Differentiability of the Value Function on $H^1(\Omega)$ of Semilinear Parabolic Infinite Time Horizon Optimal Control Problems under Control Constraints

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Abstract

An abstract framework guaranteeing the continuous differentiability of local value functions on $H^1(\Omega)$ associated with optimal stabilization problems subject to abstract semilinear parabolic equations in the presence of norm constraints on the control is established. It guarantees the local well-posedness of the associated Hamilton-Jacobi-Bellman equation in the classical sense. Examples illustrate that the assumptions imposed on the dynamical system are satisfied for practically relevant semilinear equations.

1 Introduction.

Continuous differentiability of local value functions with respect to initial data in $H^1(\Omega)$ associated with norm-constrained optimal control problems on the infinite time horizon for semilinear parabolic equations is investigated. This is an extension of our work in [BK] where a similar problem was investigated in the situation where the domain of the value function is $L^2(\Omega)$ rather than $H^1(\Omega)$. The motivation for these two different settings is the following. The class of semilinear equations for which local differentiability of the value function can be established is wider for $H^1(\Omega)$ initial data than that for $L^2(\Omega)$ initial data. The $L^2(\Omega)$ framework, on the other hand, is of independent value. It can be more flexible for consistent numerical realizations, for instance, than the setting in $H^1(\Omega)$. Concerning the structure of proofs we can frequently proceed similarly as in [BK], on a technical level, however, many differences need to be overcome.

To accomplish our goal we utilize techniques for sensitivity analysis of abstract infinite dimensional optimization problems. We consider the optimality conditions for our constraint optimal control problem as parameter dependent generalized equations and apply known results on the Lipschitz continuous dependence of their solutions [Don]. Once Lipschitz continuity of the optimal controls with respect to initial data is established, differentiability of the value function and the derivation of the Hamilton Jacobi Bellman (HJB) equation will follow by considerations which are rather straightforward by now. On a technical level, the most severe difficulties arise due to the fact that we consider infinite horizon rather the finite horizon problems. These are the natural settings

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for optimal stabilization problems. We shall assume that the linear part of the dynamical system is feedback stabilizable, see eg. [KR, Tri]. Subsequently we derive appropriate conditions which guarantee that the nonlinear equations allow stabilizing controls. These conditions will also guarantee that the adjoint state associated to a local optimal control is unique. Utilizing the transversality condition this later property is typically immediate for finite horizon problems, but it is not at all obvious in the infinite horizon case. Concerning the treatment of the norm constraints on the controls, utilizing properties of the adjoint states, we shall be able to argue that the constraints are inactive for the optimal controls beyond a certain time horizon. This property is essential (at least for our approach) to guarantee the uniqueness of the adjoint states and their Lipschitz continuous dependence on the parameters, see eg. *Step* 2 of the proof of Proposition 3.2.

Let us point out a remarkable consequence of our analysis: While differentiability of the local value function V on a subset of $H^1(\Omega)$ guarantees that its Riesz representations lie in a subset of $H^1(\Omega)'$, the use of dynamic programming will guarantee that the Riesz representations themselves are elements of $H^1(\Omega)$. This will allow that the HJB equation can in fact be interpreted on subsets of $H^1(\Omega)$, which would otherwise not be possible, due to the appearance of the action of the state equation.

The results of the paper require a smallness assumption on the initial data. While this is restrictive we conjecture that it is inevitable. Under structural assumptions, on the nonlinearity, for instance monotonicity, or assumptions on the stabilizability of the nonlinear equation, rather than only on the leading linear part as will be used in this paper, it may be possible to obtain semi-global or global results, in the sense that for every local optimal solution satisfying a second order condition, the value function is differentiable in a neighborhood. This could be of interest for future work.

Concerning the literature, there appears to be very little focusing on the C^1 property of the value function for infinite dimensional systems. For the finite dimensional case we can mention [Fra, Chapter 5], [CF], [Goe]. It should be recalled that regularity of the value function is a special case, since in general we can only expect its Lipschitz continuity, see for instance [BC], and more recent results in [BF]. On the other hand there are many papers on the sensitivity analysis of finite horizon optimal control problems with pointwise, and thus affine, control constraints. We quote some of them [BM], [Gri], [GV], [GHH], [Mal], [MT], [Tro], [Wac]. These papers are not written with the intention of application towards the HJB equation. The [BKP1], [BKP2] Taylor functions expansions are provided for optimal stabilization problems without norm constraints related to bilinear problems and the Navier-Stokes equation, respectively.

The following sections are structured as follows. Section 2 contains the precise problem statement, the assumptions which are postulated throughout the remainder of the paper, and the statement of the two main theorems. Section 3 contains the technical developments which lead up to the analysis of the adjoint equations, the transversality condition, and second order optimality. Lipschitz continuous dependence of the optimal controls, and the associated states and adjoint states with respect to the initial condition is contained in Section 4. This is proved by verifying the Dontchev-Robinson strong regular point condition. The local C^1-property of the value function and the HJB equation are treated Section 5. Section 6 contains concrete problems, which illustrate the applicability of the assumptions with respect to specific nonlinearities.

2 Differentiability of value function for optimal stabilization

2.1 Problem formulation

Let Ω be an open connected bounded subset of $\mathbb{R}^d$ with a $C^{1,1}$ boundary Γ. The associated space-time cylinder is denoted by $Q = \Omega \times (0,\infty)$ and the associated lateral boundary by $\Sigma = \Gamma \times (0,\infty)$. We consider the following the optimal stabilization problem in abstract form with associated value function,

$$\mathcal{V}(y_0) = \inf_{\substack{y \in W_\infty(\mathcal{D}(A),Y) \\ u(t) \in \mathcal{U}_{ad}}} \frac{1}{2}\int_0^\infty \|y(t)\|_Y^2\,dt + \frac{\alpha}{2}\int_0^\infty \|u(t)\|_{\mathcal{U}}^2\,dt, \tag{P}$$

subject to the semilinear parabolic equation

$$\begin{cases} y_t = Ay + F(y) + Bu & \text{in } L^2(I;Y), \quad (2.1a)\\ y(0) = y_0 & \text{in } V. \quad (2.1b)\end{cases}$$

Here $I = (0,\infty)$, $Y = L^2(\Omega)$, $V = H^1(\Omega)$, and $W_\infty(\mathcal{D}(A),Y)$ denotes the classical space for strong solutions of semilinear parabolic problems. Together with the operator A it will be introduced below. Further $\mathcal{U}_{ad} \subset \mathcal{U}$ stands for the set of admissible controls which is defined as $\mathcal{U}_{ad} = \{v \in \mathcal{U} : \|v\|_{\mathcal{U}} \le \eta\}$, where $\eta > 0$, and $\mathcal{U}$ is a Hilbert space which will be identified with its dual. By $\mathbb{P}_{\mathcal{U}_{ad}}$ we denote the Hilbert space projection of $\mathcal{U}$ onto $\mathcal{U}_{ad}$, $U = L^2(I;\mathcal{U})$, and

$$U_{ad} = \{u \in U : \|u(t)\|_{\mathcal{U}} \le \eta, \text{ for } a.e.\ t > 0\}, \tag{2.2}$$

and finally $B \in \mathcal{L}(\mathcal{U}, Y)$. For this choice of admissible controls the dynamical system can be stabilized for all sufficiently small initial data in V, see Corollary 3.3 and Remark 3.1, provided that the pair (A, B) is exponentially stabilizable.

Throughout F stands for the substitution operator associated to a mapping $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$, so that $(Fy)(t) = \mathfrak{f}(y(t))$. Sufficient conditions which guarantee the existence of solutions to (2.1) as well as solutions $(\bar{y}, \bar{u})$ to (P), with $y_0 \in V$ sufficiently small, will be given below. We shall also make use of the adjoint equation associated to an optimal state $\bar{y}$, given by

$$-p_t - A'p - F'(\bar{y})^*p = -\bar{y} \qquad \text{in } L^2(I;[\mathcal{D}(A)]'), \tag{2.3a}$$

$$\lim_{t\to\infty} p(t) = 0 \qquad \text{in } V'. \tag{2.3b}$$

Its solution p will be considered in $L^2(I;Y)$ as well as in $W_\infty(V,V') = \{v \in L^2(I;V) : v_t \in L^2(I;V')\}$, and eventually in $W_\infty(\mathcal{D}(A),Y)$.

2.2 Further notation and assumptions on A

Since V is continuously and densely embedded in Y the inclusions $V \subset Y \subset V'$ constitute a Gelfand triple with Y as pivot space. Let a be a continuous bilinear form on V which is $V - Y$ coercive:

$$\exists \rho \in \mathbb{R} \text{ and } \exists \theta > 0 : \quad a(v,v) + \rho|v|_Y^2 \ge \theta\,\|v\|_V^2\,, \text{ for all } v \in V. \tag{2.4}$$

This bilinear form induces the operator A by means of

$$\mathcal{D}(A) = \{v \in V : w \to a(v,w) \text{ is } Y - \text{continuous}\},$$
$$(Av, w) = -a(v,w), \forall v \in \mathcal{D}(A), \forall w \in V.$$

This operator is closed and densely defined in Y, and it can be uniquely extended to an operator $A \in \mathcal{L}(V, V')$. It generates an analytic, exponentially bounded semigroup e^{At} with $\|e^{At}\|_{\mathcal{L}(Y)} \le e^{\rho t}$, see eg. [BPDM, Part II, Chapter 1, pg 115]. The adjoint of A, considered as operator in Y will be denoted by A^*. We assume that $\mathcal{D}(A) = \mathcal{D}(A^*)$, algebraically and topologically. We shall also consider A as a bounded linear operator in $\mathcal{L}(\mathcal{D}(A), Y)$, with dual $A' \in \mathcal{L}(Y, [\mathcal{D}(A)]')$. Since $\mathcal{D}(A^*)$ is dense in Y, we recall from eg. [LT, Section 0.3, pg 6] that A' is the unique extension of the operator $A^* \in \mathcal{L}(\mathcal{D}(A^*), Y)$ to an element in $\mathcal{L}(Y, [\mathcal{D}(A)]')$. For any $T \in (0, \infty)$, we define the space $W(0, T; \mathcal{D}(A), Y)$ which we endow with the norm

$$\|y\|_{W(0,T;\mathcal{D}(A),Y)} := \left(\|Ay\|^2_{L^2(0,T;Y)} + \|y\|_{L^2(0,T;Y)} + \left\| \frac{dy}{dt} \right\|^2_{L^2(0,T;Y)} \right)^{1/2}, \; y \in W(0, T; \mathcal{D}(A), Y). \tag{2.5}$$

Generally, given $T > 0$ and two Hilbert spaces $X \subset Y$, by $W(0, T; X, Y)$ we denote the space

$$W(0, T; X, Y) = \left\{ y \in L^2(0, T; X);\ \frac{dy}{dt} \in L^2(0, T; Y) \right\}. \tag{2.6}$$

For $T = \infty$, we write $W_\infty(X, Y)$ and $I = (0, \infty)$. We further define $W^0_\infty = \{y \in W_\infty : y(0) = 0\}$. We also set

$$W(T, \infty; X, Y) = \left\{ y \in L^2(T, \infty; X);\ \frac{dy}{dt} \in L^2(T, \infty; Y) \right\}.$$

We shall frequently use that $W_\infty(\mathcal{D}(A), Y)$ embeds continuously into $C([0, \infty), V)$, see e.g. [LM, Theorem 4.2, Chapter 1] and that $\lim_{t \to \infty} y(t) = 0$ in V, for $y \in W_\infty(\mathcal{D}(A), Y)$, see e.g. [CK]. Further $\|\mathcal{I}\|$ denotes the norm of the embedding constant of $W_\infty(\mathcal{D}(A), Y)$ into $L^2(I; Y)$ and $\|i\|$ is the norm of the embedding V into Y.

For $\delta > 0$ and $\bar{y} \in V$, we define the open neighborhoods $B_V(\delta) = \{y \in V : \|y\|_V < \delta\}$, and $B_V(\bar{y}, \delta) = \{y \in V : \|y - \bar{y}\|_V < \delta\}$.

2.3 Assumptions for problem (P)

Here we summarize the main assumptions which will be utilized throughout the remainder of this paper.

Assumption A1. *The linear system (A, B) with $B \in \mathcal{L}(\mathcal{U}, Y)$ is exponentially stabilizable , i.e. there exists $K \in \mathcal{L}(Y, \mathcal{U})$ such that the semigroup $e^{(A-BK)t}$ is exponentially stable on Y.*

Regarding assumption (A1), we refer to e.g. [Tri].

Assumption A2. *The nonlinearity $F : W_\infty(\mathcal{D}(A), Y) \to L^2(I; Y)$ is twice continuously Fréchet differentiable, with second Fréchet derivative F'' bounded on bounded subsets of $W_\infty(\mathcal{D}(A), Y)$, and $F(0) = 0$.*

Assumption A3. *$F : W(0, T; \mathcal{D}(A), Y) \to L^1(0, T; \mathcal{H}')$ is weak-to-weak continuous for every $T > 0$, for some Hilbert space $\mathcal{H}$ which embeds continuously and densely in Y.*

Recall that $\left(L^1(0, T; \mathcal{H}')\right)' = L^\infty(0, T; \mathcal{H})$, see [Emm, Theorem 7.1.23(iv), p 164]. Moreover, $L^\infty(0, T; \mathcal{H})$ is dense in $L^2(0, T; Y)$, see [MS, Lemma A.1, p 2231].

Assumption A4. *$F'(y)$ and $F'(y)^*$ are elements of $\mathcal{L}(L^2(I;V), L^2(I;Y))$ for each $y \in W_\infty(\mathcal{D}(A), Y)$.*

Remark 2.1. The requirement that $F(0) = 0$ in (A1) is consistent with the fact that we focus on the stabilization problem with 0 as steady state for (3.19). Without loss of generality we further assume that

$$F'(0) = 0, \tag{2.7}$$

which can always be achieved by making $F'(0)$ to be perturbation of A.

Remark 2.2. Assumption (A2) is in only needed locally in the neighborhood of local solutions of (P) which will be under consideration. But it is convenient to assume this regularity globally. Assumption (A4) will only be utilized at the y-component of local solutions $(\bar{y}, \bar{u})$ of problem (P). Such local solutions $\bar{y}$ may enjoy higher regularity than being generic elements in $W_\infty(\mathcal{D}(A), Y)$. In case of finite dimensional problems with all function spaces over Ω replaced by $\mathbb{R}^d$ and A, B matrices of dimensions $d \times d$ and $d \times m$, Assumption (A1) is vacuously satisfied, Assumption (A2) becomes an assumption on the feedback stabilizability of the pair (A, B). Assumptions (A3) - (A4) need to be checked for the specific non-linearity which constitutes the control system.

2.4 Main Theorems

Now we present the main results of this paper. We shall refer to value functions associated to local minima as 'local value function'. The first theorem asserts continuous differentiability of local value functions $\mathcal{V}$ w.r.t. y_0, for all y_0 small enough. The second theorem establishes that $\mathcal{V}$ satisfies the HJB equation in the classical sense. This will be proven in Sections 4 and 5 below. Moreover we need to establish the underlying assumption that problem (P) is well-posed. This will lead to a smallness assumption on the initial states y_0.

We shall further prove the Lipschitz continuity of the state, the adjoint state, and the control with respect to the initial condition $y_0 \in V$ in the neighborhood of a locally optimal solution $(\bar{y}, \bar{u})$ corresponding to a sufficiently small reference initial state $\bar{y}_0$. This will imply the desired differentiability of the local value function associated to local minima.

