STABILIZABILITY FOR NONAUTONOMOUS LINEAR PARABOLIC EQUATIONS WITH ACTUATORS AS DISTRIBUTIONS

KARL KUNISCH[1,2], SÉRGIO S. RODRIGUES[1], DANIEL WALTER[3]

ABSTRACT. The stabilizability of a general class of abstract parabolic-like equations is investigated, with a finite number of actuators. This class includes the case of actuators given as delta distributions located at given points in the spatial domain of concrete parabolic equations. A stabilizing feedback control operator is constructed and given in explicit form. Then, an associated optimal control is considered and the corresponding Riccati feedback is investigated. Results of simulations are presented showing the stabilizing performance of both explicit and Riccati feedbacks.

1. INTRODUCTION

We consider a general class of abstract parabolic-like equations, for time $t > 0$, as

$$\dot{y} + Ay + A_{\rm rc}y = \sum_{j=1}^{M_\sigma} u_j \mathfrak{d}_j, \qquad y(0) = y_0, \tag{1.1a}$$

where A is a diffusion-like operator, $A_{\rm rc} = A_{\rm rc}(t)$ is a time-dependent reaction–convection-like operator, and $u_j = u_j(t)$ are the coordinates of the control input u used to tune the actuators $\mathfrak{d}_j$, which are assumed to be elements in the continuous dual $\mathrm{D}(A)'$ of the domain $\mathrm{D}(A)$ of the (unbounded) operator $A\colon H \to H$ in a pivot Hilbert space H. Under suitable further assumptions on these operators and actuators we shall show that the explicitly given feedback input

$$u_j(t) \coloneqq -\lambda \langle \mathfrak{d}_j, A^{-1}y\rangle_{\mathrm{D}(A)',\mathrm{D}(A)} \tag{1.1b}$$

is able to stabilize the system in the norm of a suitable space $V' \supset H$. Without entering into more details at this point the main result reads as follows.

Main Result. *Let $\mu > 0$ be given. We can find a number M_σ of actuators, and a constant $\lambda > 0$ both large enough, so that the solution of (1.1) satisfies*

$$|y(t)|_{V'} \le \mathrm{e}^{-\mu(t-s)} |y(s)|_{V'}, \quad \textit{for all } t \ge s \ge 0 \textit{ and all } y_0 \in V'.$$

In particular, $t \mapsto |y(t)|_{V'}$ is strictly decreasing at time $t = s$, if $|y(s)|_{V'} \neq 0$.

The details will be given in a more precise formulation in Theorem 3.1.

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[1] Johann Radon Institute for Comput. and Appl. Math., ÖAW, Altenbergerstr. 69, 4040 Linz, Austria.
[2] Institute for Mathematics and Scientific Computing, 621, Heinrichstrasse 36, 8010 Graz, Austria.
[3] Institute for Mathematics, Humboldt University, Rudower Chaussee 25, 10117 Berlin, Germany.
EMAILS: karl.kunisch@uni-graz.at, sergio.rodrigues@ricam.oeaw.ac.at, daniel.walter@hu-berlin.de.

1.1. **Example of application.** We shall show that we can apply the abstract result to concrete parabolic equations as

$$\tfrac{\partial}{\partial t}y - \nu\Delta y + ay + b\cdot\nabla y = -\lambda\sum_{j=1}^{M_\sigma}\langle \boldsymbol{\delta}_{x^j}, A^{-1}y(\cdot,t)\rangle_{\mathrm{D}(A)',\mathrm{D}(A)}\boldsymbol{\delta}_{x^j}, \tag{1.2a}$$

$$\mathfrak{B}y|_{\partial\Omega} = 0, \quad y(0) = y_0, \quad \text{with} \quad A \coloneqq -\nu\Delta + \mathbf{1}, \tag{1.2b}$$

evolving in rectangular spatial domains $\Omega \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$, under Dirichlet or Neumann boundary conditions, where $\mathbf{1}$ stands for the identity operator. Denoting by (x,t) a generic point in the cylinder $\Omega \times (0,+\infty)$, the reaction $a = a(x,t) \in \mathbb{R}$ and the convection $b = b(x,t) \in \mathbb{R}^d$ coefficients are known functions. In general (e.g., due to the reaction term), the norm of the state corresponding to the free dynamics (i.e., to $u = 0$) can diverge to $+\infty$ as time increases.

We shall show that system (1.2) is stable for appropriately chosen locations $x^j \in \Omega$ of the actuators, given by delta distributions $\boldsymbol{\delta}_{x^j}$ supported at the points x^j. In this concrete example, we focus on the cases $d \in \{1,2,3\}$, which are the most relevant for several real world applications. The stability result, as stated in Main Result, will hold in the norm of the continuous dual V' of the Sobolev space $V = W^{1,2}(\Omega) \subset L^2(\Omega)$. The details shall be given in Section 5.

We end this section with some remarks.

– For the moment, we may think of $M_\sigma = M$. Later on, $M_\sigma = \sigma(M)$ will be a more general strictly increasing sequence of positive integers, which will be convenient, in concrete applications, to achieve suitable auxiliary properties for(a sequence of) families of actuators.

– For simplicity, we consider the case of rectangular domains. The extension to more general convex polygonal domains shall be addressed in Remark 5.2.

– We shall check the satisfiability of the abstract assumptions, provided later on, for concrete actuators as above given as delta distributions supported at single points. We can consider more general concrete actuators in $\mathrm{D}(A)'$ (e.g., distributions supported in a curve or surface); to conclude the desired stability of the system, we will "only" need to check the satisfiability of the same assumptions by the new concrete actuators.

1.2. **Related work.** Here we focus on literature involving actuators which are not elements of the pivot space. Stabilization problems with finitely many actuators in the pivot space are considered in [1, 2, 4, 6, 18], for example.

Apparently the problem of stabilization of nonautonomous parabolic equations with actuators given by a finite number of fixed delta distributions $\boldsymbol{\delta}_{x^j}$, $1 \le j \le M_\sigma$ has not been investigated in the literature before. However, there are results on the control of parabolic equations in the one-dimensional case $\Omega = (0,l) \subset \mathbb{R}$, $l > 0$, with a mobile δ distribution $\boldsymbol{\delta}_{\overline{x}(t)}$ located at the time dependent point $\overline{x}(t) \in \Omega$. For example, see [16]. Note that, in the 1D case we have that $\boldsymbol{\delta}_{\overline{x}(t)} \in W^{-1,2}(\Omega)$, at each time instant t, where $W^{-1,2}(\Omega)$ is the continuous dual space of the Sobolev subspace $W_0^{1,2}(\Omega)$, which means that we have enough spatial regularity in order to have existence of standard weak solutions y for (1.2) (cf. [16, Appendix]). In this manuscript we consider the case of static actuators and, further, also the cases $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, where $\boldsymbol{\delta}_{x^j} \notin W^{-1,2}(\Omega)$; this means that, for actuators as $\boldsymbol{\delta}_{x^j}$ we will have to work with less regular solutions, with respect to the spatial variable. The case of delta distributions as actuators has also been considered in [10] where approximate controllability results have been derived. Note that approximate controllability to zero at a given finite time-horizon $T > 0$ does not imply stabilizability. In the context of the wave equation, for domains $\Omega \subset \mathbb{R}^2$, moving actuators $\boldsymbol{\delta}_{\mathbf{x}(t)}$ acting as a weak damping, with support is not a single point, but a closed simple curve $\mathbf{x}(t) \subset \Omega$, were considered in [15]. These distributions are more regular than point supported ones. Indeed, $\boldsymbol{\delta}_{\mathbf{x}(t)} \in W^{-1,2}(\Omega)$, $\langle \boldsymbol{\delta}_{\mathbf{x}(t)}, z\rangle \coloneqq \int_{\mathbf{x}(t)} z \,\mathrm{d}\mathbf{x}(t)$ for $z \in W^{1,2}(\Omega)$.

For $d \in \{1,2,3\}$, the point delta distributions $\boldsymbol{\delta}_{x^j} \in \mathrm{D}(A)' \subset W^{-2,2}(\Omega)$ belong to the continuous dual $\mathrm{D}(A)'$ of the domain $\mathrm{D}(A)$ of the operator A as in (1.2) (for both Dirichlet and Neumann boundary conditions). The dynamics in (1.1a) can be written as $\dot{y} = \mathcal{A}y + Bu$, with $\mathcal{A} = -A - A_{\mathrm{rc}}$ and a continuous linear control operator $B\colon \mathbb{R}^{M_\sigma} \to \mathrm{D}(A)'$. Though, we are considering the (more general) nonautonomous case $\mathcal{A} = \mathcal{A}(t)$, we would like to refer to [3] where a stabilizability criterion is presented for the autonomous case with $B\colon \mathbb{R}^{M_\sigma} \to \mathrm{D}(\mathcal{A}^*)'$, where $\mathcal{A}^*$ stands for the adjoint of $\mathcal{A}$; see also [23].

1.3. **Contents and notation.** The remaining text of the manuscript is organized as follows. In Section 2 we introduce the general assumptions defining a class of abstract parabolic-like systems under which we prove the stabilizability result. This proof is presented in Section 3. Then, we consider a related infinite time-horizon optimal control problem in Section 4 and investigate associated differential Riccati equations. The applicability of the abstract result to concrete scalar parabolic equations is shown in Section 5. Section 6 includes some comments on the discretization used to compute the solutions of the Riccati equations. The stabilizing performances of the explicit and the Riccati feedbacks are shown and discussed in Section 7 based on results of numerical simulations. We end the manuscript with concluding remarks in Section 8.

Concerning notation, we write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we set $\mathbb{R}_+ \coloneqq (0, \infty)$, and $\mathbb{N}_+ \coloneqq \mathbb{N} \setminus \{0\}$.

Given Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subseteq Y$ is a continuous inclusion. We write $X \overset{\mathrm{d}}{\hookrightarrow} Y$, and $X \overset{\mathrm{c}}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact.

We define the Bochner subspace $W(I, X, Y) \coloneqq \{f \in L^2(I, X) \mid \dot{f} \in L^2(I, Y)\}$, for a given nonempty open interval $I \subset \mathbb{R}_+$.

The space of continuous linear mappings from X into Y is denoted by $\mathcal{L}(X, Y)$. In case $X = Y$ we write $\mathcal{L}(X) \coloneqq \mathcal{L}(X, X)$. The continuous dual of X is denoted $X' \coloneqq \mathcal{L}(X, \mathbb{R})$. The adjoint of an operator $L \in \mathcal{L}(X, Y)$ will be denoted $L^* \in \mathcal{L}(Y', X')$.

By C, C_i, $i = 0, 1, \ldots$, we shall denote several positive nonessential constants (i.e., which may differ at different places in the manuscript). To underline the dependence of a constant (e.g., an upper bound) on some parameters, we use $\overline{C}_{[a_1,\ldots,a_n]}$ denoting a nonnegative function that increases in each of its nonnegative arguments a_i, $1 \le i \le n$.

2. Assumptions

We introduce the general class of evolutionary parabolic-like equations as

$$\dot{y} + Ay + A_{\rm rc}y = -\lambda \sum_{j=1}^{M_\sigma} \langle \mathfrak{d}_j, A^{-1}y\rangle \mathfrak{d}_j, \qquad y(0) = y_0, \tag{2.1}$$

where the operators A, $A_{\rm rc} = A_{\rm rc}(t)$ appearing in the system dynamics, and the actuators $\mathfrak{d}_j$ are asked to satisfy appropriate assumptions.

First of all we are given two Hilbert spaces $V \subset H = H'$. The first two assumptions require A to be a (time-independent) diffusion-like operator.

Assumption 2.1. *$A \in \mathcal{L}(V, V')$ is symmetric, and such that $(y, z) \mapsto \langle Ay, z\rangle_{V',V}$ is a complete scalar product on V.*

Hereafter, we suppose that V is endowed with the scalar product $(y, z)_V := \langle Ay, z\rangle_{V',V}$, which again makes V a Hilbert space. Necessarily, $A\colon V \to V'$ is an isometry.

Assumption 2.2. *The inclusion $V \subseteq H$ is dense, continuous, and compact.*

Necessarily, the operator A is densely defined in H, with domain $\mathrm{D}(A)$ satisfying

$$\mathrm{D}(A) \xhookrightarrow{\rm d,\,c} V \xhookrightarrow{\rm d,\,c} H \xhookrightarrow{\rm d,\,c} V' \xhookrightarrow{\rm d,\,c} \mathrm{D}(A)'.$$

Further, A has a compact inverse $A^{-1}\colon H \to H$, and we can find a nondecreasing sequence of eigenvalues $(\alpha_n)_{n\in\mathbb{N}_+}$ (repeated accordingly to their multiplicity) and a corresponding basis of eigenfunctions $(e_n)_{n\in\mathbb{N}_+}$:

$$0 < \alpha_1 \le \alpha_2 \le \dots \le \alpha_n \to +\infty, \quad Ae_n = \alpha_n e_n. \tag{2.2}$$

Observe that we have the relations

$$\begin{aligned} &\langle y, z\rangle_{X',X} = (y, z)_H, && \text{for all} \quad (y, z) \in H \times X, \qquad X \in \{V, \mathrm{D}(A)\};\\ &\langle y, z\rangle_{\mathrm{D}(A)',\mathrm{D}(A)} = \langle y, z\rangle_{V',V}, && \text{for all} \quad (y, z) \in V' \times \mathrm{D}(A). \end{aligned}$$

Hence, without ambiguity, we may omit the subscripts and will often write, for simplicity,

$$\langle y, z\rangle := \langle y, z\rangle_{X',X}, \quad \text{for } (y, z) \in X' \times X, \qquad X \in \{V, \mathrm{D}(A)\}.$$

The next assumption requires $A_{\rm rc} = A_{\rm rc}(t)$ to be a time-dependent reaction–convection-like operator.

Assumption 2.3. *For almost every $t > 0$ we have $A_{\rm rc}(t) \in \mathcal{L}(H, V')$, and we have a uniform bound as $|A_{\rm rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} =: C_{\rm rc} < +\infty$.*

Finally, we make an assumption on the sequence of the families of actuators.

Assumption 2.4. *The sequence $(U_M)_{M\in\mathbb{N}_+}$ of families $U_M := \{\mathfrak{d}^{M,j} \mid 1 \le j \le M_\sigma\}$ of actuators satisfy the following.*

1) *$M_\sigma := \sigma(M)$, where $\sigma\colon \mathbb{N}_+ \to \mathbb{N}_+$ is a strictly increasing function;*
2) *$U_M \subset \mathrm{D}(A)'$ and $\mathcal{U}_M := \operatorname{span} U_M$ has dimension $\dim \mathcal{U}_M = M_\sigma$,*
3) *there exists a sequence $(\widetilde{U}_M)_{M\in\mathbb{N}_+}$ of families $\widetilde{U}_M := \{\Psi^{M,j} \mid 1 \le j \le M_\sigma\}$ of auxiliary functions, satisfying*

a) $\widetilde U_M\subset \mathrm{D}(A)$ *and* $\widetilde{\mathcal U}_M:=\operatorname{span}\widetilde U_M$ *has dimension* $\dim\widetilde{\mathcal U}_M=M_\sigma$*,*

b) $\langle \mathfrak{d}^{M,j},\Psi^{M,i}\rangle=\begin{cases}1 & \text{if } j=i,\\ 0 & \text{if } j\neq i,\end{cases}$

c) *with* $\mathrm{D}_M:=\{z\in \mathrm{D}(A)\mid \langle \mathfrak{d}^{M,j},z\rangle=0 \text{ for all } 1\le j\le M_\sigma\}$*, the constant*

$$\xi_{M_+}:=\inf_{\Theta\in \mathrm{D}_M\setminus\{0\}}\frac{|\Theta|^2_{\mathrm{D}(A)}}{|\Theta|^2_V},\tag{2.3}$$

satisfies $\lim\limits_{M\to+\infty}\xi_{M_+}=+\infty$.

