ON THE STABILIZABILITY OF THE BURGERS' EQUATION BY RECEDING HORIZON CONTROL *

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Abstract. A receding horizon framework for stabilization of a class of infinite-dimensional controlled systems is presented. No terminal costs and constraints are used to ensure asymptotic stability of the controlled system. The key assumption is a stabilizability assumption, which can be guaranteed, for example, for the Burgers' equations with periodic and with homogeneous Neumann boundary conditions. Numerical experiments validate the theoretical results. Comparisons to the case with terminal penalties acting as control Lyapunov functions are included.

Key words. receding horizon control, model predictive control, asymptotic stability, infinite-dimensional systems.

AMS subject classifications. 49N35, 93C20, 93D20

1. Introduction.

We consider the optimal control problem

$$J_\infty(u, y_0) := \int_0^\infty \ell(y(t), u(t))dt \tag{1.1}$$

subject to

$$\begin{cases} \frac{d}{dt}y(t) = f(y(t)) + Bu(t) & \text{for } t > 0, \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where $f(0) = 0$, $\ell(0, 0) = 0$. The state $y(t)$ and the control $u(t)$ are elements of spatially dependent function spaces H and U, respectively. Furthermore, the incremental cost function $\ell(\cdot, \cdot)$ is assumed to be uniformly positive definite in both the state and control variables.

One strategy to solve problem (1.1)-(1.2) numerically employs the receding horizon control (RHC) which is also known as model predictive control (MPC). This method consists in obtaining a suboptimal solution of the infinite horizon problem by solving a series of finite horizon problems on a family of intervals which are arranged in an temporally increasing manner and which cover $[0, \infty)$. Since proceeding in this manner, the solution of (1.1)-(1.2) is not obtained, the question of justifying the RHC technique arises. This is typically addressed by analyzing whether the RHC control meets the control objective which is formulated within (1.1)-(1.2). Frequently this control objective is given by the stabilization problem. Due to replacing the infinite prediction horizon by a family of finite ones, the asymptotic stability of the receding horizon trajectory is not a-priori guaranteed. However, the succinct use of the structure of the dynamical system under consideration together with possible terminal costs and/or terminal constraints for the finite horizon problems can ensure asymptotic stability of RHC control under appropriate assumptions.

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In the past three decades, numerous results have been published on receding horizon control for finite-dimensional systems [1, 2, 12, 16, 18, 22, 23, 27] and the many references therein. Only more recently the case of infinite-dimensional systems was considered as well [17, 20, 21]. In [20] a general framework to stabilize infinite-dimensional dynamical system by receding horizon control is proposed. The stability of the receding horizon trajectory is ensured by adding control Lyapunov functions as terminal cost to the finite horizon problems. More recently several authors, see e.g [16, 17, 18, 22] managed to prove the asymptotic stability of the RHC even without use of control Lyapunov functions and terminal constraints. So far, this framework has been well studied for finite-dimensional dynamical systems [22, 29] and discrete time dynamical systems [16, 17, 18]. But as far as we know, for infinite-dimensional systems with continuous time dynamical systems there still does not exist a rigorous theory. In this paper we make a step in this direction. The present work is inspired by [29], but it differs not only in the fact that we treat partial rather than ordinary differential equations, but also we consider systems which are only locally stabilizable.

To briefly recapture the receding horizon approach, we choose a sampling time $\delta > 0$ and an appropriate prediction horizon $T > \delta$. Then sampling instances $t_k := k\delta$ for $k = 0, 1, \ldots$ are defined. At every sampling instance t_k, an open-loop optimal control problem is solved over a finite prediction horizon $[t_k, t_k + T]$. The optimal control thus obtained is applied to steer the system from time t_k with the initial state $y_{rh}(t_k)$ until time $t_{k+1} := t_k + \delta$ at which point, a new measurement of state is assumed to be available. The process is repeated starting from the new state: we obtain a new optimal control and a new predicted state trajectory by shifting the prediction horizon forward in time. Denoting the receding horizon state- and control variables by $y_{rh}(\cdot)$ and $u_{rh}(\cdot)$, respectively, we next summarize the resulting algorithm 1.

Algorithm 1 Receding Horizon Algorithm

Input: Let the prediction horizon T, the sampling time $\delta < T$, and the initial state $y_0 \in H$ be given.

1: Set $k := 0, \quad t_0 := 0$ and $y_{rh}(t_0) := y_0$.

2: Find the optimal pair $(y_T^*(\cdot; y_{rh}(t_k), t_k), u_T^*(\cdot; y_{rh}(t_k), t_k))$ over the time horizon $[t_k, t_k + T]$ by solving the finite horizon open-loop problem

$$\min_{u \in L^2(t_k, t_k+T; U)} J_T(u, y_{rh}(t_k)) := \min_{u \in L^2(t_k, t_k+T; U)} \int_{t_k}^{t_k+T} \ell(y(t), u(t)) dt,$$

$$\text{s.t} \begin{cases} \frac{d}{dt} y(t) = f(y(t)) + Bu(t) & \text{for } t \in (t_k, t_k + T), \\ y(t_k) = y_{rh}(t_k) \end{cases}$$

3: Set

$$\begin{aligned} u_{rh}(\tau) &:= u_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta), \\ y_{rh}(\tau) &:= y_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta], \\ t_{k+1} &:= t_k + \delta, \\ k &:= k + 1. \end{aligned}$$

4: Go to step 2.

The importance of RHC rests not only on speed up of the solution process, but also on its use as a state feedback mechanism which can be activated at each instance time t_k. In practice this requires to include a dynamic observer. We leave this aspect to future work.

The applicability of our theoretical work will be demonstrated for the Burgers' equation. This is a nonlinear partial differential equation (PDE) that combines both nonlinear propagation and diffusion effects. It shares some important features with the Navier-Stokes equation. The Burgers' equation has the origin as a steady state. It is asymptotically stable in the case of Dirichlet boundary conditions. For Neumann boundary conditions and periodic boundary conditions this is not the case. Control theory for the Burgers' equation was investigated, both theoretically and numerically, by many authors. From among them we only mention [3, 4, 5, 8, 9, 10, 11, 19, 24, 25, 26, 33].

The remainder of the paper is structured as follows. In Section 2 we develop an abstract setting which estimates the value of the cost J_∞ evaluated along the receding horizon control and trajectory in terms of the minimal value functional associated to (1.1)-(1.2). This result is applied in Section 3 to stabilization of the Burgers' equation with different types of boundary conditions. Section 4 contains numerical experiments which highlight the effect to the ratio $\frac{T}{\delta}$ on the stabilizing effect of the RHC strategy. Moreover comparisons are carried out comparing the effect of RHC control with and without terminal control penalty.

2. Stability of the receding horizon method. Let $V \subset H = H^* \subset V^*$ be a Gelfand triple of real Hilbert spaces with V densely contained in H. Further let U denote the control space which is also assumed to be a real Hilbert space. For any $T > 0$ and $y_0 \in H$ we consider the controlled dynamical system

$$\begin{cases} \frac{d}{dt}y(t) = f(y(t)) + Bu(t) & \text{for } t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $f \in C(V, V^*)$, $f(0) = 0$, and $B \in \mathcal{L}(U, V^*)$. Throughout the paper, it is assumed that for any triple $(T, y_0, u) \in \mathbb{R}^+ \times H \times L^2(0,T;U)$ there exists a unique $y \in W(0,T)$, where

$$W(0,T) = L^2(0,T;V) \cap H^1(0,T;V^*), \tag{2.2}$$

satisfying

$$y(t) - y(0) = \int_0^t (f(y(s)) + Bu(s))ds \quad \text{in } V^*$$

for $t \in [0,T]$. For sufficient conditions on f we refer to e.g. [31], chapter II. 3. We recall that $W(0,T)$ is continuously embedded in $C([0,T];H)$, see e.g. [31, 35].

To define the optimal control problems we introduce the continuous incremental function $\ell : H \times U \to \mathbb{R}^+$ satisfying

$$\ell(y,u) \geq \alpha_\ell(\|y\|_H^2 + \|u\|_U^2) \tag{2.3}$$

for a number $\alpha_\ell > 0$ independent of $y \in H$ and $u \in U$, and $\ell(0,0) = 0$. For every

$T > 0$ and $y_0 \in H$ consider the finite horizon optimal control problem

$$(P_T) \qquad \begin{aligned} &\min_{u \in L^2(0,T;U)} J_T(u, y_0) := \min_{u \in L^2(0,T;U)} \int_0^T \ell(y(t), u(t))dt, \\ &\text{subject to} \\ &\begin{cases} \frac{d}{dt} y(t) = f(y(t)) + Bu(t) & \text{for } t \in (0,T), \\ y(0) = y_0. \end{cases} \end{aligned}$$

Throughout we fix a neighborhood $\mathcal{N}_0$ of the origin in H. We assume that

(A1) $\qquad (P_T)$ admits an optimal pair $(y_T^*(\cdot; y_0, 0), u_T^*(\cdot; y_0, 0))$ for any $y_0 \in \mathcal{N}_0$ and $T > 0$.

Conditions on ℓ and f which imply (A1) are well-known from e.g. [32]. The functional J_∞ is defined as J_T in (P_T) with T replaced by ∞. With (A1) holding the following definition is well posed.

DEFINITION 2.1. *For any $y_0 \in \mathcal{N}_0$ the infinite horizon value function $V_\infty(\cdot)$ is defined by*

$$V_\infty(y_0) := \inf_{u \in L^2(0,\infty;U)} \{J_\infty(u, y_0) \text{ subject to } (2.1)\}.$$

Similarly, the finite horizon value function $V_T(\cdot)$ is defined by

$$V_T(y_0) := \min_{u \in L^2(0,T;U)} \{J_T(u, y_0) \text{ subject to } (2.1)\}.$$

The following notion of local stabilizability will be used.

DEFINITION 2.2 (Local stabilizability). *The dynamical system (2.1) is called locally stabilizable if for every positive T and initial function $y_0 \in \mathcal{N}_0$ there exists a control $\hat{u}(\cdot, y_0) \in L^2(0,T;U)$ with*

$$V_T(y_0) \leq J_T(\hat{u}, y_0) \leq \gamma(T)\|y_0\|_H^2, \tag{2.4}$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, non-decreasing and bounded function.

If $\mathcal{N}_0$ can be chosen to be all of H then we call (2.1) globally stabilizable. We shall require the following two assumptions:

(A2) The dynamical system (2.1) is locally stabilizable for the neighborhood $\mathcal{N}_0$.

