DIRECTIONAL DIFFERENTIABILITY FOR SHAPE OPTIMIZATION WITH VARIATIONAL INEQUALITIES AS CONSTRAINTS

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Abstract. For equilibrium-constrained optimization problems subject to nonlinear state equations, the property of directional differentiability with respect to a parameter is studied. An abstract class of parameter dependent shape optimization problems is investigated with penalty constraints linked to variational inequalities. Based on the Lagrange approach, on smooth penalties due to Lavrentiev regularization, and on adjoint operators, a shape derivative is obtained. The explicit formula provides a descent direction for the gradient algorithm identifying the shape of the breaking-line from a boundary measurement. A numerical example is presented for a nonlinear Poisson problem modeling Barenblatt's surface energies and non-penetrating cracks.

1. Introduction

In this paper we prove a directional derivative of parameter-dependent objective functions for a class of nonlinear equilibrium constraints. In particular, the penalty constraint linked to variational inequalities (VI) is investigated within Lavrentiev's regularization. The problem describes the identification of a breaking line with contact and cohesion in the frame of quasi-brittle fracture and destructive physical analysis (DPA).

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The research belongs to the fields of optimal control, shape and topology optimization [4, 34]. For optimal control of VI we cite [1, 31], for quasi- and hemi-VI see [11, 32, 36], and for optimal control of cracks we refer to [13, 22, 26]. In order to find an optimal shape, we generalize the optimization approach for semi-linear equilibrium equations from [7, 20] by adopting results on directional differentiability of Lagrangians. The main difficulty here concerns nonlinearity of state equations. In our earlier works, the shape derivative was obtained for free-boundary problems of Bernoulli type [8], nonlinear crack problems [17, 18] and Barenblatt's cracks in plane setting [20], inverse problems of shape identification [19] and breaking-line identification [6], for the Stokes flow [23] and the Stokes–Forchheimer flow [7].

The classical theory of state-constrained optimization problems deals with linear equations, typically, by partial differential equations [27, 35]. In our consideration we study state constraints, given by variational inequalities and their penalization. The challenge consists in the fact that the latter are not Fréchet differentiable (see [28, 33]). As a consequence, the directional derivative of Lagrangians and related shape differentiability fails. We suggest a novel approximation for the shape derivative along specifically linearized directions and based on generalized adjoint techniques (see [29, 30]). In particular cases, our approach is closely related to the method of averaged adjoints developed by [24].

Our research addresses the following features:

(i) Optimization subject to nonlinear and nonsmooth equilibrium constraints. Within the Lagrange multiplier approach (see [12]), in Section 2 we consider a convex objective function $\mathcal{J}$ with a nonlinear equation as constraint. The linearized Lagrangian $\mathcal{L}$ is well-posed when using the associated adjoint operator provided in Lemma 2.2.

(ii) Penalty optimization linked to variational inequalities (VI). In order to treat VI, in Section 3 we extend $\mathcal{L}$ to a penalized Lagrangian $\mathcal{L}^\varepsilon$ for $\varepsilon > 0$ (see Lemma 3.2), thus reducing the variational inequalities to case (i). Using adjoints we derive necessary and sufficient optimality conditions for a saddle-point problem providing the optimal value $l(\varepsilon)$.

(iii) Directional differentiability. We consider optimal value objective and Lagrange functions $j(\varepsilon, s) = l(\varepsilon, s)$ depending on a parameter $s > 0$. Following the concepts in [3, 4], the directional derivative $\partial_+ j(0)$ at $\varepsilon = 0$ is obtained in Theorem 2.1, and in Theorem 3.1 it is extended to $\partial_+ j(\varepsilon, 0)$ by using a differentiable Lavrentiev's ε-regularization [25].

(iv) Limit as $\varepsilon \to 0^+$. Taking the limit as the penalty parameter $\varepsilon \to 0^+$, in Theorem 3.2 we derive the reference variational inequality, its

adjoint equation, as well as primal and adjoint Lagrange multipliers for inequality constraints. However, a limit directional derivative fails since the VIs are not differentiable.

(v) Shape derivative. In Section 4 we introduce states depending on a family of geometries Ω_t parameterized by t. The diffeomorphic perturbations Ω_{t+s} are defined by a kinematic velocity Λ (see e.g. [9, 21]). They are used to characterize the shape derivative using the bijection property of the function spaces $V(\Omega_{t+s}) \mapsto V(\Omega_t)$.

