$\bar u(x) - \mathcal{I}^{\xi'}[\bar u](x) - \mathcal{I}_{\xi'}[\psi(\cdot,y)](x) + \mu H\left(x, \mu^{-1}D[\psi(\cdot,y)](x)\right) \le 0. \tag{5.78}$$

Note that

$$|D[\psi(\cdot,y)](x)| \ge \varepsilon^{-1} - C, \tag{5.79}$$

where $C > 0$ is a constant independent of ε. For the integral terms in (5.78) we proceed as in Lemma 5.1 by using Lemma 5.3. For the Hamiltonian terms we use assumption (H1) toghether with (5.79) and we get

$$\varepsilon^{-m}\left(-\varepsilon^{m-\sigma} + c_0\mu^{1-m}\right) \le C,$$

which is a contradiction for ε small, since $\sigma < m$. □

Step 2. -*Writing the viscosity inequalities* We write the viscosity inequalities for u and v for any $0 < \xi' < 1$

$$\begin{aligned}\bar u(\bar x) - v(\bar y) \le \mu H(\bar y, \mu^{-1}D_x\tilde\phi(\cdot,\bar y)(\bar x)) \quad &- \quad H(\bar x, -D_y\tilde\phi(\bar x,\cdot)(\bar y)) \\ &+ \quad \mathcal{I}^{\xi'}[\bar u](\bar x) - \mathcal{I}^{\xi'}[v](\bar y) + o_{\xi'}(1),\end{aligned} \tag{5.80}$$

where we estimated the $\mathcal{I}_{\xi'}[\phi]$ terms by $o_{\xi'}(1)$ (independent of δ) as in (5.25). Denote

$$\mathcal{H} = \mu H(\bar y, \mu^{-1}D[\tilde\phi(\cdot,\bar y)](\bar x)) - H(\bar x, -D[\tilde\phi(\bar x,\cdot)](\bar y)). \tag{5.81}$$

We recall that

$$D[\tilde\phi(\cdot,\bar y)](\bar x) = \varepsilon^{-1}[\chi'_\varepsilon(|\bar x-\bar y|)\hat p - 2\chi'_\delta(|d(\bar x)-d(\bar y)|)\tilde p n(\bar x)] + D\phi((x+y)/2)/2, \tag{5.82}$$

$$D[-\tilde\phi(\bar x,\cdot)](\bar y) = \varepsilon^{-1}[\chi'_\varepsilon(|\bar x-\bar y|)\hat p - 2\chi'_\delta(|d(\bar x)-d(\bar y)|)\tilde p n(\bar y)] - D\phi((x+y)/2)/2, \tag{5.83}$$

where

$$\hat p = \frac{\bar x - \bar y}{|\bar x - \bar y|}, \quad \tilde p = \frac{d(\bar x) - d(\bar y)}{|d(\bar x) - d(\bar y)|}. \tag{5.84}$$

By the smoothness of the distance function

$$\left| D[\tilde\phi(\cdot,\bar y)](\bar x) - D[-\tilde\phi(\bar x,\cdot)](\bar y) \right| \le 4\varepsilon^{-1}|\bar x - \bar y| + |D\phi((\bar x+\bar y)/2)|, \tag{5.85}$$

and

$$|D[\tilde\phi(\cdot,\bar y)](\bar x)|, |D[-\tilde\phi(\bar x,\cdot)](\bar y)| \le q, \quad q = 3\varepsilon^{-1} + 2^{-1}||D\phi||_{L^\infty(B_r(x_0))}.$$

Thanks to (Hb) and (Hc), we get for ε small

$$\begin{aligned}\mathcal{H} \ge (1-\mu)(m-1)\bar C q^m - A(1-\mu) - \omega_1(|\bar x - \bar y|)(1+q^m) \\ - C|D\phi((\bar x+\bar y)/2)|q^{m-1} - 4C\varepsilon^{-1}|\bar x - \bar y|q^m.\end{aligned}$$

Note that, by (5.77) and (iii) of Lemma 5.7, we can take $\epsilon = \epsilon(\mu), \delta = \delta(\mu)$ small enough so that

$$(1-\mu)(m-1)\bar C - \omega_1(|\bar x - \bar y|) - 4C\varepsilon^{-1}|\bar x - \bar y| > 0$$

and we can write

$$\begin{aligned}\mathcal{H} &\geq (1-\mu)(m-1)\bar{C}q^m/2 - C|D\phi((\bar{x}+\bar{y})/2)|q^{m-1} - A(1-\mu) - o_{\delta,\varepsilon}(1)\\ &\geq \inf_{q\geq 0}\{(1-\mu)(m-1)\bar{C}q^m/2 - C(|D\phi((\bar{x}+\bar{y})/2)|)q^{m-1}\} - A(1-\mu) - o_{\delta,\varepsilon}(1),\end{aligned}$$

where $o_{\delta,\varepsilon}(1)$ means that $\lim_{\delta\to 0} o_{\delta,\varepsilon}(1) = o_\varepsilon(1)$. Note that the infimum in the previous expression is attained and therefore

$$\mathcal{H} \geq -C_{m,\mu}|D\phi((\bar{x}+\bar{y})/2)|^m - A(1-\mu) - o_{\delta,\varepsilon}(1), \tag{5.86}$$

where $C_{m,\mu} = \bar{C}^{1-m}C^m 2^{m-1}m^{-m}(1-\mu)^{1-m}$. Then we couple (5.86), (5.81) and (5.80), we let $\delta \to 0$ and we get

$$\begin{aligned}\bar{u}(x) - v(y) \quad &- \quad C_m(1-\mu)^{1-m}|D\phi((x+y)/2)|^m - A(1-\mu) - o_\varepsilon(1)\\ &\leq \quad \mathcal{I}^{\xi'}[\bar{u}](x) - \mathcal{I}^{\xi'}[v](y) + o_{\xi'}(1).\end{aligned} \tag{5.87}$$

Step 3. *Estimate of the nonlocal terms* In order to estimate the nonlocal terms we use the following lemma. We omit the proof since it is exactly the same as that of Lemma 5.4. We remark that in the proof we deeply rely on the assumption $\sigma \in (0,1)$.