(A3) $\left.\begin{array}{l} \text{For every } T > 0 \text{ there exists a constant } c_T \geq 0 \text{ such that for every} \\ y_0 \in \mathcal{N}_0, \text{and } u \text{ with } \|u\|_{L^2(0,T;U)} \leq \sqrt{\gamma(T)/\alpha_\ell}\,\|y_0\|_H \text{ we have} \\ \|y(t)\|_H^2 \leq \|y_0\|_H^2 + c_T \int_0^t \|y(s)\|_H^2 ds + c_T \int_0^t \|u(s)\|_U^2\, ds \quad \text{for all } t \in [0,T]. \end{array}\right\}$

Below $\mathcal{B}_{d_1}(0)$ denotes a ball in H centered at 0 with radius d_1.

LEMMA 2.3. *If (A1)-(A3) hold, and $T > 0$, then there exists a neighborhood $\mathcal{B}_{d_1}(0) \subset \mathcal{N}_0$ with $d_1 = d_1(T) > 0$ such that for every $y_0 \in \mathcal{B}_{d_1}(0)$ the following inequalities hold*

$$\begin{aligned} V_T(y_T^*(\delta; y_0, 0)) \leq &\int_\delta^{t^*} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \\ &+ \gamma(T + \delta - t^*)\|y_T^*(t^*; y_0, 0)\|_H^2 \quad \textit{for all } t^* \in [\delta, T], \end{aligned} \tag{2.5}$$

and

$$\int_{t^*}^{T} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \le \gamma(T-t^*)\|y_T^*(t^*;y_0,0)\|_H^2 \quad \text{for all } t^* \in [0,T]. \tag{2.6}$$

Proof. First observe that due to (2.3) for every $y_0 \in \mathcal{N}_0$ and $t^* \in [0,T]$ we have

$$\begin{aligned}\alpha_\ell \int_0^{t^*} (\|y_T^*(t;y_0,0)\|_H^2 + \|u_T^*(t;y_0,0)\|_U^2)dt &\le \int_0^{t^*} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\ &= V_T(y_0) - V_{T-t^*}(y_T^*(t^*;y_0,0)),\end{aligned}$$

and as a consequence $\|u_T^*\|_{L^2(0,T;U)}^2 \le \frac{\gamma(T)}{\alpha_\ell}\|y_0\|_H^2$. By (2.4), (A3) and the above inequality we have

$$\begin{aligned}\|y_T^*(t^*;y_0,0)\|_H^2 &\le \|y_0\|_H^2 + c_T\int_0^{t^*}\|y_T^*(t;y_0,0)\|_H^2 dt + c_T\int_0^{t^*}\|u_T^*(t;y_0,0)\|_U^2 dt \\ &\le \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell}(V_T(y_0) - V_{T-t^*}(y_T^*(t^*;y_0,0)) \\ &\le \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell}V_T(y_0) \le (1+\frac{c_T}{\alpha_\ell}\gamma(T))\|y_0\|_H^2.\end{aligned}$$

Since $\mathcal{N}_0$ is a neighborhood of zero, it follows that there exists a ball $\mathcal{B}_{\delta_1}(0) \subseteq \mathcal{N}_0$. Choosing $d_1 := \sqrt{(1+\frac{c_T}{\alpha_\ell}\gamma(T))^{-1}\delta_1^2}$ we obtain that for every $y_0 \in \mathcal{B}_{d_1}(0)$ we have

$$y_T^*(t^*;y_0,0) \in \mathcal{N}_0 \qquad \text{for all } t^* \in [0,T].$$

Since $\gamma(T)$ is bounded from above with respect to T, the ball $\mathcal{B}_{d_1}(0)$ is independent of T.

We turn to the verification of (2.5). For simplicity of notation, we denote $y_T^*(\delta;y_0,0)$ by $y^*(\delta)$, where $y_0 \in \mathcal{B}_{d_1}(0)$. Due to Bellman's optimality principle, we have for every $t^* \in [\delta,T]$

$$\begin{aligned}V_T(y^*(\delta)) &= \int_\delta^{T+\delta} \ell(y_T^*(t;y^*(\delta),\delta),u_T^*(t;y^*(\delta),\delta))dt \\ &= \int_\delta^{t^*} \ell(y_T^*(t;y^*(\delta),\delta),u_T^*(t;y^*(\delta),\delta))dt + V_{T+\delta-t^*}(y_T^*(t^*;y^*(\delta),\delta)).\end{aligned} \tag{2.7}$$

By optimality of $y_T^*(\cdot;y^*(\delta),\delta)$ as a solution on $[\delta,T+\delta]$ with initial state $y^*(\delta) \in \mathcal{N}_0$ at $t=\delta$ we obtain

$$\begin{aligned}V_T(y^*(\delta)) &\le \int_\delta^{t^*} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + V_{T+\delta-t^*}(y_T^*(t^*;y_0,0)) \\ &\le \int_\delta^{t^*} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + \gamma(T+\delta-t^*)\|y_T^*(t^*;y_0,0)\|_H^2,\end{aligned}$$

where for the last inequality we used (2.4).

To prove the second inequality let $t^* \in [0,T]$ be arbitrary. By Bellman's principle and (2.4), we have

$$\begin{aligned} V_T(y_0) &= \int_0^{t^*} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + \int_{t^*}^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\ &= \int_0^{t^*} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + V_{T-t^*}(y_T^*(t^*;y_0,0)) \\ &\leq \int_0^{t^*} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + \gamma(T-t^*)\|y_T^*(t^*;y_0,0)\|_H^2. \end{aligned} \tag{2.8}$$

Therefore,

$$\int_{t^*}^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \leq \gamma(T-t^*)\|y_T^*(t^*;y_0,0)\|_H^2 \quad \text{for all } t^* \in [0,T],$$

as desired. ∎

Lemma 2.4. *Suppose that for some initial function $y_0 \in H$, properties* (2.5) *and* (2.6) *of Lemma 2.3 hold. Then for the choice of*

$$\theta_1 := 1 + \frac{\gamma(T)}{\alpha_\ell(T-\delta)}, \qquad \theta_2 := \frac{\gamma(T)}{\alpha_\ell \delta},$$

we have the following estimates

$$V_T(y_T^*(\delta;y_0,0)) \leq \theta_1 \int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt, \tag{2.9}$$

and

$$\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \leq \theta_2 \int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt. \tag{2.10}$$

Proof. To verify the inequality (2.9) recall that $y_T^*(\cdot;y_0,0) \in C([0,T];H)$. Hence there is a $\bar{t} \in [\delta,T]$ such that

$$\bar{t} = \arg\min_{t\in[\delta,T]} \|y_T^*(t;y_0,0)\|_H^2.$$

By (2.5) we have

$$\begin{aligned} &V_T(y_T^*(\delta;y_0,0)) \\ &\quad\leq \int_\delta^{\bar{t}} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + \gamma(T+\delta-\bar{t})\|y_T^*(\bar{t};y_0,0)\|_H^2 \\ &\quad\leq \int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + \gamma(T)\|y_T^*(\bar{t};y_0,0)\|_H^2 \\ &\quad\leq \int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt + \frac{\gamma(T)}{T-\delta}\int_\delta^T \|y_T^*(t;y_0,0)\|_H^2 dt. \end{aligned} \tag{2.11}$$

Furthermore,

$$\int_\delta^T \|y_T^*(t;y_0,0)\|_H^2 dt \leq \frac{1}{\alpha_\ell}\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt. \tag{2.12}$$

Now by using (2.12) and (2.11), we have

$$V_T(y_T^*(\delta;y_0,0)) \le (1+\frac{\gamma(T)}{\alpha_\ell(T-\delta)})\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt.$$

Turning to (2.10) we define

$$\hat{t} = \arg\min_{t\in[0,\delta]} \|y_T^*(t;y_0,0)\|_H^2.$$

Then by (2.6) we have

$$\begin{aligned}\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \le & \int_{\hat{t}}^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\ \le & \gamma(T-\hat{t})\|y_T^*(\hat{t};y_0,0)\|_H^2 \\ \le & \gamma(T)\|y_T^*(\hat{t};y_0,0)\|_H^2 \\ \le & \frac{\gamma(T)}{\delta}\int_0^\delta \|y_T^*(t;y_0,0)\|_H^2 dt.\end{aligned} \tag{2.13}$$

Moreover, we have

$$\frac{\gamma(T)}{\delta}\int_0^\delta \|y_T^*(t;y_0,0)\|_H^2 dt \le \frac{\gamma(T)}{\alpha_\ell\delta}\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt. \tag{2.14}$$

By (2.13) and (2.14) we can estimate

$$\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \le \frac{\gamma(T)}{\alpha_\ell\delta}\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt. \qquad \square$$

PROPOSITION 2.5. *Suppose that for some initial function $y_0 \in H$, conditions (2.5) and (2.6) of Lemma 2.3 hold. Then for every $\delta > 0$, there exist positive numbers $T^* > \delta$ and $\alpha \in (0,1)$ such that the following inequality is satisfied*

$$V_T(y_T^*(\delta;y_0,0)) \le V_T(y_0) - \alpha\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt, \tag{2.15}$$

for any $T \ge T^*$.

Proof. From the definition of $V_T(y_0)$ we have

$$\begin{aligned}V_T(y_T^*(\delta;y_0,0)) - V_T(y_0) = & V_T(y_T^*(\delta;y_0,0)) - \int_0^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\ \le & (\theta_1-1)\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt - \int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\ \le & (\theta_2(\theta_1-1)-1)\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt,\end{aligned}$$

where θ_1 and θ_2 are defined in Lemma 2.4. Since

$$1-\theta_2(\theta_1-1) = 1-\frac{\gamma^2(T)}{\alpha_\ell^2\delta(T-\delta)} \to 0 \text{ for } T\to\infty,$$

there exist $T^* > \delta$ and $\alpha \in (0,1)$ such that $1 - \theta_2(\theta_1 - 1) \geq \alpha$ for all $T \geq T^*$. This implies (2.15). ◻

THEOREM 2.6 (Suboptimality). *Suppose that* (A1)-(A3) *hold and let a sampling time* $\delta > 0$ *be given. Then there exist numbers* $T^* > \delta$*, and* $\alpha > 0$*, such that for every prediction horizon* $T \geq T^*$*, the receding horizon control* u_{rh} *satisfies*

$$\alpha V_\infty(y_0) \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq V_\infty(y_0) \tag{2.16}$$

for all $y_0 \in \mathcal{B}_{d_2}(0)$ *with some* $d_2 = d_2(T) > 0$ *.*

Proof. The right and left inequalities are obvious, therefore we only need to verify the middle one. For fixed $\delta > 0$ choose T^* and α according to Proposition 2.5. Define $d_2 := \sqrt{((1 + \frac{c_T}{\alpha\alpha_\ell}\gamma(T))^{-1} d_1^2}$ where $T \geq T^*$ and d_1 is defined in Lemma 2.3. We proceed by induction with respect to the receding horizon sampling index k.