Theorem 2.1. *Let (A1)-(A4) hold. Then associated to each local solution $(\bar{y}(y_0), \bar{u}(y_0))$ of (P) - of which there exists at least one - there exists a neighborhood $U(y_0)$ in V such that each local value function $\mathcal{V} : U(y_0) \subset V \to \mathbb{R}$ is continuously differentiable, provided that y_0 is sufficiently close to the origin in V. Moreover the Riesz representor of $\mathcal{V}'$ is an element of $C(U(y_0), V)$.*

Theorem 2.2. *Let (A1)-(A4) hold, and let $(\bar{y}(y_0), \bar{u}(y_0)) \in W_\infty(\mathcal{D}(A), Y) \times U_{ad}$ denote a local solution of (P) for y_0 with sufficiently small norm in V. Assume that for some $T_0 > 0$ we have $F(y(\hat{y}_0, u)) \in C([0, T_0); V')$ where $y(\hat{y}_0, u)$ denotes the solution to (2.1) on $[0, T_0)$ with arbitrary $u \in C([0, T_0); Y)$ and sufficiently small $\hat{y}_0$. Then the following Hamilton-Jacobi-Bellman equation holds in a neighborhood $\widetilde{U}(y_0)$ of y_0 in V:*

$$\mathcal{V}'(y)(Ay + F(y)) + \frac{1}{2}\|y\|_Y^2 + \frac{\alpha}{2}\left\|\mathbb{P}_{\mathcal{U}_{ad}}\left(-\frac{1}{\alpha}B^*\mathcal{V}'(y)\right)\right\|_Y^2 + \left(B^*\mathcal{V}'(y), \mathbb{P}_{\mathcal{U}_{ad}}\left(-\frac{1}{\alpha}B^*\mathcal{V}'(y)\right)\right)_Y = 0. \tag{2.8}$$

Moreover the optimal feedback law is given by

$$u = \mathbb{P}_{\mathcal{U}_{ad}}\left(-\frac{1}{\alpha}B^*\mathcal{V}'(y)\right). \tag{2.9}$$

The condition on the smallness of y_0 will be discussed in Remark 4.1 below. Roughly it involves well-posedness of the optimality system and second order sufficient optimality at local solutions. A more detailed statement of these two theorems will be given in Theorem 5.1 and Theorem 5.2 below.

3 Well-posedness of (P) and optimality conditions

3.1 Well-posedness of (P)

Here we prove well-posedness for (P) with small initial data. We recall the following consequence of the fact that under our general assumptions A is the generator of an analytic semigroup.

Consequence 1. *For all $y_0 \in V, f \in L^2(0,T;Y)$, and $T > 0$, there exists a unique solution $y \in W(0,T;\mathcal{D}(A),Y)$ to*

$$\dot{y} = Ay + f, \quad y(0) = y_0. \tag{3.1}$$

Furthermore, y satisfies

$$\|y\|_{W(0,T;\mathcal{D}(A),Y)} \leq c(T)\left(\|y_0\|_V + \|f\|_{L^2(0,T;Y)}\right) \tag{3.2}$$

for a continuous function c. Assuming that $y \in L^2(I;Y)$, consider the equation

$$\dot{y} = \underbrace{(A-\rho I)y}_{A_\rho} + \underbrace{\rho y + f}_{f_\rho}, \quad y(0) = y_0,$$

where $f_\rho \in L^2(I;Y)$. Then the operator A_ρ generates a strongly continuous analytic semigroup on Y which is exponentially stable, see [BPDM, Theorem II.1.3.1]. It follows that $y \in W_\infty(\mathcal{D}(A),Y)$, that there exists M_ρ such that

$$\|y\|_{W_\infty(\mathcal{D}(A),Y)} \leq M_\rho\left(\|y_0\|_V + \|f_\rho\|_{L^2(I;Y)}\right), \tag{3.3}$$

and that y is the unique solution to (3.1) in $W_\infty(\mathcal{D}(A),Y)$, see [BKP2, Section 2.2] .

Lemma 3.1. *There exists a constant $C > 0$, such that for all $\delta \in (0,1]$ and for all y_1 and y_2 in $W_\infty(\mathcal{D}(A),Y)$ with $\|y_1\|_{W_\infty(\mathcal{D}(A),Y)} \leq \delta$ and $\|y_2\|_{W_\infty(\mathcal{D}(A),Y)} \leq \delta$, it holds that*

$$\|F(y_1) - F(y_2)\|_{L^2(I;Y)} \leq \delta C \|y_1 - y_2\|_{W_\infty(\mathcal{D}(A),Y)}. \tag{3.4}$$

Proof. Let y_1, y_2 be as in the statement of the lemma. Using (A2) and Remark 2.1 we obtain the estimate

$$\begin{aligned}
\|F(y_1) - F(y_2)\|_{L^2(I,Y)} &\leq \int_0^1 \left\|F'(y_1 + t(y_2-y_1)) - F'(0)\right\|_{\mathcal{L}(W_\infty(\mathcal{D}(A),Y),L^2(I,Y))} dt \, \|y_2 - y_1\|_{W_\infty(\mathcal{D}(A),Y)} \\
&\leq \int_0^1\int_0^1 \left\|F''(s(y_1 + t(y_2-y_1)))(ty_2 + (1-t)y_1)\right\|_{\mathcal{L}(W_\infty(\mathcal{D}(A),Y),L^2(I,Y))} dsdt \, \|y_2 - y_1\|_{W_\infty(\mathcal{D}(A),Y)}.
\end{aligned}$$

Now the claim follows by Assumption (A2). ∎

Lemma 3.2. *Let A_s be the generator of an exponentially stable analytic semigroup $e^{A_s t}$ on Y. Let C denote the constant from Lemma 3.1. Then there exists a constant M_s such that for all $y_0 \in V$ and $f \in L^2(I;Y)$ with*

$$\tilde{\gamma} := \|y_0\|_V + \|f\|_{L^2(I;Y)} \leq \frac{1}{4CM_s^2}$$

the system

$$y_t = A_s y + F(y) + f, \quad y(0) = y_0 \tag{3.5}$$

has a unique solution $y \in W_\infty(\mathcal{D}(A), Y)$, which satisfies

$$\|y\|_{W_\infty(\mathcal{D}(A),Y)} \leq 2M_s \tilde{\gamma}.$$

Utilizing Lemma 3.1, this lemma can be verified in the same manner as [BKP2, Lemma 5, p 6]. In the following corollary we shall use Lemma 3.2 with $A_s = A - BK$, and the constant corresponding to M_s will be denoted by M_K. We also recall the constant η from (2.2).

Corollary 3.3. *For all $y_0 \in Y$ with*

$$\|y_0\|_V \leq \min\left\{\frac{1}{4CM_K^2}, \frac{\eta}{2M_K \|K\|_{\mathcal{L}(Y)} \|\mathcal{I}\|}\right\}$$

there exists a control $u \in U_{ad}$ such that the system

$$y_t = Ay + F(y) + Bu, \quad y(0) = y_0 \tag{3.6}$$

has a unique solution $y \in W_\infty$ satisfying

$$\begin{aligned} \|y\|_{W_\infty(\mathcal{D}(A),Y)} &\leq 2M_K \|y_0\|_V, \text{ and} \\ \|u\|_U \leq \|K\|_{\mathcal{L}(Y,\mathcal{U})} \|\mathcal{I}\| \|y\|_{W_\infty(\mathcal{D}(A),Y)} &\leq 2M_K \|y_0\|_V \|K\|_{\mathcal{L}(Y,\mathcal{U})} \|\mathcal{I}\|. \end{aligned} \tag{3.7}$$

Proof. By Assumption (A1), there exists K such that $A - BK$ generates an exponentially stable analytic semigroup on Y. Taking $u = -Ky$, equation (3.6) becomes

$$y_t = (A - BK)y + F(y), \quad y(0) = y_0. \tag{3.8}$$

Then by Lemma 3.2 with $\tilde{\gamma} = \|y_0\|_V$ there exists M_K such that (3.8) has a solution $y \in W_\infty(\mathcal{D}(A), Y)$ satisfying

$$\|y\|_{W_\infty(\mathcal{D}(A),Y)} \leq 2M_K \|y_0\|_V,$$

and thus the first inequality in (3.7) holds. For the feedback control we obtain

$$\|u\|_U = \|Ky\|_U \leq \|K\|_{\mathcal{L}(Y,\mathcal{U})}\|y\|_{L^2(I;Y)} \leq \|K\|_{\mathcal{L}(Y,\mathcal{U})}\|\mathcal{I}\|\|y\|_{W_\infty} \leq 2M_K\|y_0\|_V\|K\|_{\mathcal{L}(Y,\mathcal{U})}\|\mathcal{I}\|, \tag{3.9}$$

and thus the second inequality in (3.7) holds. We still need to assert that $u \in U_{ad}$. This follows from the second smallness condition on $\|y_0\|_V$ and (3.9). □

Remark 3.1. In the above proof stabilization was achieved by the feedback control $u = -Ky$. For this u to be admissible it is needed that $\mathcal{U}_{ad}$ has nonempty interior. The upper bound η could be allowed to be time dependent as long as it satisfies $\inf_{t \geq 0} |\eta(t)| > 0$.

Corollary 3.4. *Let $y_0 \in V$ and let $u \in U_{ad}$ be such that the system*

$$y_t = Ay + F(y) + Bu, \quad y(0) = y_0 \tag{3.10}$$

has a unique solution $y \in L^2(I;V)$. If

$$\gamma := \|y_0\|_V + \|\rho y + Bu\|_{L^2(I;Y)} \leq \min\left\{\frac{1}{4CM_\rho^2}, \frac{\eta}{2M_\rho \|K\|_{\mathcal{L}(Y,U)} \|\mathcal{I}\|}\right\},$$

then $y \in W_\infty(\mathcal{D}(A), Y)$ and it holds that

$$\|y\|_{W_\infty(\mathcal{D}(A),Y)} \leq 2M_\rho \gamma.$$

Proof. Since $y \in L^2(I;Y)$, we can apply Lemma 3.2 to the equivalent system

$$y_t = (A - \rho I)y + F(y) + \tilde{f},$$

where $\tilde{f} = \rho y + Bu$. This proves the assertion. □

Lemma 3.5. *There exists $\delta_1 > 0$ such that for all $y_0 \in B_V(\delta_1)$, problem (P) possesses a solution $(\bar{y}, \bar{u}) \in W_\infty(\mathcal{D}(A), Y) \times U_{ad}$. Moreover, there exists a constant $M > 0$ independent of y_0 such that*

$$\max\left\{ \|\bar{y}\|_{W_\infty(\mathcal{D}(A),Y)}, \|\bar{u}\|_U \right\} \leq M \|y_0\|_V. \tag{3.11}$$

Proof. The proof of this lemma follows with analogous argumentation as provided in [BKP2, Lemma 8]. Let us choose, $\delta_1 \leq \min\left\{\frac{1}{4CM_K^2}, \frac{\eta}{2M_K \|K\|_{\mathcal{L}(Y,U)} \|\mathcal{I}\|}\right\}$, where C as in Lemma 3.1 and M_K denotes the constant from Corollary 3.3. We obtain from Corollary 3.3 that for each $y_0 \in B_V(\delta_1)$, there exists a control $u \in U_{ad}$ with associated state y satisfying

$$\max\left\{ \|u\|_U, \|y\|_{W_\infty(\mathcal{D}(A),Y)} \right\} \leq \tilde{M} \|y_0\|_V, \tag{3.12}$$

where $\tilde{M} = 2M_K \max\left(1, \|\mathcal{I}\| \|K\|_{\mathcal{L}(Y,\mathcal{U})}\right)$. We can thus consider a minimizing sequence $(y_n, u_n)_{n\in\mathbb{N}} \in W_\infty(\mathcal{D}(A), Y) \times U_{ad}$ with $J(y_n, u_n) \leq \frac{1}{2}\tilde{M}^2 \|y_0\|_Y^2 (1 + \alpha)$. Consequently for all $n \in \mathbb{N}$ we have

$$\|y_n\|_{L^2(I;Y)} \leq \tilde{M} \|y_0\|_Y \sqrt{1+\alpha} \quad \text{and} \quad \|u_n\|_{L^2(I;\mathcal{U})} \leq \tilde{M} \|y_0\|_Y \sqrt{\frac{1+\alpha}{\alpha}}. \tag{3.13}$$

We set $\eta(\alpha, \tilde{M}) = \|i\|\left[1 + \tilde{M}\sqrt{(1+\alpha)}\left(\rho + \frac{\|B\|_{\mathcal{L}(\mathcal{U},Y)}}{\sqrt{\alpha}}\right)\right]$. Then we have $\|y_0\|_V + \|\rho y_n + Bu_n\|_{L^2(I;Y)} \leq \eta(\alpha, \tilde{M}) \|y_0\|_V$. After further reduction of δ_1, we obtain for M_ρ from Corollary 3.4:

$$\gamma = \|y_0\|_V + \|\rho y_n + Bu_n\|_{L^2(I;Y)} \leq \frac{1}{4CM_\rho^2}, \quad \text{if } y_0 \in B_V(\delta_1).$$

It follows from this corollary that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $W_\infty(\mathcal{D}(A), Y)$ with

$$\sup_{n\in\mathbb{N}} \|y_n\|_{W_\infty(\mathcal{D}(A),Y)} \leq 2M_\rho \eta(\alpha, \tilde{M}) \|y_0\|_V. \tag{3.14}$$

Extracting if necessary a subsequence, there exists $(\bar{y}, \bar{u}) \in W_\infty(\mathcal{D}(A), Y) \times U$ such that $(y_n, u_n) \rightharpoonup (\bar{y}, \bar{u}) \in W_\infty(\mathcal{D}(A), Y) \times U$, and $(\bar{y}, \bar{u})$ satisfies (3.12).

Let us prove that $(\bar{y}, \bar{u})$ is feasible and optimal. Since U_{ad} is weakly sequentially closed and $u_n \in U_{ad}$, we find that $\bar{u} \in U_{ad}$. For each fixed $T > 0$ and arbitrary $z \in L^\infty(0, T; \mathcal{H}) \subset L^2(0, T; Y)$, we have for all $n \in \mathbb{N}$ that

$$\int_0^T \langle \dot{y}_n(t), z(t) \rangle_Y dt = \int_0^T \langle Ay_n(t) - F(y_n(t)) + Bu_n(t), z(t) \rangle_Y dt. \tag{3.15}$$

Since $\dot{y}_n \rightharpoonup \dot{y}$ in $L^2(0, T; Y)$, we can pass to the limit in the l.h.s. of the above equality. Moreover, since $Ay_n \rightharpoonup Ay$ in $L^2(0, T; Y)$,

$$\int_0^T \langle Ay_n(t), z(t) \rangle_Y dt \xrightarrow[n \to \infty]{} \int_0^T \langle A\bar{y}(t), z(t) \rangle_Y dt.$$

Analogously, we obtain that

$$\int_0^T \langle Bu_n(t), z(t) \rangle_Y dt \xrightarrow[n \to \infty]{} \int_0^T \langle B\bar{u}(t), z(t) \rangle_Y dt, \quad \text{for each } z \in L^2(0, T; Y).$$

If moreover $z \in L^\infty(0, T; \mathcal{H}) \subset L^2(0, T; Y)$, we assert by (A3) that

$$\int_0^T \langle F(y_n(t)) - F(\bar{y}(t)), z(t) \rangle_{\mathcal{H}', \mathcal{H}} dt \xrightarrow[n \to \infty]{} 0.$$

Thus we have for all $z \in L^\infty(0, T; \mathcal{H})$

$$\int_0^T \langle \dot{y}(t) - Ay(t) - Bu(t), z(t) \rangle_Y dt = \int_0^T \langle F(y(t)), z(t) \rangle_Y dt. \tag{3.16}$$

Since $\dot{y} - Ay - Bu \in L^2(0, T; Y)$ and $L^\infty(0, T; \mathcal{H})$ is dense in $L^2(0, T; Y)$ we conclude that (3.16) holds for all $z \in L^2(0, T; Y)$. Thus $(\bar{y}, \bar{u})$ is feasible. By weak lower semicontinuitiy of norms it follows that $J(\bar{y}, \bar{u}) \leq \liminf_{n \to \infty} J(\bar{y}_n, \bar{u}_n)$, which proves the optimality of $(\bar{y}, \bar{u})$, and (3.11) follows from (3.13). $\square$

For the derivation of the optimality system for (P), we need the following lemma which is taken from [BK, Lemma 10].