Remark 2.5. The satisfiability of Assumptions 2.1–2.4 for scalar parabolic systems as (1.2) is addressed in Section 5.

3. The main stabilizability result

The following Theorem 3.1 is a precise statement of Main Result stated in the Introduction. This section is mainly dedicated to its proof.

We start by introducing the mapping

$$\Psi^\diamond\colon \mathbb{R}^{M_\sigma}\to\widetilde{\mathcal U}_M,\qquad \Psi^\diamond v:=\sum_{j=1}^{M_\sigma}v_j\Psi^{M,j},\tag{3.1}$$

where $\widetilde{\mathcal U}_M=\operatorname{span}\widetilde U_M$ is the linear span of the auxiliary functions as in Assumption 2.4. Since the family $\widetilde U_M$ is linearly independent, we have that $|\cdot|_{\mathbb{R}^{M_\sigma}}$, $|\Psi^\diamond\cdot|_V$, and $|\Psi^\diamond\cdot|_{\mathrm{D}(A)}$, are three norms in $\mathbb{R}^{M_\sigma}$. Hence, there exists a constant $\varpi_M>0$ such that

$$|\Psi v|^2_{\mathrm{D}(A)}\le\varpi_M\,|\Psi^\diamond v|^2_V\quad\text{and}\quad|\Psi^\diamond v|^2_V\le\varpi_M\,|v|^2_{\mathbb{R}^{M_\sigma}},\qquad\text{for every}\quad v\in\mathbb{R}^{M_\sigma}.\tag{3.2}$$

Theorem 3.1. *Let Assumptions 2.1– 2.4 hold true. Then, for every* $\mu>0$ *we can find if* $M\in\mathbb{N}_+$ *and* $\lambda>0$ *large enough, so that the solution of system* (2.1) *satisfies*

$$y\in W((0,+\infty),L^2,\mathrm{D}(A)')\quad\text{and}$$
$$|y(t)|_{V'}\le \mathrm{e}^{-\mu(t-s)}\,|y(s)|_{V'}\,,\quad\text{for all } t\ge s\ge 0 \text{ and all } y_0\in V'.$$

Furthermore, M *and* λ *can be chosen as* $M=\overline C_{[\mu,C_{\mathrm{rc}}]}$ *and* $\overline\lambda=\overline C_{[\mu,C_{\mathrm{rc}}]}$*. In addition, we have the bound* $|y|^2_{L^2((s,+\infty),L^2)}\le(2+\mu^{-1}C^2_{\mathrm{rc}})\,|y(s)|^2_{V'}$*, for all* $s\ge0$*.*

Remark 3.2. In particular, the estimate in Theorem 3.1 implies that $t\mapsto|y(t)|_{V'}$ is strictly decreasing at time $t=s$, if $|y(s)|_{V'}\neq0$; see [24, Lem. 3.3].

For the proof of Theorem 3.1 we will need an auxiliary result. For this purpose, for a given $M\in\mathbb{N}_+$ and $z\in\mathrm{D}(A)$, we denote the vector

$$\mathfrak{d}(z):=\big(\langle\mathfrak{d}^{M,1},z\rangle,\langle\mathfrak{d}^{M,2},z\rangle,\;\ldots\;,\langle\mathfrak{d}^{M,M_\sigma},z\rangle\big)\in\mathbb{R}^{M_\sigma}$$

where the $\mathfrak{d}^{M,j}\in\mathrm{D}(A)'$ are the actuators.

Lemma 3.3. *Every* $z\in\mathcal{D}(A)$*, can be written in a unique way as*

$$z=\vartheta+\Theta,\quad\text{with}\quad\vartheta:=\Psi^\diamond\mathfrak{d}(z)\in\widetilde{\mathcal U}_M\quad\text{and}\quad\Theta:=z-\vartheta\in\mathrm{D}_M,$$

where $\widetilde{\mathcal U}_M$ *and* D_M *are as in Assumption 2.4.*

Proof. We have that $\vartheta\in\widetilde{\mathcal U}_M$ due to the definition of $\Psi^\diamond$ in (3.1). Next, by the relations

$$\langle\mathfrak{d}^{M,i},\Theta\rangle=\left\langle\mathfrak{d}^{M,i},\,z-\sum_{j=1}^{M_\sigma}\langle\mathfrak{d}^{M,j},z\rangle\Psi^{M,j}\right\rangle,$$

and by **3)b)** in Assumption 2.4, we find $\langle\mathfrak{d}^{M,i},\Theta\rangle=0$ for all $\mathfrak{d}^{M,i}$, hence $\Theta\in\mathrm{D}_M$. It remains to show that $\widetilde{\mathcal U}_M\bigcap D_M=\{0\}$, for this purpose let us be given $z\in\widetilde{\mathcal U}_M\bigcap D_M$, then it follows that there exists $\overline{z}\in\mathbb{R}^{M_\sigma}$ such that

$$z=\sum_{j=1}^{M_\sigma}\overline{z}_j\Psi^{M,j}\quad\text{and}\quad\left\langle\mathfrak{d}^{M,i},\,z\right\rangle=0\quad\text{for all}\quad 1\le i\le M_\sigma.$$

Again by **3)b)** in Assumption 2.4, we find $\overline{z}_i=\left\langle\mathfrak{d}^{M,i},\,z\right\rangle$, which leads us to $z=0$. □

Lemma 3.4. *Let Assumptions 2.1, 2.2 and 2.4 hold true. Then, for every $\zeta>0$ we can find M and λ large enough such that*

$$|z|^2_{\mathrm{D}(A)}+2\lambda\,|\mathfrak{d}(z)|^2_{\mathbb{R}^{M_\sigma}}\ge\zeta\,|z|^2_V\,,\quad\textit{for all}\quad z\in\mathrm{D}(A).\tag{3.3}$$

Furthermore $M=\overline{C}_{[\zeta]}$ and $\lambda=\overline{C}_{[\zeta,\varpi_M]}$, with ϖ_M as in (3.2).

Proof. Recall the sets U_M and $\widetilde U_M$ of actuators $\mathfrak{d}^{M,j}$ and auxiliary functions $\Psi^{M,j}$ and their linear spans as $\mathcal U_M$ and $\widetilde{\mathcal U}_M$, as well as the space D_M in Assumption 2.4.

For each $z\in\mathcal{D}(A)$, due to Lemma 3.3, we can write

$$z=\vartheta+\Theta,\quad\text{with}\quad\vartheta:=\Psi^\diamond\mathfrak{d}(z)\in\widetilde{\mathcal U}_M\quad\text{and}\quad\Theta:=z-\vartheta\in\mathrm{D}_M.$$

Now, by direct computations, we find

$$\begin{aligned}|z|^2_{\mathrm{D}(A)}+2\lambda\,|\mathfrak{d}(z)|^2_{\mathbb{R}^{M_\sigma}}&=|\Theta+\vartheta|^2_{\mathrm{D}(A)}+2\lambda\,|\mathfrak{d}(z)|^2_{\mathbb{R}^{M_\sigma}}\\&=|\Theta|^2_{\mathrm{D}(A)}+2(\Theta,\vartheta)_{\mathrm{D}(A)}+|\vartheta|^2_{\mathrm{D}(A)}+2\lambda\,|\mathfrak{d}(z)|^2_{\mathbb{R}^{M_\sigma}}\\&\ge\tfrac12\,|\Theta|^2_{\mathrm{D}(A)}-|\vartheta|^2_{\mathrm{D}(A)}+2\lambda\,|\mathfrak{d}(z)|^2_{\mathbb{R}^{M_\sigma}}\end{aligned}$$

and, with ϖ_M as in (3.2) and ξ_{M_+} as in (2.3), we arrive at

$$|z|^2_{\mathrm{D}(A)}+2\lambda\,|\mathfrak{d}(z)|^2_{\mathbb{R}^{M_\sigma}}\ge\tfrac12\xi_{M_+}\,|\Theta|^2_V+(2\lambda\varpi_M^{-1}-\varpi_M)\,|\vartheta|^2_V\,.$$

Hence, for given $\zeta>0$, by choosing M and λ so that

$$\xi_{M_+}\ge4\zeta\quad\text{and}\quad\lambda\ge\zeta\varpi_M^2$$

we arrive at

$$|y|^2_{\mathrm{D}(A)}+2\lambda\,|\mathfrak{d}(z)|^2_{\mathbb{R}^{M_\sigma}}\ge2\zeta\left(|\Theta|^2_V+|\vartheta|^2_V\right)\ge\zeta\,|\Theta+\vartheta|^2_V=\zeta\,|z|^2_V\,,$$

which ends the proof. □

Proof of Theorem 3.1. Let $X_{\mathrm{rc}}:=A^{-1}A_{\mathrm{rc}}A$ and $z=A^{-1}y$ where y satisfies (2.1). Thus, z satisfies

$$\dot z+Az+X_{\mathrm{rc}}z=-\lambda\sum_{j=1}^{M_\sigma}\langle\mathfrak{d}^{M,j},z\rangle A^{-1}\mathfrak{d}^{M,j},\qquad z(0)=A^{-1}y_0\in V.\tag{3.4}$$

Multiplying the dynamics in (3.4) by $2Az$, we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z\right|_V^2 \le -2\left|z\right|_{\mathrm{D}(A)}^2 - 2(X_{\mathrm{rc}}z, Az)_{L^2} - 2\lambda\left|\mathfrak{d}(z)\right|_{\mathbb{R}^{M_\sigma}}^2$$

and, using Assumption 2.3 and the Young inequality, we obtain,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z\right|_V^2 &\le -2\left|z\right|_{\mathrm{D}(A)}^2 - 2\lambda\left|\mathfrak{d}(z)\right|_{\mathbb{R}^{M_\sigma}}^2 + 2C_{\mathrm{rc}}\left|Az\right|_{L^2}\left|z\right|_V &(3.5)\\ &\le -\left|z\right|_{\mathrm{D}(A)}^2 - 2\lambda\left|\mathfrak{d}(z)\right|_{\mathbb{R}^{M_\sigma}}^2 + C_{\mathrm{rc}}^2\left|z\right|_V^2, &(3.6)\end{aligned}$$

where we have used $(X_{\mathrm{rc}}z, Az)_{L^2} = \langle A_{\mathrm{rc}}Az, A^{-1}Az\rangle_{V',V} = \langle A_{\mathrm{rc}}Az, z\rangle_{V',V}$. Now, for a given $\mu > 0$, by Lemma 3.4 we can choose $M = \overline{C}_{[\mu,C_{\mathrm{rc}}]}$ and $\lambda = \overline{C}_{[\mu,C_{\mathrm{rc}},\varpi_M]}$, such that

$$\left|z\right|_{\mathrm{D}(A)}^2 + 2\lambda\left|\mathfrak{d}(z)\right|_{\mathbb{R}^{M_\sigma}}^2 \ge (2\mu + C_{\mathrm{rc}}^2)\left|z\right|_V^2 \tag{3.7}$$

which gives us

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z\right|_V^2 \le -2\mu\left|z\right|_V^2$$

and consequently

$$\left|z(t)\right|_V^2 \le \mathrm{e}^{-2\mu(t-s)}\left|z(s)\right|_V^2, \quad \text{for all}\quad t \ge s \ge 0, \tag{3.8}$$

which, due to $y = Az$, is equivalent to

$$\left|y(t)\right|_{V'}^2 \le \mathrm{e}^{-2\mu(t-s)}\left|y(s)\right|_{V'}^2, \quad \text{for all}\quad t \ge s \ge 0. \tag{3.9}$$

Finally, by (3.5), Young inequality, and (3.7) ,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z\right|_V^2 \le -\tfrac{3}{2}\left|z\right|_{\mathrm{D}(A)}^2 - 2\lambda\left|\mathfrak{d}(z)\right|_{\mathbb{R}^{M_\sigma}}^2 + 2C_{\mathrm{rc}}^2\left|z\right|_V^2 \le -\tfrac{1}{2}\left|z\right|_{\mathrm{D}(A)}^2 + C_{\mathrm{rc}}^2\left|z\right|_V^2$$

which, after time integration, gives us, for each $t \ge s \ge 0$,

$$\left|z(t)\right|_V^2 - \left|z(s)\right|_V^2 + \tfrac{1}{2}\left|z\right|_{L^2((s,t),\mathrm{D}(A))}^2 \le C_{\mathrm{rc}}^2\left|z\right|_{L^2((s,t),V)}^2,$$

and taking the limit as $t \to +\infty$, and using (3.8),

$$\left|z\right|_{L^2((s,+\infty),\mathrm{D}(A))}^2 \le 2\left|z(s)\right|_V^2 + 2C_{\mathrm{rc}}^2\left|z\right|_{L^2((s,+\infty),V)}^2 \le (2 + 2C_{\mathrm{rc}}^2(2\mu)^{-1})\left|z(s)\right|_V^2. \tag{3.10}$$

Finally, with $\mathcal{Y} \coloneqq L^2((s,+\infty), L^2)$, by combining (3.10) with (3.4), we arrive at

$$\begin{aligned}\left|\dot z\right|_{\mathcal{Y}} &\le \left|Az\right|_{\mathcal{Y}} + \left|X_{\mathrm{rc}}z\right|_{\mathcal{Y}} + \left|\lambda\textstyle\sum_{j=1}^{M_\sigma}\langle\mathfrak{d}^{M,j}, z\rangle A^{-1}\mathfrak{d}^{M,j}\right|_{\mathcal{Y}}\\ &\le \left(1 + \left|\mathbf{1}\right|_{\mathcal{L}(V',\mathrm{D}(A)')}C_{\mathrm{rc}} + \lambda M_\sigma\left\|\mathfrak{d}\right\|\right)\left|z\right|_{L^2((s,+\infty),\mathrm{D}(A))},\end{aligned}$$

with $\left\|\mathfrak{d}\right\| \coloneqq \max\limits_{1\le j\le M_\sigma}\left\|\mathfrak{d}^{M,j}\right\|_{\mathrm{D}(A)'}$. Therefore, $z \in W((s,+\infty),\mathrm{D}(A),L^2)$, which gives us that $y = A^{-1}z \in W((s,+\infty),L^2,\mathrm{D}(A)')$ and, by (3.10), also that $\left|y\right|_{L^2((s,\infty),L^2)} \le (2 + C_{\mathrm{rc}}^2\mu^{-1})\left|y(s)\right|_{V'}^2$ for all $s \ge 0$. □

Remark 3.5 (On the existence and uniqueness of solutions)**.** Within the proof of Theorem 3.1, we have shown the stability of system (3.4) for M and λ large enough, where we have implicitly assumed that strong solutions do exist. The existence and uniqueness of these solutions can be proven by classical arguments based on Galerkin approximations. Since the procedure is standard, we simply recall the main steps. The estimates in the proof of Theorem 3.1 also hold for Galerkin approximations of system (3.4) given by

$$\dot z^N + Az^N + P_{\mathcal{E}^{\rm f}_N} A_{\rm rc}(t) z^N = -\lambda P_{\mathcal{E}^{\rm f}_N} \sum_{j=1}^{M_\sigma} \langle \mathfrak{d}^{M,j}, z\rangle A^{-1}\mathfrak{d}^{M,j}, \tag{3.11}$$

$$z^N(0) = P_{\mathcal{E}^{\rm f}_N}(A^{-1}y_0), \tag{3.12}$$

where $P_{\mathcal{E}^{\rm f}_N} \in \mathcal{L}(H)$ stands for the orthogonal projection in H onto the subspace $\mathcal{E}^{\rm f}_N = \operatorname{span}\{e_i \mid 1 \le i \le N\}$ spanned by the first eigenfuntions of A.