First, since $d_2 \leq d_1$ the assumptions of Proposition 2.5 are applicable due to Lemma 2.3, and we have

$$V_T(y_{rh}(t_1)) \leq V_T(y_0) - \alpha \int_0^{t_1} \ell(y_{rh}(t), u_{rh}(t))dt. \tag{2.17}$$

Preceding by induction we assume that

$$y_{rh}(t_k) \in \mathcal{B}_{d_1}(0) \quad \text{for all } k = 0, \ldots, k', \tag{2.18}$$

and that

$$V_T(y_{rh}(t_{k'})) \leq V_T(y_0) - \alpha \int_0^{t_{k'}} \ell(y_{rh}(t), u_{rh}(t))dt \tag{2.19}$$

for $k' \in \mathbb{N}$.

Since $y_{rh}(t_{k'}) \in \mathcal{B}_{d_1}(0)$, by Lemma 2.3 and Proposition 2.5 we have

$$V_T(y_{rh}(t_{k'+1})) \leq V_T(y_{rh}(t_{k'})) - \alpha \int_{t_{k'}}^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt. \tag{2.20}$$

Combined with (2.19) this gives

$$V_T(y_{rh}(t_{k'+1})) \leq V_T(y_0) - \alpha \int_0^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt. \tag{2.21}$$

Moreover, by repeated use of (A3) which is applicable by (2.18) and due to (2.21) we have

$$\begin{aligned}
\|y_{rh}(t_{k'+1})\|_H^2 &\leq \|y_{rh}(t_{k'})\|_H^2 + c_T \int_{t_{k'}}^{t_{k'+1}} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2)dt \\
&\leq \|y_{rh}(0)\|_H^2 + c_T \int_0^{t_{k'+1}} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2)dt \\
&\leq \|y_{rh}(0)\|_H^2 + \frac{c_T}{\alpha_\ell} \int_0^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt \\
&\leq \|y_0\|_H^2 + \frac{c_T}{\alpha\alpha_\ell}(V_T(y_0) - V_T(y_{th}(t_{k'+1}))) \\
&\leq \|y_0\|_H^2 + \frac{c_T}{\alpha\alpha_\ell} V_T(y_0) \leq (1 + \frac{c_T}{\alpha\alpha_\ell}\gamma(T))\|y_0\|_H^2 \leq d_1^2.
\end{aligned}$$

Hence $y_{rh}(t_{k'+1}) \in \mathcal{B}_{d_1}(0)$ which concludes the induction step. Taking the limit $k' \to \infty$ we find

$$\alpha J_\infty(u_{rh}(\cdot), y_0) = \alpha \int_0^\infty \ell(y_{rh}(t), u_{rh}(t))dt \le V_T(y_0), \tag{2.22}$$

which concludes the proof. □

REMARK 2.1. If (2.1) is globally stabilizable, i.e. (2.4) holds with $\mathcal{N}_0$ replaced by H and if also in (A1) is satisfied for all $y_0 \in H$, then Theorem 2.6 holds for all $y_0 \in H$, without the need of (A3). In fact (A3) was only used in the proof of Lemma 2.3 for the construction of $\mathcal{B}_{d_1}(0)$ which is not needed any more if (A2) holds globally.

In the following Theorem, we will show that the value function $V_{T-\delta}$ exponentially decays along the the receding horizon trajectory y_{rh}.

THEOREM 2.7 (Exponentially decay). *Suppose that* (A1)-(A3) *hold and let a sampling time $\delta > 0$ be given. Then there exist numbers $T^* > \delta$, $\alpha > 0$ such that for every prediction horizon $T \ge T^*$, and every $y_0 \in \mathcal{B}_{d_2}(0)$ with $d_2(T) > 0$, the receding horizon trajectory $y_{rh}(\cdot)$ satisfies*

$$V_T(y_{rh}(t_k)) \le e^{-\zeta t_k} V_T(y_0), \tag{2.23}$$

where ζ is a positive number depending on α, δ and T but independent of y_0 . Moreover, for every positive t we have

$$V_{T-\delta}(y_{rh}(t)) \le c e^{-\zeta t} V_T(y_0) \tag{2.24}$$

with a positive constant c depending on α, δ and T, but independent of y_0

Proof. Let $\delta > 0$ be arbitrary. Then according to Theorem 2.6 and (2.15), there exists a positive number T^* such that for every $T \ge T^*$ and $y_0 \in \mathcal{B}_{d_2}(0)$ with $d_2 > 0$ we have

$$V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \le -\alpha \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))dt \quad \text{for every } k \in \mathbb{N}, \tag{2.25}$$

with a positive $\alpha < 1$. Moreover, by using (2.9) and (2.10) we have

$$\begin{aligned} V_T(y_{rh}(t_{k+1})) \le& \theta_1 \int_{t_{k+1}}^{t_k+T} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))\, dt \\ \le& \theta_1\theta_2 \int_{t_k}^{t_{k+1}} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))\, dt \\ =& \theta_1\theta_2 \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))\, dt, \end{aligned} \tag{2.26}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are defined in Lemma 2.4. Now by using (2.25) and (2.26) we obtain

$$V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \le \frac{-\alpha}{\theta_1\theta_2} V_T(y_{rh}(t_{k+1})) \quad \text{for every } k \in \mathbb{N}.$$

Therefore, by defining $\eta := (1 + \frac{\alpha}{\theta_1\theta_2})^{-1}$ for every $k \in \mathbb{N}$ we can write

$$V_T(y_{rh}(t_k)) \le \eta V_T(y_{rh}(t_{k-1})) \le \eta^2 V_T(y_{rh}(t_{k-2})) \le \cdots \le \eta^k V_T(y_0). \tag{2.27}$$

Now by defining $\zeta := \frac{|\ln \eta|}{\delta}$, we obtain the inequality (2.23).

Turning to the inequality (2.24) with $t > 0$ arbitrary, then there exists an index k such that $t \in [t_k, t_{k+1}]$. Now since $T - \delta \leq T + t_k - t$ and by using Bellman's optimality principle we have

$$\begin{aligned} V_{T-\delta}(y_{rh}(t)) &\leq V_{T+t_k-t}(y_{rh}(t)) \\ &= V_T(y_{rh}(t_k)) - \int_{t_k}^{t} \ell(y_T^*(s; y_{rh}(t_k), t_k), u_T^*(s; y_{rh}(t_k), t_k))\, ds \\ &\leq V_T(y_{rh}(t_k)). \end{aligned} \tag{2.28}$$

By using (2.27) and (2.28) we obtain

$$V_{T-\delta}(y_{rh}(t)) \leq V_T(y_{rh}(t_k)) \leq \frac{\eta^{k+1}}{\eta} V_T(y_0) = \frac{1}{\eta} e^{-\zeta t_{k+1}} V_T(y_0) \leq \frac{1}{\eta} e^{-\zeta t} V_T(y_0). \qquad \square$$

REMARK 2.2. The above result is similar to the result obtained in [20] (Theorem 2.4), if the value function $V_{T-\delta}$ is considered as a control Lyapunov function G. At every iteration k of Algorithm 1 for every open-loop optimal control problem we have

$$\begin{aligned} &\min_{u \in L^2(t_k, t_k+T; U)} J_T(u, y_{rh}(t_k)) \\ &= \int_{t_k}^{T+t_k} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))dt \\ &= \int_{t_k}^{\delta+t_k} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))dt + V_{T-\delta}(y_T^*(\delta + t_k; y_{rh}(t_k), t_k)) \\ &= \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))dt + V_{T-\delta}(y_{rh}(t_{k+1})). \end{aligned}$$

This means that the terminal cost $V_{T-\delta}$ is implicitly added to the objective function of every open-loop optimal control problem. Indeed $V_{t-\delta}$ can be interpreted as an approximation of the infinite horizon value function V_∞ which is incorporated in the objective function of every open-loop problem.

REMARK 2.3. Note that the inequality (2.24) does not imply the asymptotic stability of receding horizon trajectory $y_{rh}(\cdot)$ in the space H, unless the finite horizon value function $V_{t-\delta}$ is uniformly positive on the level-sets of $V_{t-\delta}$. That is, for every positive $r > 0$ we have

$$V_{T-\delta}(y) \geq C\|y\|_H^2 \quad \text{for all } y \in \Pi_r,$$

where C is a positive constant depending on the time horizon $T - \delta$ and Π_r is defined by

$$\Pi_r := \{y \in H \mid V_{T-\delta}(y) \leq r\}.$$

In the case of finite-dimensional controlled systems, the above condition was investigated in [30]. However for infinite-dimensional controlled systems, this condition only holds in special cases [14].

3. Stabilization of Burgers' Equation. Here we apply results of the previous section to the stabilization of the viscous Burgers' equation with periodic and Neumann boundary conditions, respectively. For these boundary conditions the origin of the uncontrolled system is stable but not asymptotically stable. In the case of Dirichlet boundary conditions, on the other hand, the origin is asymptotically stable.

We shall investigate Assumptions (A1)-(A3) and show that in the case of periodic boundary conditions Algorithm 1 provides globally stabilizing controls, while for Neumann boundary conditions we obtain locally stabilizing controls.

3.1. Burgers' Equation with Periodic Boundary Conditions. For an arbitrary finite horizon $T > 0$, we consider the controlled Burgers' equation with the periodic boundary conditions of the form

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu(t) & \text{in } (0,T)\times(0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases} \tag{3.1}$$

Throughout $\mu > 0$ and $y_0 \in L^2(0,1)$ are fixed, and the control operator B is the extension-by-zero operator given by

$$(Bu)(x) = \begin{cases} u(x) & x \in \hat{\Omega}, \\ 0 & x \in (0,1)\backslash\hat{\Omega}, \end{cases}$$

with $\hat{\Omega}$ a nonempty open subset of $(0,1)$.