(vi) Application to non-penetrating Barenblatt's cracks. We apply shape perturbations to the nonlinear crack problem (4.11) under non-penetration [14, 15] in the anti-plane setting (see [10]). Beyond the classic Griffith's brittle fracture, Barenblatt's cohesion (see [2, 16]) allows crack faces to close smoothly and determines a-priori unknown cracks by those points where opening occurs along a breaking line Σ_t.

(vii) Hadamard formula and descent directions. In Theorem 4.1 we specify the shape derivative for the nonlinear Poisson problem described in (vi), and express it by a Hadamard formula over the moving boundary in Theorem 4.2. This formula provides kinematic velocities Λ for a descent direction $\partial_+ j(\varepsilon, 0) < 0$ within a gradient method.

(viii) Identification of breaking lines. Finally, in Section 5 we present a numerical simulation of the gradient descent algorithm for the inverse problem of identification of the breaking line Σ_t, which minimizes the objective $\mathcal{J}$ of least-square misfit from a boundary observation. We report that the faces need to be open for identification within DPA.

2. Directional differentiability of Lagrangians for equilibrium constraints

In separable Banach spaces V and X, let a linear operator $M : V \mapsto X$ map the space of *states* $u \in V$ to *observations* $z \in X$. We consider an abstract *objective function* dependent on a positive parameter $s \in I := [0, s_0)$, $s_0 > 0$:

$$\mathcal{J}(s, z) : \ I \times X \mapsto \overline{\mathbb{R}}. \tag{2.1}$$

Next we introduce our state constraint. Let the continuous function $\mathcal{E}(s, u) : I \times V \mapsto \overline{\mathbb{R}}$ be the *energy functional.* For every fixed s we assume that its is differentiable, i.e.

(E1) $\mathcal{E}$ possesses the Gateaux derivative $\mathcal{E}'(s, u) \in V^\star$ such that

$$\langle \mathcal{E}'(s, u), v\rangle = \lim_{r\to 0} \frac{\mathcal{E}(s, u + rv) - \mathcal{E}(s, u)}{r} \quad \text{for } u, v \in V,\ s \in I.$$

We define the reference state as a solution $u_0 \in V$ at $s = 0$ to the equilibrium equation expressed in the variational form:

$$\langle \mathcal{E}'(0, u_0), v\rangle = 0 \quad \text{for all } v \in V. \tag{2.2}$$

Here and in what follows the brackets $\langle \cdot, \cdot \rangle$ stands for the duality pairing between V and its dual space $V^\star$. The variational equation (2.2) constitutes the optimality condition for the minimum

$$\mathcal{E}(0, u_0) = \min_{u \in V} \mathcal{E}(0, u). \tag{2.3}$$

Lemma 2.1. *Let assumption (E1) and the following hold:*

(E2) $\mathcal{E}'$ at $s = 0$ is coercive: there exist $\underline{a} > 0$ and $f \in V^\star$ such that

$$\langle \mathcal{E}'(0, u), u\rangle \geq \underline{a}\|u\|_V^2 - \langle f, u\rangle \quad \textit{for } u \in V;$$

(E3) $[u \mapsto \mathcal{E}'(0, u)] : V \mapsto V^\star$ is weak-to-weak continuous: if $u^k \rightharpoonup u_0$ weakly in V as $k \to \infty$, then $\mathcal{E}'(0, u^k) \rightharpoonup \mathcal{E}'(0, u_0)$ $\star$-weakly in $V^\star$.

Then there exists a solution $u_0 \in V$ to (2.2).