Lemma 3.6. *Let $G \in \mathcal{L}(W_\infty(\mathcal{D}(A), Y), L^2(I; Y))$ such that $\|G\| < \frac{1}{M_K}$, where $\|G\|$ denotes the operator norm of G. Then for all $f \in L^2(I; Y)$ and $y_0 \in V$, there exists a unique solution to the problem:*

$$y_t = (A - BK)y(t) + (Gy)(t) + f(t), \quad y = y_0.$$

Moreover,

$$\|y\|_{W_\infty(\mathcal{D}(A), Y)} \leq \frac{M_K}{1 - M_K \|G\|} \left(\|f\|_{L^2(I;Y)} + \|y_0\|_V \right).$$

3.2 Regular point condition and first order optimality condition

To establish optimality conditions for (P), we consider (P) as a special case of the following abstract optimization problem

$$\begin{cases} \min \ f(y,u), \\ e(y,u;y_0) = 0, \quad y \in W_\infty(\mathcal{D}(A),Y), \ u \in U_{ad}. \end{cases} \tag{3.17}$$

for $y_0 \in V$. Thus e describes a parametric equality constraint. Indeed, the relationship between these two problems is provided by

$$f(y,u) = \frac{1}{2}\int_0^\infty \|y(t)\|_Y^2 \, dt + \frac{\alpha}{2}\int_0^\infty \|u(t)\|_{\mathcal{U}}^2 \, dt, \tag{3.18}$$

and

$$e(y,u,y_0) = \begin{pmatrix} y_t - Ay - F(y) - Bu \\ y(0) - y_0, \end{pmatrix} \tag{3.19}$$

with $f : W_\infty(\mathcal{D}(A),Y) \times U_{ad} \longrightarrow \mathbb{R}^+$ and $e : W_\infty(\mathcal{D}(A),Y) \times U_{ad} \times V \longrightarrow L^2(I;Y) \times V$. In what follows,

i. $y_0 \in V$ denotes a nominal reference parameter, and

ii. $(\bar{y},\bar{u})$ is a local solution ($P_{\bar{y}_0}$).

With Assumption (A1) - (A3) holding it follows that

iii. $f : W_\infty(\mathcal{D}(A),Y) \times U_{ad} \longrightarrow \mathbb{R}^+$ is twice continuously differentiable in a neighborhood of $(\bar{y},\bar{u})$,

iv. $e : W_\infty(\mathcal{D}(A),Y) \times U_{ad} \times V \longrightarrow L^2(I;Y) \times V$ is continuous, and twice continuously differentiable w.r.t. (y,u), with first and second derivative Lipschitz continuous in a neighborhood of $(\bar{y},\bar{u},y_0)$.

We introduce the Lagrangian $\mathcal{L} : W_\infty(\mathcal{D}(A),Y) \times U_{ad} \times L^2(I;Y) \times V' \longrightarrow \mathbb{R}$ associated to (3.17) by

$$\mathcal{L}(y,u,y_0,\lambda) = f(y,u) + (\lambda, e(y,u,y_0))_{L^2(I;Y)\times V', L^2(I;Y)\times V}. \tag{3.20}$$

Here the initial condition $y_0 \in V$ enters as an index. We say that the regular point condition is satisfied at $(\bar{y},\bar{u},y_0) \in W_\infty(\mathcal{D}(A),Y) \times U_{ad} \times V$, if

$$0 \in \text{int} \left\{ e'(\bar{y},\bar{u},y_0) \begin{pmatrix} W_\infty(\mathcal{D}(A),Y) \\ U_{ad} - \bar{u} \end{pmatrix} \right\}, \tag{3.21}$$

where *int* denotes the interior in the $L^2(I;Y) \times V$ topology, and the prime stands for the derivative with respect to (y,u). If this condition holds then the existence of a Lagrange multiplier $\lambda_0 \in L^2(I;Y) \times V'$ is guaranteed such that the following first order optimality condition holds, see e.g. [MZ]:

$$\begin{cases} \mathcal{L}_y(\bar{y},\bar{u},y_0,\lambda_0) = 0, \\ (\mathcal{L}_u(\bar{y},\bar{u},y_0,\lambda_0), u - \bar{u})_U \geq 0, \ \forall u \in U_{ad} \\ e(\bar{y},\bar{u},y_0) = 0. \end{cases} \tag{3.22}$$

This is equivalent to

$$\begin{cases} \mathcal{L}_y(\bar y,\bar u,y_0,\lambda_0)=0,\\ 0\in \mathcal{L}_u(\bar y,\bar u,y_0,\lambda_0)+\partial \mathbf{I}_{U_{ad}}(\bar u),\\ e(\bar y,\bar u,y_0)=0 \end{cases} \tag{3.23}$$

where

$$\partial \mathbf{I}_{U_{ad}}(\bar u)=\{\tilde u\in U: (\tilde u(t),v(t)-\bar u(t))_{\mathcal{U}}\le 0, \forall t\in I,\ v\in U_{ad}\}\,. \tag{3.24}$$

In the next proposition the regular point condition is expressed for our particular constraint $e=0$ and the first order optimality conditions for problem (P) are established.

Proposition 3.1. *There exists $\delta_2\in(0,\delta_1]$ such that each local solution $(\bar y,\bar u)$ with $y_0\in B_V(\delta_2)$ is a regular point, i.e. (3.21) is satisfied, and there exists an adjoint state $(\bar p,\bar p_1)\in L^2(I;Y)\times V'$ satisfying*

$$(v_t-Av-F'(\bar y)v,\bar p)_{L^2(I;Y)}+\langle v(0),\bar p_1\rangle_{V,V'}+(\bar y,v)_{L^2(I;Y)}=0 \text{ for all } v\in W_\infty(\mathcal{D}(A),Y), \tag{3.25}$$

$$\langle \alpha\bar u-B^*\bar p,u-\bar u\rangle_U\ge 0 \text{ for all } u\in U_{ad}. \tag{3.26}$$

Moreover $\bar p$ satisfies

$$\begin{gathered} -\bar p_t-A'\bar p-F'(\bar y)^*\bar p=-\bar y \quad \text{in } L^2(I;[\mathcal{D}(A)]'),\\ \bar p\in W_\infty(Y,[\mathcal{D}(A)]'),\ \bar p_1=\bar p(0), \text{ and } \lim_{t\to\infty}\bar p(t)=0 \quad \text{in } V'. \end{gathered} \tag{3.27}$$

In addition there exists $\widetilde M>0$, independent of $y_0\in B_V(\delta_2)$, such that

$$\|\bar p\|_{W_\infty(Y,[\mathcal{D}(A)]')}\le \widetilde M\,\|y_0\|_V\,. \tag{3.28}$$

Proof. To verify the regular point condition, we evaluate e defined in (3.19) at $(\bar y,\bar u,y_0)$. To check the claim on the range of $e'(\bar y,\bar u,y_0)$ we consider for arbitrary $(r,s)\in L^2(I,Y)\times V$ the equation

$$z_t-Az-F'(\bar y)z-B(w-\bar u)=r,\ z(0)=s \tag{3.29}$$

for unknowns $(z,w)\in W_\infty(\mathcal{D}(A),Y)\times U_{ad}$. By taking $w=-Kz\in U$ we obtain

$$z_t-(A-BK)z-F'(\bar y)z+B\bar u=r,\ z(0)=s.$$

We apply Lemma 3.6 to this equation with $G=-F'(\bar y)$ and $f=r-B\bar u$. By Lemma 3.5 and (2.7) in Remark 2.1 there exists $\delta_2\in(0,\delta_1]$ such that $\|F'(\bar y)\|_{\mathcal{L}(W_\infty(\mathcal{D}(A),Y),L^2(I;Y))}\le \frac12 M_K$. Consequently by Lemma 3.6 there exists $\widetilde M$ such that

$$\begin{aligned} \|z\|_{W_\infty(\mathcal{D}(A),Y)} &\le \widetilde M\big(\,\|r\|_{L^2(I;Y)}+\|s\|_V+\|B\|_{\mathcal{L}(\mathcal{U},Y)}\,\|\bar u\|_U\,\big)\\ &\le \widetilde M\big(\,\|r\|_{L^2(I;Y)}+\|s\|_V+\|B\|_{\mathcal{L}(\mathcal{U},Y)}\,M\,\|y_0\|_V\,\big) \end{aligned} \tag{3.30}$$

with M as in (3.11). We still need to check whether $w=-Kz$ is feasible, which will be the case if $w(t)\le\eta$ for a.e. $t\in I$. Indeed we have

$$\|w(t)\|_Y\le \|K\|_{\mathcal{L}(Y,\mathcal{U})}\,\|z(t)\|_Y\le \|K\|_{\mathcal{L}(Y,\mathcal{U})}\,\|\mathcal{I}\|\,\widetilde M\big(\,\|r\|_{L^2(I;Y)}+\|s\|_V+\|B\|_{\mathcal{L}(\mathcal{U},Y)}\,M\,\|y_0\|_V\,\big).$$

Consequently, possibly after further reducing δ_2, and choosing $\tilde\delta>0$ sufficiently small we have

$$\|w\|_{L^\infty(I;Y)}\le\eta \text{ for all } y_0\in B_Y(\delta_2) \text{ and all } (r,s) \text{ satisfying } \|(r,s)\|_{L^2(I;Y)\times V}\le\tilde\delta. \tag{3.31}$$

Consequently the regular point condition is satisfied. Hence there exists a multiplier $\lambda = (\bar{p}, \bar{p}_1) \in L^2(I;Y) \times V'$ satisfying,

$$\begin{aligned} \langle \mathcal{L}_y(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1), v \rangle_{W_\infty(\mathcal{D}(A),Y)', W_\infty(\mathcal{D}(A),Y)} = 0,\ \forall v \in W_\infty(\mathcal{D}(A), Y), \\ \langle \mathcal{L}_u(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1), u - \bar{u} \rangle_U \geq 0,\ \forall u \in U_{ad}, \end{aligned} \tag{3.32}$$

where

$$\mathcal{L}(y, u, y_0, p, p_1) = J(y, u) + \int_0^\infty (p, y_t - Ay - F(y) - Bu)_Y dt + \langle p_1, y(0) - y_0 \rangle_{V,V'}.$$

This implies that (3.25) and (3.26) hold.

By (A4), we have $F'(\bar{y})^* \bar{p} \in L^2(I; [\mathcal{D}(A)]')$. Thus $-A'\, \bar{p} - F'(\bar{y})^* \bar{p} + \bar{y} \in L^2(I; [\mathcal{D}(A)]')$, and (3.25) implies that $\bar{p} \in W_\infty(Y, [\mathcal{D}(A)]')$. Next we verify that $\lim_{t\to\infty} p(t) = 0$ in V'. For this purpose, we consider $A_\rho^{-1/2} p$ where $A_\rho = (A - \rho I)$ is exponentially stable. Since $p \in W_\infty(Y, [\mathcal{D}(A)]')$, we have $A_\rho^{-1/2} p \in W_\infty(V, V') \subset C(I; Y)$. Then by [CK], we have $\lim_{t\to\infty} A_\rho^{-1/2} p(t) = 0$ in Y. This yields $\lim_{t\to\infty} p(t) = 0$ in V'. Thus we derived

$$-\bar{p}_t - A^* \bar{p} - F'(\bar{y})^* \bar{p} = -\bar{y} \text{ in } L^2(I; [\mathcal{D}(A)]') \text{ and } \lim_{t\to\infty} \bar{p}(t) = 0 \text{ in } V',$$

and (3.25)-(3.27) hold. Testing the first identity in (3.32) with $v \in L^2(I;Y)$ we also have $\bar{p}_1 = \bar{p}(0) \in V'$, which is well-defined since $\bar{p} \in W_\infty(Y, [\mathcal{D}(A)]') \subset C(I; V')$.
It remains to estimate $\bar{p} \in W_\infty(Y, [\mathcal{D}(A)]')$. Let $r \in L^2(I;Y)$ with $\|r\|_{L^2(I;Y)} \leq \tilde{\delta}$, and consider

$$z_t - Az - F'(\bar{y})z - B(w - \bar{u}) = -r,\ z(0) = 0. \tag{3.33}$$

Arguing as in (3.29)-(3.30) there exists a solution to (3.33) with $w = -Kz$ such that

$$\|z\|_{W_\infty(\mathcal{D}(A),Y)} \leq \widetilde{M}\big(\tilde{\delta} + \|B\|_{\mathcal{L}(\mathcal{U},Y)} M \|y_0\|_V\big) \leq \widetilde{M}\big(\tilde{\delta} + \|B\|_{\mathcal{L}(\mathcal{U},Y)} M\delta_2\big) =: C_1. \tag{3.34}$$

From (3.31) we have that $\|w\|_{L^\infty(I,\mathcal{U})} \leq \eta$. Let us now observe that by (3.25) with $v = z,\ v(0) = z(0) = 0$,

$$\begin{aligned} (\bar{p}, r)_{L^2(I,Y)} &= (\bar{p}, -z_t + Az + F'(\bar{y})z)_{L^2(I,Y)} + (\bar{p}, B(w - \bar{u}))_{L^2(I;Y)}, \\ &= (\bar{y}, z)_{L^2(I;Y)} + (B^* \bar{p}, w - \bar{u})_U. \end{aligned} \tag{3.35}$$

We next estimate using (3.34)

$$\langle \bar{p}, r \rangle_{L^2(I;Y)} \leq \|\bar{y}\|_{L^2(I,Y)} \|z\|_{L^2(I;Y)} + \alpha \langle \bar{u}, w - \bar{u} \rangle_U \quad \leq \Big(\|\bar{y}\|_{L^2(I;Y)} + \alpha \|\bar{u}\|_U\Big)\Big(\tilde{C}_1 + \eta + \|\bar{u}\|_U\Big)$$

where $\tilde{C}_1$ depends on C_1 and the embedding $W_\infty(\mathcal{D}(A), Y)$ into $L^2(I;Y)$. By (3.11), this implies the existence of a constant C_2 such that

$$\sup_{\|r\|_{L^2(I;Y)} \leq \tilde{\delta}} (\bar{p}, r)_{L^2(I;Y)} \leq C_2 \|y_0\|_V,$$

and thus

$$\|\bar{p}\|_{L^2(I;Y)} \leq \frac{C_2}{\tilde{\delta}} \|y_0\|_V, \quad \text{for all } y_0 \in B_V(\delta_2). \tag{3.36}$$

We estimate, again using (A4)

$$\|\bar{p}_t\|_{L^2(I;[\mathcal{D}(A)]')} \le \left\|A^*\bar{p} + F'(\bar{y})^*\bar{p} - \bar{y}\right\|_{L^2(I;[\mathcal{D}(A)]')} \le C_3 \|\bar{p}\|_{L^2(I;Y)} + C_4(\|\bar{p}\|_{L^2(I;Y)} + \|\bar{y}\|_{L^2(I;Y)}).$$

By (3.11) and (3.36) we obtain $\|\bar{p}_t\|_{L^2(I;[\mathcal{D}(A)]')} \le C_5 \|y_0\|_V$. Combining this estimate with (3.36) yields (3.28). □

In the following result we obtain stronger properties for the adjoint states $\bar{p}$.

Proposition 3.2. *For each local solution $(\bar{y}, \bar{u})$ with $y_0 \in B_V(\delta_2)$ the associated adjoint state $\bar{p}$ is unique. It satisfies $\bar{p} \in W_\infty(\mathcal{D}(A), Y)$,*

$$-\bar{p}_t - A^*\bar{p} - F'(\bar{y})^*\bar{p} = -\bar{y} \quad \text{in } L^2(I;Y), \tag{3.37}$$

and

$$\lim_{t\to\infty} \bar{p}(t) = 0 \quad \text{in } V. \tag{3.38}$$

Moreover, there exists $\hat{M} > 0$, independent of $y_0 \in B_V(\delta_2)$, such that

$$\|\bar{p}\|_{W_\infty(\mathcal{D}(A),Y)} \le \hat{M} \|y_0\|_V, \text{ and } \bar{u} \in C(\bar{I};\mathcal{U}). \tag{3.39}$$

Proof. Throughout we fix an adjoint state $\bar{p}$ associated to a local solution $(\bar{y}, \bar{u})$ with $y_0 \in B_V(\delta_2)$.