For the function space setting of (3.1) we introduce the spaces

$$V := \{y \in H^1(0,1) \mid y(0) = y(1)\}, \quad H := L^2(0,1),$$

and

$$W(0,T) := \{\phi : \phi \in L^2(0,T;V), \quad \frac{d}{dt}\phi \in L^2(0,T;V^*)\},$$

where V^* is the adjoint space of V. The spaces H and V are endowed with the usual norms $\|\cdot\|_H := \|\cdot\|_{L^2(0,1)}$ and $\|\cdot\|_V := \|\cdot\|_{H^1(0,1)}$. Further $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{V^*,V}$ denote the inner product in H and the duality pairing between V and V^*. We recall that $W(0,T)$ is continuously embedded into $C([0,T];H)$, see e.g. [31]. It will be convenient to define the continuous trilinear form $b : V \times V \times V \to \mathbb{R}$ by

$$b(\varphi,\psi,\phi) = \int_0^1 \varphi\psi_x\phi dx.$$

We shall frequently use the property that

$$b(y,y,y) = \int_0^1 y_x y^2 dx = \frac{1}{3}(y^3(1) - y^3(0)) = 0 \quad \text{for all } y \in V. \tag{3.2}$$

It is well-known that for every control $u \in L^2(0,T;L^2(\hat{\Omega}))$, equation (3.1) admits a unique weak solution $y \in W(0,T)$, i.e. y satisfies $y(0) = y_0$ in H, and for a.e. $t \in (0,T)$,

$$\frac{d}{dt}\langle y(t),\varphi\rangle_{V^*,V} + \mu\langle y(t),\varphi\rangle_V - \mu\langle y(t),\varphi\rangle_H + b(y(t),y(t),\varphi) = \langle Bu(t),\varphi\rangle_H \tag{3.3}$$

holds for all $\varphi \in V$. Using (3.2) and Gronwall's lemma it can easily be shown that there exists a constant C_T such that

$$|y(\cdot; y_0, u)|_{W(0,T)} \leq C_T(|y_0|_H + |u|_{L(0,T;L^2(\hat{\Omega}))}), \tag{3.4}$$

where $y(\cdot; y_0, u)$ indicates the dependence of the solution on y_0 and u. The running cost will be taken of the form

$$l(y, u) := \frac{1}{2}\|y\|_H^2 + \frac{\beta}{2}\|u\|_{L^2(\hat{\Omega})}^2, \tag{3.5}$$

where $\beta > 0$.

We have now specified all items of the finite horizon problem (P_T). Using (3.4) it follows from standard subsequential limit arguments that (P_T) with the control system given by (3.1) admits a solution for each $y_0 \in H$. In particular (A1) holds with $\mathcal{N}_0 = H$. In the following lemma we show that Assumption (A2) holds as well.

Lemma 3.1 (Global stabilizability). *For each $T > 0$ and initial state $y_0 \in H$ there exists a control $\hat{u}(\cdot; y_0) \in L^2(0, T; L^2(\hat{\Omega}))$ such that*

$$V_T(y_0) \leq J_T(\hat{u}, y_0) \leq \gamma(T)\|y_0\|_H^2, \tag{3.6}$$

for a continuous, non-decreasing and bounded function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. Set $u(t) := -y(t)|\hat{\Omega}$ and consider

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) - By(t)|\hat{\Omega} & \text{in } (0,T) \times (0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases} \tag{3.7}$$

By multiplying the first equation in (3.7) with $y(t)$ from both sides and taking the L^2-scalar product we have for a.e. $t \in (0,T)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu(\|y_x(t)\|_H^2 - y_x(t,1)y(t,1) + y_x(t,0)y(t,0)) \\ + b(y(t), y(t), y(t)) + \|y(t)\|_{L^2(\hat{\Omega})}^2 = 0. \end{aligned} \tag{3.8}$$

Taking into account the boundary conditions and (3.2) one can express (3.8) as

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + \|y(t)\|_{L^2(\hat{\Omega})}^2 = 0.$$

One can easily show that $\|y\|_1^2 := \|y_x\|_H^2 + \frac{1}{\mu}\|y\|_{L^2(\hat{\Omega})}^2$ is a norm which is equivalent to the H^1-norm [28], Page 26. Thus there exist positive constants $c_2 > c_1 > 0$ such that

$$c_1\|y\|_V^2 \leq \|y\|_1^2 \leq c_2\|y\|_V^2 \quad \text{for all } y \in V,$$

and consequently

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu c_1\|y(t)\|_V^2 \leq 0,$$

and therefore,

$$\frac{d}{dt}\|y(t)\|_H^2 + 2\mu c_1\|y(t)\|_H^2 \leq 0 \quad \text{for all } t \in [0,T].$$

Multiplying both sides of the above equation by $e^{2\mu c_1 t}$ and integrating from 0 to t we obtain

$$\|y(t)\|_H^2 \leq \|y_0\|_H^2 e^{-2\mu c_1 t} \quad \text{for all } t \in [0,T].$$

By integrating the above inequality over the interval $[0,T]$, we obtain

$$\int_0^T \|y(t)\|_H^2 dt \leq \frac{1}{2\mu c_1}(1-e^{-2\mu c_1 T})\|y_0\|_H^2. \tag{3.9}$$

By the definition of the value function $V_T(\cdot)$ and (3.9) we have

$$V_T(y_0) \leq \int_0^T (\frac{1}{2}\|y(t)\|_H^2 + \frac{\beta}{2}\| - y(t)\|_{L^2(\hat{\Omega})}^2)\, dt \leq \frac{1+\beta}{4\mu c_1}(1-e^{-2\mu c_1 T})\|y_0\|_H^2,$$

and (A2) follows with $\gamma(T) := \frac{1+\beta}{4\mu c_1}(1-e^{-2\mu c_1 T})$, and $\mathcal{N}_0 = H$. □

From Lemma 3.1 assumption (A2) holds globally. Thus by Remark 2.1 we can directly apply Theorem 2.6 without addressing (A3) and conclude that: for any arbitrary sampling time δ, there exists a positive T^* such that for every $T \geq T^*$ the receding horizon control u_{rh} is globally suboptimal (within H) with suboptimality factor $\alpha > 0$, and we have

$$\alpha V_\infty(y_0) \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq V_\infty(y_0) \tag{3.10}$$

for every $y_0 \in H$. Now it remains for us to show that the receding horizon control u_{rh} computed by Algorithm 1 is globally stabilizing. This property will be verified by means of the following theorem.

THEOREM 3.2. *Let $y_0 \in H$ and a sampling time $\delta > 0$ be given, and suppose that Algorithm 1 is applied for the stabilization of the Burgers' equation* (3.1) *with a prediction horizon $T \geq T^*$, where T^* is introduced by Proposition 2.5. Then the receding horizon trajectory y_{rh} is asymptotically stable.*

Proof. First, we show that

$$\|y_{rh}\|_{L^\infty(0,\infty;H)} \leq \nu \|y_0\|_H \tag{3.11}$$

for a constant $\nu > 0$.

Due to (3.5), (3.6) and (3.10), we have

$$\frac{\alpha \min\{1,\beta\}}{2} \int_0^\infty \left(\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2\right) dt$$
$$\leq \alpha J_\infty(u_{rh}(\cdot), y_0) \leq V_T(y_0) \leq \gamma(T)\|y_0\|_H^2.$$

Therefore by choosing $\sigma_1 := \frac{2\gamma(T)}{\alpha \min\{1,\beta\}}$, we obtain

$$\int_0^\infty \|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt \leq \sigma_1 \|y_0\|_H^2. \tag{3.12}$$

Moreover, the receding horizon state given by Algorithm 1 satisfies $y_{rh} \in C([0,\infty), H)$, for every $k \in \mathbb{N}$ we have

$$y_{rh}\mid_{(t_k,t_{k+1})} \in L^2(t_k,t_{k+1};V), \quad \frac{d}{dt} y_{rh}\mid_{(t_k,t_{k+1})} \in L^2(t_k,t_{k+1};V^*), \tag{3.13}$$

and y_{rh} is the solution of

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu_{rh}(t) & \text{in } (t_k, t_{k+1}) \times (0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (t_k, t_{k+1}), \\ y(t_k,\cdot) = y_{rh}(t_k) \text{ for } k > 0, \text{ and } y(0,\cdot) = y_0 \text{ for } k = 0 & \text{in } (0,1). \end{cases}$$

By multiplying the above equation by $y_{rh}(\cdot)$ and integrating over the interval $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 = \langle Bu_{rh}(t), y_{rh}(t)\rangle_H \quad \text{a.e. } t \in (t_k, t_{k+1}).$$

From Cauchy-Schwarz and Young's inequalities we infer that

$$\frac{d}{dt}\|y_{rh}(t)\|_H^2 + 2\mu\|(y_{rh})_x(t)\|_H^2 \leq \|u_{rh}(t)\|^2_{L^2(\hat{\Omega})} + \|y_{rh}(t)\|_H^2 \quad \text{a.e. } t \in (t_k, t_{k+1}).$$

Integrating from t_k to t for every $t \in (t_k, t_{k+1})$ we have

$$\begin{aligned} &\|y_{rh}(t)\|_H^2 \\ &\qquad \leq \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|^2_{L^2(\hat{\Omega})} ds + \int_{t_k}^t \|y_{rh}(s)\|_H^2 \, ds \quad \text{for all } t \in (t_k, t_{k+1}). \end{aligned}$$

By the same estimate as above for the interval (t_{k-1}, t_k) we have

$$\|y_{rh}(t_k)\|_H^2 \leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t_k} \|u_{rh}(s)\|^2_{L^2(\hat{\Omega})} ds + \int_{t_{k-1}}^{t_k} \|y_{rh}(s)\|_H^2 \, ds. \tag{3.14}$$

Moreover by the above two estimates we have

$$\|y_{rh}(t)\|_H^2 \leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t} \|u_{rh}(s)\|^2_{L^2(\hat{\Omega})} ds + \int_{t_{k-1}}^{t} \|y_{rh}(s)\|_H^2 \, ds.$$

By repeating the above argument for $k-2, k-3, \ldots, 0$, one can show that

$$\begin{aligned} \|y_{rh}(t)\|_H^2 \leq& \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|^2_{L^2(\hat{\Omega})} ds + \int_{t_k}^t \|y_{rh}(s)\|_H^2 \, ds \\ \leq& \|y_0\|_H^2 + \int_0^t \|u_{rh}(s)\|^2_{L^2(\hat{\Omega})} ds + \int_0^t \|y_{rh}(s)\|_H^2 \, ds \\ \leq& \|y_0\|_H^2 + \int_0^\infty \|u_{rh}(s)\|^2_{L^2(\hat{\Omega})} ds + \int_0^\infty \|y_{rh}(s)\|_H^2 \, ds \\ \leq& (1+\sigma_1)\|y_0\|_H^2, \end{aligned}$$

where in the last line (3.12) has been used. Choosing $\nu := \sqrt{1+\sigma_1}$ we obtain (3.11).