Proof. We introduce a Galerkin approximation of (2.2) by nonlinear equations in subspaces $V^n \subset V$ of finite dimension $n \in \mathbb{N}$ as follows

$$\langle \mathcal{E}'(0, u^n), v^n\rangle = 0 \quad \text{for all } v^n \in V^n.$$

Since the strong and weak convergences coincide in finite-dimensional spaces, under the coercivity and continuity assumptions (E2) and (E3) solutions $u^n \in V^n$ exist according to the Brouwer fixed point theorem, see e.g. [5]. The solutions are uniformly bounded in V due to (E2). Hence there exists a weakly convergent subsequence u^{n_k} and an accumulation point $u_0 \in V$. Taking the limit as $n_k \to \infty$ due to (E3) the assertion of the theorem follows. □

Induced by the *state equation* (2.2), we have the optimal value

$$j(0) = \mathcal{J}(0, Mu_0) \quad \text{for } u_0 \in V \text{ solving } \mathcal{E}'(0, u_0) = 0. \tag{2.4}$$

Our aim is to extend the state-constrained optimization (2.4) to a well-posed *optimal value objective function* $j : I \subset \mathbb{R} \mapsto \overline{\mathbb{R}}$ in such a way that it has a *directional derivative* at $s = 0$:

$$\partial_+ j(0) = \lim_{s \to 0^+} \frac{j(s) - j(0)}{s} \quad \text{(one sided derivative)}. \tag{2.5}$$

Further we linearize the mapping $u \mapsto \mathcal{E}'$ around the reference solution u_0 to (2.2) and use a Lagrange method [12]. For this task we employ for fixed $(s,u_0)\in I\times V$ an '*associated to adjoint*' linear operator $(\mathcal{E}')^\star(s,u_0)\in\mathscr{L}(V,V^\star)$, which is defined by means of the *Lagrange identity* (see [30, Chapter 1]):

$$\langle(\mathcal{E}')^\star(s,u_0)v,u_0\rangle=\langle\mathcal{E}'(s,u_0)-\mathcal{E}'(s,0),v\rangle\quad\text{for } v\in V,\ s\in I. \tag{2.6}$$

Lemma 2.2. *If the following assumption holds:*

(E*1) the second Gateaux derivative $\mathcal{E}''(s,ru_0)\in\mathscr{L}(V,V^\star)$ exists:

$$\langle\mathcal{E}''(s,ru_0)w,v\rangle=\lim_{\xi\to0}\Big\langle\frac{\mathcal{E}'(s,ru_0+\xi w)-\mathcal{E}'(s,ru_0)}{\xi},v\Big\rangle,\ \ v,w\in V,$$

for $s\in I$, and $r\mapsto\mathcal{E}''(s,ru_0)$ is continuous for $r\in[0,1]$,

then an associated to adjoint operator in (2.6) is given by

$$\langle(\mathcal{E}')^\star(s,u_0)v,w\rangle:=\int_0^1\langle\mathcal{E}''(s,ru_0)w,v\rangle\,dr. \tag{2.7}$$

Proof. From the Newton–Leibniz axiom we have

$$\langle\mathcal{E}'(s,u_0),v\rangle=\langle\mathcal{E}'(s,0),v\rangle+\int_0^1\langle\mathcal{E}''(s,ru_0)u_0,v\rangle\,dr. \tag{2.8}$$

Inserting $w=u_0$ into (2.7) and using (2.8) implies (2.6). ☐

Based on Lemma 2.2, a linearized *Lagrange function* $\mathcal{L}:I\times V^3\mapsto\overline{\mathbb{R}}$ is well defined for $u,v\in V$ as follows

$$\mathcal{L}(s,u_0,u,v):=\mathcal{J}(s,Mu)-\langle(\mathcal{E}')^\star(s,u_0)v,u\rangle-\langle\mathcal{E}'(s,0),v\rangle. \tag{2.9}$$

For the Lagrangian $\mathcal{L}$ we consider the *saddle-point (minimax) problem*:

$$\mathcal{L}(s,u_0,u_s,v)\le\mathcal{L}(s,u_0,u_s,v_s)\le\mathcal{L}(s,u_0,u,v_s) \tag{2.10}$$

for all $(u,v)\in V^2$. Following [3], we introduce the *optimal values*:

$$l_s:=\sup_{v\in V}\inf_{u\in V}\mathcal{L}(s,u_0,u,v)\le\inf_{u\in V}\sup_{v\in V}\mathcal{L}(s,u_0,u,v)=:l^s$$

and the corresponding *solution sets*:

$$\begin{aligned}K^s&:=\{u\in V|\ \sup_{v\in V}\mathcal{L}(s,u_0,u,v)=l^s\},\\ K_s&:=\{v\in V|\ \inf_{u\in V}\mathcal{L}(s,u_0,u,v)=l_s\},\end{aligned} \tag{2.11}$$

which determine a multi-valued function $[s\rightrightarrows K^s\times K_s]:I\rightrightarrows V^2$. Later we shall prove that these sets are not empty.