Therefore, for arbitrary given $T > 0$, by the analogue to (3.8) we will have that for M and λ large enough it holds that

$$\left|z^N\right|^2_{L^\infty((0,T),V)} \le \left|z^N(0)\right|^2_V \le |z(0)|^2_V\,. \tag{3.13}$$

Then, by the analogue of (3.6), we can find that

$$\left|z^N\right|^2_{L^2((0,T),{\rm D}(A))} \le C_2 \left|z^N(0)\right|^2_V, \tag{3.14}$$

with C_2 independent of N. Next, we can use the dynamics (3.11) to obtain

$$\left|\dot z^N\right|^2_{L^2((0,T),L^2)} \le C_3 \left|z^N(0)\right|^2_V \tag{3.15}$$

with C_2 independent of N. These uniform (in N) estimates above, allow us to show that a weak limit of a suitable subsequence of such Galerkin approximations is a strong solution for system (3.4) defined on the time interval $I_T := (0, T)$. The uniqueness of the solution can be concluded by (3.8), which clearly also holds for the difference of two solutions, due to the linearity of the dynamics. Finally, since $T > 0$ is arbitrary, we can conclude the existence and uniqueness of a strong solution defined for all time $t > 0$.

4. Optimal control and Riccati equations

In some applications we may be interested in stabilizing controls minimizing a given cost functional, in this section we discuss such an optimal control problem.

We shall address the Riccati equations associated to the optimal controls as well. Though the procedure is classical, this is not a standard problem in our setting due to the low spatial regularity of the state and control.

We fix a general set of M_0 actuators

$$U_{M_0} := \{\mathfrak{d}_j \mid 1 \le j \le M_0\} \subset {\rm D}(A)', \qquad \mathcal{U}_{M_0} := \operatorname{span} U_{M_0}, \qquad \dim \mathcal{U}_{M_0} = M_0,$$

and consider the open-loop version of system (2.1), for time $t \ge s$, as

$$\dot y + Ay + A_{\rm rc} y = Bu, \qquad y(s) = y_s, \qquad \text{with} \qquad Bu := \sum_{j=1}^{M_0} u_j \mathfrak{d}_j. \tag{4.1}$$

Note that $B \in \mathcal{L}(\mathbb{R}^{M_0}, {\rm D}(A)')$. We assume that such actuators allow us to stabilize the system with a control input $u = Ky$ given in feedback form, for example, we can take actuators as given in Theorem 3.1, with $M_0 = M_\sigma$ large enough and $K = (K_1, K_2, \dots, K_{M_\sigma})$, with $K_j y = -\lambda\langle \mathfrak{d}_j, A^{-1}y\rangle$ as in (2.1). In particular, for any fixed $\beta > 0$, the energy functional

$$\widehat{\mathcal{J}}_s(y_s; y, u) := \tfrac{1}{2}\, |y|^2_{L^2((s,+\infty),H)} + \tfrac{1}{2}\beta\, |u|^2_{L^2((s,+\infty),\mathbb{R}^{M_0})} \tag{4.2}$$

will be bounded for the stabilizing feedback control input $u = Ky$ and corresponding state y. Thus, it makes sense to look for controls minimizing this functional. Here, we shall relax this goal and shall rather be looking for controls minimizing the "relaxed" functional

$$\mathcal{J}_s(y_s; y, u) := \tfrac{1}{2} \left| P_{\mathcal{E}^{\mathrm{f}}_{M_1}} y \right|^2_{L^2((s,+\infty),H)} + \tfrac{1}{2}\beta \left| u \right|^2_{L^2((s,+\infty),\mathbb{R}^{M_0})} \tag{4.3}$$

where $P_{\mathcal{E}^{\mathrm{f}}_{M_1}}$ is the orthogonal projection in H onto the space

$$\mathcal{E}^{\mathrm{f}}_{M_1} = \operatorname{span}\{e_i \mid 1 \le i \le M_1\},$$

spanned by the first eigenfunctions e_i of the operator A; see Section 2. We shall show in Theorem 4.1 that, for large enough M_1, the boundedness of the relaxed cost (4.3) implies the boundedness of the original cost (4.2). This relaxation can be important for numerical computations, We shall revisit this point in Section 7.3.2.

Theorem 4.1. *Let every solution (y, u) of system (4.1) satisfy $\mathcal{J}_s(y_s; y, u) \le C_J \left| y_s \right|^2_{V'}$ with a constant $C_J = C_J(\beta) > 0$ independent of y_s. If M_1 is large enough, then we also have $\widehat{\mathcal{J}}_s(y_s; y, u) \le \widehat{C}_J \left| y_s \right|^2_{V'}$, with a constant $\widehat{C}_J = \widehat{C}_J(\beta) > 0$ independent of y_s.*

Proof. With $q := P_{\mathcal{E}^{\mathrm{f}}_{M_1}} y$ and $Q := y - P_{\mathcal{E}^{\mathrm{f}}_{M_1}} y$, we find

$$\dot{Q} + AQ + (\mathbf{1} - P_{\mathcal{E}^{\mathrm{f}}_{M_1}}) A_{\mathrm{rc}} Q = F$$

with $F := (\mathbf{1} - P_{\mathcal{E}^{\mathrm{f}}_{M_1}}) B u - (\mathbf{1} - P_{\mathcal{E}^{\mathrm{f}}_{M_1}}) A_{\mathrm{rc}} q$ and

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} \left| Q \right|^2_{V'} + 2 \left| Q \right|^2_H &= -2\langle (\mathbf{1} - P_{\mathcal{E}^{\mathrm{f}}_{M_1}}) A_{\mathrm{rc}} Q, A^{-1} Q \rangle_{V',V} + 2\langle F, A^{-1} Q \rangle_{\mathrm{D}(A)',\mathrm{D}(A)} \\
&\le 2C_{\mathrm{rc}} \left| Q \right|_{L^2} \left| Q \right|_{V'} + 2 \left| F \right|_{\mathrm{D}(A)'} \left| Q \right|_H \\
&\le \left| Q \right|^2_H + 2C_{\mathrm{rc}}^2 \left| Q \right|^2_{V'} + 2 \left| F \right|^2_{\mathrm{D}(A)'}
\end{aligned}$$

where $\left| F \right|_{\mathrm{D}(A)'} \le C_B \left| u \right|_{\mathbb{R}^{M_0}} + C_{\mathrm{rc}} \left| q \right|_H$ for a suitable constant $C_B > 0$, which gives us

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left| Q \right|^2_{V'} + \tfrac{1}{2} \left| Q \right|^2_H \le -\tfrac{1}{2} \left| Q \right|^2_H + 2C_{\mathrm{rc}}^2 \left| Q \right|^2_{V'} + 4(C_B^2 \left| u \right|^2_{\mathbb{R}^{M_0}} + C_{\mathrm{rc}}^2 \left| q \right|^2_H).$$

By denoting the eigenvalue of A defined as

$$\alpha_{M_1+} := \min_{Q \in \mathrm{D}(A) \bigcap \mathcal{E}^{\perp}_{M_1}} \tfrac{\left| AQ \right|_H}{\left| Q \right|_H} = \min_{Q \in \mathcal{E}^{\perp}_{M_1}} \tfrac{\left| Q \right|^2_H}{\left| Q \right|^2_{V'}},$$

where $\mathcal{E}^{\perp}_{M_1}$ is the orthogonal space to $\mathcal{E}^{\perp}_{M_1}$ in H, we find that

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left| Q \right|^2_{V'} + \tfrac{1}{2} \left| Q \right|^2_H \le -(\tfrac{1}{2}\alpha_{M_1+} - 2C_{\mathrm{rc}}^2) \left| Q \right|^2_{V'} + 4(C_B^2 \left| u \right|^2_{\mathbb{R}^{M_0}} + C_{\mathrm{rc}}^2 \left| q \right|^2_H).$$

Now, for M_1 is large enough such that $\alpha_{M_1+} \ge 4C_{\mathrm{rc}}^2$, we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left| Q \right|^2_{V'} + \tfrac{1}{2} \left| Q \right|^2_H \le 4(C_B^2 \left| u \right|^2_{\mathbb{R}^{M_0}} + C_{\mathrm{rc}}^2 \left| q \right|^2_H),$$

and time integration gives us for all $t \ge s \ge 0$,

$$\begin{aligned}
\left| Q(t) \right|^2_{V'} + \tfrac{1}{2} \left| Q \right|^2_{L^2((s,t),H)} &\le \left| Q(s) \right|^2_{V'} + 4(C_B^2 \left| u \right|^2_{L^2((s,t),\mathbb{R}^{M_0})} + C_{\mathrm{rc}}^2 \left| q \right|^2_{L^2((s,t),H)}), \\
&\le \left| Q(s) \right|^2_{V'} + 8(C_B^2 + C_{\mathrm{rc}}^2) C_J \left| y_s \right|^2_{V'},
\end{aligned}$$

which implies

$$|Q|^2_{L^\infty((s,+\infty),V')} + \tfrac{1}{2}|Q|^2_{L^2((s,+\infty),H)} \le (1 + 8(C_B^2 + C_{\rm rc}^2)C_J)\,|y_s|^2_{V'}\,.$$

In particular,

$$\begin{aligned}\widehat{\mathcal{J}}_s(y_s;y,u) &= \tfrac{1}{2}|y|^2_{L^2((s,+\infty),H)} + \tfrac{1}{2}\beta\,|u|^2_{L^2((s,+\infty),\mathbb{R}^{M_0})}\\ &= \tfrac{1}{2}|q|^2_{L^2((s,+\infty),H)} + \tfrac{1}{2}\beta\,|u|^2_{L^2((s,+\infty),\mathbb{R}^{M_0})} + \tfrac{1}{2}|Q|^2_{L^2((s,+\infty),H)}\\ &\le (C_J + 1 + 8(C_B^2 + C_{\rm rc}^2)C_J)\,|y_s|^2_{V'}\,,\end{aligned}$$

and the result follows with $\widehat{C}_J \coloneqq 1 + (1 + 8C_B^2 + 8C_{\rm rc}^2)C_J$. □

4.1. **The optimal control problem.** Recalling Theorems 3.1 and 4.1, for M_0 and M_1 large enough, it is justified to consider the following optimal control problem.

Problem 4.2. *For given $s \ge 0$ and $y_s \in V'$, find a control $\overline{u} \in L^2(s + \mathbb{R}_+, \mathbb{R}^{M_0})$ and corresponding state $\overline{y} \in L^2((s,+\infty),H)$ such that $(y,u) = (\overline{y},\overline{u})$ satisfies (4.1), for $t \ge s$, and minimizes the cost functional $\mathcal{J}_s$ as in (4.3).*

By the linearity of the dynamics and by the convexity of $(y,u) \mapsto \mathcal{J}_s(y_s;y,u)$ it follows that the optimal pair $(\overline{y},\overline{u}) = (\overline{y},\overline{u})(y_s)$ solving Problem 4.2 is unique. The existence of an optimal pair can be shown by a classical minimizing sequence argument.

We shall show that the optimal control $\overline{u}$ can be taken in feedback form $\overline{u} = -B^*\Pi\overline{y}$, where the operator $\Pi = \Pi(t)$, can be found by solving a suitable Riccati operator equation, with $\Pi(t) \in \mathcal{L}(V',V)$ for almost all $t > 0$.

Assumption 4.3. *We assume that the optimal pair solving Problem 4.2 satisfies the bound $\mathcal{J}_s(y_s;\overline{y},\overline{u}) \le C_J\,|y_s|^2_{V'}$ with C_J independent of (s,y_s).*

Observe that Assumption 4.3 is satisfied if we take actuators as in Theorem 3.1. Indeed, we have $\mathcal{J}_s(y_s;\overline{y},\overline{u}) \le \mathcal{J}_s(y_s;y,u)$ where (y,u) is the state-control pair in Theorem 3.1, with $u_j = -\lambda\langle\mathfrak{d}_j, A^{-1}y\rangle$. This theorem gives us $\left|P_{\mathcal{E}^{\rm f}_{M_1}} y\right|^2_{L^2((s,+\infty),L^2)} \le |y|^2_{L^2((s,+\infty),L^2)} \le C_0\,|y_s|^2_{V'}$, with C_0 independent of (s,y_s). Then, for the control input we also find $|u|^2_{L^2((s,+\infty),\mathbb{R}^{M_\sigma})} \le C_1\,|y_s|^2_{V'}$, with C_1 independent of (s,y_s).

Next, observe that by using the optimal pair $(\overline{y},\overline{u}) = (\overline{y},\overline{u})(y_s)$ we can define a semi-scalar product on V' (cf. [12, Ch. I, Def. 1.1]) as follows

$$0 \le ((y_s^1,y_s^2)) \coloneqq (P_{\mathcal{E}^{\rm f}_{M_1}}\overline{y}(y_s^1), P_{\mathcal{E}^{\rm f}_{M_1}}\overline{y}(y_s^2))_{L^2((s,+\infty),L^2)} + \beta(\overline{u}(y_s^1),\overline{u}(y_s^2))_{L^2((s,+\infty),\mathbb{R}^{M_0})}.$$

The optimal cost, associated to the initial state y_s, reads $\mathcal{J}_s(y_s;\overline{y},\overline{u}) = \frac{1}{2}((y_s,y_s))$. Furthermore, for each fixed y_s^1 the mapping $y_s^2 \mapsto \xi_{y_s^1}(y_s^2) \coloneqq ((y_s^1,y_s^2))$ is linear and bounded, and $\xi_{y_s^1} \in V = V'' = \mathcal{L}(V',\mathbb{R})$. Therefore, we have the representation

$$((y_s^1,y_s^2)) = \langle\Pi_s y_s^1, y_s^2\rangle_{V,V'}, \qquad \Pi_s y_s^1 \coloneqq \xi_{y_s^1}, \qquad \Pi_s \in \mathcal{L}(V',V).$$

Also, note that Π_s is well defined, because if Ξ_s has the same properties it follows that $0 = \langle(\Pi_s - \Xi_s)y_s^1, y_s^2\rangle_{V,V'}$ for all $(y_s^1,y_s^2) \in V' \times V'$ which implies that $\Pi_s - \Xi_s = 0$. We can also see that Π_s is linear. Further, it is bounded due to $\langle\Pi_s y_s^1, y_s^2\rangle_{V,V'} =$

$((y_s^1, y_s^2)) \le 2\mathcal{J}_s(y_s^1; \overline{y}^1, \overline{u}^1)^{\frac{1}{2}} \mathcal{J}_s(y_s^2; \overline{y}^2, \overline{u}^2)^{\frac{1}{2}} \le 2C_J \left|y_s^1\right|_{V'} \left|y_s^2\right|_{V'}$, with C_J as in Assumption 4.3. Thus, with $\Pi_s \in \mathcal{L}(V', V)$ we have the representation

$$0 \le \tfrac{1}{2}\langle \Pi_s y_s, y_s\rangle_{V,V'} = \tfrac{1}{2}((y_s, y_s)) = \mathcal{J}_s(y_s; \overline{y}, \overline{u}) \tag{4.4}$$

for the optimal cost associated with Problem 4.2.