Next we are in the position to prove

$$\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 = 0.$$

For every $t'' \geq t'$ we have

$$\begin{aligned}\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 &= \int_{t'}^{t''} \frac{d}{dt}\|y_{rh}(t)\|_H^2\,dt \\ &= 2\int_{t'}^{t''} \langle y_{rh}(t), \mu(y_{rh})_{xx}(t) - (y_{rh})_x(t)y_{rh}(t) + Bu_{rh}(t)\rangle_{V,V^*}\,dt \\ &= -2\mu\int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H\,dt \\ &\leq 2\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})}\|y_{rh}(t)\|_H\,dt \\ &\leq 2\big(\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt\big)^{\frac{1}{2}}\big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\big)^{\frac{1}{2}},\end{aligned}$$

and thus

$$\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 \leq 2\sqrt{\sigma_1}\nu\|y_0\|_H^2(t''-t')^{\frac{1}{2}}. \tag{3.15}$$

For the last inequality, (3.11) and (3.12) have been used. Moreover, from (3.5), (3.6) and (3.10) we have

$$\frac{\alpha}{2}\int_0^\infty \|y_{rh}(t)\|_{L^2(\hat{\Omega})}^2 \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq \gamma(T)\|y_0\|_H^2 < \infty.$$

This estimate implies that

$$\lim_{t\to\infty}\int_{t-L}^{t} \|y_{rh}(s)\|_H^2 ds = 0 \tag{3.16}$$

for all $L > 0$. Suppose to the contrary that

$$\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 \neq 0.$$

Then there exists an $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^\infty$ with $t_n > 0$ and $\lim_{n\to\infty} t_n = \infty$ for which

$$\|y_{rh}(t_n)\|_H^2 > \epsilon \quad \text{for all } n = 1, 2, \ldots. \tag{3.17}$$

It follows from (3.15) and (3.17) that for every $L > 0$ and $n = 1, 2, \ldots$

$$\begin{aligned}&\int_{t_n-L}^{t_n} \|y_{rh}(t)\|_H^2 dt \\ &= \int_{t_n-L}^{t_n} \|y_{rh}(t_n)\|_H^2 dt - \int_{t_n-L}^{t_n} \big(\|y_{rh}(t_n)\|_H^2 - \|y_{rh}(t)\|_H^2\big)dt, \\ &> L\epsilon - 2\sqrt{\sigma_1}\nu\|y_0\|_H^2\int_{t_n-L}^{t_n}(t_n-t)^{\frac{1}{2}}dt = L\epsilon - \frac{4}{3}\sqrt{\sigma_1}\nu\|y_0\|_H^2 L^{\frac{3}{2}}.\end{aligned} \tag{3.18}$$

Setting $\omega := \frac{4}{3}\sqrt{\sigma_1}\nu\|y_0\|_H^2$, and choosing $L := (\frac{\epsilon}{2\omega})^2$, we obtain

$$\int_{t_n-L}^{t_n} \|y_{rh}(t)\|_H^2 dt > \frac{L\epsilon}{2} \quad \text{for all } n = 1, 2, \ldots.$$

This contradicts (3.16). Hence $\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 = 0$ and the proof is complete.
∎

3.2. Burgers' Equation with Homogeneous Neumann Boundary Conditions. Here we consider the controlled Burgers' equation with homogenous Neumann boundary conditions of the form

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu(t) & \text{in } (0,T)\times(0,1), \\ y_x(t,0) = y_x(t,1) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases} \tag{3.19}$$

We can utilize the same notation as in Section 3.1, except for the energy space which is now chosen to be

$$V := \{y \in H^1(0,1) \mid y_x(0) = y_x(1) = 0\}.$$

Again $V \subset H \subset V^*$ is a Gelfand triple and $W(0,T)$ is continuously embedded in $C([0,T];H)$.

The significant difference between (3.19) and (3.1) is given by the fact that in the case of periodic boundary conditions the nonlinearity is conservative, i.e. we have that $b(\phi,\phi,\phi) = 0$ for all $\phi \in V$, which is not the case for Neumann boundary conditions. As a consequence we have to rely on the local version of the results of Section 2.

Again we use the weak or variational solution concept of (3.3). Due to the fact that the nonlinearity is not conservative the verification of a global weak solution is not trivial. We have the following result.

Lemma 3.3. *For every $T > 0$, and for every $y_0 \in H$, and $u \in L^2(0,T;L^2(\hat{\Omega}))$ there exists a unique solution $y(\cdot;y_0,u) \in W(0,T)$ to* (3.19)*. Moreover there exists a constant C_T such that*

$$|y(\cdot;y_0,u)|_{W(0,T)} \leq C_T(1 + |y_0|_H + |u|_{L(0,T;L^2(\hat{\Omega}))}),$$

for all $y_0 \in H$, and $u \in L^2(0,T;L^2(\hat{\Omega}))$.

For the proof we refer to [34]. For the step that the local solution can be extended to a global one we prefer the argument given in [15] for which it is useful to recall that for a measurable function, which will be u in our case, the function $E \to \int_E |u(t)|_{L^2(\hat{\Omega})}\, dt$, with E a measurable subset of $(0,T)$, is absolutely continuous.

The running cost will again be taken to be of the form (3.5). It is now standard to argue the existence of a solution to (P_T) with the control system given by (3.19). In particular (A1) holds with $\mathcal{N}_0 = H$. In the following lemmas we show that Assumptions (A2) and (A3) holds as well.

Lemma 3.4 (Local stabilizability). *There exists a neighborhood $\mathcal{B}_{\delta_1}(0) \subset H$ such that for every $T > 0$ and every $y_0 \in \mathcal{B}_{\delta_1}(0)$ there exists a control $\hat{u}(\cdot,y_0) \in L^2(0,T;L^2(\hat{\Omega}))$ with*

$$V_T(y_0) \leq J_T(\hat{u},y_0) \leq \gamma(T)\|y_0\|_H^2,$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, non-decreasing and bounded function.

Proof. Setting $\hat{u}(t) := -y(t)|\hat{\Omega}$ in the first equation of (3.19), multiplying $y(t)$ and taking the L^2-scalar product we obtain

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + b(y(t),y(t),y(t)) + \|y(t)\|^2_{L^2(\hat{\Omega})} = 0 \tag{3.20}$$

As in the case of periodic boundary conditions one can argue that $\|y\|_1^2 := \|y_x\|_H^2 + \frac{1}{\mu}\|y\|^2_{L^2(\hat{\Omega})}$ is an equivalent norm to H^1-norm, see e.g. [28], Page 26, and hence there exist positive constants $c_2 > c_1 > 0$ such that $c_1\|y\|_V^2 \le \|y\|_1^2 \le c_2\|y\|_V^2$ for all $y \in V$. The nonlinearity satisfies the following equality

$$b(y,y,y) = \int_0^1 y_x y^2 dx \le \|y\|_{L^\infty(0,1)}\|y_x\|_H\|y\|_H \le c_a\|y\|_1^2\|y\|_H \quad \text{for all } y \in V,$$

where c_a, depends on the embedding constant of V into $L^\infty(0,1)$ and c_1. From (3.20) we therefore deduce that

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + \|y(t)\|_{L^2(\hat{\Omega})}^2 \le c_a\|y(t)\|_1^2\|y(t)\|_H,$$

and consequently

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y(t)\|_1^2 \le c_a\|y(t)\|_1^2\|y(t)\|_H.$$

Now let us choose $\|y_0\|_H$ sufficiently small, say $\|y_0\|_H \le \frac{\mu}{4c_a}$. Then by continuity of the solution for a short interval of time $[0,T^*]$, we have $\|y(t)\|_H \le \frac{\mu}{2c_a}$ for all $t \in [0,T^*]$ and further

$$\frac{d}{dt}\|y(t)\|_H^2 + \mu c_1\|y(t)\|_H^2 \le 0 \quad \text{for all } t \in [0,T^*].$$

Multiplying both sides of the above equation by $e^{\mu c_1 t}$ and integrating from 0 to t we obtain

$$\|y(t)\|_H^2 \le \|y_0\|_H^2 e^{-\mu c_1 t} \le (\frac{\mu}{4c_a})^2 \quad \text{for all } t \in [0,T^*]. \tag{3.21}$$

Repeating the above argument implies that $\|y(t)\|_H \le \frac{\mu}{4c_a}$ will remain small for all $t \in [0,T]$ and moreover we have

$$\|y(t)\|_H^2 \le \|y_0\|_H^2 e^{-\mu c_1 t} \quad \text{for all } t \in [0,T].$$

Integration over $[0,T]$ implies that

$$\int_0^T \|y(t)\|_H^2 dt \le \frac{1}{\mu c_1}(1 - e^{-\mu c_1 T})\|y_0\|_H^2. \tag{3.22}$$

By the definition of the value function $V_T(\cdot)$ and (3.22) we have

$$V_T(y_0) \le \int_0^T \frac{1}{2}\|y(t)\|_H^2 + \frac{\beta}{2}\|y(t)\|_{L^2(\hat{\Omega})}^2 dt \le \frac{1+\beta}{2\mu c_1}(1 - e^{-\mu c_1 T})\|y_0\|_H^2,$$

where $\gamma(T) := \frac{1+\beta}{2\mu c_1}(1 - e^{-\mu c_1 T})$ is a nondecreasing, continuous and bounded function, as desired, and $\delta_1 := \frac{\mu}{4c_a}$. ∎

Lemma 3.5. *Assumption* (A3) *holds for* (3.19) *with* $\mathcal{N}_0 = \mathcal{B}_{\delta_1}(0)$ *defined in Lemma 3.4.*

Proof. For every $y_0 \in \mathcal{B}_{\delta_1}(0)$ we have from (3.19) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y(t)\|_V^2 \\ &\quad \le \mu\|y(t)\|_H^2 + |b(y(t),y(t),y(t))| + |\langle y(t), Bu(t)\rangle_H| \quad \text{a.e } t \in [0,T]. \end{aligned} \tag{3.23}$$

From Agmon's inequality we recall that there exists a constant c_A such that

$$\|\phi\|_{L^\infty(0,1)} \leq c_A \|\phi\|_H^{\frac{1}{2}} \|\phi\|_V^{\frac{1}{2}} \quad \text{for all } \phi \in V,$$

and consequently there exits a constant c_I such that

$$b(\phi,\phi,\phi) \leq \|\phi\|_{L^\infty(0,1)} \|\phi\|_H \|\phi\|_V \leq c_A \|\phi\|_H^{\frac{3}{2}} \|\phi\|_V^{\frac{3}{2}} \leq \mu \|\phi\|_V^2 + c_I \|\phi\|_H^6 \quad \text{for all } \phi \in V.$$

Utilizing the above inequality and (3.23) we obtain

$$\frac{d}{dt}\|y(t)\|_H^2 \leq 2\mu \|y(t)\|_H^2 + 2c_I \|y(t)\|_H^6 + 2\|y(t)\|_H \|u(t)\|_{L^2(\hat{\Omega})} \qquad \text{for a.e } t \in [0,T].$$

Upon integration we obtain

$$\|y(t)\|_H^2 \leq \|y(0)\|_H^2 + (2\mu + 1 + 2c_I \|y\|^4_{C([0,T];H)}) \int_0^t \|y(s)\|_H^2 \, ds + \int_0^t \|u(s)\|^2_{L^2(\hat{\Omega})} ds.$$

By Lemma 3.3 the family

$$\{\|y(\cdot;y_0,u)\|_{C([0,T];H)} \,|\, y_0 \in \mathcal{B}_{\delta_1}(0), \|u\|_{L^2(0,T;L^2(\hat{\Omega}))} \leq \sqrt{2\gamma(T)/\min\{1,\beta\}}\, \|y_0\|_H\}$$

is bounded, and hence the desired estimate follows. ∎

Now we are in the position that we can apply Theorem 2.6 and it remains for us to show that the receding horizon control u_{rh} computed by Algorithm 1 is stabilizing. This will be accomplished in the following theorem. It uses the quantifier $d_2(T)$ for the size of the neighborhood of the initial data. Recall that $d_2(T)$ depends on $\gamma(T)$, which was given explicitly in the proof of Lemma 3.4 and on c_T, the existence of which was provided in the proof of Lemma 3.5.