Lemma 5.8. *Under the above notations, we have*

$$\mathcal{I}^{\xi'}[\bar{u}](x) - \mathcal{I}^{\xi'}[v](y) \leq C\varepsilon^{-1}|x-y| + \mathcal{P}_\xi + \mathcal{K}^\xi + o_\varepsilon(1) + o_{\xi'}(1), \tag{5.88}$$

where $C > 0$ is independent of all the parameters.

Then the rest of the proof easily follows as in Lemma 5.1, step 5. □

6. Applications to evolutive problems: existence, uniqueness and asymptotic behavior

In this section we present some applications of our results proved in the previous sections for the stationary case to the evolutive setting and we consider the associated Cauchy problem

$$\begin{cases}\partial_t u - \mathcal{I}[u(\cdot,t)](x) + H(x,t,u,Du) = 0 & \text{in } \Omega\times(0,+\infty),\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x) & \text{in } \bar{\Omega},\end{cases} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^N$ satisfies assumption (O), $u_0 \in C(\Omega)$, $\mathcal{I}[u]$ is an integro-partial differential operator of censored type and of order striclty less than 1, defined as

$$\mathcal{I}[u(\cdot,t)](x) = \lim_{\delta\to 0^+}\int_{\substack{|z|>\delta,\\ x+j(x,z)\in\bar{\Omega}}} [u(x+j(x,z),t) - u(x,t)]d\mu_x(z), \tag{6.2}$$

where μ_x is a singular nonnegative Radon measure satisfying (M), and $j(x,z)$ is a jump function satisfying (J0), (J1) (see Section 2) . The main example are measures μ_x with density $\frac{d\mu_x}{dz} = g(x,z)|z|^{-(N+\sigma)}$ with $\sigma < 1$ and g a nonnegative bounded function Lipschitz in x uniformly with respect to z. Note that the operator (6.2) is the natural extension to the evolutive case of the nonlocal operator considered in the stationary case and defined in (1.2). Moreover $H : \bar{\Omega}\times[0,+\infty)\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function whose growth in the gradient makes it the leading order term in the equation, and can be coercive or of Bellman type. We refer to the following section for the precise assumptions on the Hamiltonian. The well-posedness of problems as (6.1) follows from analogous arguments used for the stationary problem with some standard adaptations. We refer to Theorem 6.1 and Theorem 6.2 for further details and proofs of uniqueness and existence.

We study two different kind of asymptotic behaviour of the solutions of (6.1). First we consider an Hamiltonian either of Bellman type either coercive under assumptions ensuring uniqueness of the solution of the associated stationary problem. We prove, by classical methods based on the weak-relaxed semilimits, the convergence as $t \to +\infty$ of the solution of (6.1) to the unique solution of the associated stationary problem. On the other hand, when the associated stationary problem has not unique solution, we consider an Hamiltonian with superfractional coercive growth and we study the so-called *ergodic large time behaviour*, proving that the solution of (6.1) approaches a solution of the so-called *ergodic problem* as $t \to +\infty$. We follow the methods of [11], which rely on the Hölder regularity in Ω for subsolutions of the associated ergodic problem. We refer to subsection 6.4 for more details.

6.1. **Assumptions.** We are going to consider the finite time horizon problem associated to (6.1)

$$(6.3) \qquad \begin{cases} \partial_t u - \mathcal{I}[u(\cdot,t)](x) + H(x,t,u,Du) = 0 & \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(t) & \text{in } \bar{\Omega}, \end{cases}$$

The definition of viscosity solutions to (6.3) (and then to (6.1)) is the natural extension of Definition 2.1 to the corresponding Cauchy problem.

We assume the following condition.

(H') For all $R > 0$, there exists $\gamma_R \geq 0$ such that for all $x \in \bar{\Omega}, u, v \in \mathbb{R}, |u|, |v| \leq R, 0 \leq t \leq R$ and $p \in \mathbb{R}^n$, we have

$$H(x,t,u,p) - H(x,t,v,p) \geq \gamma_R(u-v).$$

Remark 10. Note that in the comparison principle (Theorem 6.1), assumption (H') is assumed mainly for simplicity of exposition and can be relaxed assuming only $\gamma_R \geq 1$. Indeed if $\gamma_R < 1$ we perform the change $\tilde{u} = ue^{-(\gamma_R - 1)t}$ (analogously for the supersolution) and prove Theorem 6.1 for $\tilde{u}$ and $\tilde{v}$.

As it is classical in viscosity solutions theory, the comparison principle allows the application of Perron's method to conclude the existence. To this end, we introduce the following assumption, which will allows us to build sub and supersolutions:

(E') For all $T > 0, R > 0$ there exists a constant $H_R > 0$ such that

$$||H(x,t,r,p)||_\infty \leq H_R \ \forall x \in \Omega, t \in [0,T], r, p \in \mathbb{R}, |r|, |p| \leq R.$$

We consider Hamiltonian either of Bellman type either coercive.

We say that the Hamiltonian H is of Bellman type if for $t \in [0,+\infty), x \in \bar{\Omega}, p \in \mathbb{R}^N, H(x,t,r,p)$ can be written as

$$(6.4) \qquad H(x,t,r,p) = \sup_{\alpha \in \mathcal{A}} \{\lambda(x,t,\alpha)r - b(x,t,\alpha)\cdot p - l(x,t,\alpha)\},$$

where $b, \lambda : \bar{\Omega} \times [0,+\infty) \times \mathcal{A} \to \mathbb{R}^N$ and $l : \bar{\Omega} \times [0,+\infty) \times \mathcal{A} \to \mathbb{R}$, are continuous and bounded functions and satisfy the following properties.