<u>*Step 1*: ($W_\infty(V, V')$-regularity)</u>. Since $\bar{p} \in L^2(I;Y)$ there exists a monotonically increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ and $\bar{p}_n = \bar{p}(t_n) \to 0$ in Y. Now consider following problem.

$$-p_t - A^*p - F'(\bar{y})^*p = -\bar{y} \text{ in } L^2(0, t_n; V'), \quad p(t_n) = \bar{p}_n \text{ in } Y. \tag{3.40}$$

Since $F'(\bar{y})^* \in \mathcal{L}(L^2(I;V), L^2(I;Y))$ this problem admits a unique solution in $W(0, t_n; V, V')$ which coincides with $\bar{p}$ on $[0, t_n]$. Using that $\lim_{n\to\infty} t_n = \infty$ this implies that $\bar{p} \in W_{loc}(0, \infty; V, V')$. Next we derive a bound for $\bar{p} \in W_\infty(V, V')$. By (3.40) we obtain,

$$-\frac{1}{2}\frac{d}{dt}\|\bar{p}(t)\|_Y^2 + a(\bar{p}(t), \bar{p}(t)) + \rho\|\bar{p}(t)\|_Y^2 = \langle F'(\bar{y})^*\bar{p}(t), \bar{p}(t)\rangle_{V',V} + \rho\|\bar{p}(t)\|_Y^2 + (\bar{y}(t), \bar{p}(t))_Y.$$

By integrating w.r.t. t on $(0, t_n)$ we obtain,

$$\begin{aligned}
&\frac{1}{2}\|\bar{p}(0)\|_Y^2 + \int_0^{t_n} a(\bar{p},\bar{p})\,dt + \rho\int_0^{t_n}\|\bar{p}(t)\|_Y^2\,dt \\
&\qquad \le \frac{1}{2}\|\bar{p}(t_n)\|_Y^2 + \int_0^{t_n}\|F'(\bar{y})^*\bar{p}\|_{V'}\|\bar{p}\|_V\,dt + \int_0^{t_n}\left(\rho + \frac{1}{2}\right)\|\bar{p}\|_Y^2\,dt + \frac{1}{2}\int_0^{t_n}\|\bar{y}\|_Y^2\,dt, \\
&\qquad\qquad \le \frac{1}{2}\|\bar{p}(t_n)\|_Y^2 + \frac{\theta}{2}\int_0^\infty\|\bar{p}\|_V^2\,dt + \int_0^{t_n}\left(\frac{c_1^2}{2\theta} + \rho + \frac{1}{2}\right)\|\bar{p}\|_Y^2\,dt + \frac{1}{2}\int_0^{t_n}\|\bar{y}\|_Y^2\,dt,
\end{aligned}$$

where c_1 denotes the norm of $F'(y) \in \mathcal{L}(L^2(I;V), L^2(I;Y))$ according to (A4). Taking the limit $n \to \infty$ and using (2.4) we obtain

$$\theta\int_0^\infty\|\bar{p}\|_V^2\,dt \le \left(\frac{c_1^2}{\theta} + 2\rho + 1\right)\|\bar{p}\|_{L^2(I;Y)}^2 + 2\|\bar{y}\|_{L^2(I;Y)}^2 < \infty. \tag{3.41}$$

This estimate, together with (3.11) and (3.28) implies the existence of a constant c_2 independent of $y_0 \in B_V(\delta_2)$ such that $\|\bar{p}\|_{L^2(I;V)} \le c_2 \|y_0\|_V$. Combining this with

$$-\bar{p}_t - A^*\bar{p} - F'(\bar{y})^*\bar{p} = -\bar{y} \quad \text{in } L^2(0,\infty;V'), \tag{3.42}$$

we obtain $\bar{p} \in W_\infty(V,V')$ and the existence of a constant c_3 such that

$$\|\bar{p}\|_{W_\infty(V,V')} \le c_3 \|y_0\|_V, \forall y_0 \in B_V(\delta_2) \tag{3.43}$$

follows. By [CK] this implies that $\lim_{t\to\infty} \bar{p}(t) = 0$ in Y. The optimality condition (A3) implies that $\bar{u}(t) = \frac{1}{\alpha}\mathbb{P}_{\mathcal{U}_{ad}}(B^*\bar{p}(t))$, where $\mathbb{P}_{\mathcal{U}_{ad}}$ denotes the projection onto $\mathcal{U}_{ad}$. Together with $\bar{p} \in W_\infty(V,V') \subset C(I;Y)$ this implies that $\bar{u} \in C(\bar{I},\mathcal{U})$, which is the second claim in (3.39).

Step 2: (Uniqueness of the multiplier). Let $\bar{p}$ and $\bar{q}$ be two possibly different adjoint states and denote by $\delta\bar{p} = \bar{q} - \bar{p}$. We shall utilize the fact that there exists T such that $\|B^*\bar{p}(t)\|_{\mathcal{U}} \le \eta$ and $\|B^*\bar{q}(t)\|_{\mathcal{U}} \le \eta$ for all $t \ge T$. Consequently $B^*\bar{p}(t) = \mathbb{P}_{\mathcal{U}_{ad}}(B^*\bar{p}(t)) = \mathbb{P}_{\mathcal{U}_{ad}}(B^*\bar{q}(t)) = B^*\bar{q}(t) = \bar{u}(t)$ for all $t \ge T$. Let us now consider the construction utilized in (3.33), now with $r \in S := \{r \in L^2(T,\infty;Y) : \|r\|_{L^2(T,\infty;Y)} \le \tilde{\delta}\}$. We construct function pairs (z,w) with $z \in W^0_\infty(T,\infty;\mathcal{D}(A),Y) = \{z \in W_\infty(T,\infty;\mathcal{D}(A),Y) : z(T) = 0\}$ and $w = -Kz$ with $\|w\|_{L^\infty(T,\infty;\mathcal{U})} \le \eta$ by means of

$$z_t - Az - F'(\bar{y})z - B(w - \bar{u}) = -r, \; z(T) = 0,$$

for any $r \in S$. Note that $\|F'(\bar{y})\|_{\mathcal{L}(W^0_\infty(T,\infty;\mathcal{D}(A),Y),L^2(I;Y))} \le \|F'(\bar{y})\|_{\mathcal{L}(W_\infty(\mathcal{D}(A),Y),L^2(I;Y))}$. Consequently as in (3.34) we obtain existence of a solution to the above equation with $\|z\|_{W^0_\infty(T,\infty;\mathcal{D}(A),Y)} \le C_1$, with C_1 independent of $r \in S$. Combining this with the equations satisfied by $\bar{q}$ and $\bar{p}$ we obtain for all $r \in S$, using that $z \in W^0_\infty(T,\infty;\mathcal{D}(A),Y)$,

$$\begin{aligned}(\delta p, r)_{L^2(I,Y)} &= (\delta p, -z_t + Az + F'(\bar{y})z)_{L^2(T,\infty;Y)} + (\bar{p}, B(w - \bar{u}))_{L^2(T,\infty;Y)}, \\ &= (\bar{y} - \bar{y}, z)_{L^2(T,\infty;Y)} + (B^*\delta p, w - \bar{u})_{L^2(T,\infty;\mathcal{U})} = (\bar{u} - \bar{u}, w - \bar{u})_{L^2(T,\infty;\mathcal{U})} = 0.\end{aligned}$$

Here we used (3.26) and $\|B^*\bar{p}(t)\|_{\mathcal{U}} \le \eta$, $\|B^*\bar{q}(t)\|_{\mathcal{U}} \le \eta$ for all $t \ge T$ in an essential manner. The above equality implies that $\bar{q} = \bar{p}$ on $[0,\infty)$. Next we observe that $\bar{q}$ and $\bar{p}$ satisfy (3.42) on $[0,T]$ with the same terminal value $\bar{p}(T)$ at $t = T$. Consequently $\bar{q} = \bar{p}$ on $[0,T]$ and the uniqueness of the adjoint state follows.

Step 3: ($W_\infty(\mathcal{D}(A),Y)$-regularity). The proof is very similar to that of *Step* 1. Since $\bar{p} \in L^2(I;V)$ there exists a monotonically increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ and $\bar{p}_n = \bar{p}(t_n) \to 0$ in V. Now consider following problem.

$$-p_t - A^*p - F'(\bar{y})^*p = -\bar{y} \; \text{ in } L^2(0,t_n;Y), \quad p(t_n) = \bar{p}_n \text{ in } V. \tag{3.44}$$

Since $F'(\bar{y})^* \in \mathcal{L}(L^2(I;V), L^2(I;Y))$ this problem admits a unique solution in $W(0,t_n;\mathcal{D}(A),Y)$ which coincides with $\bar{p}$ on $[0,t_n]$. Using that $\lim_{n\to\infty} t_n = \infty$ this implies that $\bar{p} \in W_{loc}(0,\infty;\mathcal{D}(A),Y)$. Next we obtain a bound for $\bar{p} \in W(0,\infty;\mathcal{D}(A),Y)$. Taking the inner product in (3.44) with $-A^*_\rho\bar{p} = (-A^* + \rho I)\bar{p}$ we obtain,

$$-\frac{1}{2}\frac{d}{dt}(a(\bar{p}(t),\bar{p}(t)) + \rho\|\bar{p}(t)\|^2_Y) + \|A_\rho\bar{p}(t)\|^2_Y \le \frac{1}{2}\|A_\rho\bar{p}(t)\|^2_Y + \frac{3}{2}\left(\|F'(\bar{y})^*\bar{p}(t)\|^2_Y + \rho\|\bar{p}(t)\|^2_Y + \|\bar{y}(t)\|^2_Y\right).$$

By integrating w.r.t. t on $(0, t_n)$ we obtain,

$$\frac{\theta}{2}\|\bar{p}(0)\|_V^2 + \frac{1}{2}\int_0^{t_n} \|A_\rho^*\bar{p}\|_Y^2 \, dt \\ \leq \frac{1}{2}a(\bar{p}(t_n), \bar{p}(t_n)) + \rho\|\bar{p}(t_n)\|_Y^2 + \frac{3}{2}\int_0^{t_n} \left(\|F'(\bar{y})^*\bar{p}(t)\|_Y^2 + \rho\,\|\bar{p}(t)\|_Y^2 + \|\bar{y}(t)\|_Y^2\right) dt.$$

Taking the limit $n \to \infty$ implies that

$$\theta\,\|\bar{p}(0)\|_V^2 + \int_0^{\infty} \|A_\rho^*\bar{p}\|_Y^2 \, dt \;\leq\; 3\int_0^{\infty} \left(\|F'(\bar{y})^*\bar{p}(t)\|_Y^2 + \rho\,\|\bar{p}(t)\|_Y^2 + \|\bar{y}(t)\|_Y^2\right) dt.$$

Thus by (3.11), (3.43), and (A4) there exists a constant c_4 independent of $y_0 \in B_V(\delta_2)$ such that $\|\bar{p}\|_{L^2(I;D(A))} \leq c_4\,\|y_0\|_V$. Combining this with (3.44) we obtain $\bar{p} \in W_\infty(\mathcal{D}(A), Y)$ and the existence of a constant c_5 such that

$$\|\bar{p}\|_{W_\infty(\mathcal{D}(A),Y)} \leq c_5\,\|y_0\|_V\,, \forall y_0 \in B_V(\delta_2)$$

follows. Finally $A_\rho\,\bar{p} \in W_\infty(I; Y, \mathcal{D}(A'))$ and thus $\lim_{t\to\infty} A_\rho\,\bar{p}(t) = 0$ in V' by (3.27). This implies that $\lim_{t\to\infty} \bar{p}(t) = 0$ in V.

□

3.3 Second order optimality condition

Let $(\bar{y}, \bar{u}) \in W_\infty(\mathcal{D}(A), Y) \times U_{ad}$ be a local solution to (P) with $y_0 \in B_V(\delta_2)$, so that the results of the previous section are available, and let $\mathcal{A} \in \mathcal{L}(W_\infty(\mathcal{D}(A), Y) \times U_{ad}, L^2(I;Y) \times U_{ad})$ denote the operator representation of $\mathcal{L}''(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1)$, i.e.

$$(\mathcal{A}(v_1, w_1), (v_2, w_2))_{L^2(I;Y)\times U_{ad}} = \mathcal{L}''(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1)((v_1, w_1), (v_2, w_2)) \tag{3.45}$$

where $(v_i, w_i) \in W_\infty(\mathcal{D}(A), Y) \times U_{ad}$ for $i = 1, 2$, and define

$$\mathcal{E} = e'(\bar{y}, \bar{u}, y_0) \in \mathcal{L}(W_\infty(\mathcal{D}(A), Y) \times U_{ad}, L^2(I;Y) \times V). \tag{3.46}$$

Above again, the primes denote differentiation with respect to (y, u).

We say that $\mathcal{A}$ is positive definite on $\ker \mathcal{E}$ if

$$\exists \kappa > 0 : (\mathcal{A}(v, w), (v, w))_{L^2(I;Y)\times U_{ad}} \geq \kappa\,\|(v, w)\|^2_{W_\infty(\mathcal{D}(A),Y)\times U_{ad}}\,, \quad \forall (v, w) \in \ker \mathcal{E}. \tag{3.47}$$

The following proposition derives the second order sufficient optimality conditions for (P).

Proposition 3.3. *Consider problem (P) with (A1)-(A4) holding. Then there exists $\delta_3 \in (0, \delta_2]$ such that the second order sufficient optimality condition (3.47) is satisfied for (P) uniformly for all local solutions with $y_0 \in B_V(\delta_3)$.*

Proof. The second derivative of e is given by

$$e''(\bar{y}, \bar{u}, y_0)((v_1, w_1), (v_2, w_2)) = \begin{pmatrix} F''(\bar{y})(v_1, v_2) \\ 0 \end{pmatrix}, \quad \forall\, v_1, v_2 \in W_\infty(\mathcal{D}(A), Y),\ \forall w_1, w_2 \in U. \tag{3.48}$$

For the second derivative of $\mathcal{L}$ w.r.t. (y, u), we find

$$\mathcal{L}''(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1)((v_1, w_1), (v_2, w_2)) = \int_0^\infty (v_1, v_2)_Y dt + \alpha \int_0^\infty (w_1, w_2)_Y dt - \int_0^\infty (\bar{p}, F''(\bar{y})(v_1, v_2))_Y dt. \quad (3.49)$$

By (A2) for F'' and Lemma 3.5 , there exists M_1 such that for all $v \in W_\infty(\mathcal{D}(A), Y)$,

$$\int_0^\infty |(\bar{p}, F''(\bar{y})(v, v))_Y| dt \leq \int_0^\infty \|\bar{p}\|_Y \left\|F''(\bar{y})(v, v)\right\|_Y dt \quad (3.50)$$
$$\leq \|\bar{p}\|_{L^2(I;Y)} \left\|F''(\bar{y})(v, v)\right\|_{L^2(I;Y)} \leq M_1 \|\bar{p}\|_{L^2(I;Y)} \|v\|^2_{W_\infty(\mathcal{D}(A),Y)},$$

for each solution $(\bar{y}, \bar{u})$ of (P) with $y_0 \in B_V(\delta_2)$. Then we obtain

$$\mathcal{L}''(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1)((v, w), (v, w)) \geq \int_0^\infty \|v\|^2_Y dt + \alpha \int_0^\infty \|w\|^2_{\mathcal{U}} dt - \widetilde{M}_1 \|\bar{p}\|_{L^2(I;Y)} \|v\|^2_{W_\infty(\mathcal{D}(A),Y)}. \quad (3.51)$$