Theorem 3.6. *Let a sampling time $\delta > 0$ be given, and suppose that Algorithm 1 is applied for the stabilization of the Burgers' equation* (3.19) *with a perdition horizon $T \geq T^*$, where T^* is introduced by Proposition 2.5. Then the receding horizon trajectory y_{rh} is asymptotically stable provided that $|y_0|_H \leq d_2(T)$.*

Proof. We recall that $\delta_1 = \frac{\mu}{4c_a}$ depends on embedding constants and was introduced in the proof of Lemma 3.4. Furthermore, we have $d_2(T) \leq d_1(T) \leq \delta_1$. To verify the claim we can follow for the most part the proof of Theorem 3.2. Again we first show that there exists some $\nu > 0$ such that

$$\|y_{rh}\|_{L^\infty(0,\infty;H)} \leq \nu \|y_0\|_H, \tag{3.24}$$

for each $y_0 \in \mathcal{B}_{d_2}(0)$. By construction we have that $y_{rh} \in C([0,\infty),H)$, that (3.13) holds, and that

$$\int_0^\infty \|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|^2_{L^2(\hat{\Omega})} dt \leq \sigma_1 \|y_0\|_H^2, \tag{3.25}$$

where $\sigma_1 := \frac{2\gamma(T)}{\alpha \min\{1,\beta\}}$. For any $k = 0,1,\dots$ we have

$$\begin{aligned}\frac{d}{2dt}\|y_{rh}(t)\|_H^2 + \mu \|(y_{rh})_x(t)\|_H^2 &+ b(y_{rh}(t), y_{rh}(t), y_{rh}(t)) \\ &= \langle B u_{rh}(t), y_{rh}(t)\rangle_H \quad \text{a.e. } t \in (t_k, t_k+1).\end{aligned}$$

Furthermore, d_2 in Theorem 2.6 has been chosen in such way that for every $t > 0$, the receding horizon trajectory $y_{rh}(t)$ stays in the neighborhood $\mathcal{B}_{\delta_1}(0)$. In other word, we have

$$\|y_{rh}(t)\|_H \leq \delta_1 = \frac{\mu}{4c_a} < \frac{\mu}{2c_a} \quad \text{for all } t > 0, \tag{3.26}$$

where c_a defined in Lemma 3.4. Now by Cauchy-Schwarz and Young's inequalities, by (3.26) and the definition of $\|\cdot\|_1$, we infer that

$$\begin{aligned} b(y_{rh}(t),&y_{rh}(t), y_{rh}(t)) \\ &\leq \|y_{rh}(t)\|_{L^\infty(0,1)} \|y_{rh}(t)\|_H \|(y_{rh})_x(t)\|_H \\ &\leq c_a \|y_{rh}(t)\|_H \|y_{rh}(t)\|_1^2 \\ &\leq \frac{\mu}{2}\|y_{rh}(t)\|_1^2 \leq \frac{\mu}{2}\|(y_{rh})_x(t)\|_H^2 + \frac{1}{2}\|y_{rh}(t)\|_H^2 \quad \text{for a.e. } t \in (t_k, t_{k+1}). \end{aligned}$$

Thus for every $k \in \mathbb{N}$ we have

$$\frac{d}{dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 \leq \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 + 2\|y_{rh}(t)\|_H^2 \quad \text{for a.e. } t \in (t_k, t_{k+1}),$$

and therefore for $t \in (t_k, t_{k+1})$

$$\begin{aligned} \|y_{rh}(t)\|_H^2 + &\mu \int_{t_k}^t \|(y_{rh})_x(s)\|_H^2 ds \\ &\leq \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_k}^t \|y_{rh}(s)\|_H^2 ds. \end{aligned} \tag{3.27}$$

By the same estimate as above for the interval (t_{k-1}, t_k) we have

$$\begin{aligned} \|y_{rh}(t_k)\|_H^2 + &\mu \int_{t_{k-1}}^{t_k} \|(y_{rh})_x(s)\|_H^2 ds \\ &\leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t_k} \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_{k-1}}^{t_k} \|y_{rh}(s)\|_H^2 ds. \end{aligned} \tag{3.28}$$

By summing (3.27) and (3.28) we have

$$\begin{aligned} \|y_{rh}(t)\|_H^2 + &\mu \int_{t_{k-1}}^t \|(y_{rh})_x(s)\|_H^2 ds \\ &\leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_{k-1}}^t \|y_{rh}(s)\|_H^2 ds. \end{aligned}$$

Repeating the above argument for $k-2, k-3, \ldots, 0$, it follows that

$$\begin{aligned} \|y_{rh}(t)\|_H^2 \leq& \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_k}^t \|y_{rh}(s)\|_H^2 ds \\ \leq& \|y_0\|_H^2 + \int_0^\infty \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_0^\infty \|y_{rh}(s)\|_H^2 ds \\ \leq& (1 + 2\sigma_1)\|y_0\|_H^2, \end{aligned}$$

where for the last inequality (3.25) has been used. Choosing $\nu := \sqrt{1+2\sigma_1}$ we obtain (3.24).

Now we are in the position to prove

$$\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 = 0.$$

For every $t'' \geq t'$ we have

$$\begin{aligned}
\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 &= \int_{t'}^{t''} \frac{d}{dt}\|y_{rh}(t)\|_H^2 dt \\
&= 2\int_{t'}^{t''} \langle y_{rh}(t), \mu(y_{rh})_{xx}(t) - (y_{rh})_x(t)y_{rh}(t) + Bu_{rh}(t)\rangle_{V,V^*} dt \\
&\leq -2\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + 2\int_{t'}^{t''} c_a \|y_{rh}(t)\|_H \|y_{rh}(t)\|_1^2 dt \\
&\quad + 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H dt \\
&\leq -\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + \int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt + 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H dt \\
&\leq 2\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})} \|y_{rh}(t)\|_H dt + \int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt \\
&\leq 2\Big(\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}} \\
&\leq 3\sqrt{\sigma_1}\nu \|y_0\|_H^2 (t''-t')^{\frac{1}{2}},
\end{aligned}$$

where (3.25) and (3.24) were used to obtain the last inequality. Now the proof can be completed as the one for Theorem 3.2, except that the factor 2 in (3.18) has to be replaced by the factor 3 which appeared in the last estimate. ∎

4. Numerical Results. We present numerical results to illustrate the theoretical findings of the previous sections. For the Burgers' equation with periodic -, or homogeneous Neumann boundary conditions, every nonzero constant function is a steady state of the uncontrolled equation. Hence the origin is stable, but it is not asymptotically stable. Consequently it is of interest to force the state of these equations to the steady state by an external control which is computed on the basis of the receding horizon control.

Our numerical experiments will also include a comparison of the performance of the receding horizon control scheme with and without terminal penalty term. The latter case was investigated in the previous section, the former one in [20], where it was shown that the quadratic penalty term $G(y) = \frac{1}{2}\|y\|_{L^2(\Omega)}^2$, can be used as a control Lyapunov function for the Navier-Stokes eqaution.

Our numerical tests utilize the following Algorithm 2, where G is chosen as one of the two cost functionals:

$$\text{(4.1)} \qquad \textit{Zero: } G(y) = 0, \text{ or } \textit{Quadratic: } G(y) = \frac{1}{2}\|y\|_{L^2(0,1)}^2.$$

Algorithm 2

Input: Let a final computational time horizon T_∞, and an initial state $y_0 \in L^2(0,1)$ be given.

1: Choose a prediction horizon $T < T_\infty$, and a sampling time $\delta \in (0, T]$.

2: Consider a grid $0 = t_0 < t_1 \cdots < t_r = T_\infty$ on the interval $[0, T_\infty]$, where $t_i := i\delta$ for $i = 0, \ldots, r$.

3: Solve successively the open-loop subproblem on $[t_i, t_i + T]$:

$$\min \frac{1}{2} \int_{t_i}^{t_i+T} \|y(t)\|^2_{L^2(0,1)} dt + \frac{\beta}{2} \int_{t_i}^{t_i+T} \|u(t)\|^2_{L^2(\hat{\Omega})} dt + G(y(t_i + T)), \tag{4.2}$$

subject to Burger' equations (3.19) (or (3.1)) for the initial condition

$$y(t_i) = y^*_T(t_i) \text{ if } i \geq 1 \text{ and } y(t_i) = y_0 \text{ if } i = 0,$$

where $y^*_T(\cdot)$ is the solution to the preceding subproblem on $[t_{i-1}, t_{i-1} + T]$.

4: The receding horizon optimal pair $(y_{rh}(\cdot), u_{rh}(\cdot))$ is obtained by concatenation of the optimal pairs $(y^*_T(t), u^*_T(t))$ of the finite horizon subproblems on $[t_i, t_{i+1}]$ for $i = 0, \ldots, r-1$.

For $G(y) = 0$, Algorithm 2 essentially coincides with Algorithm 1, except for the fact that we need to terminate our computations at some $T_\infty < \infty$.

The numerical simulations were carried out on the MATLAB platform. Throughout, the spatial discretization was done by the standard Galerkin method based on piecewise linear basis functions with mesh-size $h = 0.0125$. The ordinary differential equations resulting after spatial discretization were solved by the implicit Euler method with step-size $\Delta t = 0.0125$, where the nonlinear systems of equations within the implicit Euler method were solved by Newton's method. Every open-loop problem was solved by applying the Barzilai-Browein (BB) gradient steps [6] with a nonmonotone line search [13] on the reduced problem in the "first optimize, then discretize" manner. For every open-loop problem, the optimization algorithm was terminated when L^2-norm of the gradient for the reduced objective function was less than the tolerance 10^{-6}. Furthermore in all examples, we set $\delta = 0.25$ and $\beta = 10^{-3}$.