(C') *Uniform continuity of l and λ*:
There exist modulus of continuity ω_l, ω_λ such that such that $\forall \alpha \in \mathcal{A}, \forall x, y \in \bar{\Omega}, t, s \in [0, +\infty)$

$$|l(x,t,\alpha) - l(y,s,\alpha)| \leq \omega_l(|x-y| + |t-s|);$$

$$|\lambda(x,t,\alpha) - \lambda(y,s,\alpha)| \leq \omega_\lambda(|x-y| + |t-s|);$$

(L') *Uniform Lipschitz continuity of the drift b*: There exists $C > 0$ such that $\forall \alpha \in \mathcal{A}\, \forall (x,s), (y,t) \in \bar{\Omega} \times [0+\infty)$

$$|b(x,s,\alpha) - b(y,t,\alpha)| \leq C(|x-y| + |s-t|).$$

Remark 11. Note that assumption (L') may seem unusual since it requires also the uniform Lipschitz continuity of the drift b in the time variable. This is due to the fact that in the proof of the comparison principle (Theorem 6.1) we need to double also the time variable in the test function, which as a consequence will have the same dependence on x and t, which we estimate by assumption (L'). We refer to Remark 12.

We assume that the components of the boundary are of three different types whose definition is analogous to that of $\Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma$ defined in (2.3), (2.4), (2.5) in the stationary case, with the only difference that now they are part of the parabolic boundary. For simplicity of exposition, we do not repeat the definition. For the sake of simplicity and for the rest of the paper, we adopt the following notation.

(B') Assumption (B) where $\Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma$ are considered as parts of the parabolic boundary.

In the case of coercive Hamiltonians, we restrict the time dependence of H by the assumption:

(H0') There exists $H_0 : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ continuous and $f : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ unformly continuous and bounded such that for all $x \in \bar{\Omega}, r \in \mathbb{R}, p \in \mathbb{R}^N$

$$H(x,t,r,p) = H_0(x,r,p) - f(x,t).$$

We assume that the Hamiltonian satisfies in x uniformly in t and uniformly on compact sets in r the assumptions made in the stationary case, namely, (H1), (Ha) (sublinear coercivity) or (H1), (Hb), (Hc) (superlinear coercivity). For the sake of brevity and simplicity of exposition, we omit to repeat them and for the rest of the paper we adopt the following notations.

(He') The Hamiltonian satisfies (H1), (Ha) in x uniformly in t and uniformly on compact sets in r.

(He") The Hamiltonian satisfies (H1), (Hb), (Hc) in x uniformly in t and uniformly on compact sets in r.

6.2. **Existence and uniqueness.** In the following theorem we state the comparison principle for the problem (6.1).

Theorem 6.1. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying (O), $u_0 \in C(\bar{\Omega})$. Assume (M), (J0), (J1). Let H be an Hamiltonian either of Bellman type as in* (6.4) *satisfying (C'), (L'), (B') or a coercive Hamiltonian satisfying (H0') and either*

(He') or (He"). Assume also (H'). Let $u, v \in L^\infty(\bar\Omega \times [0,T])$ for all $T > 0$ be respectively a usc sub and lsc supersolution of (6.1)*. Then*

$$u \le v \quad \text{in } \bar\Omega \times [0,+\infty).$$

Remark 12. We omit the proof of Theorem 6.1, since it follows closely the arguments presented in Lemma 5.1 and Lemma 5.6 for the stationary case adapted by standard arguments to the evolution setting. We only remark that, since the strategy of the proof of Lemma 5.1 relies on an asymmetric use of the viscosity inequalities satisfied by the sub- and supersolution (see in particular in Step 1), the standard approach by the Ishi's Lemma for evolutive equation is not applicable. Then, we need to double also the time variable in the test function, which as a consequence will have the same dependence on x and t. Because of this, in particular in order to estimate the Hamiltonian terms in the viscosity inequalities satisfied by u and v (in the Bellman case), we need the uniform continuity also with respect to time as stated in assumption (L').

For both the coercive and Bellman case, the application of Perron's method on a sequence of finite-time horizon problems with the form (6.3) with $T \to +\infty$ and the strong comparison principle allows us to get the existence of a solution which is defined for all time. Note that, in order to apply Perron's method, we ask the initial datum to be bounded, i.e. $u_0 \in BC(\Omega)$.

Moreover, in order to have the uniform boundedness in T of the solutions of (6.1), we suppose the following assumption.

(H") There exists $\gamma_0 > 0$ such that $\gamma_R \ge \gamma_0$ for all $R > 0$ where γ_R is defined in (H').

Theorem 6.2. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying (O), $u_0 \in BC(\mathbb{R}^N)$. Assume (M), (J0), (J1). Let H be an Hamiltonian either of Bellman type as in* (6.4) *satisfying (C'), (L'), (B') or a coercive Hamiltonian satisfying (H0') and either (He') or (He"). Assume also (H'), (E'). Then, there exists a unique viscosity solution to problem* (6.1) *in $C(\bar\Omega \times [0,+\infty)) \cap L^\infty(\bar\Omega \times [0,T])$. In addition, if (H") holds, then the unique solutions $u \in C(\bar\Omega \times [0,+\infty)) \cap L^\infty(\bar\Omega \times [0,T])$ for all $T > 0$ is uniformly bounded in $\bar\Omega \times [0,+\infty)$.*

6.3. **Large time behavior I: convergence in the classical sense.** We address the question of the asymptotic behaviour as $t \to \infty$ of the solution of (6.1), under assumption (H"), which ensures existence and uniqueness for the associated stationary problem. The main result is the following theorem.

Theorem 6.3. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying (O), $u_0 \in BC(\Omega)$. Assume (M), (J0), (J1). Let H be an Hamiltonian either of Bellman type as in* (6.4) *satisfying (C'), (L'), (B') or a coercive Hamiltonian satisfying (H0') and either (He') or (He"). Assume (H"), (E'). Assume also that there exists a continuous function $\bar H : \bar\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying*

$$H(\cdot, t, \cdot, \cdot) \to \bar H \quad \text{locally uniformly in } \bar\Omega \times \mathbb{R} \times \mathbb{R}^N,$$

as $t \to \infty$. Then, there exists a unique bounded viscosity solution u for the following problem

$$\begin{cases} -\mathcal{I}(u) + \bar H(x, u, Du) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Moreover, the unique viscosity solution u of (6.1) *converges uniformly on compact sets in $\bar{\Omega}$ to u_∞, the unique viscosity solution of the problem* (6.5).