4.2. **Dynamic programming principle.** Let us consider the following auxiliary optimal control problem.

Problem 4.4. *Given $s \ge 0$ and $y_0 \in V'$, find a control $\underline{u} \in L^2((0,s), \mathbb{R}^{M_0})$ and corresponding state $\underline{y} \in L^2((0,s), H)$ such that $(y,u) = (\underline{y}, \underline{u})$ satisfies (4.1), for $t \in (0,s)$, with $y(0) = y_0$ and minimizes the cost functional*

$$\mathcal{J}_0^s(y_0; y, u) \coloneqq \tfrac{1}{2}\left|P_{\mathcal{E}_{M_1}^{\mathrm{f}}} y\right|^2_{L^2((0,s),H)} + \beta\tfrac{1}{2}\left|u\right|^2_{L^2((0,s),\mathbb{R}^{M_0})} + \tfrac{1}{2}\langle \Pi_s y(s), y(s)\rangle_{V,V'}. \tag{4.5}$$

Note that (4.5) includes a cost as (4.3), now in the finite time interval $(0,s)$, and the optimal cost-to-go $\frac{1}{2}\langle \Pi_s y(s), y(s)\rangle_{V,V'}$ after time $t = s$ as in (4.4).

Let $(\overline{y}_{[0]}, \overline{u}_{[0]}) = (\overline{y}, \overline{u})(y_0)$ be the optimal pair associated to Problem 4.2 in the case $s = 0$ with initial state y_0. The dynamic programming principle tells us that this pair is related to the pair solving Problem 4.4 as follows

$$(\overline{y}_{[0]}, \overline{u}_{[0]})|_{(0,s)} = (\underline{y}, \underline{u})(y_0), \qquad (\overline{y}_{[0]}, \overline{u}_{[0]})|_{(s,+\infty)} = (\overline{y}, \overline{u})(\underline{y}(s)), \tag{4.6}$$

where $(\underline{y}, \underline{u})(\underline{y}(s))$ is the pair solving Problem 4.2, with initial state $y_s = \underline{y}(s) = \overline{y}_{[0]}(s)$.

4.3. **Optimal feedback.** We show that the optimal control $\overline{u}$ solving Problem 4.2 can be taken in feedback form $\overline{u}(t) = -B^*\Pi_t\overline{y}(t)$. For this purpose, we use standard tools from optimal control, namely, the Karush-Kuhn-Tucker conditions associated with Problem 4.4. For this purpose we define the spaces

$$\mathcal{X}_0^s \coloneqq W((0,s), L^2, \mathrm{D}(A)') \times L^2((0,s), \mathbb{R}^{M_0}), \tag{4.7}$$

$$\mathcal{Y}_0^s \coloneqq V' \times L^2((0,s), \mathrm{D}(A)') \tag{4.8}$$

together with the mapping

$$\Psi\colon \mathcal{X}_0^s \to \mathcal{Y}_0^s, \qquad (y,u) \mapsto (y(0), \dot{y} + Ay + A_{\mathrm{rc}}y - Bu). \tag{4.9}$$

Note that Ψ is surjective, because system (4.1), with h in the place of Bu and with $s = 0$,

$$\dot{y} + Ay + A_{\mathrm{rc}}y = h, \quad y(0) = y_0, \tag{4.10}$$

has a solution $y \in \mathcal{X}_0^s$ (again, we can reason through Galerkin approximations. In particular, we can take $u = 0$).

By the Karush-Kuhn-Tucker Theorem [5, Sect. A.1] [26, Thm. 3.1; Eqs. (1.1) and (1.4)], we have that there exists $(\xi, p) \in (\mathcal{Y}_0^s)' = V \times L^2((0,s), \mathrm{D}(A))$ such that

$$\mathrm{d}\mathcal{J}_0^s|_{(\underline{y},\underline{u})} = (\xi, p) \circ \Psi,$$

which implies that

$$(P_{\mathcal{E}^{\mathrm{f}}_{M_1}}\underline{y}, P_{\mathcal{E}^{\mathrm{f}}_{M_1}} w)_{L^2((0,s),L^2)} + \beta(\underline{u}, v)_{L^2((0,s),\mathbb{R}^{M_0})} + \langle \Pi_s \underline{y}(s), w(s)\rangle_{V,V'}$$
$$= \langle w(0), \xi\rangle_{V',V} + \langle \dot{w} + Aw + A_{\mathrm{rc}}w - Bv, p\rangle_{L^2((0,s),\mathrm{D}(A)'),L^2((0,s),\mathrm{D}(A))}$$

for all $(w, v) \in \mathcal{X}_0^s$. After standard arguments, for time $t \in (0, s)$, we find that

$$\dot{p} - Ap - (A_{\mathrm{rc}})^* p + P_{\mathcal{E}^{\mathrm{f}}_{M_1}}\underline{y} = 0, \qquad p(s) = \Pi_s \underline{y}(s),$$
$$\beta\underline{u} = -B^* p = -B^*\Pi_s\underline{y}(s).$$

Recalling (4.6), the dynamics of system (4.1) with the optimal pair $(y, u) = (\overline{y}_{[0]}, \overline{u}_{[0]})$ solving Problem 4.2 for time $t \geq 0$ and with initial state $y_0 \in V$ reads

$$\dot{y} = -Ay - A_{\mathrm{rc}}y - \beta^{-1}BB^*\Pi y, \qquad y(0) = y_0, \tag{4.11}$$

with $\Pi(t) = \Pi_t$. Further, since $p \in L^2((0, s), \mathrm{D}(A))$, from the relation $p(s) = \Pi_s\underline{y}(s) = \Pi_s\overline{y}_{[0]}(s)$, we conclude that in (4.11) we will have that $\Pi_t y(t) \in \mathrm{D}(A)$, for almost all $t > 0$. In particular, we have that the vector $B^*\Pi y \in \mathbb{R}^{M_\sigma}$, with coordinates $(B^*\Pi y)_j = \langle \mathfrak{d}_j, \Pi y\rangle_{\mathrm{D}(A)',\mathrm{D}(A)}$, is well defined for almost all $t > 0$.

4.4. **Imposing the exponential stability rate.** Let us fix $\mu_{\mathrm{ric}} > 0$ and introduce the shifted reaction-convection term as

$$A_{\mathrm{rc}}^{\mu_{\mathrm{ric}}} \coloneqq A_{\mathrm{rc}} - \mu_{\mathrm{ric}}\mathbf{1}, \qquad \mu_{\mathrm{ric}} > 0.$$

Since $A_{\mathrm{rc}}^{\mu_{\mathrm{ric}}}$ satisfies Assumption 2.3 (with a new bound $C_{\mathrm{rc}}^{\mu_{\mathrm{ric}}} = C_{\mathrm{rc}} + \mu_{\mathrm{ric}}\left|\mathbf{1}\right|_{\mathcal{L}(H,V')}$) we can apply Theorem 3.1 (with some $\mu > 0$) to guarantee the existence of a stabilizing control for M and λ large enough. Then, considering the analogue to Problem 4.2 and following the discussion in previous sections, we arrive at the corresponding cost-to-go

$$\tfrac{1}{2}\langle \Pi_s^{\mu_{\mathrm{ric}}}\widetilde{y}_s, \widetilde{y}_s\rangle_{V,V'} = \mathcal{J}_s(\widetilde{y}_s; \overline{y}, \overline{u}), \tag{4.12}$$

with the solution of the analogue of (4.11)

$$\dot{\widetilde{y}} = -A\widetilde{y} - (A_{\mathrm{rc}} - \mu_{\mathrm{ric}}\mathbf{1})\widetilde{y} - \beta^{-1}BB^*\Pi^{\mu_{\mathrm{ric}}}\widetilde{y}, \qquad y(0) = y_0, \tag{4.13}$$

giving us, by taking $\widetilde{u} = \beta^{-1}B^*\Pi^{\mu_{\mathrm{ric}}}\widetilde{y}$, the minimizer of the cost functional

$$\mathcal{J}_s(\widetilde{y}_s; \widetilde{y}, \widetilde{u}) = \tfrac{1}{2}\left|P_{\mathcal{E}^{\mathrm{f}}_{M_1}}\widetilde{y}\right|^2_{L^2((s,+\infty),H)} + \beta\tfrac{1}{2}\left|\widetilde{u}\right|^2_{L^2((s,+\infty),\mathbb{R}^{M_0})}. \tag{4.14}$$

Now observe that by Theorem 4.1, $\left|\widetilde{y}\right|^2_{L^2((s,+\infty),H)}$ will be bounded for large M_1, and by using Datko Theorem [13, Thm. 1] we can derive that $\left|\widetilde{y}\right|_{V'}$ is exponentially decreasing with some small rate $\mu_0 > 0$. Therefore, $\left|y\right|_{V'} \coloneqq \mathrm{e}^{-\mu_{\mathrm{ric}}t}\left|\widetilde{y}\right|_{V'}$ is exponentially decreasing with rate at least μ_{ric} and this y solves

$$\dot{y} = -Ay - A_{\mathrm{rc}}y - \beta^{-1}BB^*\Pi^{\mu_{\mathrm{ric}}}y, \qquad y_s = \mathrm{e}^{-\mu_{\mathrm{ric}}s}\widetilde{y}_s. \tag{4.15}$$

Further, note that the minimization of (4.14) subject to

$$\dot{\widetilde{y}} + A\widetilde{y} + (A_{\mathrm{rc}} - \mu_{\mathrm{ric}}\mathbf{1})\widetilde{y} = B\widetilde{u}, \quad \widetilde{y}(s) = \widetilde{y}_s. \tag{4.16}$$

is equivalent to the minimization of

$$\mathcal{J}_s^{\mu_{\rm ric}}(y_s;y,u) \coloneqq \tfrac{1}{2}\left|\mathrm{e}^{\mu_{\rm ric}\cdot}P_{\mathcal{E}^{\rm f}_{M_1}}y\right|^2_{L^2((s,+\infty),H)} + \beta\tfrac{1}{2}\left|\mathrm{e}^{\mu_{\rm ric}\cdot}u\right|^2_{L^2((s,+\infty),\mathbb{R}^{M_0})}, \tag{4.17}$$

subject to (4.1).

4.5. **Riccati.** It is well known that with actuators in the pivot space H the cost-to-go operator $\Pi^{\mu_{\rm ric}}$ satisfies a Riccati equation. We confirm here that this is also the case for the "less regular" actuators that we consider, in the space $\mathrm{D}(A)'\supset H$.

From (4.12) and (4.14), with $\overline{y}(s)=w$, for a generic $w\in V'$, we find

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\Pi\overline{y},\overline{y}\rangle_{V,V'}\big|_{t=s} &= 2\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_s(w;\overline{y},\overline{u}) = -\left|P_{\mathcal{E}^{\rm f}_{M_1}}w\right|^2_H - \beta\left|\widetilde{u}(s)\right|^2_{\mathbb{R}^{M_0}}\\ &= -(P_{\mathcal{E}^{\rm f}_{M_1}}w,w)_H - \beta^{-1}(B^*\Pi^{\mu_{\rm ric}}w,B^*\Pi^{\mu_{\rm ric}}w)_{\mathbb{R}^{M_0}}\\ &= -\langle P_{\mathcal{E}^{\rm f}_{M_1}}w,w\rangle_{V,V'} - \beta^{-1}\langle(B^*\Pi^{\mu_{\rm ric}})^*B^*\Pi^{\mu_{\rm ric}}w,w\rangle_{V,V'}.\end{aligned} \tag{4.18}$$

Let us denote

$$L \coloneqq -A - A_{\rm rc} + \mu_{\rm ric}\mathbf{1}. \tag{4.19}$$

Using the dynamics (4.13), we expand the left-hand side of (4.18) (proceeding analogously as in [5, Sect. 3.2, Rem. 3.11(b)]). Formally, we find

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\Pi^{\mu_{\rm ric}}\overline{y},\overline{y}\rangle &= \langle\dot\Pi^{\mu_{\rm ric}}\overline{y},\overline{y}\rangle + \langle\dot{\overline{y}},\Pi^{\mu_{\rm ric}}\overline{y}\rangle + \langle\Pi^{\mu_{\rm ric}}\overline{y},\dot{\overline{y}}\rangle\\ &= \langle\dot\Pi^{\mu_{\rm ric}}\overline{y},\overline{y}\rangle + \langle L\overline{y}-\beta^{-1}BB^*\Pi^{\mu_{\rm ric}}\overline{y},\Pi^{\mu_{\rm ric}}\overline{y}\rangle + \langle\Pi^{\mu_{\rm ric}}\overline{y},L\overline{y}-\beta^{-1}BB^*\Pi^{\mu_{\rm ric}}\overline{y}\rangle,\end{aligned}$$

hence, at $t=s$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\Pi^{\mu_{\rm ric}}\overline{y},\overline{y}\rangle|_{t=s} = \langle(\dot\Pi^{\mu_{\rm ric}} + \Pi^{\mu_{\rm ric}}L + L^*\Pi^{\mu_{\rm ric}} - 2\beta^{-1}(B^*\Pi^{\mu_{\rm ric}})^*B^*\Pi^{\mu_{\rm ric}})w,w\rangle. \tag{4.20}$$

Combining (4.18) and (4.20), we obtain the Riccati equation

$$-\dot\Pi = (L^*\Pi^{\mu_{\rm ric}})^* + L^*\Pi^{\mu_{\rm ric}} - \beta^{-1}(B^*\Pi^{\mu_{\rm ric}})^*B^*\Pi^{\mu_{\rm ric}} + P_{\mathcal{E}^{\rm f}_{M_1}}. \tag{4.21}$$

As we have seen, in (4.11) we will have $\Pi^{\mu_{\rm ric}}(t)y(t)\in\mathrm{D}(A)$ for almost all $t>0$. Let

$$Z(t) \coloneqq \{w\in H \mid \Pi^{\mu_{\rm ric}}(t)w\in\mathrm{D}(A)\}.$$

We have $B^*\Pi^{\mu_{\rm ric}}(t)\colon Z(t)\to\mathbb{R}^{M_0}$ and $L^*(t)\Pi^{\mu_{\rm ric}}(t)\colon Z(t)\to H$. Thus $(L^*\Pi^{\mu_{\rm ric}})^*\colon H\to Z(t)'$ and $(B^*\Pi^{\mu_{\rm ric}})^*\colon\mathbb{R}^{M_0}\to Z(t)'$ are well defined. This implies that all the operators on the right hand side of (4.21) are well defined from $Z(t)$ into $Z(t)'$. Thus also $\dot\Pi^{\mu_{\rm ric}}$ maps $Z(t)$ into $Z(t)'$. Keeping in mind that we should understand it as (4.21), we shall still write the Riccati equation in the canonical form

$$\dot\Pi^{\mu_{\rm ric}} + \Pi^{\mu_{\rm ric}}L + L^*\Pi^{\mu_{\rm ric}} - \Pi^{\mu_{\rm ric}}B_\beta B_\beta^*\Pi^{\mu_{\rm ric}} + C^*C = 0 \tag{4.22}$$

with $B_\beta \coloneqq \beta^{-\frac{1}{2}}B$ and $C = P_{\mathcal{E}^{\rm f}_{M_1}}$.