Now let $0 \neq (v, w) \in \ker \mathcal{E} \subset W_\infty(\mathcal{D}(A), Y) \times U_{ad}$, where $\mathcal{E}$ as defined in (3.46) is evaluated at $(\bar{y}, \bar{u})$. Then,

$$v_t - Av - F'(\bar{y})v - Bw = 0, \quad v(0) = 0.$$

Next choose $\rho > 0$, such that the semigroup generated by $(A - \rho I)$ is exponentially stable. This is possible due to (A1). We equivalently write the system in the previous equation as,

$$v_t - (A - \rho I)v - F'(\bar{y})v - \rho v - Bw = 0, \quad v(0) = 0.$$

Now, we invoke Lemma 3.6 with $A - BK$ replaced by $A - \rho I$, $G = F'(\bar{y})$, and $f(t) = \rho v(t) + Bw(t)$, and the role of the constant M_K will now be assumed by a parameter M_ρ. By selecting $\delta_3 \in (0, \delta_2]$ such that $\|\bar{y}\|_{W_\infty(\mathcal{D}(A),Y)}$ sufficiently small, we can guarantee that $\left\|F'(\bar{y})\right\|_{\mathcal{L}(W_\infty(\mathcal{D}(A),Y);L^2(I;Y))} \leq {}^1/_{2M_\rho}$, see (3.11) and (2.7) in Remark 2.1. Then the following estimate holds,

$$\|v\|_{W_\infty(\mathcal{D}(A),Y)} \leq 2M_\rho \|v + Bw\|_{L^2(I;Y)}.$$

This implies that

$$\|v\|^2_{W_\infty(\mathcal{D}(A),Y)} \leq \widetilde{M}_2 \left(\|v\|^2_{L^2(I;Y)} + \|w\|^2_{L^2(I;Y)}\right). \quad (3.52)$$

for a constant $\widetilde{M}_2$ depending on $M_\rho, \|B\|$. These preliminaries allow the following lower bound on $\mathcal{L}''$:

$$\mathcal{L}''(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1)((v, w), (v, w)) \geq \int_0^\infty \|v\|^2_Y dt + \alpha \int_0^\infty \|w\|^2_Y dt - \widetilde{M}_1 \|\bar{p}\|_{L^2(I;Y)} \|v\|^2_{W_\infty(\mathcal{D}(A),Y)}$$
$$\text{by (3.52)} \quad \geq \int_0^\infty \|v\|^2_Y + \alpha \int_0^\infty \|w\|^2_Y - \widetilde{M}_1\widetilde{M}_2 \|\bar{p}\|_{L^2(I;Y)} \left[\|v\|^2_{L^2(I;Y)} + \|w\|^2_{L^2(I;Y)}\right]$$
$$= \left(1 - \widetilde{M}_1\widetilde{M}_2 \|\bar{p}\|_{L^2(I;Y)}\right) \|v\|^2_{L^2(I;Y)} + \left(\alpha - \widetilde{M}_1\widetilde{M}_2 \|\bar{p}\|_{L^2(I;Y)}\right) \|w\|^2_{L^2(I;Y)}$$
$$\geq \tilde{\gamma} \left[\|v\|^2_{L^2(I;Y)} + \|w\|^2_{L^2(I;Y)}\right] \quad (3.53)$$

where $\tilde{\gamma} = \min\left\{1 - \widetilde{M}_1\widetilde{M}_2 \|\bar{p}\|_{L^2(I;Y)}, \alpha - \widetilde{M}_1\widetilde{M}_2 \|\bar{p}\|_{L^2(I;Y)}\right\}$. By possible further reduction of δ_3 it can be guaranteed that $\tilde{\gamma} > 0$, see (3.36). Then by (3.52), we obtain,

$$\begin{aligned}\mathcal{L}''(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1)((v, w), (v, w)) &\geq \frac{\tilde{\gamma}}{2}\left[\|v\|^2_{L^2(I;Y)} + \|w\|^2_{L^2(I;Y)}\right] + \frac{\tilde{\gamma}}{2\widetilde{M}_2}\|v\|^2_{W_\infty(\mathcal{D}(A),Y)} \\ &\geq \frac{\tilde{\gamma}}{2\widetilde{M}_2}\|v\|^2_{W_\infty(\mathcal{D}(A),Y)} + \frac{\tilde{\gamma}}{2}\|w\|^2_{L^2(I;Y)}.\end{aligned}$$

By selecting $\bar{\gamma} = \min\left\{\frac{\tilde{\gamma}}{2\widetilde{M}_2}, \frac{\tilde{\gamma}}{2}\right\}$, we obtain the positive definiteness of $\mathcal{L}''$, i.e.

$$\mathcal{L}''(\bar{y}, \bar{u}, y_0, \bar{p}, \bar{p}_1)((v, w), (v, w)) \geq \bar{\gamma}\|(v, w)\|^2_{W_\infty(\mathcal{D}(A),Y)\times U}, \; y_0 \in B_Y(\delta_3), \; (v, w) \in \ker \mathcal{E}. \tag{3.54}$$

□

4 Lipschitz stability of optimal controls

4.1 Generalized equations

We recall a result on parametric Lipschitz stability of solutions of generalized equations in a form due to Dontchev [Don]. For this purpose we consider

$$0 \in \mathcal{F}(x) + \mathcal{N}(x), \tag{4.1}$$

where $\mathcal{F}$ is a C^1-mapping between two Banach spaces $\mathcal{X}$ and $\mathcal{Z}$, and $\mathcal{N} : \mathcal{X} \mapsto 2^{\mathcal{Z}}$ is a set-valued mapping with a closed graph. Let $\bar{x}$ be a solution of (4.1). The generalized equation is said to be strongly regular at $\bar{x}$, if there exist open balls $B_{\mathcal{X}}(\bar{x}, \delta_x)$ and $B_{\mathcal{Z}}(0, \delta_z)$ such that for all $\boldsymbol{\beta} \in B_{\mathcal{Z}}(0, \delta_z)$ the linearized and perturbed equation

$$\boldsymbol{\beta} \in \mathcal{F}(\bar{x}) + \mathcal{F}'(\bar{x})(x - \bar{x}) + \mathcal{N}(x) \tag{4.2}$$

admits a unique solution $x = x(\boldsymbol{\beta})$ in $B_{\mathcal{X}}(\bar{x}, \delta_x)$, and the mapping $\boldsymbol{\beta} \mapsto x$ is Lipschitz continuous from $B_{\mathcal{Z}}(0, \delta_z)$ to $B_{\mathcal{X}}(\bar{x}, \delta_x)$. We have the following result which allows to conclude local stability of the perturbed nonlinear problem from the stability of the linearized one.

Theorem 4.1. *Let $\bar{x}$ be a solution of (4.1) and assume that (4.1) is strongly regular $\bar{x}$. Then there exist open balls $B_{\mathcal{X}}(\bar{x}, \delta'_x)$ and $B_{\mathcal{Z}}(0, \delta'_z)$ such that for all $\boldsymbol{\beta} \in B_{\mathcal{Z}}(0, \delta_z)$, the perturbed equation*

$$\boldsymbol{\beta} \in \mathcal{F}(x) + \mathcal{N}(x)$$

has a unique solution $x = x(\boldsymbol{\beta})$ in $B_{\mathcal{X}}(\bar{x}, \delta'_x)$, and the solution mapping $\boldsymbol{\beta} \mapsto x(\boldsymbol{\beta})$ is Lipschitz continuous from $B_{\mathcal{Z}}(0, \delta'_z)$ to $B_{\mathcal{X}}(\bar{x}, \delta'_x)$.

4.2 The perturbed optimal control problem

To cast the first order optimality system (3.23) as a special case of (4.1), let $(\bar{y}, \bar{u})$ be a local solution of (P) with initial datum $y_0 \in B_V(\delta_3)$, and let $\bar{p}$ denote the associated adjoint state. Then optimality system for (P) can be expressed as:

$$0 \in \mathcal{F}(\bar{y}, \bar{u}, \bar{p}) + (0, 0, 0, \partial \mathbf{I}_{U_{ad}}(\bar{u}))^T, \tag{4.3}$$

where the function $\mathcal{F}:\mathcal{X}\to\mathcal{Z}$ with

$$\begin{aligned}\mathcal{X}&=W_\infty(\mathcal{D}(A),Y)\times(U\cap C(\bar{I};\mathcal{U}))\times W_\infty(\mathcal{D}(A),Y),\ \text{and}\\ \mathcal{Z}&=L^2(I;Y)\times(U\cap C(\bar{I};\mathcal{U}))\times L^2(I;Y)\times V\end{aligned}\tag{4.4}$$

is given by

$$\mathcal{F}(y,u,p)=\begin{pmatrix} y_t-Ay-F(y)-Bu\\ \alpha u-B^*p\\ -p_t-A^*p-F'(y)^*p+y\\ y(0)-y_0\end{pmatrix},\tag{4.5}$$

and

$$\partial\mathbf{I}_{U_{ad}}(u)=\left\{\tilde{u}\in U\cap C(\bar{I};\mathcal{U}):(\tilde{u}(t),v-u(t))_{\mathcal{U}}\le 0,\,\forall\,t\in I,\ v\in B_\eta(0)\right\},\tag{4.6}$$

with $B_\eta(0)=\{v\in\mathcal{U}:\|v\|_{\mathcal{U}}\le\eta\}$. In order to apply Theorem 4.1 to (4.3), we need to show strong regularity of this equation at the reference solution $(\bar{y},\bar{u},\bar{p})$ of (4.3). First we note that $\mathcal{F}$ is continuously differentiable by (A3). Observe also that for $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3,\beta_4)\in\mathcal{Z}$ the generalized equation

$$\boldsymbol{\beta}\in\mathcal{F}(y,u,p)+(0,0,0,\partial\mathbf{I}_{U_{ad}}(u)).\tag{4.7}$$

is the first order optimality system for

$$\inf_{\substack{y\in W_\infty(\mathcal{D}(A),Y)\\ u\in U_{ad}}}\hat{J}(y,u)=\inf_{\substack{y\in W_\infty(\mathcal{D}(A),Y)\\ u\in U_{ad}}}\frac{1}{2}\int_0^\infty\|y\|_Y^2\,dt+\frac{\alpha}{2}\int_0^\infty\|u\|_{\mathcal{U}}^2\,dt-\int_0^\infty(y,\beta_3)_Y dt-\int_0^\infty(u,\beta_2)_Y dt,\tag{4.8a}$$

subject to

$$\begin{cases} y_t=Ay+Bu+F(y)+\beta_1 & \text{in } L^2(I;Y), \quad (4.8b)\\ y(0)=y_0+\beta_4 & \text{in } V. \quad (4.8c)\end{cases}$$

The linearized version of (4.7) is given by

$$\boldsymbol{\beta}\in\mathcal{F}(\bar{y},\bar{u},\bar{p})+\mathcal{F}'(\bar{y},\bar{u},\bar{p})(y-\bar{y},u-\bar{u},p-\bar{p})+(0,0,0,\partial\mathbf{I}_{U_{ad}}(u)),\tag{4.9}$$

or equivalently

$$\begin{cases} y_t=Ay+Bu+F'(\bar{y})(y-\bar{y})-F(\bar{y})+\beta_1, & (4.10a)\\ \alpha u-B^*p+\partial\mathbf{I}_{U_{ad}}(u)\ni\beta_2, & (4.10b)\\ -p_t-A^*p-F'(\bar{y})^*\,p+y-[F'(\bar{y})^*\,\bar{p}]'(y-\bar{y})=\beta_3, & (4.10c)\\ y(0)=y_0+\beta_4. & (4.10d)\end{cases}$$

This is the optimality system of the following perturbed linear-quadratic optimization problem

$$\inf_{\substack{y\in W_\infty(\mathcal{D}(A),Y)\\ u\in U_{ad}}}\hat{J}(y,u)=\inf_{\substack{y\in W_\infty(\mathcal{D}(A),Y)\\ u\in U_{ad}}}\frac{1}{2}\int_0^\infty\|y\|_Y^2\,dt+\frac{\alpha}{2}\int_0^\infty\|u\|_{\mathcal{U}}^2\,dt,$$
$$-\frac{1}{2}\int_0^\infty([F'(\bar{y})^*\bar{p}]'y-\bar{y},y-\bar{y})_Y dt-\int_0^\infty(y,\beta_3)_Y dt-\int_0^\infty(u,\beta_2)_{\mathcal{U}}\,dt,\tag{4.11a}$$

subject to

$$\begin{cases} y_t = Ay + Bu + F'(\bar{y})(y - \bar{y}) - F(\bar{y}) + \beta_1 & \text{in } L^2(I;Y) \quad (4.11b) \\ y(0) = y_0 + \beta_4 & \text{in } V. \quad (4.11c) \end{cases}$$

Remark 4.1. Concerning some basic properties of problem (4.11) we can proceed as in Lemma 4.7, Remark 4.2, and *Step* (ii) of Theorem 2.1 of [KP1]. First, note that the second order sufficient optimality condition in the sense of (3.47) for (P) and for (4.9) coincide. By Proposition 3.3 it is satisfied by each local solution to (P) if $y_0 \in B_V(\delta_3)$. Then there exists a bounded neighborhood $\hat{V}$ of the origin in $\mathcal{Z}$ such that for each $\boldsymbol{\beta} \in \hat{V}$ there exists a unique solution $(y_{(\boldsymbol{\beta})}, u_{(\boldsymbol{\beta})}, p_{(\boldsymbol{\beta})}) \in W_\infty(\mathcal{D}(A), Y) \times U \times W_\infty(\mathcal{D}(A), Y)$ to the perturbed linearized problem (4.11) or equivalently of (4.10). Moreover $(y_{(\boldsymbol{\beta})}, u_{(\boldsymbol{\beta})}) \in W_\infty(\mathcal{D}(A), Y) \times U$ depends Lipschitz continuously on $\boldsymbol{\beta} \in \hat{V}$. For the latter we can proceed as in Step (iii) of the proof in [KP1, Theorem 2.1]. Note, however, that at this point we have not yet guaranteed that $\boldsymbol{\beta} \to u_{(\boldsymbol{\beta})}$ is Lipschitz continuous with values in $C(\bar{I};\mathcal{U})$, which is necessary due to the norm on $\mathcal{X}$. This, and the Lipschitz continuity of $\boldsymbol{\beta} \to u_{(\beta)}$ will be established in the following subsection. In the proof of Lemma 4.1 we shall also require that $\|\beta_2\|_{C(\bar{I};\mathcal{U})} \le \frac{\alpha\eta}{2}$, which is henceforth assumed to hold.

4.3 Lipschitz stability of optimal controls, states and adjoint states

Now we will show Lipschitz stability of optimal control, state and adjoint state, in a neighborhood of a local solution $(\bar{y}, \bar{u})$ of (P) with initial datum $y_0 \in B_V(\delta_3)$, and associated adjoint state $\bar{p}$. This will be achieved once Lipschitz continuity of the solution mapping $\boldsymbol{\beta} \in \hat{V} \mapsto (y_{(\boldsymbol{\beta})}, u_{(\boldsymbol{\beta})}, p_{(\boldsymbol{\beta})})$ of the perturbed linearized problem (4.11) is proven. For this purpose we require the following result for the adjoint states.

Lemma 4.1. *Let* (A1)-(A4) *hold and let* $(\bar{y}, \bar{u})$*, and* $\bar{p}$ *be a local solution and associated adjoint state to* (P) *corresponding to an initial datum* $y_0 \in B_V(\delta_3)$*. Then the mapping* $\boldsymbol{\beta} \mapsto p_{(\boldsymbol{\beta})}$ *is continuous from* $\hat{V} \subset \mathcal{Z}$ *to* $W_\infty(\mathcal{D}(A), Y)$*.*

Proof. The proof related to that of Proposition 4.8 in [KP2], but it is sufficiently different so that we prefer to provide it here.

Step 1: For $\boldsymbol{\beta} \in \hat{V}$, with $\hat{V}$ as in Remark 4.1, there exists a unique solution $(y_{(\boldsymbol{\beta})}, u_{(\boldsymbol{\beta})}, p_{(\boldsymbol{\beta})})$ to the perturbed linear system (4.11). The perturbed costate equation, and the constraint on the control can be expressed as

$$-\partial_t p_{(\boldsymbol{\beta})} - A^* p_{(\boldsymbol{\beta})} - F'(\bar{y})^* p_{(\boldsymbol{\beta})} + y_{(\boldsymbol{\beta})} - [F'(\bar{y})^* \bar{p}]'(y_{(\boldsymbol{\beta})} - \bar{y}) = \beta_1 \quad \text{in } L^2(I;Y), \tag{4.12a}$$

$$\langle \alpha u_{(\boldsymbol{\beta})} - B^* p_{(\boldsymbol{\beta})} - \beta_2, w - u_{(\boldsymbol{\beta})} \rangle_U \ge 0 \quad \text{for all } w \in U_{ad}. \tag{4.12b}$$

Since $p_{(\boldsymbol{\beta})} \in W_\infty(\mathcal{D}(A), Y) \subset C(\bar{I};Y)$ and $\beta_2 \in C(\bar{I};Y)$, this implies that $u_{(\boldsymbol{\beta})} \in C(\bar{I};\mathcal{U})$.

Step 2: (Boundedness of $\{p_{(\boldsymbol{\beta})} : \boldsymbol{\beta} \in \hat{V}\}$ in $W_\infty(\mathcal{D}(A), Y)$.) Since $\hat{V}$ is assumed to be bounded, by Remark 4.1 there exists a constant M such that

$$\|y_{(\boldsymbol{\beta})}\|_{W_\infty(\mathcal{D}(A),Y)} + \|u_{(\boldsymbol{\beta})}\| \le M \quad \text{for all } \boldsymbol{\beta} \in \hat{V}.$$

To argue the boundedness of $p_{(\boldsymbol{\beta})}$, replacing $\bar{y}$ by $y_{(\boldsymbol{\beta})} - [F'(\bar{y})^*\bar{p}]'(y_{(\boldsymbol{\beta})} - \bar{y})$ we can first proceed as in *Step 3* of proof of Proposition 3.1 to obtain the boundedness of $\{\|p_{(\boldsymbol{\beta})}\|_{W_\infty(Y,[\mathcal{D}(A)]')} : \beta \in \hat{\mathbf{V}}\}$. Subsequently we proceed as in *Steps 1* and *3* of Proposition 3.2 to obtain the boundedness of $\{\|p_{(\boldsymbol{\beta})}\|_{W_\infty(\mathcal{D}(A),Y)} : \beta \in \hat{\mathbf{V}}\}$.