For the existence and uniqueness for the problem (6.5) we refer to Theorem 3.1 and Corollary 3.2. We omit the proof of Theorem 6.3 since it is rather classical and follows the same arguments used in [15], where that the same kind of results have been given in the case of the Dirichlet problem for nonlocal equation (fractional laplacian) with Hamiltonian both coercive both of Bellman type.

6.4. **Large time behavior II: convergence to the ergodic problem.** In this subsection we prove large time behavior for the problem

$$\begin{cases} \partial_t u(x) - \mathcal{I}[u(\cdot,t)](x) + H(x,Du) = 0 & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \bar{\Omega}. \end{cases} \tag{6.6}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ satisfying (O), $u_0 \in C(\Omega)$ and H is an Hamiltonian in superfractional coercive form, that is, $m > 1$ in (H1').

Existence and uniqueness for the problem (6.6) follow from Theorem 6.2. Note that H does not depend on u, so it does not satisfy (H').

The main result of this section is Theorem 6.8, namely the convergence as $t \to +\infty$ of the solution of (6.6) to a solution of the ergodic problem, which we solve in Proposition 6.7. The proof of Theorem 6.8 strongly relies on the Hölder regularity up to the boundary for subsolutions of (6.6) and on the control of their oscillation, proved by Barles, Ley Koike, Topp in [11], Corollary 2.14 (see also Barles and Topp [14]), in the case of censored operators and coercive Hamiltonian with $m > 1$. We recall this result in Proposition 6.6. We remark that, differently form [13], where Lipschitz regularity of the solutions is used to linearize the equations in order to apply the Strong Maximum Principle, our proof relies mainly on the use of a Strong Maximum Principle à la Coville [18], [19]) (see also Ciomaga [16]). This means that it relies mainly on a topological property of the the support of the measure defining the nonlocal operator. Note that in this final part we assume Ω bounded for techincal reasons related to the proof of the Strong maximum principle, we refer to the proof of Proposition 6.4.

6.4.1. *A strong maximum principle.* We need some notation for the statement of the Strong Maximum Principle. Let μ, j be as in the definition of the nonlocal operator $\mathcal{I}$, that is, satisfying (M), (J0), (J1) and denote by suppμ the support of the measure μ. For $x \in \mathbb{R}^n$ we define inductively

$$X_0(x) = \{x\}; \quad X_{r+1}(x) = \cup_{\xi \in X_r(x)} \{\xi + j(\xi, \text{supp}\{\mu_x\})\} \cap \bar{\Omega}, \quad \text{for } r \in \mathbb{N},$$

and

$$\mathcal{X}(x) = \overline{\cup_{r\in\mathbb{N}} X_r}.$$

The Strong Maximum Principle presented in this paragraph relies in the nonlocality of the operator under the "iterative covering property"

$$\mathcal{X}(x) = \Omega, \quad \text{for all } x \in \Omega. \tag{6.7}$$

The most basic example is the case where $j(x,z) = z$ and there exists $r > 0$ such that $B_r \subset \text{supp}\{\mu\}$. For further details and examples we refer to [11].

The following proposition states the Strong Maximum Principle.

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain satisfying (O). Let H be a coercive Hamiltonian satisfying (H0') and (He") with $m > 1$. Assume (M), (J0), (J1), (E') and* (6.7)*. Let u, v be respectively a sub and a supersolution of* (6.6)*, such that there exists $(x_0, t_0) \in \bar{\Omega} \times (0, +\infty)$ satisfying*

$$(u - v)(x_0, t_0) = \sup_{\bar{\Omega}\times(0,+\infty)} \{u - v\}.$$

Then, the function $u - v$ is constant in $\bar{\Omega} \times [0, t_0]$. Moreover we have

$$(u - v)(x, t) = \sup_{x\in\bar{\Omega}} \{u(x, 0) - v(x, 0)\}, \text{ for all } (x, t) \in \bar{\Omega} \times [0, t_0].$$

The proof of Proposition 6.4 uses the following lemma, which is is a consequence of the comparison principle, see [13], Theorem 4.1.

Lemma 6.5. *Let assumptions of Proposition 6.4 hold. Let u, v be locally bounded sub and supersolution to equation* (6.6) *and for $t \in [0, +\infty)$ define*

$$k(t) = \max_{\bar{\Omega}} \{u(x, t) - v(x, t)\}.$$

Then, for all $0 \le s \le t$, we have $k(t) \le k(s)$.

We prove the strong maximum principle. We essentially follow the argument of [11], Proposition 4.1, with some changes due to presence of the Neumann boundary condition. We give a sketch of the proof, focusing on the main differences.

Proof (Proof of Proposition 6.4). We divide the proof into several steps.

Step 1. *-Preliminaries* We want to prove that $(u - v)(x, t) = k(0)$ for each $(x, t) \in \bar{\Omega} \times [0, t_0]$. Since $k(t_0)$ is a global maximum value of k in $[0, +\infty)$, by Lemma 6.5 we have $k(t) = k(0)$ for all $t \in [0, t_0]$. Then, we have just to prove that

$$u(x, \tau) - v(x, \tau) = k(\tau), \ \forall x \in \bar{\Omega}.$$

for each $\tau \in (0, t_0)$. By upper-semicontinuity, we derive the result up to $\tau = 0$ and $\tau = t_0$. Fix $\tau \in (0, t_0)$ and define the set

$$\mathcal{B}_\tau = \{x \in \bar{\Omega} \, : \, (u - v)(x, \tau) = k(\tau)\}. \tag{6.8}$$

We observe that by the upper-semicontinuity of $u - v$, $\mathcal{B}_\tau$ is nonempty. Then, the claim of the proposition follows once proved that $\mathcal{B}_\tau = \bar{\Omega}$.