For every example, we implemented the receding horizon strategy for different choices of the prediction horizon T, and the two terminal costs G in (4.1). Furthermore, in order to have a measure for the performance of the receding horizon strategy, we consider

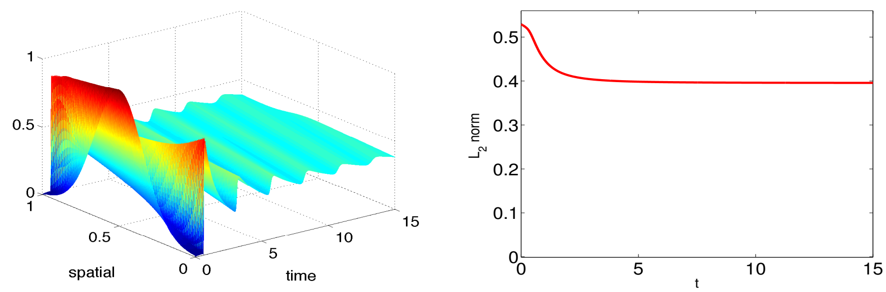
1. $J_{T_\infty}(u_{rh}, y_0) := \frac{1}{2} \int_0^{T_\infty} \|y_{rh}(t)\|^2_{L^2(0,1)} dt + \frac{\beta}{2} \int_0^{T_\infty} \|u_{rh}(t)\|^2_{L^2(\hat{\Omega})} dt$,
2. $\|y_{rh}\|_{L^2(Q)}$ with $Q := (0, T_\infty) \times (0, 1)$,
3. $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$,
4. $T.I$: The total number of iterations (BB-gradient steps) that the optimizer needs for all open-loop problems on the intervals (t_i, t_{i+1}) for $i = 0, \ldots, r-1$.

EXAMPLE 4.1. We considered the Burgers' equation (3.1) with periodic boundary conditions. We chose $y_0(x) = \exp(-20(x - 0.5)^2)$ as the initial function, $\mu = 10^{-3}$ as a viscosity parameter, and $T_\infty = 15$. Further the receding horizon control acts only on the set

$$\hat{\Omega} = (0.1, 0.2) \cup (0.4, 0.6) \cup (0.8, 0.9) \subset (0, 1).$$

Figures 1(a) and 1(b) depict, respectively, the solution and the evolution of the $L^2(0,1)$-norm for the state of the uncontrolled Burgers' equation (3.1). For the uncontrolled solution y^u we have

$$\|y^u\|_{L^2(Q)} = 1.5768, \qquad \|y^u(T_\infty)\|_{L^2(0,1)} = 0.3958.$$



(a) State (b) Evolution of $L^2(0,1)$-norm of state

Fig. 1. *Uncontrolled solution for Example 4.1*

The results of Algorithm 2 for different choices of T and G and the fixed sampling time $\delta = 0.25$ are summarized in Table 1. Figure 2 shows the results for the receding horizon pairs (y_{rh}, u_{rh}) if $G = 0$ and $T = 1$.

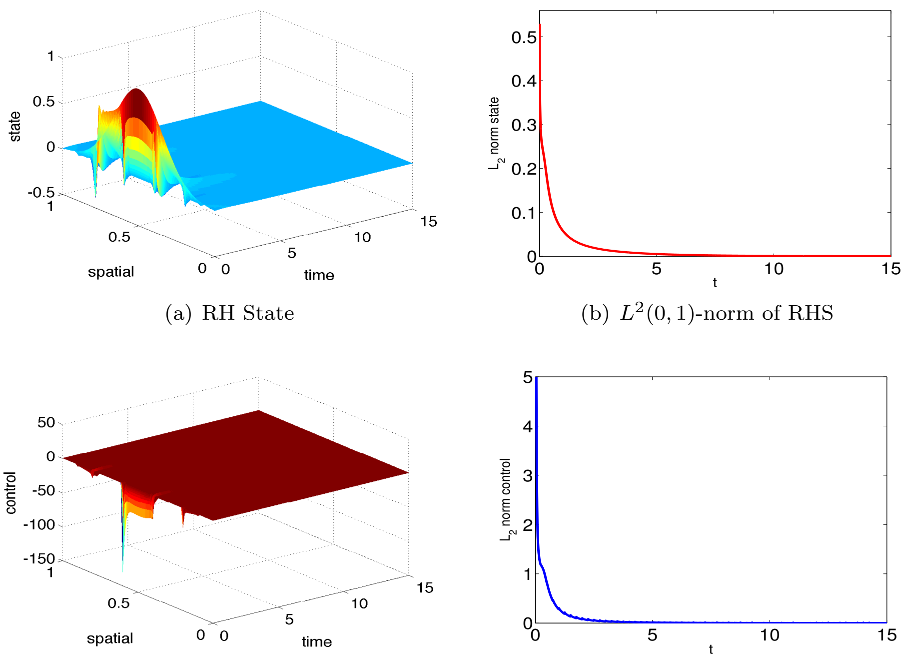
G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	$T.I$
Quadratic	$T = 1$	0.021891	0.1799	2.62×10^{-5}	11861
	$T = 0.5$	0.023196	0.1820	3.95×10^{-5}	8352
	$T = 0.25$	0.027547	0.1828	1.32×10^{-4}	6041
Zero	$T = 1$	0.021886	0.1805	8.41×10^{-5}	6220
	$T = 0.5$	0.021893	0.1818	2.10×10^{-4}	3139
	$T = 0.25$	0.021943	0.1856	4.26×10^{-4}	1467

Table 1
Numerical results for Example 4.1

As expected, increasing the prediction horizon T results in a decrease of the stabilizing measures $\|y_{rh}\|_{L^2(Q)}$ and $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$, both for quadratic and zero terminal penalties. The quadratic terminal penalty term results in smaller values of the stabilization measures, with the difference in the $\|y_{rh}\|_{L^2(Q)}$-norm less pronounced than in the $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$-norm. Using a non-trivial terminal penalty results in higher iteration numbers for the optimizer. In view of the fact that the choice of T has only little effect on the stabilizing measures, but significant effect on the number of iterations in the optimization algorithm, small T is preferable for this class of problems. It should also be of interest to search for methods which adaptively tune the prediction horizon.

Example 4.2. Here we considered the stabilization of the Burgers' equation (3.19) with homogeneous Neumann boundary conditions. We set $y_0(x) = \cos(\pi x)$ as the initial function and chose $T_\infty = 10$. The spatial support for the controls is

$$\hat{\Omega} = (0, 0.15) \cup (0.85, 1) \subset (0, 1).$$



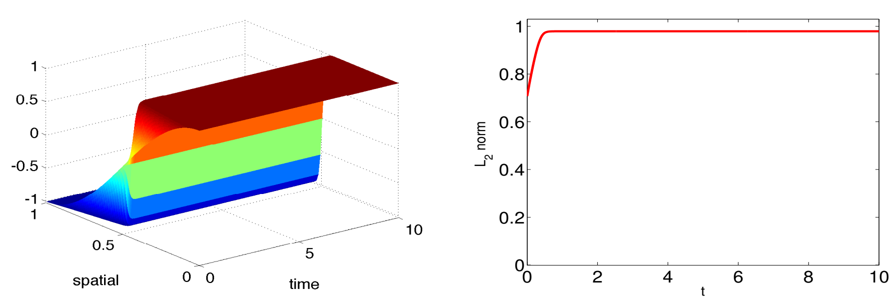
(a) RH State (b) $L^2(0,1)$-norm of RHS

(c) RH Control (d) $L^2(\hat{\Omega})$-norm of RHC

FIG. 2. *Receding horizon trajectories for Example 4.1*

Furthermore, $\mu = 0.01$. Note that for this small viscosity parameter and the above anti-symmetric initial function, the uncontrolled numerical solution of (3.19) approaches a non-constant, time independent steady state, see [7]. The uncontrolled solution y^u is illustrated in Figure 3 and we have

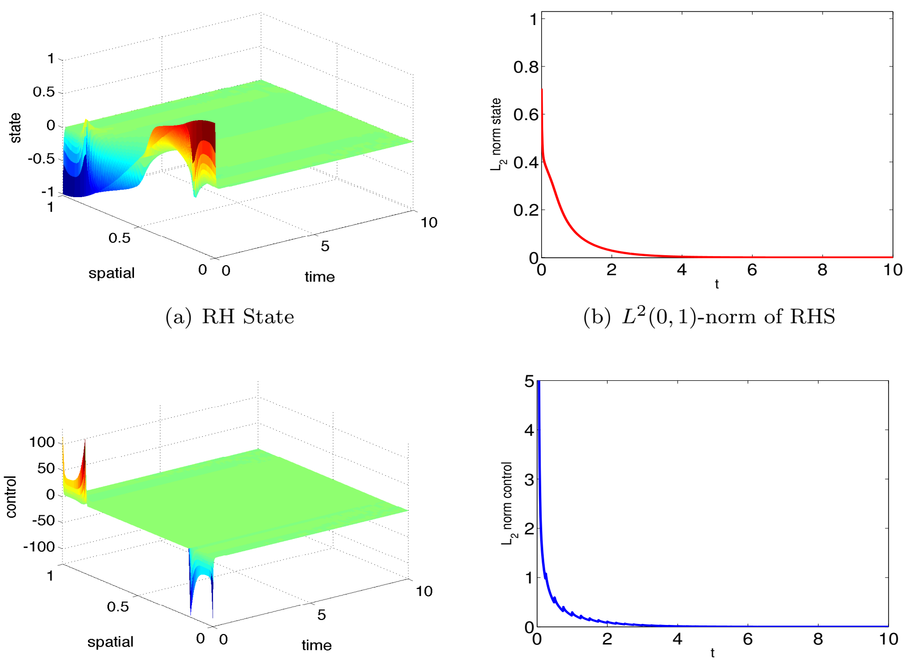
$$\|y^u\|_{L^2(Q)} = 3.08, \qquad \|y^u(T_\infty)\|_{L^2(0,1)} = 0.9791.$$



(a) State (b) Evolution of $L^2(0,1)$-norm of state

FIG. 3. *Uncontrolled solution for Example 4.2*

Table 2 reveals the numerical results of Algorithm 2 for different choices of the prediction horizon T and the terminal cost G. Figure 4 shows the results for the receding horizon pairs (y_{rh}, u_{rh}) in the case that zero terminal cost and $T = 1$ were chosen.