Step 3: (Continuity of $p_{(\boldsymbol{\beta})}$ in $W_\infty(\mathcal{D}(A),Y)$). Let $\{\boldsymbol{\beta}_n\}$ be a convergent sequence in $\hat{V}$ with limit $\boldsymbol{\beta}$. Since $\left\{\|p_{(\boldsymbol{\beta}_n)}\|_{W_\infty(\mathcal{D}(A),Y)} : n \in \mathbb{N}\right\}$ is bounded, there exists a subsequence $\{\boldsymbol{\beta}_{n_k}\}$ such that $p_{(\boldsymbol{\beta}_{n_k})} \rightharpoonup \tilde{p}$ weakly in $W_\infty(\mathcal{D}(A),Y)$ and strongly $L^2(0,T;V)$ for every $T \in (0,\infty)$, see e.g. [Emm, Satz 8.1.12, p 213]. Passing to the limit in the variational form of

$$-\partial_t p_{(\boldsymbol{\beta}_{n_k})} - A^* p_{(\boldsymbol{\beta}_{n_k})} - F'(\bar{y})^* p_{(\boldsymbol{\beta}_{n_k})} + y - [F'(\bar{y})^*\bar{p}]'\left(y_{(\boldsymbol{\beta}_{n_k})} - \bar{y}\right) = (\boldsymbol{\beta}_{n_k})_1,$$

we obtain

$$-\partial_t \tilde{p} - A^*\tilde{p} - F'(\bar{y})^*\tilde{p} + y_{(\boldsymbol{\beta})} - [F'(\bar{y})^*\bar{p}]'\left(y_{(\boldsymbol{\beta})} - \bar{y}\right) = (\boldsymbol{\beta})_1. \tag{4.13}$$

Since the solution to this equation is unique we have $p_{(\boldsymbol{\beta}_n)} \rightharpoonup p_{(\boldsymbol{\beta})}$ weakly in $W_\infty(\mathcal{D}(A),Y)$. To obtain strong convergence we set $\delta\boldsymbol{\beta} = \boldsymbol{\beta}_n - \boldsymbol{\beta}$, $\delta p = p_{(\boldsymbol{\beta}_n)} - p_{(\boldsymbol{\beta})}$, $\delta y = y_{(\boldsymbol{\beta}_n)} - y_{(\boldsymbol{\beta})}$. From (4.12a) we derive that

$$-\partial_t(\delta p) - A^*(\delta p) - F'(\bar{y})^*(\delta p) + (I - [F'(\bar{y})^*\bar{p})]'(\delta y) = (\delta\boldsymbol{\beta})_1, \tag{4.14}$$

holds in $L^2(I;Y)$. We shall employ a duality argument to obtain a bound on δp. Moreover we shall argue that the constraint$\|u_{(\boldsymbol{\beta})}(t)\|_{\mathcal{U}} \le \eta$ is inactive for all t sufficiently large. Indeed, since $p_{(\boldsymbol{\beta})} \in W_\infty(\mathcal{D}(A),Y)$ we have $\lim_{t\to\infty} p_{(\boldsymbol{\beta})}(t) = 0$ in V. Hence there exists $\hat{T}$ such that $\frac{1}{\alpha}\left\|B^*p_{(\boldsymbol{\beta})}(t)\right\|_{\mathcal{U}} \le \frac{\eta}{4}$ for all $t \ge \hat{T}$, and by the choice of $\hat{V}$, where we assumed that $\|\beta_2\|_{C(\bar{I};\mathcal{U})} \le \frac{\alpha\eta}{2}$, see Remark 4.1, we also have that

$$\left\|u_{(\boldsymbol{\beta})}(t)\right\|_{\mathcal{U}} = \left\|\mathbb{P}_{\mathcal{U}_{ad}}\left[\frac{1}{\alpha}\left(B^*p_{(\boldsymbol{\beta})}(t) + \beta_2(t)\right)\right]\right\|_{\mathcal{U}} = \frac{1}{\alpha}\left\|B^*p_{(\boldsymbol{\beta})}(t) + \beta_2(t)\right\|_{\mathcal{U}} \le \frac{3\eta}{4}, \tag{4.15}$$

i.e. the constraint is inactive for $t \ge \hat{T}$.

Henceforth let $r, z, w = -Kz$ and $\tilde{\tilde{\delta}}$ be as introduced in (3.33) and recall that $w \in U_{ad}$. Then we obtain

$$\begin{aligned}
\langle \delta p, B(Kz - u_{(\boldsymbol{\beta}_{n_k})})\rangle_{L^2(I;Y)} &\le \int_0^{\hat{T}} \langle p_{(\boldsymbol{\beta}_{n_k})}(t) - p_{(\boldsymbol{\beta})}(t), B(Kz(t) - u_{(\boldsymbol{\beta}_{n_k})}(t))\rangle_Y dt \\
&\qquad + \int_{\hat{T}}^\infty \langle B^*(p_{(\boldsymbol{\beta}_{n_k})}(t) - p_{(\boldsymbol{\beta})}(t)), Kz(t) - u_{(\boldsymbol{\beta}_{n_k})}(t)\rangle_{\mathcal{U}}\, dt \\
&\le \int_0^{\hat{T}} \left\|B^*(p_{(\boldsymbol{\beta}_{n_k})}(t) - p_{(\boldsymbol{\beta})}(t))\right\|_Y \left\|Kz(t) - u_{(\boldsymbol{\beta}_{n_k})}(t)\right\|_{\mathcal{U}} dt \\
&\quad + \int_{\hat{T}}^\infty \langle (\alpha u_{(\boldsymbol{\beta}_{n_k})}(t) - \beta_{n_k,2}(t)) - (\alpha u_{(\boldsymbol{\beta})}(t) - \beta_2(t)), Kz(t) - u_{(\boldsymbol{\beta}_{n_k})}(t)\rangle_{\mathcal{U}}\, dt,
\end{aligned}$$

where we used that $w \in U_{ad}$ and feasibility of $u_{(\boldsymbol{\beta})}(t)$ for $t \ge \hat{T}$. Consequently we have

$$\begin{aligned}
\langle \delta p, B(Kz - u_{(\boldsymbol{\beta}_{n_k})})\rangle_{L^2(I;Y)} \le \Big(&\|B\|_{\mathcal{L}(\mathcal{U},Y)} \left\|p_{(\boldsymbol{\beta}_{n_k})} - p_{(\boldsymbol{\beta})}\right\|_{L^2(0,\hat{T};Y)} + \alpha\left\|u_{(\boldsymbol{\beta}_{n_k})} - u_{(\boldsymbol{\beta})}\right\|_U \\
&+ \|\beta_{n_k,2} - \beta_2\|_U\Big)\left(\|K\|_{\mathcal{L}(Y,\mathcal{U})}\|z\|_{W_\infty} + \left\|u_{(\boldsymbol{\beta}_{n_k})}\right\|_U\right).
\end{aligned} \tag{4.16}$$

Let $R_{\tilde{\delta}} = \left\{ r \in L^2(I;Y) : \|r\|_{L^2(I;Y)} \le \delta \right\}$. For arbitrary $r \in R_{\tilde{\delta}}$ let $z = z(r)$ denote the solution to (3.33) with $\boldsymbol{\beta} \in \hat{V}$. We find

$$\langle \delta p, r\rangle_{L^2(I;Y)} = \langle (I - [F'(\bar{y})^*\bar{p}]')(\delta y) - \delta\beta_1, z\rangle_{L^2(I;Y)} + \langle \delta p, B(Kz_{(\boldsymbol{\beta})} - u_{(\boldsymbol{\beta})})\rangle_{L^2(0,\hat{T};Y)},$$

and thus, using (4.16) and (3.34), we obtain for some $C_2 > 0$,

$$\begin{aligned}\|\delta p\|_{L^2(I;Y)} &= \sup_{r \in R_{\tilde{\delta}}} \langle \delta p, r\rangle_{L^2(I;Y)} \\ &\le C_2 \left(\|\delta y\|_{W_\infty(\mathcal{D}(A),Y)} + \|(\delta\beta_1, \delta\beta_2)\|_{L^2(I;Y)\times U} + \|\delta p\|_{L^2(0,\hat{T};Y)} + \|\delta u\|_U \right). \end{aligned} \tag{4.17}$$

Since $\|\delta y\|_{W_\infty(\mathcal{D}(A),Y)} \to 0$, $\|\delta p\|_{L^2(0,\hat{T};Y)} \to 0$, $\|(\delta\beta_1, \delta\beta_2)\|_{L^2(I;Y)\times U} \to 0$ for $n \to 0$ this implies that $\|\delta p\|_{L^2(I;Y)} \to 0$. To obtain convergence to 0 of δp in $W_\infty(V, V')$ we can proceed similarly as in Step 1 of the proof of Proposition 3.2. For a moment we now emphasize the dependence of δp on n and write δp_n instead. Since $\delta p_n \in L^2(I;Y)$ of each n there exists a monotonically increasing sequence $\{t(n)_k\}_{n\in\mathbb{N}}$ with $\lim_{k\to\infty} t(n)_k = \infty$ and $\lim_{k\to\infty} \delta p_n(t(n)_k) = 0$ in Y. Now, replacing $-\bar{y}$ by$-(I - [F'(\bar{y})^*\bar{p}]'\delta y_n + (\delta\beta_n)_1$ we obtain analogously to (3.41)

$$\theta \int_0^\infty \|\delta p_n\|_V^2 \le \left(\frac{c_1^2}{\theta} + 1 + 2\rho \right) \|\delta p_n\|^2_{L^2(I;Y)} + 2 \left\| -(I - [F'(\bar{y})^*\bar{p}]'\delta y_n + (\delta\beta_n)_1 \right\|^2_{L^2(I;Y)}.$$

The right hand side tends to 0 for $n \to \infty$, and hence $\lim_{n\to\infty} \|\delta p_n\|_{L^2(I;V)} = 0$. Utilizing this fact in (4.14) the convergence $\lim_{n\to\infty} \|\delta p_n\|_{W_\infty(V,V')} = 0$ follows. Since $\delta p_n \in L^2(I;V)$ we also have $\lim_{k\to\infty} \delta p_n(t(n)_k) = 0$ in V, for some monotonically increasing sequence $\{t(n)_k\}_{n\in\mathbb{N}}$ with $\lim_{k\to\infty} t(n)_k = \infty$, for each n.

By taking the inner product in (4.14) with $-A^*_\rho = -A^* + \rho I$ and continuing as in the *Step 3* of Proposition 3.2, we obtain,

$$\int_0^\infty \|A^*_\rho(\delta p)(t)\|_Y^2 \, dt \le \\ C_3 \int_0^\infty \left(\|F'(\bar{y})^*(\delta p)(t)\|_Y^2 + \left\|[F'(\bar{y})^*\bar{p}]'(\delta y)(t)\right\|_Y^2 + \rho \|(\delta p)(t)\|_Y^2 + \|(\delta\boldsymbol{\beta})_1(t)\|_Y^2 \right) dt,$$

for a constant C_3 independent of n. Since $\|\delta y\|_{W_\infty(\mathcal{D}(A),Y)} \to 0$, $\|\delta p\|_{L^2(I;V)} \to 0$, $\|(\delta\boldsymbol{\beta})_1\|_{L^2(I;Y)} \to 0$ for $n \to 0$ this implies that $\|\delta p\|_{L^2(I;\mathcal{D}(A))} \to 0$. Utilizing this fact in (4.14) we find that $\lim_{n\to\infty} \|\delta p\|_{W_\infty(\mathcal{D}(A),Y)} = 0$.

□

Proposition 4.1. *Let (A1)-(A4) hold and let $(\bar{y}, \bar{u})$, and $\bar{p}$ denote a local solution and associated adjoint state to (P) corresponding to an initial condition $y_0 \in B_V(\delta_3)$. Then there exists $\varepsilon > 0$ and $C > 0$ such that for all $\hat{\boldsymbol{\beta}}$ and $\boldsymbol{\beta} \in \hat{V} \cap B_{\mathcal{Z}}(\varepsilon)$*

$$\left\|\hat{y}_{(\hat{\boldsymbol{\beta}})} - y_{(\boldsymbol{\beta})}\right\|_{W_\infty(\mathcal{D}(A),Y)} + \left\|\hat{p}_{(\hat{\boldsymbol{\beta}})} - p_{(\boldsymbol{\beta})}\right\|_{W_\infty(\mathcal{D}(A),Y)} + \left\|u_{(\hat{\boldsymbol{\beta}})} - u_{(\boldsymbol{\beta})}\right\|_{C(\bar{I};\mathcal{U})} \le C \left\|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right\|_{\mathcal{Z}} \tag{4.18}$$

holds.

Proof. The Lipschitz continuity of $(y_{(\boldsymbol{\beta})}, u_{(\boldsymbol{\beta})}) \in W_\infty(\mathcal{D}(A), Y) \times U$ for $\boldsymbol{\beta}$ in a neighborhood of the origin was already addressed in Remark 4.1. We need to assert the extra Lipschitz continuity of $u_{\boldsymbol{\beta}} \in C(\bar{I}; \mathcal{U})$ and the Lipschitz continuity of $p_{\boldsymbol{\beta}}$.

Let us henceforth set $(y, u, p) = \left(y_{(\boldsymbol{\beta})}, u_{(\boldsymbol{\beta})}, p_{(\boldsymbol{\beta})}\right)$, and $(\hat{y}, \hat{u}, \hat{p}) = \left(\hat{y}_{(\hat{\boldsymbol{\beta}})}, \hat{u}_{(\hat{\boldsymbol{\beta}})}, \hat{p}_{(\hat{\boldsymbol{\beta}})}\right)$. We also set $\delta\boldsymbol{\beta} = \boldsymbol{\beta} - \hat{\boldsymbol{\beta}}$, $\delta p = p_{(\boldsymbol{\beta})} - \hat{p}_{(\hat{\boldsymbol{\beta}})}$, $\delta y = y_{(\boldsymbol{\beta})} - \hat{y}_{(\hat{\boldsymbol{\beta}})}$. Then we obtain the equation

$$-\partial_t(\delta p) - A^*(\delta p) - F'(\bar{y})^*(\delta p) + (I - [F'(\bar{y})^*\bar{p}])'(\delta y) = (\delta\beta_1) \in L^2(I; Y) \subset L^2(I; V'). \tag{4.19}$$

Since $\left\|[F'(\bar{y})^*\bar{p}]'(\delta y)\right\|_{L^2(I;Y)} \lesssim \|\delta y\|_{W_\infty(\mathcal{D}(A),Y)}$ equation (4.19) is well-defined on $L^2(I; V')$. Hence we can at first apply the same technique as in [KP1, Proposition 2, pg. 32] to obtain the Lipschitz continuous dependence of δp with respect to $\boldsymbol{\beta}$. Indeed since $\bar{p} \in W_\infty(V, V')$ there exists T such that $\frac{1}{\alpha}\|B^*\bar{p}(t)\|_Y \le \frac{\eta}{2}$, for all $t \ge T$, and thus the control constraint is inactive on $[T, \infty$. Utilizing the continuity established in Lemma 4.1 there exists $\epsilon > 0$ such that $\frac{1}{\alpha}\|B^*p_{(\boldsymbol{\beta})}(t) + \beta_2(t)\|_Y \le \frac{3\eta}{4}$, for all $t \ge T$ and all $\boldsymbol{\beta} \in \hat{V} \cap B_{\mathcal{Z}}(\varepsilon)$, and thus by (4.12b) the control $u_{\boldsymbol{\beta}}$ is inactive for these values of $\boldsymbol{\beta}$ and t. We can now proceed as in the mentioned reference to assert that there exists a constant C_1 independent of $\boldsymbol{\beta} \in \hat{V} \cap B_{\mathcal{Z}}(\varepsilon)$ such that

$$\|\delta p\|_{W_\infty(V,V')} \le C_1 \left(\|\delta y\|_{W_\infty(\mathcal{D}(A),Y)} + \|\delta u\|_U + \|\delta\boldsymbol{\beta}\|_{\mathcal{Z}}\right). \tag{4.20}$$