5. Application to scalar parabolic equations

We consider system (1.2), in a rectangular domain $\Omega = \times_{n=1}^{d}(0,l_n) \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, with side-lengths $l_n > 0$, $1 \le n \le d$, which we rewrite in evolutionary form as

$$\dot y + Ay + A_{\rm rc}y = -\lambda \sum_{j=1}^{M_\sigma} \langle \boldsymbol{\delta}_{x^j}, A^{-1}y(\cdot,t)\rangle_{\mathrm{D}(A)',\mathrm{D}(A)} \boldsymbol{\delta}_{x^j}, \qquad y(0) = y_0, \tag{5.1a}$$

with actuators given by delta distributions $\boldsymbol{\delta}_{x^{M,j}}$, and with

$$A = -\nu\Delta + \mathbf{1} \quad \text{and} \quad A_{\rm rc} \coloneqq (a-1)\mathbf{1} + b\cdot\nabla \tag{5.1b}$$

under either Dirichlet or Neumann boundary conditions $\mathfrak{B} \in \{\mathbf{1}, \mathbf{n}\cdot\nabla\}$, where $\mathbf{n} = \mathbf{n}(\overline{x})$ stands for the unit outward normal vector at a point $\overline{x} \in \partial\Omega$ on the boundary of Ω.

For brevity, we denote Lebesgue and Sobolev spaces of scalar functions defined in Ω, as $L^p \coloneqq L^p(\Omega)$ and $W^{s,p} \coloneqq W^{s,p}(\Omega) \subset L^p$, for $s \ge 0$ and $p \ge 1$.

More precisely, we take

$$H \coloneqq L^2, \qquad H' = H,$$

as the pivot Hilbert space, and define the space

$$V = W^{1,2}, \qquad \text{for Neumann bcs;}$$
$$V = \left\{h \in W^{1,2} \,\middle|\, h|_{\partial\Omega} = 0\right\} \qquad \text{for Dirichlet bcs;}$$

and the operator

$$\langle Ay, w\rangle_{V',V} \coloneqq \nu(\nabla y, \nabla z)_{(L^2)^d} + (y,z)_{L^2}$$

with domain

$$\mathrm{D}(A) \coloneqq \{h \in H \mid Ah \in H\} = \left\{h \in W^{2,2} \,\middle|\, \mathfrak{B}h|_{\partial\Omega} = 0\right\}.$$

We endow the spaces H, V and $\mathrm{D}(A)$ with the scalar products

$$(y,z)_H \coloneqq (y,z)_{L^2}, \qquad (y,z)_V \coloneqq \langle Ay, w\rangle_{V',V}, \qquad (y,z)_{\mathrm{D}(A)} \coloneqq (Ay, Aw)_{L^2}.$$

For the reaction and convection terms, we assume that

$$a \in L^\infty(\mathbb{R}_+, L^3(\Omega)) \quad \text{and} \quad b \in L^\infty(\Omega\times\mathbb{R}_+)^d \quad \text{with} \quad \nabla\cdot b \in L^\infty(\mathbb{R}_+, L^3(\Omega)), \tag{5.2a}$$

and further

$$b\cdot\mathbf{n} = 0, \quad \text{if} \quad \mathfrak{B} = \mathbf{n}\cdot\nabla. \tag{5.2b}$$

5.1. **Satisfiability of Assumptions 2.1–2.3.** It is clear that Assumptions 2.1–2.2 are satisfied. We show now that Assumption 2.3 is a corollary of (5.2). Indeed, for arbitrary $w \in V$ and $y \in V$, we find that

$$\begin{aligned}
&\langle (a(t,\cdot)-1)y + b(t,\cdot)\cdot\nabla y, w\rangle_{V',V} = ((a(t,\cdot)-1)y + b(t,\cdot)\cdot\nabla y, w)_{L^2}\\
&\quad\le |a(t,\cdot)-1|_{L^3}\,|y|_{L^2}\,|w|_{L^6} - (y, \nabla\cdot(b(t,\cdot)w))_{L^2} + \int_{\partial\Omega} ywb\cdot\mathbf{n}\,\mathrm{d}\partial\Omega\\
&\quad\le C_1\,|a(t,\cdot)-1|_{L^3}\,|y|_{L^2}\,|w|_{W^{1,2}} + |b(t,\cdot)|_{(L^\infty)^d}\,|y|_{L^2}\,|\nabla w|_{(L^2)^d}\\
&\qquad + |\nabla\cdot b(t,\cdot)|_{L^3}\,|y|_{L^2}\,|w|_{L^6}\\
&\quad\le C_2(|a(t,\cdot)-1|_{L^3} + |b(t,\cdot)|_{(L^\infty)^d} + |\nabla\cdot b(t,\cdot)|_{L^3})\,|y|_{L^2}\,|w|_V\,.
\end{aligned}$$

Therefore, we can see that Assumption 2.3 holds with

$$C_{\mathrm{rc}} = C_2(|a-1|_{L^\infty(\mathbb{R}_+,L^3)} + |b(t,\cdot)|_{(L^\infty(\Omega\times\mathbb{R}_+))^d} + |\nabla\cdot b(t,\cdot)|_{L^\infty(\mathbb{R}_+,L^3)}),$$

by a continuity argument and density of $V \subset L^2$.

5.2. **Satisfiability of Assumption 2.4.** We construct, for each $M \in \mathbb{N}_+$, the supports $x^{M,j} \in \Omega = \bigtimes_{n=1}^{d} (0, l_n)$, $1 \le j \le M_\sigma$, of $M_\sigma = (d+1)M^d$ delta distributions $\boldsymbol{\delta}_{x^{M,j}}$.

For $M = 1$, in the case $d = 1$ we take two distinct points in Ω; in the case $d = 2$ we take 3 noncolinear points in Ω; in the case $d = 3$ we take 4 noncoplanar points in Ω. Let us denote those points as

$$x^{1,j} \in \Omega, \qquad 1 \le j \le d+1. \tag{5.3a}$$

For $M > 1$, we partition the rectangle Ω into M^d rescaled copies $\Omega^{M,k}$, $1 \le k \le M^d$, of Ω. In each copy $\Omega^{M,k} = v^k + \frac{1}{M}\Omega$, $v^k \in \{v = (v_1, \dots, v_d) \in \mathbb{R}^d \mid v_n = (i-1)\frac{l_n}{M},\ 1 \le i \le M,\ 1 \le n \le d\}$, we select $d+1$ points

$$x^{M,(d+1)(k-1)+m} = v^k + \frac{1}{M}x^{1,m} \in \Omega^{M,k}, \qquad 1 \le m \le d+1. \tag{5.3b}$$

In this way, $\Omega^{M,k}\backslash\{x^{M,(d+1)(k-1)+m} \mid 1 \le m \le d+1\} = v^k + \frac{1}{M}\Omega\backslash\{x^{1,m} \mid 1 \le m \le d+1\}$.

Thus we arrive at the set of actuators, for a fixed $M \in \mathbb{N}_+$, given by

$$U_M := \{\boldsymbol{\delta}_{x^{M,j}} \mid 1 \le j \le M_\sigma\}, \qquad \mathcal{U}_M := \operatorname{span} U_M. \tag{5.3c}$$

Now let us fix an arbitrary set of auxiliary functions

$$\widetilde{U}_M := \{\Psi^{M,j} \mid 1 \le j \le M_\sigma\} \subset \mathrm{D}(A), \qquad \widetilde{\mathcal{U}}_M := \operatorname{span} \widetilde{U}_M \tag{5.4a}$$

satisfying

$$\Psi^{M,j}(x^{M,j}) = 1 \quad \text{and} \quad \Psi^{M,j}(x^{M,i}) = 0 \quad \text{for} \quad i \ne j. \tag{5.4b}$$

Finally, we denote the subspace

$$\mathrm{D}_M := \{z \in \mathrm{D}(A) \mid z(x^{M,i}) = 0 \text{ for all } 1 \le i \le M_\sigma\}. \tag{5.5}$$

Note that, for the delta distribution actuators $\mathfrak{d}^{M,j} := \boldsymbol{\delta}_{x^{M,j}}$, we have the relation $\langle \mathfrak{d}^{M,j}, z\rangle = z(x^{M,i})$. Therefore, to show that Assumption 2.4 holds true it remains to show that the constant, in (2.3),

$$\xi_{M_+} = \inf_{\Theta\in\mathrm{D}_M\backslash\{0\}} \frac{|\Theta|^2_{\mathrm{D}(A)}}{|\Theta|^2_V},$$

diverges to $+\infty$ as M increases. For this purpose, we follow a slight variation of the arguments in [25, Sect. 4.2].

Firstly, we observe that, due to the choice of the $x^{1,j}$, in the case $M = 1$ the function $q \mapsto \left|(q(x^{1,1}), \dots, q(x^{1,d+1}))\right|_{\mathbb{R}^{d+1}}$ is a seminorm on the set of polynomials of degree less or equal to 1. Therefore, we can follow the steps as in [25, Sect. 4.2] to conclude that $|\nabla\nabla z|_{(L^2)^{d^2}}$ is a norm equivalent to the usual $W^{2,2}$-norm in the space D_1 in (5.5). In particular $|\nabla\nabla z|^2_{(L^2)^{d^2}} \geq C_0 \, |z|^2_{W^{1,2}}$ for all $z \in \mathrm{D}_1$.

Next, again following [25, Sect. 4.2], using the fact that for $M > 1$ the domain Ω is partitioned in rescaled (congruent) copies of itself, we can conclude that

$$|\nabla\nabla z|^2_{(L^2)^{d^2}} \geq C_1 M^2 \, |z|^2_{W^{1,2}}\,, \quad \text{for all} \quad z \in \mathrm{D}_M.$$

Now, using also the equivalence of the norms $\mathrm{D}(A)$ and $W^{2,2}$ in $\mathrm{D}(A) \subseteq W^{2,2}$ and the equivalence of the norms of V and $W^{1,2}$ in $V \subseteq W^{1,2}$, we obtain for an appropriate constant $C_2 > 0$,

$$\begin{aligned}\lim_{M\to+\infty} \xi_{M_+} &\geq \lim_{M\to+\infty} \inf_{\Theta\in \mathrm{D}_M\setminus\{0\}} C_2 \frac{|\Theta|^2_{W^{2,2}}}{|\Theta|^2_{W^{1,2}}} \geq C_2 \lim_{M\to+\infty} \inf_{\Theta\in \mathrm{D}_M\setminus\{0\}} \frac{|\nabla\nabla\Theta|^2_{(L^2)^{d^2}}}{|\Theta|^2_{W^{1,2}}}\\ &\geq C_2 C_1 \lim_{M\to+\infty} M^2 = +\infty.\end{aligned}$$

Therefore, Assumption 2.4 holds true.

We have shown that Assumptions 2.1–2.4 are satisfiable. Thus the abstract result in Theorem 3.1 can be applied to obtain the following result.

Corollary 5.1. *Let $\mu > 0$ be given. If we construct the actuators locations $x^{M,j}$ as in Section 5.2 and choose the positive integer M and the constant $\lambda > 0$ large enough, then the solution of (1.2) satisfies, under Dirichlet or Neumann bcs, the estimate*

$$|y(t)|_{V'} \leq \mathrm{e}^{-\mu(t-s)} \, |y(s)|_{V'}\,, \quad \textit{for all } t \geq s \geq 0 \textit{ and all } y_0 \in V'.$$

Remark 5.2. We applied the strategy above to a rectangular (box) domain $\Omega \subset \mathbb{R}^d$. The strategy can be applied to more general convex polygonal domains $\mathbf{P} \subset \mathbb{R}^2$. Indeed, the strategy is based on the fact that a rectangular domain $\mathbf{R} \subset \mathbb{R}^2$ can be decomposed into M^2 similar re-scaled copies of itself. Now, we recall that a triangle $\mathbf{T} \subset \mathbb{R}^2$ can also be (up to a rotation) decomposed into 4^{M-1} similar re-scaled copies of itself. Thus, the arguments can be applied to triangular domains. Hence, after a triangulation, the strategy can also be applied to general convex polygonal domains $\mathbf{P} \subset \mathbb{R}^2$. Now using these 2D polygonal domains, we can see that we can apply the strategy to cylindrical domains as $\mathbf{C} = \mathbf{P} \times (0, L) \subset \mathbb{R}^3$. The extension/applicability of the strategy to more general 3D domains is not clear yet and may require extra work.

6. Discretization of the Riccati equation

We briefly address the discretization of the Riccati equation in a piecewise linear finite-elements context. For a more general context we refer the reader to [20] and references therein.

Let $Q \in \mathcal{L}(X, X')$, with X a Hilbert space be given. We assume that $V = W^{1,2}(\Omega)$ is dense in X and approximate elements in X by elements in the finite-dimensional space

$$H_N := \mathrm{span}\{\mathbf{h}_n \mid 1 \le n \le N\}$$

where the $\mathbf{h}_n$ are the (piecewise linear) hat functions, forming a basis for the finite-element space H_N, defined by

$$\mathbf{h}_n(p_n) = 1 \quad \text{and} \quad \mathbf{h}_n(p_m) = 0, \quad \text{for} \quad 1 \le n, m \le N, \quad n \ne m,$$

where the p_n are the points in a given triangulation $\mathcal{T}$ of $\overline{\Omega}$. Since elements in X are approximated by elements in H_N, we consider now an arbitrary pair

$$(y, z) \in H_N \times H_N. \tag{6.1}$$

Let $[Q] \in \mathbb{R}^{N\times N}$ be the matrix with entries in the ith row and jth column given by

$$[Q]_{ij} := \langle Qh_j, h_i\rangle_{X',X}.$$

Note that, $\langle Qy, z\rangle_{X',X} =: \overline{z}^\top [Q]\overline{y}$, where $\overline{v} \in \mathbb{R}^{N\times 1}$ is the column vector whose entries $\overline{v}_{n,1}$, for $1 \le n \le N$, are the finite-element coordinates of $v = \mathbf{h}(\overline{v}) := \sum_{n=1}^{N} \overline{v}_{n,1}\mathbf{h}_n$. Next, let us denote

$$\overline{Q} := \mathbf{M}^{-1}[Q] \in \mathbb{R}^{N\times N} \tag{6.2}$$

where $\mathbf{M} = [\mathbf{1}]$ is the mass matrix, $(y, z)_{L^2(\Omega)} = \overline{z}^\top \mathbf{M}\overline{y}$, for (y, z) as in (6.1).