Step 2. *-Localization on time τ* For $\eta > 0$, define the function

$$(x, t) \to \Phi(x, t) := u(x, t) - v(x, t) - \eta(t - \tau)^2$$

and note that for each $(x, t) \in \bar{\Omega} \times (0, +\infty)$ and for $\tilde{x} \in \mathcal{B}_\tau$ where $\mathcal{B}_\tau$ is defined in (6.8), we have $\Phi(x, t) \le k(t) - \eta(t - \tau)^2 \le k(\tau) = (u - v)(\tilde{x}, \tau) = \bar{W}(\tilde{x}, \tau)$. Then the supremum of Φ in $\bar{\Omega} \times (0, +\infty)$ is achieved and

$$\sup_{(x,t)\in\Omega\times(0,+\infty)} \Phi(x, t) = k(\tau).$$

Step 3. *-Localization around a point in $\mathcal{B}_\tau$* From now on we fix $x_\tau \in \mathcal{B}_\tau$, where $\mathcal{B}_\tau$ is defined in (6.8). We define for $\varepsilon, \alpha > 0$

$$\psi_{\varepsilon,\alpha}(x) = e^{-Kd(x)} \frac{|x - x_\tau|^2}{\varepsilon^2} - \alpha d(x),$$

where d is the signed distance from the boundary (see Remark 1) and $K > 0$ satisfies $K > ||D^2 d||_\infty + 1$. Note that $\psi_{\varepsilon,\alpha}(x_\tau) = -\alpha d(x_\tau)$. Moreover for each $\varepsilon > 0$ the first derivatives of $\psi_{\varepsilon,\alpha}$ are bounded, depending on ε and α.

For $0 < \mu < 1$ we denote $\omega_\mu = \mu u - v$ and we consider

$$(x,t) \to \Phi_\mu(x,t) := \omega_\mu(x,t) - \eta|t-\tau|^2 - (1-\mu)\psi_{\varepsilon,\alpha}(x).$$

By the upper-semicontinuity of Φ_μ, there exists $(x_\mu, t_\mu) \in \bar{\Omega} \times [0, t_0+1]$ such that

$$\Phi_\mu(t_\mu, x_\mu) = \max_{\bar{\Omega}\times[t_0,t_0+1]} \Phi_\mu.$$

Since $\Phi_\mu \to \Phi$ locally uniformly on $\bar{\Omega}\times[0,+\infty)$ as $\mu \to 1$, we get up to subsequences

$$(x_\mu, t_\mu) \to (\bar{x}, \tau) \quad \text{as } \mu \to 1.$$

Not also that for any α small enough

$$\bar{x} = \bar{x}_\varepsilon \to x_\tau \quad \text{as } \varepsilon \to 0. \tag{6.9}$$

Indeed, by using the maximum point inequality for Φ_μ, we have

$$\begin{aligned} \Phi_\mu(x_\mu, t_\mu) &= (u-v)(x_\mu,t_\mu) + (\mu-1)(u+\psi_{\varepsilon,\alpha})(x_\mu,t_\mu) - \eta|t_\mu-\tau|^2 \\ &\geq k(\tau) + (\mu-1)u(x_\tau,\tau) - \alpha(\mu-1)d(x_\tau), \end{aligned} \tag{6.10}$$

where we used the definition of $k(\tau)$. Since $t_\mu \in [t_0, t_0+1]$ for all μ close to 1, we have $(u-v)(x_\mu,t_\mu) \leq k(t_\mu) \leq k(\tau)$. Coupling the previous inequality with (6.10) we get $\psi_{\varepsilon,\alpha}(x_\mu) + \alpha d(x_\tau) \leq u(x_\tau,\tau) - u(x_\mu,t_\mu)$. Then, by the boundedness of u and of d, for α small, we deduce that $|x_\mu - x_\tau| \leq C\varepsilon$ for some $C > 0$ independent on μ, which implies (6.9).

Step 4. -*Writing the viscosity inequality for* ω_μ Denote

$$\phi(x,t) := (1-\mu)\psi_{\varepsilon,\alpha}(x) + \eta(t-\tau)^2.$$

We test ω_μ with the function ϕ in (x_μ, t_μ). We suppose $x_\mu \in \partial\Omega$, since the other case being analogous and even simpler. By the Taylor's formula for the distance function, we have

$$n(x_\mu)(x_\mu - x_\tau) + \frac{1}{2}(x_\mu - x_\tau)^T D^2 d(x_\mu)(x_\mu - x_\tau) + o(|x_\mu - x_\tau|^2) = d(x_\tau) \geq 0$$

and then

$$n(x_\mu)(x_\mu - x_\tau) \geq -||D^2 d||_\infty |x_\mu - x_\tau|^2/2 + o(|x_\mu - x_\tau|^2). \tag{6.11}$$

Take μ, ε small enough so that

$$1 + \frac{o(|x_\mu - x_\tau|^2)}{|x_\mu - x_\tau|^2} \geq 0. \tag{6.12}$$

By (6.11), by the definition of K and (6.12), we have

$$\begin{aligned} \frac{\partial \phi_{\varepsilon,\alpha}}{\partial n}(x_\mu, t) &= \frac{\partial \psi_{\varepsilon,\alpha}}{\partial n}(x_\mu) \\ &\geq e^{-Kd(x_\mu)} \frac{|x_\mu - x_\tau|^2}{\varepsilon^2}\left[K - ||D^2 d||_\infty + \frac{o(|x_\mu - x_\tau|^2)}{|x_\mu - x_\tau|^2}\right] + \alpha \\ &\geq e^{-Kd(x_\mu)} \frac{|x_\mu - x_\tau|^2}{\varepsilon^2}\left[1 + \frac{o(|x_\mu - x_\tau|^2)}{|x_\mu - x_\tau|^2}\right] + \alpha > 0. \end{aligned}$$

Then, the rest of the proof follows exactly as in [11], by writing the viscosity inequality for ω_μ, letting first $\mu \to 1$, then $\varepsilon \to 0$ and using the iterative covering property (6.7). We omit the details since they are exactly the same as those of Proposition 4.1 of [11]. □

6.4.2. *The ergodic problem.* Roughly speaking, solving the *ergodic problem* means pass to the limit as $\delta \to 0$ in the stationary problem

$$\begin{cases} \delta u(x) - \mathcal{I}[u(\cdot)](x) + H(x, Du) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.13}$$

whose existence and uniqueness for $\delta > 0$ holds by Theorem 3.1.