Next we need to obtain the Lipschitz continuity of p in $W_\infty(\mathcal{D}(A), Y)$. For this purpose we take the inner product in (4.19) with $-A_\rho^* p = (-A^* + \rho I)p$ and continue as in the *Step 3* of Proposition 3.2. We find

$$\|\delta p\|_{L^2(I;\mathcal{D}(A))} \le C_2 \left(\|\delta p\|_{L^2(I;Y)} + \|\delta y\|_{W_\infty(\mathcal{D}(A),Y)} + \|\delta\beta_1\|_{L^2(I;Y)}\right). \tag{4.21}$$

Combining the two inequalities from above, we obtain

$$\|\delta p\|_{L^2(I;\mathcal{D}(A))} \le C_3 \left(\|\delta y\|_{W_\infty(\mathcal{D}(A),Y)} + \|\delta u\|_U + \|\delta\boldsymbol{\beta}\|_{\mathcal{Z}}\right). \tag{4.22}$$

Then by (4.19), we obtain the Lipschitz continuity of p_t in $L^2(I; Y)$ and of δy. Combining these results we deduce that

$$\|\delta p\|_{W_\infty(\mathcal{D}(A),Y)} \le C_3 \left(\|\delta y\|_{W_\infty(\mathcal{D}(A),Y)} + \|\delta u\|_U + \|\delta\boldsymbol{\beta}\|_{\mathcal{Z}}\right). \tag{4.23}$$

We also have

$$u_{(\boldsymbol{\beta})} = \mathbb{P}_{\mathcal{U}_{ad}}\left[-\frac{1}{\alpha}\left(B^*p_{(\boldsymbol{\beta})} + \beta_2\right)\right] \in U \cap C(\bar{I}; \mathcal{U}),$$

and thus

$$\begin{aligned}\|\delta u(t)\|_{\mathcal{U}} &\le \left\|\mathbb{P}_{\mathcal{U}_{ad}}\left[-\frac{1}{\alpha}\left(B^*\hat{p}_{(\hat{\boldsymbol{\beta}})}(t) + \hat{\beta}_2(t)\right)\right] - \mathbb{P}_{\mathcal{U}_{ad}}\left[-\frac{1}{\alpha}\left(B^*p(t)_{(\boldsymbol{\beta})} + \beta_2(t)\right)\right]\right\|_{\mathcal{U}} \\ &\le \frac{1}{\alpha}\left(\|B^*\|\,\|\delta p(t)\|_Y + \|\delta\beta(t)\|_{\mathcal{U}}\right).\end{aligned}$$

This yields

$$\|\delta u\|_{C(\bar{I};\mathcal{U})} \le C_4 \left(\|\delta p\|_{W_\infty(\mathcal{D}(A),Y)} + \|\delta\beta_2\|_{C(\bar{I};\mathcal{U})}\right), \tag{4.24}$$

Combining (4.23) and (4.24), there exists a constant L such that

$$\left\|\hat{y}_{(\hat{\boldsymbol{\beta}})} - y_{(\boldsymbol{\beta})}\right\|_{W_\infty(\mathcal{D}(A),Y)} + \left\|\hat{p}_{(\hat{\boldsymbol{\beta}})} - p_{(\boldsymbol{\beta})}\right\|_{W_\infty(\mathcal{D}(A),Y)} + \left\|\hat{u}_{(\hat{\boldsymbol{\beta}})} - u_{(\boldsymbol{\beta})}\right\|_{U\cap C(\bar{I};Y)} \le L\left\|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right\|_{\mathcal{Z}} \tag{4.25}$$

for all $\hat{\boldsymbol{\beta}}$ and $\boldsymbol{\beta} \in \hat{V} \cap B_{\mathcal{Z}}(\varepsilon)$. □

We have concluded the verification of the strong regularity condition for problem (4.3) and can conclude the following result from Theorem 4.1.

Corollary 4.2. *Let the assumptions (A1)-(A4) hold and let $(\bar{y}, \bar{u})$ be a local solution of (P) corresponding to an initial datum $y_0 \in B_V(\delta_3)$. Then there exist $\delta_4 > 0$, a neighborhood $\hat{U} = \hat{U}(\bar{y}, \bar{u}, \bar{p}) \subset W_\infty(\mathcal{D}(A), Y) \times (U \cap C(\bar{I}; \mathcal{U})) \times W_\infty(\mathcal{D}(A), Y)$, and a constant $\mu > 0$ such that for each $\tilde{y}_0 \in B_V(y_0; \delta_4)$ there exists a unique $(y(\tilde{y}_0), u(\tilde{y}_0), p(\tilde{y}_0)) \in \hat{U}(\bar{y}, \bar{u}, \bar{p})$ satisfying the first order condition, and*

$$\|(y(\hat{y}_0), u(\hat{y}_0), p(\hat{y}_0)) - (y(\check{y}_0), u(\check{y}_0), p(\check{y}_0))\|_{W_\infty(\mathcal{D}(A),Y)\times(U\cap C(\bar{I};\mathcal{U}))\times W_\infty(\mathcal{D}(A),Y)} \le \mu \|\hat{y}_0 - \check{y}_0\|_V\,, \tag{4.26}$$

for all $\hat{y}_0, \check{y}_0 \in B_V(y_0, \delta_4)$. Moreover $(y(\tilde{y}_0), u(\tilde{y}_0))$ is a local solution of (P).

Above $B_V(y_0, \delta_4)$ denotes the ball of radius δ_4 and center y_0 in V.

5 Differentiability of the cost-functional and HJB equation

As a consequence of the Corollary 4.2 and Proposition 3.3 concerning the second order sufficient optimality condition, for each $y_0 \in B_V(\delta_3)$ there exists a neighborhood within which the local minimal value function $\mathcal{V}$ is well-defined and the corresponding controls, states and adjoint states depend Lipschitz continuously on the initial data. Consequently the local value function itself is locally Lipschitz continuous. Exploiting the structure of the cost functional in (P) Fréchet differentiability of the minimal value function can be obtained. We continue to use the notation for $B_V(y_0, \delta_4)$ of Corollary 4.2 above and locally optimal solutions are understood in the sense of $\hat{U}$.

Theorem 5.1. *(Sensitivity of Cost) Let the assumptions (A1)-(A4) hold and let $(\bar{y}, \bar{u})$ be a local solution of (P) corresponding to an initial datum $y_0 \in B_V(\delta_3)$. Then for each $\hat{y}_0 \in B_V(y_0, \delta_4)$ the local minimal value function associated to (P) is Fréchet differentiable from $B_V(y_0, \delta_4)$ to $\mathbb{R}$ with derivative given by*

$$\mathcal{V}'(\hat{y}_0) = -p(0; \hat{y}_0), \tag{5.1}$$

and the Riesz representor of $\mathcal{V}'$ lies in $C(U(y_0), V)$.

With Corollary 4.2 available (5.1) can be verified with the same techniques as the analogous one with V replaced by $L^2(\Omega)$ given in [KP1, Theorem 4.10]. The claim concerning the Riesz representor follows from (5.1) and Corollary 4.2.

For the final theorem we need an additional assumption.

Assumption A5. *With the notation of the previous theorem there exists $T_0 \in (0, \infty)$ such that $F(y(\hat{y}_0, u)) \in C([0, T_{y_0}); V')$ for all $\hat{y}_0 \in B(y_0, \delta_4)$ and all $u \in C([0, T_0); \mathcal{U})$, where $y(\hat{y}_0, u)$ denotes the solution to (2.1) on $[0, T_0)$ with initial condition $\hat{y}_0$ and control u.*

Theorem 5.2. *Let assumptions (A1)-(A5) hold and let $(\bar{y}, \bar{u})$ be a local solution of (P) corresponding to an initial datum $y_0 \in B_V(\delta_3)$. Then the following Hamilton-Jacobi-Bellman equation holds for the local minimal value function (in the sense of $\hat{U}$ from Corollary 4.2) on $B_V(y_0, \delta_4)$:*

$$\mathcal{V}'(y)(Ay + F(y)) + \frac{1}{2}\|y\|_Y^2 + \frac{\alpha}{2}\left\|\mathbb{P}_{\mathcal{U}_{ad}}\left(-\frac{1}{\alpha}B^*\mathcal{V}'(y)\right)\right\|_Y^2 + \left(B^*\mathcal{V}'(y), \mathbb{P}_{\mathcal{U}_{ad}}\left(-\frac{1}{\alpha}B^*\mathcal{V}'(y)\right)\right)_Y = 0, \tag{5.2}$$

and the feedback law is given by

$$u = \mathbb{P}_{\mathcal{U}_{ad}}\left(-\frac{1}{\alpha}B^*\mathcal{V}'(y)\right). \tag{5.3}$$

Proof. The structure of the proof is rather standard, [KP1, Theorem 5.1]. But due to regularity issues special treatment is required. Also compared to [KP1] we modify some arguments to allow local rather than global solutions. Let $\hat{y}_0 \in B_V(y_0, \delta_4)$ with associated local solution and adjoint state $(\hat{y}, \hat{u}, \hat{p}) \in \hat{U}$. In particular we have that $\hat{u}(t) = \mathbb{P}_{\mathcal{U}_{ad}}\left(\frac{1}{\alpha}B^*\hat{p}(t)\right)$, and since $\hat{p} \in C([0,\infty); V)$ it holds $\hat{u} \in C([0,\infty);\mathcal{U})$. Let $\hat{u}_0$ denote the limit of $\hat{u}$ as time t tends to 0. Since $\hat{y} \in C([0,\infty); V)$ and since $B_V(y_0, \delta_4)$ is open there exists $\tau_{\hat{y}_0} \in (0, T_0]$ such that $\hat{y}(t) \in B_V(y_0, \delta_4)$, for all $t \in [0, \tau_{\hat{y}_0})$, and $\mathcal{V}$ is well-defined there.

Step 1. Let us first prove that

$$\mathcal{V}'(\hat{y}_0)\big(A\hat{y}_0 + F(\hat{y}_0) + B\hat{u}_0\big) + \ell(\hat{y}_0, \hat{u}_0) = 0, \tag{5.4}$$

where $\ell(y, u) = \frac{1}{2}\|y\|_Y^2 + \frac{\alpha}{2}\|u\|_{\mathcal{U}}^2$. Since by the previous theorem the Riesz representor of $\mathcal{V}'(\hat{y}_0)$ is an element of V, and since the arguments of $\mathcal{V}'(\hat{y}_0)$ are all contained in V', the left hand side of the above equality is well-defined. Here we note that (A5) implies that $F(\hat{y}_0) \in V'$.

To verify the equality we use the dynamic programming principle in the form

$$\frac{1}{\tau}\int_0^\tau \ell(\hat{y}(s), \hat{u}(s))ds + \frac{1}{\tau}\big(\mathcal{V}(\hat{y}(\tau)) - \mathcal{V}(\hat{y}_0)\big) = 0, \tag{5.5}$$

for $\tau \in (0, \tau_{\hat{y}_0}))$. By continuity of $\hat{y}$ and $\hat{u}$ in Y, respectively $\mathcal{U}$ at time 0, the first term converges to $\ell(\hat{y}_0, \hat{u}_0)$ as $\tau \to 0$. To take $\tau \to 0$ in the second term we first consider

$$\frac{1}{\tau}\big(\hat{y}(\tau) - \hat{y}_0\big) = \frac{1}{\tau}\big(e^{A_{ext}\tau}\hat{y}_0 - \hat{y}_0\big) + \frac{1}{\tau}\int_0^\tau e^{A_{ext}(\tau - s)}\big[F(\hat{y}(s)) + B\hat{u}(s)\big]ds, \tag{5.6}$$

where $e^{A_{ext}t}$ denotes the extension of the semigroup e^{At} on Y, to V'. This follows by [EN, Theorem 5.5] and an application of interpolation theory. Using (A5) and $B\hat{u} \in C([0,\infty); Y)$ we can pass to the limit in V' in (5.6) to obtain that

$$\lim_{\tau \to 0^+}\frac{1}{\tau}\big(\hat{y}(\tau) - \hat{y}_0\big) = A\hat{y}_0 + F(\hat{y}_0) + B\hat{u}_0 \text{ in } V'. \tag{5.7}$$

Now we return to the second term in (5.5) which we express as

$$\begin{aligned}\frac{1}{\tau}\big(\mathcal{V}(\hat{y}(\tau)) - \mathcal{V}(\hat{y}_0)\big) &= \int_0^1 \mathcal{V}'\big(\hat{y}_0 + s(\hat{y}(\tau) - \hat{y}_0)\big)\,\frac{1}{\tau}(\hat{y}(\tau) - \hat{y}_0)\,ds \\ &= \int_0^1 \langle \hat{p}(0; (y_0 + s(\hat{y}(\tau) - \hat{y}_0))\,, \frac{1}{\tau}(\hat{y}(\tau) - \hat{y}_0)\rangle_{V,V'}\,ds.\end{aligned} \tag{5.8}$$

Using (5.7) and since $y \to \hat{p}(0;y)$ is continuous from V to itself at $\hat{y}_0$ by Proposition 4.2, we can pass to the limit in (5.8) to obtain

$$\lim_{\tau\to 0^+} \frac{1}{\tau}\big(\mathcal{V}(\hat{y}(\tau)) - \mathcal{V}(\hat{y}_0)\big) = \mathcal{V}'(\hat{y}_0)\big(A\hat{y}_0 + F(\hat{y}_0) + B\hat{u}_0\big). \tag{5.9}$$

Now we can pass to the limit in (5.5) and obtain (5.4).

Step 2: For $u \in \mathcal{U}_{ad}$ we define $\tilde{u} \in U_{ad}$ by,

$$\tilde{u}(t) = \begin{cases} u & \text{for } \tau \in (0,\hat{\tau}), \\ \hat{u}(t) & \text{for } \tau \in [\hat{\tau},\infty) \end{cases}$$

and set $\tilde{y} = y(\hat{y}_0, \tilde{u})$ as the solution to (2.1). Here $0 < \hat{\tau} \le \min(T_{y_0})$ is chosen sufficiently small so that $\tilde{u}$ lies in the region of locality of the local solution $\hat{u}$. Also note that $\tilde{y}(t) \in B_V(\hat{y}_0, \delta_4)$, for all t sufficiently small, and hence $\mathcal{V}(\tilde{y}(t))$ is well-defined for all small t. By local optimality of $\hat{u}$ we have

$$\frac{1}{\tau}\int_0^\tau \ell(\tilde{y}(s), \tilde{u}(s))ds + \frac{1}{\tau}\big(\mathcal{V}(\tilde{y}(\tau)) - \mathcal{V}(\hat{y}_0)\big) \ge 0,$$

for all τ sufficiently small. We next pass to the limit $\tau \to 0^+$ in the above inequality. This is trivial for the first term which tends to $\ell(\hat{y}_0, u)$. For the second one we can argue as above, replacing $(\hat{y}, \hat{u})$ by $(\tilde{y}, \tilde{u})$ in (5.6) and using (A5), resulting in

$$\mathcal{V}'(\hat{y}_0)\big(A\hat{y}_0 + \mathcal{F}(\hat{y}_0) + Bu\big) + \ell(\hat{y}_0, u) \ge 0, \tag{5.10}$$

for arbitrary $u \in \mathcal{U}_{ad}$. This inequality is an equality if $u = \hat{u}_0$, and thus the quadratic function on the left had side of (5.10) reaches its minimum 0 at $u = \hat{u}_0$. This implies that $\hat{u}_0 = \mathbb{P}_{\mathcal{U}_{ad}}\left(-\frac{1}{\alpha}B^*\mathcal{V}'(\hat{y}_0)\right)$. Inserting this expression into (5.4) we obtain (5.2) at $y = \hat{y}_0$, and (5.3) at $(y,u) = (\hat{y}_0, \hat{u}_0)$ holds as well. Since $\hat{y}_0 \in B_V(y_0, \delta_4)$ was chosen arbitrarily the proof is complete. □

6 Applications

In the final section we discuss the applicability of the presented theory for selected examples. Throughout Ω denotes an open connected bounded subset of $\mathbb{R}^d$, with Ω convex or with a $C^{1,1}$ boundary Γ. The associated space-time cylinder is denoted by $Q = \Omega \times (0,\infty)$ and the associated lateral boundary by $\Sigma = \Gamma \times (0,\infty)$. By ν and ∂_ν, we denote the outward unit normal and the associated outward normal derivative on Γ. The symbol $\lesssim$ means an inequality up to some constant $C > 0$.