For a given $y \in H_N$, let us define the unique vector $\overline{Qy} \in \mathbb{R}^{N\times 1}$ defined by

$$\langle Qy, z\rangle_{X',X} =: \overline{z}^\top \mathbf{M}\overline{Qy} \quad \text{for all} \quad z \in H_N.$$

Then, we can write $\langle Qy, z\rangle_{X',X} = \overline{z}^\top [Q]\overline{y} = \overline{z}^\top \mathbf{M}\overline{Q}\overline{y}$, which gives us

$$\overline{Qy} = \overline{Q}\overline{y}, \quad \text{for all} \quad \overline{y} =\in \mathbb{R}^{N\times 1} \quad \text{with} \quad y = \mathbf{h}(\overline{y}) \in H_N. \tag{6.3}$$

Further, from $\langle Q^* y, z\rangle_{X',X} = \langle Qz, y\rangle_{X',X} = \overline{y}^\top [Q]\overline{z} = \overline{z}^\top [Q]^\top \overline{y}$, we find

$$[Q]^\top = [Q^*]. \tag{6.4}$$

Now, for a given product $L_1L_2 \in \mathcal{L}(X, X')$ with $L_2 \in \mathcal{L}(X, X_1')$ and $L_1 \in \mathcal{L}(X_1', X')$, for another Hilbert spaces X_1, and with dense inclusions $V \subset X_1$ and $V \subset X_1'$, we write

$$\begin{aligned} \overline{z}^\top [L_1L_2]\overline{y} &= \langle L_1L_2y, z\rangle_{X',X} = \langle L_2y, L_1^* z\rangle_{X_1',X_1} \\ &= \overline{L_1^* z}^\top \mathbf{M}\overline{L_2}\overline{y} = \overline{z}^\top \overline{L_1^*}^\top \mathbf{M}\overline{L_2}\overline{y} = \overline{z}^\top (\mathbf{M}^{-1}[L_1^*])^\top \mathbf{M}\overline{L_2}\overline{y} \\ &= \overline{z}^\top [L_1^*]^\top \overline{L_2}\overline{y} = \overline{z}^\top \mathbf{M}\overline{L_1}\,\overline{L_2}\overline{y} \end{aligned}$$

which gives us

$$[L_1L_2] = \mathbf{M}\overline{L_1}\,\overline{L_2} \quad \text{and} \quad \overline{L_1L_2} = \overline{L_1}\,\overline{L_2}. \tag{6.5}$$

Taking $X = V'$ and denoting by $\mathbf{\Pi} = [\Pi^{\mu_{\rm ric}}]$ the finite-element matrix associated with the solution of the Riccati equation (4.22), we find

$$\begin{aligned}
0 &= [\dot{\Pi}^{\mu_{\rm ric}}] + [\Pi^{\mu_{\rm ric}} L] + ([\Pi^{\mu_{\rm ric}} L])^\top - [\Pi^{\mu_{\rm ric}} B_\beta B_\beta^* \Pi^{\mu_{\rm ric}}] + [C^* C]\\
&= \dot{\mathbf{\Pi}} + \mathbf{M}\overline{\Pi^{\mu_{\rm ric}} L} + (\mathbf{M}\overline{\Pi^{\mu_{\rm ric}} L})^\top - \mathbf{M}\overline{\Pi^{\mu_{\rm ric}} B_\beta B_\beta^* \Pi^{\mu_{\rm ric}}} + \mathbf{M}\overline{C^* C}\\
&= \dot{\mathbf{\Pi}} + \mathbf{\Pi}\overline{L} + \overline{L}^\top \mathbf{\Pi} - \mathbf{\Pi}\overline{B_\beta B_\beta^*}\mathbf{M}^{-1}\mathbf{\Pi} + \mathbf{M}\overline{C^* C}\\
&= \dot{\mathbf{\Pi}} + \mathbf{\Pi}\overline{L} + \overline{L}^\top \mathbf{\Pi} - \mathbf{\Pi}\mathbf{M}^{-1}[B_\beta B_\beta^*]\mathbf{M}^{-1}\mathbf{\Pi} + \mathbf{M}\overline{C^* C}
\end{aligned}$$

Recalling B in (4.1), we note that $\langle B_\beta B_\beta^* y, z\rangle_{\mathrm{D}(A)',\mathrm{D}(A)} = (B_\beta^* y, B_\beta^* z)_{\mathbb{R}^{M_0}}$. Hence, with

$$\mathbf{B}_\beta := \beta^{-\frac{1}{2}}[b_{(n,j)}] \in \mathbb{R}^{N\times M_0},$$

where $[b_{(n,j)}]$ is the matrix with columns containing the finite-element vectors $b_{(:,j)}$ corresponding to the delta actuators δ_{x^j}, $1 \le j \le M_0$, we obtain

$$\langle B_\beta B_\beta^* y, z\rangle_{\mathrm{D}(A)',\mathrm{D}(A)} = (B_\beta^* y, B_\beta^* z)_{\mathbb{R}^{M_0}} = (\mathbf{B}_\beta^\top \overline{z})^\top \mathbf{B}_\beta^\top \overline{y} = \overline{z}^\top \mathbf{B}_\beta \mathbf{B}_\beta^\top \overline{y}.$$

Thus, we find $[B_\beta B_\beta^*] = \mathbf{B}_\beta \mathbf{B}_\beta^\top$.

Next for $C = C^* = C^*C = P_{\mathcal{E}^{\rm f}_{M_1}}$, the orthogonal projection onto $\mathcal{E}^{\rm f}_{M_1}$, we recall that

$$\overline{P_{\mathcal{E}^{\rm f}_{M_1}} y} = [c_{(n,j)}]\mathbf{\Theta}\,[c_{(n,j)}]^\top \mathbf{M}\overline{y} = [c_{(n,j)}]\mathbf{\Theta}_{\mathbf{c}}^\top \mathbf{\Theta}_{\mathbf{c}}\,[c_{(n,j)}]^\top \mathbf{M}\overline{y}$$

where the jth column of $[c_{(n,j)}]$ contains the finite-element vector corresponding to the eigenfunction e_j of A, $1 \le j \le M_1$, and $\mathbf{\Theta}_{\mathbf{c}}$ is the Cholesky factor of $\mathbf{\Theta} := ([c_{(n,j)}]^\top \mathbf{M}[c_{(n,j)}])^{-1}$.

Therefore, we arrive at the discretization of (4.22) as

$$0 = \dot{\mathbf{\Pi}} + \mathbf{\Pi}\overline{L} + \overline{L}^\top \mathbf{\Pi} - \mathbf{\Pi}\mathfrak{B}_\beta \mathfrak{B}_\beta^\top \mathbf{\Pi} + \mathfrak{C}^\top \mathfrak{C} \tag{6.6a}$$

with

$$\overline{L} = \mathbf{M}^{-1}[L^{\mu_{\rm ric}}], \quad \mathfrak{B}_\beta = \mathbf{M}^{-1}\mathbf{B}_\beta \quad \text{and} \quad \mathfrak{C} = \mathbf{\Theta}_{\mathbf{c}}[c_{(n,j)}]^\top \mathbf{M}, \tag{6.6b}$$

where $[L^{\mu_{\rm ric}}]$ is the matrix corresponding to a spatial finite-dimensional discretization of the diffusion-reaction-convection terms. That is, with (y,z) as in (6.1), $(Ly,z)_{V',V} = \overline{z}^\top [L^{\mu_{\rm ric}}]\overline{y}$. Namely, for the diffusion, reaction, and convection terms we take

$$\begin{aligned}
\langle(-\nu\Delta + \mathbf{1})y, z\rangle_{V',V} &= \overline{z}^\top(\nu\mathbf{S} + \mathbf{M})\overline{y},\\
\langle(a - 1 - \mu_{\rm ric})y, z\rangle_{V,V'} &= \overline{z}^\top[\tfrac{1}{2}(\mathcal{D}_{[\overline{a-1-\mu_{\rm ric}}]}\mathbf{M} + \mathbf{M}\mathcal{D}_{[\overline{a-1-\mu_{\rm ric}}]}\overline{y},\\
\langle b\cdot\nabla y, z\rangle_{X',X} &= \overline{z}^\top(\mathcal{D}_{[\overline{b}_1]}\mathbf{D}_{x_1} + (\mathcal{D}_{[\overline{b}_2]}\mathbf{D}_{x_2})\overline{y},
\end{aligned}$$

where $\mathbf{S}$ is the stiffness matrix, $\mathcal{D}_{[\overline{v}]}$ is the diagonal matrix with entries given by the vector $\overline{v}$, and $\mathbf{D}_{x_i}$ is the finite-element matrix corresponding to the discretization of the spatial partial derivative $\frac{\partial}{\partial x_i}$, that is, $\langle\frac{\partial}{\partial x_i}y, z\rangle_{V',V} = \overline{z}^\top \mathbf{D}_{x_i}\overline{y}$, for (y,z) as in (6.1).

7. Numerical results

We consider actuators constructed as in Section 5.2. The results of the following simulations utilize a spatial discretization based on piecewise linear finite elements (hat functions) associated to an unstructured triangulation $\mathcal{T}$ of the spatial rectangular domain and to a temporal discretization based on an implicit-explicit scheme. An implicit Crank–Nicolson scheme is used to discretize the symmetric diffusion and reaction terms and an explicit Adams–Bashford extrapolation is used to discretize the nonsymmetric convection term and the feedback control term.

7.1. **Explicit feedback control.** We present the results of simulations showing the stability of system (1.2) under Neumann boundary conditions,

$$\tfrac{\partial}{\partial t} y + (-\nu\Delta + \mathbf{1})y + (a-1)y + b\cdot\nabla y = \sum_{j=1}^{M_0} K_j^{M,\lambda}(y)\boldsymbol{\delta}_{x^{M,j}}, \tag{7.1a}$$

$$(\mathbf{n}\cdot\nabla y)|_{\partial\Omega} = 0, \quad y(0) = y_0, \tag{7.1b}$$

with explicit feedback control

$$K^{M,\lambda} = (K_1^{M,\lambda}, \dots, K_{M_0}^{M,\lambda}), \qquad K_j^{M,\lambda}(y) \coloneqq -\lambda(-\nu\Delta + \mathbf{1})^{-1}y)(x^{M,j}). \tag{7.1c}$$

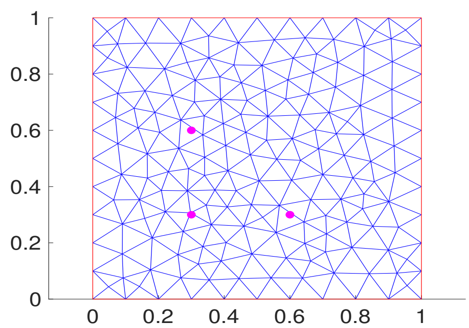
We shall use the two configurations of the actuators shown as shown in Fig. 1, corresponding to the cases $M \in \{1,2\}$, following the construction in Section 5.2, by taking $M_0 = M_\sigma \in \{3, 12\}$ actuators. In the same figure we can see the (coarsest) triangulation $\mathcal{T} = \mathcal{T}^0$ that we shall consider. The locations of the actuators do not coincide with mesh points. Thus, for a fixed M, we shall write each location $x^{M,j}$ as a convex combination of the vertices $p^1(T_k)$, $p^2(T_k)$, $p^3(T_k)$, of a (closed) triangle T_k containing $x^{M,j}$, that is,

$$x^{M,j} = \sum_{n=1}^{3} c_n p^n(T_k), \qquad \sum_{n=1}^{3} c_n = 1, \quad 0 \le c_n \le 1.$$

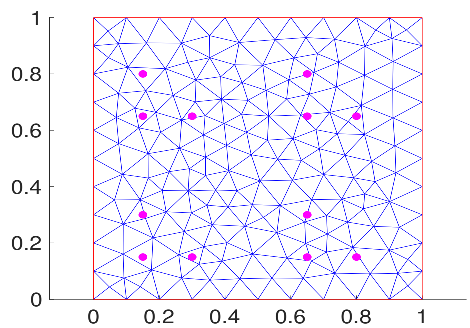
Then we approximate the actuator $\boldsymbol{\delta}_{x^{M,j}}$ as

$$\boldsymbol{\delta}_{x^{M,j}} \approx \boldsymbol{\delta}^{\mathcal{T}}_{x^{M,j}} = \sum_{n=1}^{3} c_n \boldsymbol{\delta}_{p^n(T_k)}, \tag{7.2}$$

where the superscript in $\mathfrak{d}_j^{\mathcal{T}}$ underlines that the approximation depends on the triangulation $\mathcal{T}$ of the spatial domain. Note that in the case that there is more than one triangle in $\mathcal{T}$ containing $x^{M,j}$, the approximation $\boldsymbol{\delta}^{\mathcal{T}}_{x^{M,j}}$ does not depend on the choice of the triangle $T_k \in \mathcal{T}$ containing $x^{M,j}$.



(a) Case $M = 1$.



(b) Case $M = 2$.

FIGURE 1. Initial triangulation $\mathcal{T}^0$ and actuators locations $x^{M,j}$.

As reaction and convection coefficients in system (7.1) we take

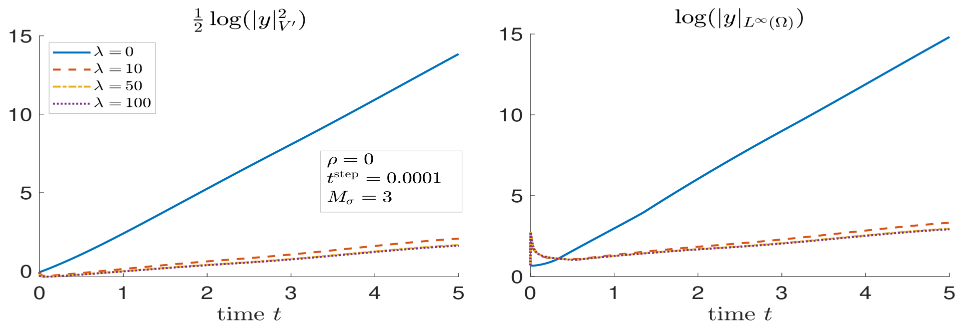
$$a(t,x) - 1 = -3 - (2 - x_1)\cos(\pi x_2) - 2\left|\sin(t + x_2)\right|_{\mathbb{R}}, \tag{7.3a}$$

$$b(t,x) = \begin{bmatrix} \frac{t+2}{t+1} x_1(x_1 - 1)x_2 \\ -\cos(t)(x_1 - \frac{1}{2})x_2(x_2 - 1) \end{bmatrix}, \tag{7.3b}$$

and as diffusion coefficient and initial state we take

$$\nu = 0.1, \quad \text{and} \quad y_0 = x_1(1 + \sin(2x_2)). \tag{7.3c}$$

In Fig. 2(a) we show results corresponding to 3 actuators located as in Fig. 1(a), corresponding to the case $M = 1$, using the time-step $t^{\text{step}} = 10^{-4}$ for temporal discretization. Fig. 2(a) shows that the free dynamics, corresponding to the case $\lambda = 0$, is unstable in the V' norm. Recall that our stabilizability result has been derived in such distribution space norm. We can see that such 3 actuators are not able to stabilize the system for feedback gain parameters $\lambda \in \{10, 50, 100\}$. Further, we observe a convergence-like behavior to an unstable regime, as λ increases, of the norm of the solution, which allows us to extrapolate that such actuators are unable to stabilize the system, no matter how large we take λ. This confirms the theoretical result on the necessity of taking M large enough. We show in Fig. 2(b) also the L^∞-norm of the solution, thus showing the behavior of the largest magnitude reached by the solution.