We solve the ergodic problem in Proposition 6.7. Note that we need the compactness of the familty of solutions $\{u_\delta\}$, which relies mainly on the regularity result for subsolutions of equation (6.13) proved in [11], Theorem 5.5 and which we recall in the following proposition.

Proposition 6.6. *Let assumptions of Proposition 6.4 hold. Then any bounded viscosity subsolution $u : \mathbb{R}^N \to \mathbb{R}$ to* (6.13) *is Hölder continuous in $\bar{\Omega}$ with Hölder exponent $\gamma_0 = \frac{m-\sigma}{m}$ and Hölder seminorm depending on Ω, the data and $osc_\Omega(u)$ and not on δ. Moreover, there exists $K > 0$ such that for any bounded viscosity subolsoution of* (6.13) *we have*

$$osc_\Omega(u) \leq K. \tag{6.14}$$

Proposition 6.7. *Under the assumptions of Proposition 6.4, there exists a unique constant $\lambda \in \mathbb{R}$ for which the stationary ergodic problem*

$$\begin{cases} \lambda - \mathcal{I}[u(\cdot)](x) - H(x, Du) = 0 & \textit{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \textit{in } \partial\Omega \end{cases} \tag{6.15}$$

has a solution $w \in C^{\frac{m-\sigma}{m}}(\bar{\Omega})$. Moreover w is the unique solution of (6.15) *up to an additive constant.*

Proof. A key ingredient is Proposition 6.6, which gives the compacteness of the family of solutions of the approximating equation (6.13). Once we have the compactness, the proof follows standard arguments which we do not repeat. The uniqueness follows by the comparison principle for (6.6) and the application of the strong maximum principle for the problem (6.6) (Proposition 6.4). □

6.4.3. *Convergence as $t \to +\infty$.*

Theorem 6.8. *Let assumptions of Proposition* (6.4) *hold. Let u be the unique solution to problem* (6.6)*. Then, there exists a pair (w, λ) solution to* (6.15) *such that*

$$u(x, t) - \lambda t - w(x) \to 0 \textit{ as } t \to +\infty,$$

uniformly on $\bar{\Omega}$.

We omit the details of the proof since we follow closely the arguments given in [11] (see also [5], [15] for the local framework and [34] for the nonlocal one).

We just observe that again a crucial ingredient of the proof is the Hölder regularity of the solutions of (6.6) (see Proposition 6.6). Once established the regularity, the proof follows by the application of the strong maximum principle proved in Proposition 6.4.

Appendix

First we prove some lemmas used in the proof of Theorem 3.1, that is, the following Lemma 6.9, Lemma 5.3 and Lemma 6.10. In Section 6.6 we prove the existence of the blow-up supersolution, stated in Remark 9, Lemma 5.5.

6.5. **Some lemmas used in the proof of Theorem 3.1.**

Lemma 6.9. *Let H be an Hamiltonian of Bellman type. For all $\hat{s} \in \partial\Omega$, there exists $r = r(\hat{s}) > 0$ and $\gamma, C_2 > 0$ constants such that for all $s \in \bar{B}_r(\hat{s}), \lambda \in \mathbb{R}, p \in \mathbb{R}^N$, it holds:*

(i) if $\hat{s} \in \Gamma_{out}$

$$H(s, p - |\lambda| n(s)) \geq \gamma|\lambda| - C_2|p| - C_2; \tag{6.16}$$

$$H(s, p + |\lambda| n(s)) \leq -\gamma|\lambda| + C_2|p| + C_2; \tag{6.17}$$

(ii) if $\hat{s} \in \Gamma$, then

$$H(s, p - |\lambda| n(s)) \geq \gamma|\lambda| - C_2|p| - C_2; \tag{6.18}$$

$$H(s, p + |\lambda| n(s)) \geq \gamma|\lambda| - C_2|p| - C_2. \tag{6.19}$$

Proof. First we prove (i). Since $\hat{s} \in \Gamma_{\text{out}}$, for $\alpha \in \mathcal{A}$, there exists $r_1, \gamma_1 > 0$ small enough such that $b(s, \alpha) \cdot n(s) \geq \gamma_1$ for any $s \in \bar{\Omega} \cap B_{r_1}(\hat{s})$. Then, by the boundedness of b and l, we get for some $C_1 > 0$

$$H(s, p - |\lambda| n(s)) \geq -b(s,\alpha) \cdot p + |\lambda| b(s,\alpha) \cdot n(s) - l(s,\alpha) \geq |\lambda|\gamma_1 - C_1|p| - C_1, \tag{6.20}$$

for any $s \in \bar{\Omega} \cap B_{r_1}(\hat{s})$. To prove (6.17), we approximate the supremum in the Hamiltonian by a sequence $\tilde{\alpha} \in \mathcal{A}$. In particular, we take $\tilde{\alpha}$ such that

$$H(s, p + |\lambda| n(s)) \leq -b(s, \tilde{\alpha}) \cdot p - \lambda b(s, \tilde{\alpha}) \cdot n(s) - l(s, \tilde{\alpha}) + 1.$$