6.1 The Schlögl model in $d \in \{1,2,3\}$.

Here we consider the optimal stabilization problem for the Schlögl model also known as Nagumo model under control constraints in dimension $d \in \{1,2,3\}$. In the earlier paper [BK], where the initial conditions where taken in $L^2(\Omega)$ the dimension was restricted to $d = 1$. Here we treat

$$\mathcal{V}(y_0) = \inf_{\substack{y \in W_\infty(\mathcal{D}(A), Y) \\ u \in U_{ad}}} \frac{1}{2}\int_0^\infty \|y(t)\|_Y^2\, dt + \frac{\alpha}{2}\int_0^\infty \|u(t)\|_{\mathcal{U}}^2\, dt, \tag{6.1}$$

subject to the semilinear parabolic equation

$$\begin{cases} y_t = \Delta y + R(y) + Bu & \text{in } Q, & (6.2)\\ \partial_\nu y = 0 & \text{on } \Sigma, & (6.3)\\ y(x,0) = y_0 & \text{in } \Omega. & (6.4)\end{cases}$$

where R is the cubic polynomial of the form,

$$R(y) = ay(y-\xi_1)(y-\xi_2), \text{with real numbers } \xi_1, \xi_2, \text{and } a < 0,$$

and $B \in \mathcal{L}(\mathcal{U}, Y)$. Note that $R(y) = ay^3 + by^2 + cy$, with $b = a(\xi_1 + \xi_2)$ and $c = a\xi_1\xi_2$. The origin of the uncontrolled system is locally unstable, if $\xi_1\xi_2 < 0$ and exponentially stable if $\xi_1\xi_2 > 0$. To cast this problem in the framework of Section 2, we set $V = H^1(\Omega)$, $a(v,w) = (\nabla v, \nabla w) - c(v,w)$ and associated operator

$$Ay = (\Delta + c\boldsymbol{I})y, \text{ with } \mathcal{D}(A) = \{y \in H^2(\Omega) : \partial_\nu y|_\Gamma = 0\} \tag{6.5}$$

and

$$F(y) = ay^3 + by^2. \tag{6.6}$$

Clearly (2.4) is satisfied. Concerning condition (A1), if $c < 0$ then the semigroup generated by A is exponentially stable. In case $c \geq 0$ feedback stabilization by finite dimensional controllers was analyzed in [Tri, Theorem 6.1], and [KR] for instance. In [KR] stabilizability by finitely many controllers was established for parabolic-like systems with the number of controllers associated to the spectral properties of A. It is left to the reader to check that the nonlinearity F is twice differentiable as a mapping $F : W_\infty(\mathcal{D}(A), Y) \to L^2(I;Y)$. For the sake of illustration, we verify the continuity of the bilinear form of F'' on $W_\infty(\mathcal{D}(A), Y) \times W_\infty(\mathcal{D}(A), Y)$. For this purpose, we take $v_1, v_2 \in W_\infty(\mathcal{D}(A), Y)$ and $y_1, y_2 \in W_\infty(\mathcal{D}(A), Y)$ and estimate, setting $\delta y = y_2 - y_1$

$$\left\|(F''(y_2) - F''(y_1))(v_1, v_2)\right\|^2_{L^2(I;Y)} \leq 6 \int_0^\infty \int_\Omega |a\, \delta y\, v_1 v_2|^2 \, dx dt$$

$$\leq 6 \int_0^\infty |a| \, \|\delta y\|^2_{L^6(\Omega)} \|v_1\|^2_{L^6(\Omega)} \|v_2\|^2_{L^6(\Omega)} \, dt \leq 6C_1 \|v_1\|^2_{W_\infty(\mathcal{D}(A),Y)} \|v_2\|^2_{W_\infty(\mathcal{D}(A),Y)} \int_0^\infty \|\delta y\|^2_{\mathcal{D}(A)} \, dt,$$

where C_1 depends on the continuous embedding $V \to L^6(\Omega)$, which holds for $d \leq 3$, and the continuous injection $W_\infty(\mathcal{D}(A), Y) \to C(I;V)$ is used. We also have $F(0) = F'(0) = 0$ and thus (2.7) is satisfied. The following Lemma together with the compact embedding $W(0,T;\mathcal{D}(A),Y) \to L^2(0,T;V)$, see e.g. [Emm, Satz 8.1.12, p213], implies that (A3) is satisfied with $\mathcal{H} = Y$.

Lemma 6.1. *For $y_1, y_2 \in W(0,T;\mathcal{D}(A),Y)$ and $z \in L^\infty(0,T;Y)$ the following estimates hold:*

$$\int_0^T |\langle y_1^3 - y_2^3, z\rangle_Y| \, dt$$

$$\leq C_2 \|z\|_{L^2(0,T;Y)} \|y_1 - y_2\|_{L^2(0,T;V)} \left[\|y_1\|^2_{W(0,T;\mathcal{D}(A),Y)} + \|y_2\|^2_{W(0,T;\mathcal{D}(A),Y)}\right], \tag{6.7}$$

$$\int_0^T |\langle y_1^2 - y_2^2, z\rangle_Y| \, dt \leq C_3 \|z\|_{L^2(0,T;Y)} \|y_1 - y_2\|_{L^2(0,T;V)} \left[\|y_1\|_{L^2(0,T;V)} + \|y_2\|_{L^2(0,T;V)}\right] \tag{6.8}$$

where $C_2, C_3 > 0$ are independent of y_1, y_2, and z.

Proof. For the first inequality, we estimate, using embedding constants C_i independent of y_1, y_2, z

$$\begin{aligned}\int_0^T |\langle y_1^3 - y_2^3, z\rangle_Y| \, dt &= \int_0^T \int_\Omega |(y_1^3 - y_2^3)z| \, dxdt \\ &\leq C_4 \int_0^T \|z\|_Y \|y_1 - y_2\|_{L^6(\Omega)} \left\|y_1^2 + y_1y_2 + y_2^2\right\|_{L^3(\Omega)} dt \\ &\leq C_5 \|z\|_{L^\infty(0,T;Y)} \|y_1 - y_2\|_{L^2(0,T;V)} \left\|y_1^2 + y_2^2\right\|_{L^2(0,T;L^3(\Omega))}.\end{aligned}$$

Then we estimate, for $i = 1, 2$,

$$\begin{aligned}\|y_i^2\|_{L^2(0,T;L^3(\Omega))} &= \left[\int_0^T \|y_i^2\|^2_{L^3(\Omega)} dt\right]^{1/2} = \left[\int_0^T \left(\int_\Omega |y_i|^6 \, dx\right)^{2/3} dt\right]^{1/2} = \left[\int_0^T \|y_i\|^4_{L^6(\Omega)} dt\right]^{1/2} \\ &\leq C_6 \|y_i\|_{C(0,T;V)} \|y_i\|_{L^2(0,T;V)} \leq C_7 \|y_i\|^2_{W(0,T;\mathcal{D}(A),Y)}.\end{aligned}$$

This proves the first inequality. The verification of the second inequality is left to the reader. □

Now we turn to (A4) and verify that $F'(y) = 3ay^2 + 2by \in \mathcal{L}\left(L^2(I;V), L^2(I,Y)\right)$. We concentrate on the term y^2 and estimate for $z \in L^2(I;V)$:

$$\begin{aligned}\int_0^\infty \int_\Omega |y|^4|z|^2 dx \, dt &\leq \int_0^\infty \left(\int_\Omega |y|^6 dx\right)^{2/3} \left(\int_\Omega |z|^6 dx\right)^{1/3} dt \leq \int_0^\infty \|y\|^4_{L^6(\Omega)} \|z\|^2_{L^6(\Omega)} dt \\ &\leq C_8 \|y\|^4_{W_\infty(I;\mathcal{D}(A),Y)} \|z\|^2_{L^2(I;V)}\end{aligned}$$

and the claim follows. Finally for (A5), we utilize the fact that $V \subset L^6(\Omega)$ for $d \leq 3$, and $y \in C(I;V)$ for $y_0 \in B_V(\delta_4)$. This implies $F(y) \in C(I;Y) \subset C(I;V')$.

6.2 Quartic nonlinearity y^4 in $d \in \{1, 2\}$.

In this case we consider a semilinear parabolic problem with $F(y) = ky^4$ in dimension $d \in \{1, 2\}$. The computations will be carried out for $d = 2$ but $d = 1$ will readily follow. Specifically the controlled system is given as follows:

$$\begin{cases} y_t = \Delta y + ky^4 + Bu & \text{in } Q, \quad (6.9a) \\ y = 0 & \text{on } \Sigma, \quad (6.9b) \\ y(x,0) = y_0 & \text{in } \Omega. \quad (6.9c) \end{cases}$$

For this model (A1) is satisfied with A the Laplacian $Y = L^2(\Omega)$ with Dirichlet boundary conditions. It generates an asymptotically stable analytic semigroup. We use Gagliardo's inequality [BF, p173] in dimension two to show that $F : W_\infty(I;\mathcal{D}(A), Y) \to L^2(I;Y)$ with $F(y) = ky^4$ is well-defined:

$$\|y^4\|^2_{L^2(I;Y)} = \int_0^\infty \|y\|^8_{L^8(\Omega)} dt \lesssim \int_0^\infty \left[\|y\|_Y^{1/4} \|y\|_V^{3/4}\right]^8 dt \leq C \|y\|^6_{W_\infty(\mathcal{D}(A),Y)} \|y\|^2_{L^2(I;Y)}.$$

Moreover, assumption (A2) requires us to show that F is twice continuously differentiable. It can be checked that F is twice continuously differentiable with derivatives given by $F'(y) = 4ky^3$ and $F''(y) = 12ky^2$. By computations as carried out in Lemma 6.1, one can deduce F' and F'' are bounded on bounded subsets of $W_\infty(\mathcal{D}(A), Y)$.

To verify (A3), we take $z \in L^\infty(0,T;Y)$ and estimate for $y_1, y_2 \in W_\infty(I;\mathcal{D}(A), Y)$:

$$\begin{aligned}\int_0^T |\langle y_1^4 - y_2^4, z\rangle| dt \lesssim \int_0^T \int_\Omega |(y_1 - y_2)\left(y_1^3 + y_2^3\right) z| \, dxdt \lesssim \int_0^\infty \|y_1 - y_2\|_{L^4(\Omega)} \left\|y_1^3 + y_2^3\right\|_{L^4(\Omega)} \|z\|_Y \, dt \\ \lesssim \|z\|_{L^\infty(0,T;Y)} \int_0^T \|y_1 - y_2\|_{L^4(\Omega)} \left(\|y_1\|^3_{L^{12}(\Omega)} + \|y_2\|^3_{L^{12}(\Omega)}\right) dt \\ \lesssim \|z\|_{L^\infty(0,T;Y)} \int_0^T \|y_1 - y_2\|_V \left(\|y_1\|^3_V + \|y_2\|^3_V\right) dt \\ \lesssim \|z\|_{L^\infty(0,T;Y)} \|y_1 - y_2\|_{L^2(0,T;V)} \left(\|y_1\|^2_{C(0,T;V)} + \|y_2\|^2_{C(0,T;V)}\right)\left(\|y_1\|_{L^2(0,T;V)} + \|y_2\|_{L^2(0,T;V)}\right).\end{aligned}$$

This implies (A3), since weak convergence in $W_\infty(I;\mathcal{D}(A), Y)$ implies strong convergence in $L^2(0,T;V)$. For (A4), we show $F'(y) = 4ky^3 \in \mathcal{L}(L^2(I;V), L^2(I;Y))$ for $y \in W_\infty(\mathcal{D}(A), Y)$. We estimate for $z \in L^2(I;V)$,

$$\left\|F'(y)z\right\|^2_{L^2(I;Y)} \lesssim \int_0^\infty \int_\Omega \left|y^3 z\right|^2 dxdt \lesssim \int_0^\infty \|y\|^6_{L^8(\Omega)} \|z\|^2_{L^8(\Omega)} \, dt \le C \|y\|^6_{W_\infty(\mathcal{D}(A),Y)} \|z\|^2_{L^2(I;V)}.$$

Assumption (A5) follows by an analogous argumentation as presented in Section 6.1.

With similar arguments the quintic nonlinearity can be considered in dimension 1.

6.3 Nonlinearities induced by functions with globally Lipschitz continuous second derivative.

Consider the system (P) with A associated to a strongly elliptic second order operator with domain $H^2(\Omega) \cap H^1_0(\Omega)$, so that (A1) are satisfied. Let $F : W_\infty(\mathcal{D}(A), Y) \to L^2(I;Y)$ be the Nemytskii operator associated to a mapping $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ which is assumed to be $C^2(\mathbb{R})$ with first and second derivatives globally Lipschitz continuous. We discuss assumption (A2)-(A5) for such an F, and show that they are satisfied for dimensions $d \in \{1,2,3\}$. By direct calculation it can be checked that F is continuously Fréchet differentiable for $d \in \{1,2,3\}$. We leave this part to the reader and immediately turn to the second derivative. For $y, h_1, h_2 \in W_\infty(\mathcal{D}(A), Y)$ the relevant expression is given by

$$\begin{aligned}&\left\|F'(y+h_2)h_1 - F'(y)h_1 - F''(y)(h_1,h_2)\right\|^2_{L^2(I;Y)} \\ &\quad = \int_0^\infty \int_\Omega \left|(\mathfrak{f}'(y(t,x)+h_2(t,x)) - \mathfrak{f}'(y(t,x)) - \mathfrak{f}''(y(t,x))h_2(t,x))h_1(t,x)\right|^2 dxdt \\ &\quad = \int_0^\infty \int_\Omega |g(t,x)\ h_2(t,x)h_1(t,x)|^2 \, dxdt,\end{aligned}$$

where $g(t,x) = \displaystyle\int_0^1 (\mathfrak{f}''(y(t,x)+sh_2(t,x)) - \mathfrak{f}''(y(t,x)))ds$. Let us denote the global Lipschitz constant of $\mathfrak{f}''$ by L. Then we estimate

$$\int_0^\infty \int_\Omega |gh_1h_2|^2 \, dxdt \le \frac{L^2}{4} \int_0^\infty \|h_2(t)\|^4_{L^6(\Omega)} \|h_1(t)\|^2_{L^6(\Omega)} \, dt$$

$$\leq \frac{L^2}{4} \int_0^\infty \|h_2(t)\|_V^4 \|h_1(t)\|_V^2 \, dt \leq C \|h_2\|^4_{W_\infty(\mathcal{D}(A),Y)} \int_0^\infty \|h_1(t)\|_V^2 \, dt$$
$$\leq \tilde{C} \|h_2\|^4_{W_\infty(\mathcal{D}(A),Y)} \|h_1\|^2_{W_\infty(\mathcal{D}(A),Y)} .$$

This implies that F admits a second Fréchet derivative which is bounded on bounded subsets of $W_\infty(\mathcal{D}(A), Y)$. Its continuity with respect to y can be checked with similar arguments. In order to verify (A3), we set $\mathcal{H} = Y = L^2(\Omega)$ and consider a sequence $y_n \rightharpoonup \hat{y}$ in $W(0, T; \mathcal{D}(A), Y)$ and let $z \in L^\infty(0, T; \mathcal{H}) \subset L^2(0, T; Y)$ be given. Then we estimate

$$\int_0^T |\langle F(y_n) - F(\hat{y}), z\rangle_{\mathcal{H}',\mathcal{H}}|dt = \int_0^T \int_\Omega |(\mathfrak{f}(y_n) - \mathfrak{f}(\hat{y}))z| \, dxdt \leq C \|y_n - \hat{y}\|_{L^2(0,T;Y)} \|z\|_{L^2(0,T;Y)} .$$

Then by the compactness of $W(0, T; \mathcal{D}(A), Y)$ in $L^2(0, T; Y)$, we obtain (A3).
To verify (A4) we proceed with $y \in W_\infty(\mathcal{D}(A), Y)$, $z \in L^2(I; V)$ and estimate

$$\|F'(y)z\|^2_{L^2(I;Y)} = \int_0^\infty \int_\Omega \left(\mathfrak{f}'(y) - \mathfrak{f}'(0)\right)^2 z^2 \, dxdt \leq C \|y\|^2_{W_\infty(\mathcal{D}(A),Y)} \|z\|^2_{L^2(I;V)} .$$

This shows $F'(y)$ satisfies (A4). Assumption (A5) can be verified since $\mathfrak{f}'(y)$, is assumed to be globally Lipschitz continuous and $y \in C(I; V)$ for $y \in B_V(\delta_4)$.

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