(a) Evolution of V'-norm. (b) Evolution of L^∞-norm.

FIGURE 2. With explicit feedback; $M = 1$. Data (7.3).

Next, we increase M, by taking $M = 2$, corresponding to considering a 2×2 partition of the spatial domain into 4 rescaled copies of itself and by taking 12 actuators located as in Fig. 1(b). In addition, we shall also consider 3 consecutive refinements of the spatial triangulation $\mathcal{T}^0$ in Fig. 1(b). Hence, we will have 4 triangulations as

$$\mathcal{T}^\rho, \qquad \rho \in \{0, 1, 2, 3\},$$

where $\mathcal{T}^{\rho+1}$ is obtained from $\mathcal{T}^\rho$ by dividing each triangle $T_k^\rho \in \mathcal{T}^\rho$ into 4 congruent triangles by connecting the middle points of the edges of T_k^ρ (regular refinement). The initial mesh $\mathcal{T}^0$ was generated with the Matlab routine `initmesh`, and each refimement was done using the Matlab routine `refinemesh`.

Fig. 3 shows that delta distributions actuators located as in Fig. 1(b) are able to stabilize the system (for the given initial state). We also see that as we refine the spatial

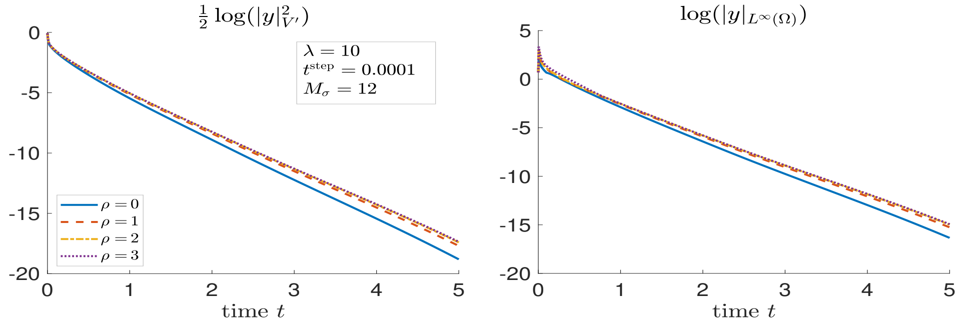


FIGURE 3. With explicit feedback; $M = 2$. Spatial refinements. Data (7.3).

mesh we observe a convergence of the evolution of both the V'-norm and the L^∞-norm to a stable behavior. With respect to the V'-norm, this confirms our theoretical stability result. With respect to the L^∞-norm, we do not have such a stability result, but we can observe that this norm also converges exponentially to zero.

We show time snapshots of the solutions in Fig. 4, where we can see the effect of the feedback control action since the sign of the solution at actuators locations tends to be opposite to the sign of the solution at neighboring points.

7.2. **Explicit versus Riccati feedback for the autonomous case.** We have seen that the explicit feedback corresponding to 3 actuators located as in Fig. 1(a) is not able to stabilize the system. However, such 3 actuators could be able to stabilize the system if we take a different feedback. We illustrate this point, in the particular autonomous setting, comparing the explicit feedback with the more computationally expensive classical Riccati feedbacks.

First of all, solving (backwards) the differential Riccati equation for $t \in [0, +\infty)$ is not possible. Secondly, solving it in a large time interval $t \in [0, T]$ (for a suitable final condition at time $t = T$, e.g., see [9, 17, 22]) can be a very time-expensive task.

Therefore, we restrict the comparison to the autonomous case, where the solution of the Riccati equation is independent of time and we have "just" to solve a single algebraic Riccati equation, namely, we shall compute the solution of

$$0 = \mathbf{\Pi}\overline{L_0} + \overline{L_0}^\top\mathbf{\Pi} - \mathbf{\Pi}\mathfrak{B}_\beta\mathfrak{B}_\beta^\top\mathbf{\Pi} + \mathfrak{C}^\top\mathfrak{C} \tag{7.4a}$$

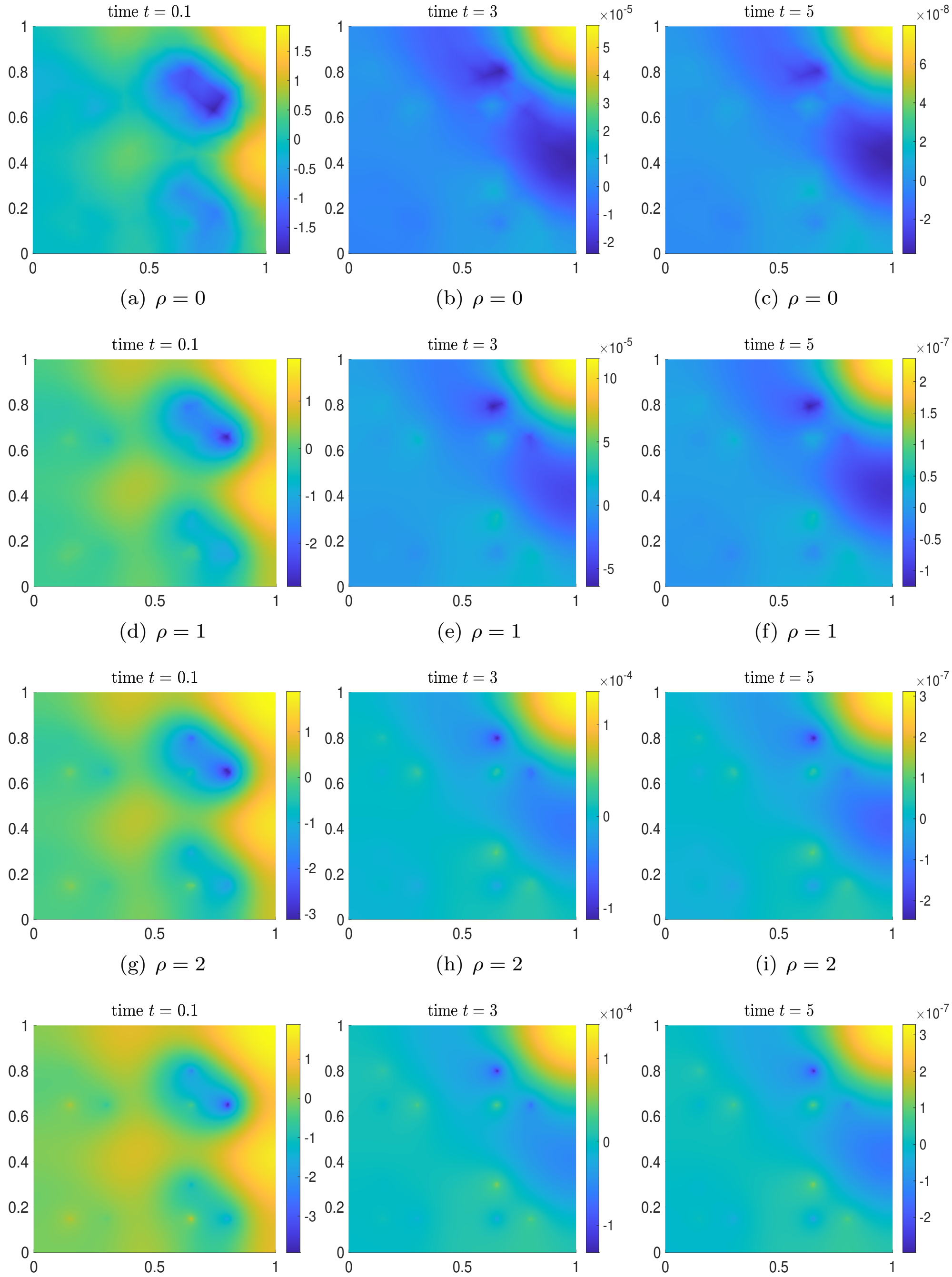
with

$$\overline{L_0} = \mathbf{M}^{-1}\mathbf{L}^{\mu_{\rm ric}}(0), \quad \mathfrak{B}_\beta = \mathbf{M}^{-1}\mathbf{B}_\beta \quad \text{and} \quad \mathfrak{C} = \mathbf{\Theta}_{\mathbf{c}}[c_{(n,j)}]^\top\mathbf{M}, \tag{7.4b}$$

corresponding to taking, as reaction and convection coefficients in system (7.1), the evaluation at initial time $t = 0$ of the functions in (7.3), that is,

$$a(x) - 1 = -3 - (2 - x_1)\cos(\pi x_2) - 2\left|\sin(x_2)\right|_{\mathbb{R}}, \quad b(x) = \begin{bmatrix} 2x_1(x_1 - 1)x_2 \\ -(x_1 - \frac{1}{2})x_2(x_2 - 1) \end{bmatrix}. \tag{7.5a}$$

The diffusion coefficient and initial state are taken as in (7.3). Finally, we have taken the 3 actuators located as in Fig. 1(a) and set $M_1 = 30$ for the dimension of the



(a) $\rho = 0$ (b) $\rho = 0$ (c) $\rho = 0$

(d) $\rho = 1$ (e) $\rho = 1$ (f) $\rho = 1$

(g) $\rho = 2$ (h) $\rho = 2$ (i) $\rho = 2$

(j) $\rho = 3$ (k) $\rho = 3$ (l) $\rho = 3$

FIGURE 4. Time snapshots for several spatial refinements; Explicit feedback; $M = 2$. Data (7.3).

observation operator $C = P_{\mathcal{E}^{\mathrm{f}}_{M_1}}$. That is,

$$M_0 = M_\sigma = 3 \quad \text{and} \quad M_1 = 30. \tag{7.5b}$$

So, the following simulations correspond to an autonomous system, with a feedback input $u(t) = u(y(t)) \in \mathbb{R}^{M_0}$ either constructed from the solution of the algebraic Riccati equation, $u = -\beta^{-1}B^*\Pi y$, or given explicitly as in (7.1).

We have computed the solution $\mathbf{\Pi}$ of the Riccati equation (6.6) only for the coarse triangulation $\mathcal{T}^0$ in Fig. 1(a), and then used such solution to construct an appropriate feedback to perform the similations in refinements of $\mathcal{T}^0$.

Let us write $\mathbf{\Pi}^0 = \mathbf{\Pi}$ to underline that $\mathbf{\Pi}^0$ corresponds to the solution computed the coarse triangulation $\mathcal{T}^0$. Following the notations in Section 6, we see that for the simulations in such triangulation the feedback control is given by

$$\mathbf{B}^0\mathbf{u}^0(t) \quad \text{with} \quad \mathbf{u}^0(t) = -\beta^{-1}\mathfrak{B}^0\mathbf{\Pi}^0\overline{y}^0(t) \tag{7.6}$$

where $\mathbf{B}^0 = [b^0_{(n,j)}] \in \mathbb{R}^{N_0\times M_0}$ is the matrix containing, in its columns, the vectors $b^0_{(n,j)}$ in the coarse mesh corresponding to the actuators, $\mathbf{M}_0$ is the mass matrix in the coarse mesh, $\mathfrak{B}^0 \coloneqq (\mathbf{M}_0^{-1}\mathbf{B}^0)^\top$, and $\overline{y}^0(t) \in \mathbb{R}^{N_0\times 1}$ is the vector state at time $t \ge 0$.

The matrix $\mathbf{M}_0^{-1}\mathbf{B}^0$ (and its transpose) as well as the matrix $\overline{L_0}$ in (7.4) have been computed prior to the time-stepping to avoid inverting the mass at each time step. The computation of $\overline{L_0}$ in fine meshes can be a demanding numerical task, as well as the computation of the solution of the algebraic Riccati equation (7.4). That is why we did such computations in the coarse triangulation $\mathcal{T}^0$ only. Next, we show how we have used the Riccati solution $\mathbf{\Pi}^0$, computed in $\mathcal{T}^0$, for simulations in refinements of $\mathcal{T}^0$.

Observe that $\mathbf{u}^0(t)$ gives us the feedback coordinates used to tune the actuators. For simulations in a refined mesh such coordinates were computed as follows. Let $\mathbf{B}^\rho = [b^\rho_{(n,j)}] \in \mathbb{R}^{N_\rho\times M_0}$, $\rho \in \{1, 2, 3\}$ be the matrix containing the actuators vectors $b^\rho_{(n,j)}$ in the mesh $\mathcal{T}^\rho$, with $N_\rho > N_0$ points, which is a refinement $\mathcal{T}^0$. Let $\overline{y}^\rho(t) \in \mathbb{R}^{N_\rho\times 1}$ be the vector state at time $t \ge 0$. We assume that the first N_0 indexes correspond to points in the coarse mesh, that is, the coordinates of $\mathcal{M}^0\overline{y}^\rho \in \mathbb{R}^{N_0\times 1}$, defined as $(\mathcal{M}^0\overline{y}^\rho)(j, 1) \coloneqq \overline{y}^\rho(j, 1)$, $1 \le j \le N_0$, correspond to the values of $\overline{y}^\rho$ at the points in the coarse mesh. Then, we compute the feedback control on the finer mesh as

$$\mathbf{B}^\rho\mathbf{u}^\rho(t) \quad \text{with} \quad \mathbf{u}^\rho(t) = -\beta^{-1}\mathfrak{B}^0\mathbf{\Pi}^0\overline{\mathcal{M}}_0\overline{y}^\rho. \tag{7.7}$$

In Fig. 5 we see that the explicit feedback with the 3 delta actuators localized as in Fig. 1(a), and with the control parameter gain $\lambda = 10$, is not able to stabilize the resulting autonomous system.

The same actuators allow us to stabilize the system if we take the classical Riccati feedback control as illustrated in Fig. 6, where we can also see that the feedback computed as in (7.7) by using the Riccati solution in the coarse triangulation $\mathcal{T}^0$ in Fig. 1(a), gives us a stabilizing control in consecutive refinements $\mathcal{T}^\rho$ of $\mathcal{T}^0$. Furthermore, we observe convergence of the norm evolution to a stable behavior as we increase the number of refinements. Hence, validating the construction we propose in (7.7). Of course, we cannot expect that such strategy/construction will work for an arbitrary initial coarse mesh, hence the simulations also show that the coarse triangulation $\mathcal{T}^0$, in Fig. 1(a), is fine enough for such strategy.