Since $\hat{s} \in \Gamma_{\text{out}}$ and using that $\mathcal{A}$ is compact, there exist $r_2, C > 0$ small enough such that $b(s, \tilde{\alpha}) \cdot n(s) \geq \gamma_2$ for any $s \in \bar{\Omega} \cap B_{r_2}(\hat{s})$. Then by the boundedness of b and l we get

$$H(s, p + |\lambda| n(s)) \leq -b(s, \tilde{\alpha}) \cdot p - |\lambda| b(s, \tilde{\alpha}) \cdot n(s) - l(s, \tilde{\alpha}) + 1 \leq -\gamma_2|\lambda| + C_1|p| + C_1 + 1, \tag{6.21}$$

for any $s \in \bar{\Omega} \cap B_{r_2}(\hat{s})$. We conclude the proof of (6.16) and (6.17) by using (6.20) and (6.21) and by denoting $r = \min\{r_1, r_2\}, \gamma = \min\{\gamma_1, \gamma_2\}$ and $C_2 = C_1 + 1$.
Now we prove (ii). Since $\hat{s} \in \Gamma$, there exist $r, \gamma > 0$ such that $b(s, \alpha_1) \cdot n(s) \geq \gamma$ and $b(s, \alpha_2) \cdot n(s) \leq -\gamma$ for any $s \in \bar{\Omega} \cap B_r(\hat{s})$. Then we get

$$H(s, p - |\lambda| n(s)) \geq -b(s, \alpha_1) \cdot p + |\lambda| b(s, \alpha_1) \cdot n(s) - l(s, \alpha_1) \geq |\lambda|\gamma - C_1|p| - C_1,$$

for any $s \in \bar{\Omega} \cap B_r(\hat{s})$, proving (6.18). Analogously, we get

$$H(s, p + |\lambda| n(s)) \geq -b(s, \alpha_2) \cdot p + |\lambda| b(s, \alpha_2) \cdot n(s) - l(s, \alpha_2) \geq |\lambda|\gamma - C_1|p| - C_1,$$

for any $s \in \bar{\Omega} \cap B_r(\hat{s})$ and by denoting $C_2 = C_1 + 1$ we conclude (6.19). □

Now we prove Lemma 5.3, whose statement is given in the proof of Theorem 3.1, Step 1.

Proof of Lemma 5.3. First we prove (i). Take $-\mathcal{I}_\xi[\phi(\cdot,y)](x)$. By the definition of $\tilde{\phi}$, since χ_ε, ϕ are Lipschitz and by (J1) we have

$$\tilde{\phi}(x+j(x,z),y) - \tilde{\phi}(x,y) \le C\varepsilon^{-1}|z| + C|z|$$

and by the first of (M) we have

$$\int_{\substack{x+j(x,z)\in\Omega, \\ |z|<\xi}} \tilde{\phi}(x+j(x,z),y) - \tilde{\phi}(x,y)d\mu_x(z) \le \varepsilon^{-1}C\xi^{1-\sigma}. \tag{6.22}$$

Take now $-\mathcal{I}_\xi[u](x)$. Note that by the boundedness of u we have

$$\int_{\substack{x+j(x,z)\in\Omega, \\ |z|>\xi}} u(x+j(x,z)) - u(x)d\mu_x(z) \le 2||u||_\infty \int_{\substack{x+j(x,z)\in\Omega, \\ |z|>\xi}} d\mu_x(z)$$

and then by the first of (M), we get for some $C > 0$

$$\int_{\substack{,x+j(x,z)\in\Omega \\ |z|>\xi}} u(x+j(x,z)) - u(x)d\mu_x(z) \le C\xi^{-\sigma}. \tag{6.23}$$

Pluggin toghether (6.22) and (6.23), we conclude (i) for some $C_1 > 0$. Similarly we prove (ii). $\square$

We state Lemma 6.10 and we omit the proof since it follows by standard arguments.

Lemma 6.10. *Let $R, \nu > 0$ and denote $\psi_R(x) = \psi(R^{-1}|x|)$ where ψ is a smooth function such that*

$$\psi(s) = \begin{cases} 0 & \text{for } 0 \le s < \frac{1}{2}, \\ \text{increasing} & \text{for } \frac{1}{2} \le s < 1, \\ ||u||_\infty + ||v||_\infty + 1 & \text{for } s \ge 1. \end{cases} \tag{6.24}$$

Let d denote the distance from the boundary of Ω in a neighbourhood V of the boundary and extend it C^1 and bounded in all the domain. Let $\mathcal{I}$ as in (1.2) *and assume (M) and (J1). Then the function $\phi = \psi_R + \nu d$ satisfies*

$$\mathcal{I}[\phi](\cdot) \le o_{\nu,R}(1), \quad |D\phi(\cdot)| \le o_{\nu,R}(1). \tag{6.25}$$

6.6. **Blow-up supersolution.** In this section we prove Lemma 5.5, used in the proof of Theorem 3.1 (see in particular Lemma 5.1, Remark 9).

We construct the function U_r as showed in the following and then we prove Lemma 6.11. Note that Lemma 5.5 follows as a consequence of Lemma 6.11 and we prove it after the statement of Lemma 6.11.

Let $\bar{x} \in \Gamma_{\text{in}}$ and $r = r(\bar{x})$ be given as in assumption (O). We recall that by (O), there exists a $W^{2,\infty}$-diffeomorphism $\psi : B_r(\bar{x}) \mapsto \mathbb{R}^N$, satisfying

$$\psi_N(s) = d(s) \text{ for any } s \in B_r(\bar{x}), \tag{6.26}$$

where d is the signed distance from the boundary of Ω.

We define $U_r$in a suitable neighbourhood of the point $\bar{x}$, where we rectify the boundary and carry on the computations. In particular we define U_r as follows

$$U_r(x) = \bar{U}_r(d(x)) \quad \text{for } x \in B_r(\bar{x}) \cap \Omega, \tag{6.27}$$

where

$$\bar{U}_r(s) = -\log(s) + \frac{3}{2}\log r \quad \text{if } 0 < s \le r.$$

Note that $\bar{U}_r \in C^\infty(0,r)$ is (nonnegative) monotone and decreasing.