(a) RH State (b) $L^2(0,1)$-norm of RHS

(c) RH Control (d) $L^2(\hat{\Omega})$-norm of RHC

FIG. 4. *Receding horizon trajectories for Example 4.2*

G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	$T.I$
Quadratic	$T = 1$	0.053394	0.2820	3.38×10^{-6}	5890
	$T = 0.5$	0.056004	0.2792	9.99×10^{-6}	3957
	$T = 0.25$	0.060580	0.2788	9.74×10^{-6}	3285
Zero	$T = 1$	0.053058	0.2835	7.23×10^{-6}	3382
	$T = 0.5$	0.052961	0.2873	1.46×10^{-5}	1698
	$T = 0.25$	0.053717	0.2977	4.00×10^{-5}	903

TABLE 2
Numerical results for Example 4.2.

Concerning the effect of different choices of T and G, the same observations as in Example 4.1 apply.

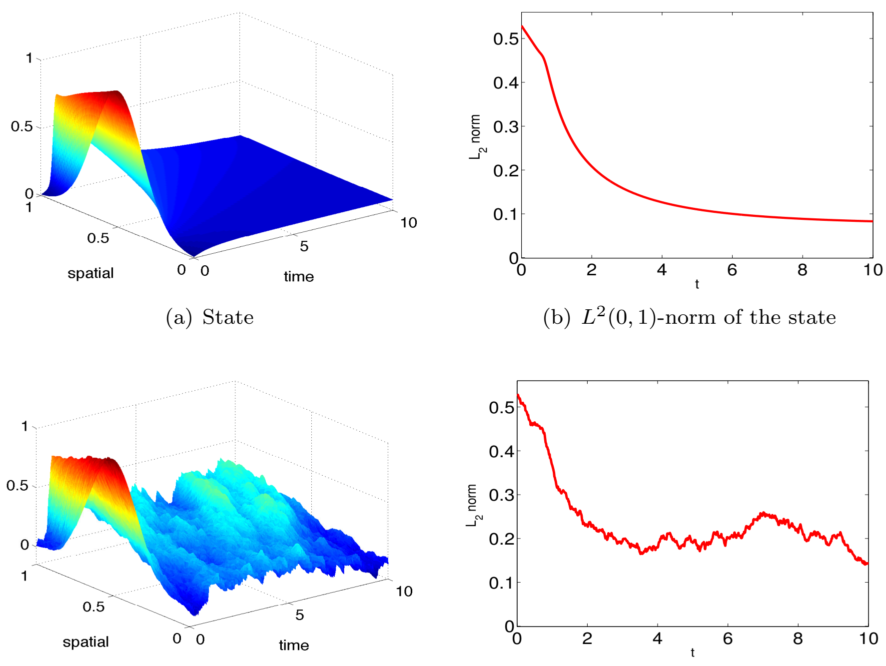
EXAMPLE 4.3. In this example, we dealt with the stabilization of a *noisy* Burgers' equation with homogeneous Neumann boundary conditions. We chose $y_0(x) = \exp(-20(x-0.5)^2)$ as the initial function, $\mu = 0.01$ as a viscosity parameter, and $T_\infty = 10$. Furthermore, the noise was simulated by generating uniformly distributed random numbers within the range $[-4, 4]$. It was added to the right-hand side of the equation (3.19) at the spatial-temporal gird points. The results correspond to uncontrolled solutions are reported in Table 3.

Problem types	$\|y^u\|_{L^2(Q)}$	$\|y^u(T_\infty)\|_{L^2(0,1)}$
Uncontrolled state without noise	0.6291	0.0829
Uncontrolled state with noise	0.7945	0.1436

TABLE 3
Uncontrolled solutions for Example 4.3

In Figure 5, we show the results for uncontrolled solution with noise and without noise. The control acts only on the set

$$\hat{\Omega} = (0.1, 0.3) \cup (0.7, 0.9) \subset (0, 1).$$



(a) State (b) $L^2(0,1)$-norm of the state

(c) Noisy state (d) $L^2(0,1)$-norm of the noisy state

FIG. 5. *Uncontrolled solutions for Example 4.3*

In implementations of Algorithm 2 on every interval $[t_i, t_i + T]$, first an open-loop optimal control u_T^* was computed for every subproblem without noise. Then the optimal control u_T^* is used to steer the noisy Burgers' equation. This process was repeated for every interval $[t_i, t_i + T]$ with $i = 0, \ldots, r-1$. Table 4 (rep. Table 5) represents the results of Algorithm 2 applied on the Burgers' equation (3.19) with noise (resp. without noise) for different choices of the prediction horizon T and the terminal cost G.

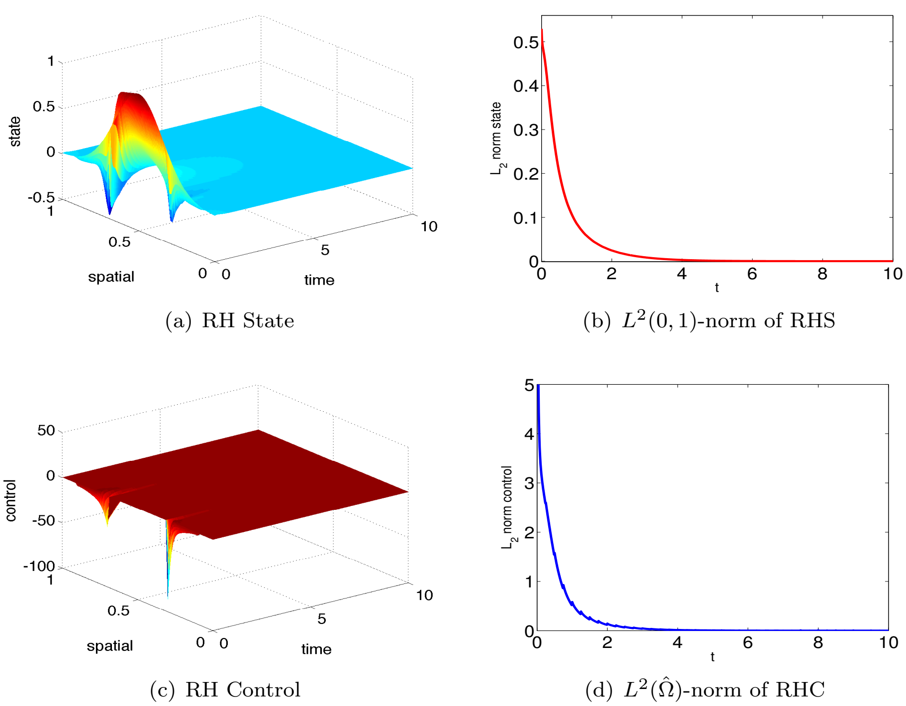
In Figures 6 and 7, we show the results for the receding horizon pairs (y_{rh}, u_{rh}) in the case that zero terminal cost and $T = 1$ were chosen.

G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	$T.I$
Quadratic	$T = 1$	0.049456	0.2996	0.0426	4327
	$T = 0.5$	0.052292	0.3031	0.0418	3489
	$T = 0.25$	0.060291	0.3063	0.0441	2739
Zero	$T = 1$	0.049545	0.3009	0.0438	2463
	$T = 0.5$	0.050112	0.3046	0.0462	1479
	$T = 0.25$	0.053008	0.3163	0.0512	880

TABLE 4
Numerical results corresponding to the noisy equation for Example 4.3

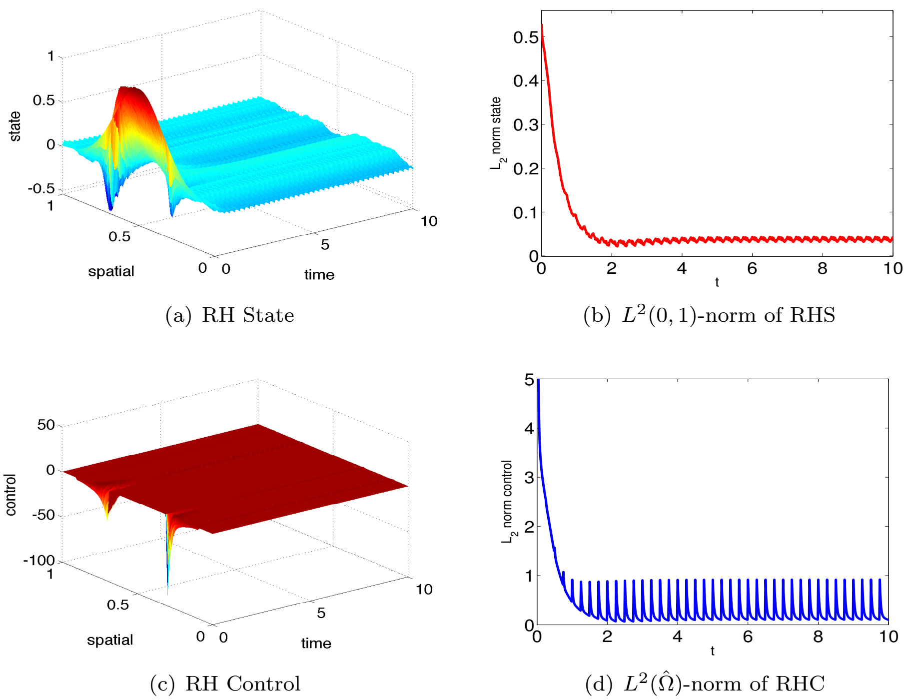
G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	$T.I$
Quadratic	$T = 1$	0.044212	0.2828	2.56×10^{-6}	4215
	$T = 0.5$	0.046700	0.2855	9.70×10^{-6}	3346
	$T = 0.25$	0.054818	0.2892	5.58×10^{-6}	2636
Zero	$T = 1$	0.044212	0.2839	9.26×10^{-6}	2532
	$T = 0.5$	0.044626	0.2870	2.79×10^{-5}	1450
	$T = 0.25$	0.047149	0.2983	1.51×10^{-4}	835

TABLE 5
Numerical results corresponding to the equation without noise for Example 4.3



(a) RH State
(b) $L^2(0,1)$-norm of RHS
(c) RH Control
(d) $L^2(\hat{\Omega})$-norm of RHC

FIG. 6. *Receding horizon trajectories for Example 4.3 without noise*



(a) RH State

(b) $L^2(0,1)$-norm of RHS

(c) RH Control

(d) $L^2(\hat{\Omega})$-norm of RHC

Fig. 7. *Receding horizon trajectories for Example 4.3 with presence of noise*

From Tables 4 and 5 we note that the stabilization quantifiers for the quadratic and zero terminal penalties differ less in the case with noise than without noise. Comparing Figures 6(d) and 7(d) we note the effect on the required control action due to noise in the equation.

Consistently over all numerical results it can be observed that a longer prediction horizon leads to smaller values of J_{T_∞}.

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