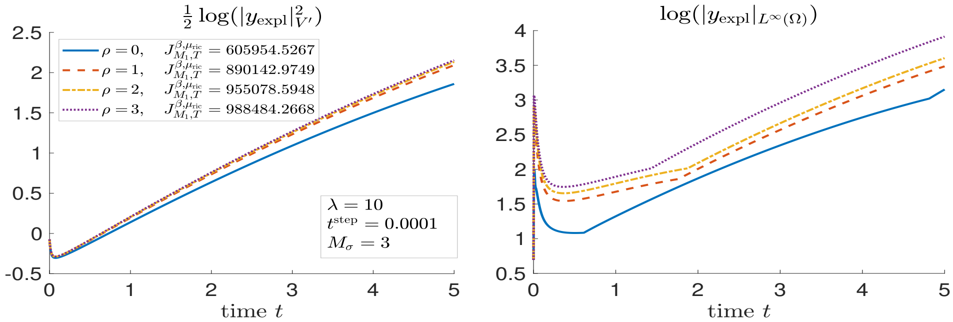


FIGURE 5. With explicit feedback; $M = 1$. Data (7.5).

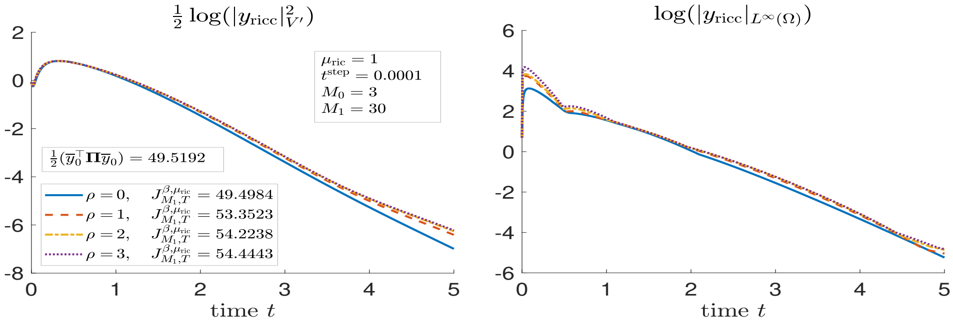


FIGURE 6. With Riccati feedback computed for $\rho = 0$; $M = 1$. Data (7.5).

Observe also that the slope of lines in Fig. 6 are close to (likely, a bit smaller than) $-1 = -\mu_{\rm ric}$ (for large time), where $\mu_{\rm ric}$ is the value we have chosen in (7.4), hence to "guarantee" a stability rate (at least) $\mu_{\rm ric}$. Note also that the evolution of the norm of the solution is not strictly decreasing. This is common for solutions $y_{\rm ricc}$ under the action of a Riccati based feedback, where the stability holds as

$$|y_{\rm ricc}(t)|_{V'} \le C\mathrm{e}^{-\mu(t-s)}\, |y_{\rm ricc}(s)|_{V'}\,,$$

with a "transient bound" constant C (often) strictly larger than 1.

In Figs. 5 and 6 we also see the value of the truncated cost

$$J^{\beta,\mu_{\rm ric}}_{M_1,T} \coloneqq \tfrac{1}{2}\left|\mathrm{e}^{\mu_{\rm ric}\cdot} P_{\mathcal{E}^{\rm f}_{M_1}} y\right|^2_{L^2((0,T),L^2(\Omega))} + \beta\tfrac{1}{2}\left|\mathrm{e}^{\mu_{\rm ric}\cdot} u\right|^2_{L^2((0,T),\mathbb{R}^{M_0})},$$

where $T = 5$ is the time we run the simulations up to; see (4.17). Furthermore, we also show the approximated value for the optimal cost $\frac{1}{2}\langle \Pi y_0, y_0\rangle_{V,V'}$ given by $\frac{1}{2}(\overline{y}_0^\top \mathbf{\Pi} \overline{y}_0)$.

Next, for the sake of completeness we computed the solution of the Riccati equation for the mesh after one refinement, $\rho = 1$, and used it for simulations on the refinements corresponding to $\rho \in \{2, 3\}$. The results are shown in Fig. 7. We see that we obtain the same qualitative behavior as in Fig. 6 where we used the Riccati solution computed in the coarser mesh corresponding to $\rho = 0$. Thus we conclude again that the coarser mesh is likely fine enough to capture the behavior of the norm of the state solving the optimal control problem. It is also interesting to observe that the corresponding truncated costs $J^{\beta,\mu_{\rm ric}}_{M_1,T}$ in Figs. 7 and 6 are close to each other.

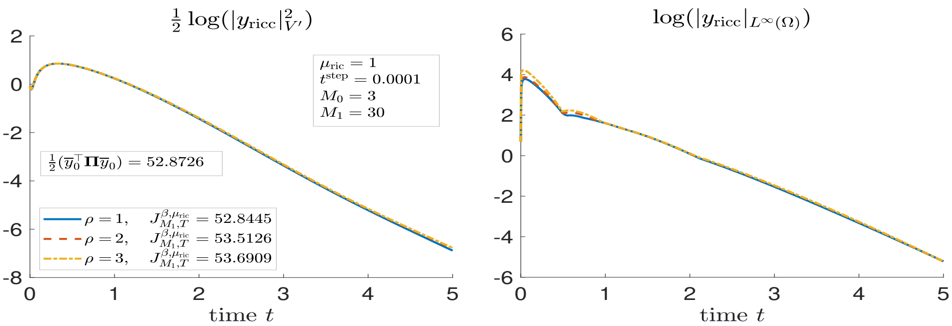


FIGURE 7. With Riccati feedback computed for $\rho = 1$; $M = 1$. Data (7.5).

7.3. **Further remarks.** We discuss points related to the Riccati and explicit feedback.

7.3.1. *On the algebraic matrix equations.* Since $\mathfrak{C}$ is positive semi-definite then $\mathbf{\Pi}$ is the unique exponentially stabilizing symmetric positive semidefinite solution of (7.4). Indeed, if $\mathbf{\Pi}_1$ is another stabilizing symmetric positive semidefinite solution, with $\mathbf{D} := \mathbf{\Pi} - \mathbf{\Pi}_1$, we find

$$\begin{aligned}
0 &= \mathbf{D}\overline{L_0} + \overline{L_0}^\top \mathbf{D} - \mathbf{\Pi}\mathfrak{B}_\beta\mathfrak{B}_\beta^\top\mathbf{\Pi} + \mathbf{\Pi}_1\mathfrak{B}_\beta\mathfrak{B}_\beta^\top\mathbf{\Pi}_1\\
&= \mathbf{D}\overline{L_0} + \overline{L_0}^\top \mathbf{D} - \mathbf{D}\mathfrak{B}_\beta\mathfrak{B}_\beta^\top\mathbf{\Pi} - \mathbf{\Pi}_1\mathfrak{B}_\beta\mathfrak{B}_\beta^\top\mathbf{D}\\
&= \mathbf{D}X + X_1^\top\mathbf{D}, \quad \text{with} \quad X := \overline{L_0} - \mathfrak{B}_\beta\mathfrak{B}_\beta^\top\mathbf{\Pi} \quad \text{and} \quad X_1 := \overline{L_0} - \mathfrak{B}_\beta\mathfrak{B}_\beta^\top\mathbf{\Pi}_1.
\end{aligned}$$

From the stability of X and X_1 we have that all the eigenvalues of X and of X_1 have negative real part which implies that $\mathbf{D} = 0$; see [11, Thm. 1] [21, Thm. 1].

7.3.2. *On the choice of M_1.* The solver we used looks for low-rank (when compared to the number N of mesh points) approximations $\mathbf{\Pi_f}$ of the Cholesky factor $\mathbf{\Pi_c}$ of the Riccati solution, more precisely, so that $\mathbf{\Pi} \approx \mathbf{\Pi_f}^\top\mathbf{\Pi_f}$. Numerical evidence shows that (in the autonomous case) the existence of such low-rank factors can be expected when the ranks of $\mathbf{B} \in \mathbb{R}^{N\times M_0}$ and $\mathbf{C} \in \mathbb{R}^{M_1\times N}$, respectively, M_0 and M_1, are small (again, when compared to N). We have chosen $M_1 = 30$ in our simulations for which we obtained a full-rank factor $\mathbf{\Pi_f} \in \mathbb{R}^{N\times N}$, with $N = 185$ for the case $\rho = 0$, this also gives us a positive definite approximation $\mathbf{\Pi_f}^\top\mathbf{\Pi_f}$ for the solution $\mathbf{\Pi}$. For the refined mesh we obtained a rectangular factor $\mathbf{\Pi_f} \in \mathbb{R}^{265\times 697}$. This of course does not give us a positive definite approximation $\widetilde{\mathbf{\Pi}} := \mathbf{\Pi_f}^\top\mathbf{\Pi_f}$, but it gives us (or, can still give us) a good approximation for $\mathbf{\Pi}$ and for the product control input operator $\mathbf{K} = -\beta^{-1}(\mathbf{M}^{-1}\mathbf{B})\widetilde{\mathbf{\Pi}}$ in (7.6). An explanation for this phenomenon of existence of such low-rank factor $\mathbf{\Pi_f}$ approximations is as follows. Note that Π associated to the cost functional (4.3) does not necessarily define a scalar product, because the optimal cost can vanish for suitable initial conditions. For example, for the autonomous dynamics (A is independent of time)

$$\dot{y} = -Ay + Bu, \qquad y(s) = y_s,$$

we can see that, if $j > M_1$ and $y_s = e_j$, then the minimum of $\mathcal{J}$ in (4.3) vanishes (by taking $u = 0$). Since the dynamics is autonomous, we know that Π is independent of time and satisfies $\langle \Pi e_j, e_j \rangle_{V,V'} = 0$. Hence, it does not define a scalar product. In this situation, we can expect that a numerical approximation $\mathbf{\Pi} \in \mathbb{R}^{N\times N}$ of Π can be itself well approximated by products as $\widetilde{\mathbf{\Pi}} = \widetilde{\mathbf{\Pi}}_{\mathbf{f}}^\top \widetilde{\mathbf{\Pi}}_{\mathbf{f}}$ of low-rank factors $\widetilde{\mathbf{\Pi}}_{\mathbf{f}} \in \mathbb{R}^{n\times N}$ (i.e., with n small when compared N); see also discussions in [7, Sect. 4] and [19].

7.3.3. *On the computation of the Riccati feedback in fine discretizations.* Recalling (6.6), we need the inverse of the mass matrix to construct $\overline{L} = \mathbf{M}^{-1}\mathbf{L}^{\mu_{\rm ric}}$ and $\mathfrak{B}_\beta = \mathbf{M}^{-1}\mathbf{B}_\beta$. The latter is not problematic if the number of actuators M_0 is small, the former is also of no concern for coarse discretizations, but it can be an issue (at least, very time expensive) when working on fine discretizations, where the number of columns of $\mathbf{L}^{\mu_{\rm ric}}$ is large. Circumventing this issue, we computed the finite-element Riccati solution on a coarse triangulation of the spatial domain, and then used this solution, in the form (6.6), for simulations on refinements of such a triangulation. Of course, the size (i.e., number of points) of such a coarse mesh is expected to depend on the parameters of the free dynamics (e.g., on how unstable the dynamics is). This means that for parameters requiring a large size coarse mesh we may need a different strategy. At this point, we must say that in order to avoid the inverse of the mass matrix in the Riccati equation we could think of computing first the product $\mathbf{\Xi} = \mathbf{M}^{-1}\mathbf{\Pi}\mathbf{M}^{-1}$, and then recover $\mathbf{\Pi} = \mathbf{M}\mathbf{\Xi}\mathbf{M}$. Note that from (6.6) we find that $\mathbf{\Xi}$ solves the more general equation as follows

$$0 \approx \mathbf{M}\dot{\mathbf{\Xi}}\mathbf{M} + \mathbf{M}\mathbf{\Xi}\mathbf{L}^{\mu_r ic} + (\mathbf{L}^{\mu_r ic})^\top \mathbf{\Xi}\mathbf{M} - \mathbf{M}\mathbf{\Xi}\mathbf{B}_\beta \mathbf{B}_\beta^\top \mathbf{\Xi}\mathbf{M} + \mathbf{C}^\top \mathbf{C}$$

(cf. [20, eq. (15)], [8, below eq. (8)], [14, eq. (4.8)]). Thus, we can write the equation avoiding the inverse of the mass matrix, but we have to compute the solution of a more general equation, which is numerically more involved. We refer the interested reader to the works mentioned above and references therein.

7.3.4. *Explicit versus Riccati.* We conclude this section with comments on a comparison between explicit and Riccati feedbacks. Some advantages of the Riccati feedback are:

- for a given number of actuators, it may succeed to stabilize the system when the explicit feedback fails;
- it gives us the solution minimizing a classical energy functional;

while some advantages of the explicit feedback are:

- it is less expensive to compute than Riccati;
- its computation does not require knowledge of the reaction-convection operator;
- its computation cost is essentially the same for autonomous and nonautonomous systems, while for general nonautonomous systems, it is impossible to solve the Riccati equation in the entire time-line $t \in [0, +\infty)$.

For the particular case of nonautonomous time-periodic dynamics, the solution of the Riccati equation will be time-periodic. In this case it is possible to solve such equation, by looking for a solution in a finite time interval with length equal to the time-period.

Computing the solution is still an expensive numerical task, but if we succeed, then the resulting feedback could (or, will likely) stabilize the system when the explicit one fails.

8. Final remarks

We have shown that stabilizability in the distribution space norm $V' = \mathrm{D}(A^{\frac{1}{2}})'$ can be achieved for parabolic-like equations, with distribution actuators in $\mathrm{D}(A)'$, where A is a diffusion-like operator in a Hilbert pivot space H, with domain $\mathrm{D}(A) \subset H$.

As an application we considered the case of scalar parabolic equations in a spatial domain $\Omega \subset \mathbb{R}^d$, with $d \in \{1, 2, 3\}$, where the actuators are delta distributions $\boldsymbol{\delta}_{x^i} \in \mathrm{D}(A)'$ of a given finite set of spatial points $x^i \in \Omega \subset \mathbb{R}^d$. For this particular setting, in the case $d = 1$, we will actually have that $\boldsymbol{\delta}_{x^i} \in V'$, and the stabilizability can be achieved in the norm of the Hilbert pivot function space $H = L^2(\Omega)$. We have seen that the numerical simulations, for domains $\Omega \subset \mathbb{R}^2$ confirm the stability in the distribution space norm V'. Besides, the same simulations suggest that the stability also holds in the function space $L^\infty(\Omega)$-norm. Of course, the simulations correspond to a set of selected parameters and, by this reason we cannot infer the validity of such stability in $L^\infty(\Omega)$-norm, in general. It would be interesting to know whether or not the stability holds in some function space norm, in the cases $d \in \{2, 3\}$.

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