Lemma 6.11. *For any $\bar{x} \in \Gamma_{in}$, let $r = r(\bar{x})$ be defined as in assumption (O). Let U_r be defined as in (6.27). Then we have for ξ small enough (with respect to r)*

$$-\mathcal{I}_\xi[U_r](x) \geq -Ad(x)^{-\sigma} \quad \text{in } B_{\frac{r}{2}}(\bar{x}) \cap \Omega. \tag{6.28}$$

In particular there exists $\tilde{r}(r, A, \sigma) = \tilde{r}$ such that $\tilde{r} \leq r$ and

$$-b(x,\alpha) \cdot DU_r(x) - \mathcal{I}_\xi[U_r](x) \geq 0 \quad \text{in } B_{\frac{\tilde{r}}{2}}(\bar{x}) \cap \Omega \quad \forall \alpha \in \mathcal{A}. \tag{6.29}$$

Remark 13. Note that the strict positivity of the drift term on the points of Γ_{in} is essential here to prove (6.29), since the drift term controls the integral term which explodes on the boundary, as (6.28) shows.

As a consequence of Lemma 6.11, we prove Lemma 5.5.

Proof of Lemma 5.5. Take $\tilde{r}$ as defined in Lemma 6.11 and let $U_{\tilde{r}}$ be defined as in (6.27) for $r = \tilde{r}$. Then $U_{\tilde{r}}$ is a nonnegative decreasing function which trivially satisfies (ii) of Lemma 5.5 with $\omega_{\tilde{r}}(s) = \frac{1}{\bar{U}_{\tilde{r}}(s)}$. Moreover, (i) Lemma 5.5 follows as a direct application of (6.29) of Lemma 6.11. □

Now we prove Lemma 6.11.

Proof of Lemma 6.11. First we prove (6.28). Let $\xi \leq C_j^{-1}\frac{r}{2}$. Then, by (J1), for $|z| \leq \xi$ and $x \in B_{\frac{r}{2}}(\bar{x})$, we have that $x + j(x,z) \in B_r(\bar{x})$. We describe the domain of integration of $\mathcal{I}_\xi[U_r]$ through the diffeomorphism ψ as $x + j(x,z) \in \bar{\Omega} = \psi_N(x + j(x,z)) \geq 0$. By the definition of U_r and (6.26)

$$\mathcal{I}_\xi[U_r](x) = -\int_{\substack{\psi_N(x+j(x,z)) \geq 0, \\ |z| \leq \xi}} [\ln(\psi_N(x+j(x,z))) - \ln(\psi_N(x))]\, d\mu_x(z).$$

We write $\mathcal{I}_\xi[U_r](x) = I^1 + I^2$, where

$$I^1 = -\int_{\substack{\psi_N(x+j(x,z)) > \psi_N(x), \\ |z| \leq \xi}} [\ln(\psi_N(x+j(x,z))) - \ln(\psi_N(x))]\, d\mu_x(z).$$

and

$$I^2 = -\int_{\substack{\psi_N(x) \geq \psi_N(x+j(x,z)) \geq 0, \\ |z| \leq \xi}} [\ln(\psi_N(x+j(x,z))) - \ln(\psi_N(x))]\, d\mu_x(z).$$

and since $I^1 \leq 0$, we get

$$\mathcal{I}_\xi[U_r](x) \leq -\int_{\substack{\psi_N(x) \geq \psi_N(x+j(x,z)) \geq 0, \\ |z| \leq \xi}} [\ln(\psi_N(x+j(x,z))) - \ln(\psi_N(x))]\, d\mu_x(z). \tag{6.30}$$

We proceed performing a change of variable in order to write the set of integration in terms of $\psi_N(x)$. In other words, we write

$$\psi(x + j(x,z)) - \psi(x) = w.$$

Then by (J0), (J1), the first of (M) and since ψ is $W^{2,\infty}$, (6.30) becomes

$$\mathcal{I}_\xi[U_r](x) \leq \bar{C} \int_{\substack{0 \geq w_N \geq -\psi_N(x), \\ |w| \leq C\xi}} \left| \ln\left(1 + \frac{w_N}{\psi_N(x)}\right) \right| \frac{dw}{|w|^{N+\sigma}},$$

for some $\bar{C}, C > 0$. By the change of variable $y = \frac{w}{\psi_N(x)}$, we get

$$\mathcal{I}_\xi[U_r](x) \leq \bar{C}\psi_N(x)^{-\sigma} \int_{0 \geq y_N \geq -1} |\ln(1 + y_N)| \frac{dy}{|y|^{N+\sigma}}. \tag{6.31}$$

Note that the integral in the right hand side is finite and does not depend on ξ. For convenience of notation we denote $A := \bar{C} \int_{0 \geq y_N \geq -1} |\ln(1 + y_N)| \frac{dy}{|y|^{N+\sigma}}$. Then (6.31) becomes

$$\mathcal{I}_\xi[U_r](x) \leq Ad(x)^{-\sigma},$$

which is exactly (6.28).

Now we prove (6.29). First note that, by the definition of U_r, we have

$$DU_r(x) = d(x)^{-1}n \quad \text{in } B_{C_j r}(\bar{x}) \cap \Omega. \tag{6.32}$$

Then, by (6.28) and (6.32), we have for all $\alpha \in \mathcal{A}$

$$b(x, \alpha) \cdot DU_r(x) + \mathcal{I}_\xi[U_r](x) \leq d(x)^{-1}(b(x, \alpha) \cdot n + d(x)^{1-\sigma} A).$$

Since we are in a neighbourhood of Γ_{in}, $\sigma < 1$ and $\mathcal{A}$ is compact, there exists $0 < \tilde{r} < r$ (depending only on A, σ and r), such that if $x \in B_{\frac{r}{2}}(\bar{x}) \cap \Omega$

$$b(x, \alpha) \cdot DU_r(x) + \mathcal{I}_\xi[U_r](x) \leq 0 \quad \forall \alpha \in \mathcal{A}.$$

Then (6.29) follows and we conclude the proof of the Lemma. □

Acknowledgment

This work has been carried out during the permanence of the author at the Laboratoire de Mathématiques de Tours, which the author wants to thank for the warm welcome. In particular the author wants to express her deepest gratitude to Prof. Guy Barles, for proposing the problem and for the important support given. Without him, this work would not have been carried out. The author kinly thanks also the anonymous referees, whose comments and suggestions were precious in the improving of the final presentation of the paper.

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