Lemma 2.3. *Let (E1)–(E3), (E*1) and the following assumptions hold*

(J1) $\mathcal{J}$ possesses a Gateaux derivative $\mathcal{J}'(s,z)\in X^\star$ such that

$$\langle\mathcal{J}'(s,z),\xi\rangle_X=\lim_{r\to0}\frac{\mathcal{J}(s,z+r\xi)-\mathcal{J}(s,z)}{r}\quad\textit{for } z,\xi\in X,\ s\in I,$$

where $\langle \cdot\,, \cdot \rangle_X$ *is the duality pairing between* X *and its dual space* $X^\star$*;*

(J2) the objective functional is convex:

$$\langle \mathcal{J}'(s,\xi), z-\xi \rangle_X \le \mathcal{J}(s,z) - \mathcal{J}(s,\xi) \quad \text{for } z,\xi \in X,\ s \in I;$$

(E**2) the associated to adjoint operator is symmetric:*

$$\langle (\mathcal{E}')^\star(s,u_0)v, u\rangle = \langle (\mathcal{E}')^\star(s,u_0)u, v\rangle \quad \text{for } u,v \in V,\ s \in I;$$

(E**3)* $(\mathcal{E}')^\star(s,u_0)$ *is coercive uniformly with respect to* s*: there exist* $\underline{a}^\star > 0$ *and* $f^\star \in V^\star$ *such that*

$$\langle (\mathcal{E}')^\star(s,u_0)u, u\rangle \ge \underline{a}^\star \|u\|_V^2 - \langle f^\star, u\rangle \quad \text{for } u \in V,\ s \in I.$$

Then for every $s \in I$ *there exists a state* $u_s \in V$ *solving the equation:*

$$\langle (\mathcal{E}')^\star(s,u_0)v, u_s\rangle + \langle \mathcal{E}'(s,0), v\rangle = 0 \quad \text{for all } v \in V, \tag{2.12}$$

and an adjoint state $v_s \in V$ *satisfying the adjoint equation:*

$$\langle (\mathcal{E}')^\star(s,u_0)v_s, u\rangle = \langle \mathcal{J}'(s,Mu_s), Mu\rangle_X \quad \text{for all } u \in V. \tag{2.13}$$

The pair $(u_s,v_s) \in K^s \times K_s$ *is a saddle point satisfying*

$$l(s) := l_s = \mathcal{L}(s,u_0,u_s,v_s) = l^s, \quad s \in I. \tag{2.14}$$

If $f^\star = 0$ *in (E***3), then the saddle-point is unique.*

The proof of Lemma 2.3 is given in Appendix A.

We note that u_0 from Lemma (2.1) is a solution to the s-dependent equation (2.12) at $s=0$. The latter also coincides with the reference equation (2.2) due to the Lagrange identity (2.6).

The next lemma establishes a sequential semi-continuity property for the solution set $K^s \times K_s$ at $s \to 0^+$.

Lemma 2.4. *Let (E1)–(E3), (E***1)–(E***3), (J1), (J2) and the following assumptions hold true:*

(E4) $\mathcal{E}'(s,0)$ *is bounded: there exist* $\overline{a} > 0$ *such that*

$$\|\mathcal{E}'(s,0)\|_{V^\star} \le \overline{a} \quad \text{for } s \in I;$$

(E5) $s \mapsto \mathcal{E}'(s,0)$ *is continuous from the right at* $u=0$ *as* $s \to 0^+$*;*

(J3) $\mathcal{J}'(s,Mu_s)$ *on solutions is bounded: there exist* $\overline{a}_{\mathcal{J}} > 0$ *such that*

$$\|\mathcal{J}'(s,Mu_s)\|_{X^\star} \le \overline{a}_{\mathcal{J}} \|u_s\|_V \quad \text{for } u_s \in K^s,\ s \in I;$$

(J4) $s \mapsto \mathcal{J}'(s,Mu_s)$ *on solutions* $u_s \in K^s$ *is continuous as* $s \to 0^+$*;*

(E**4)* $(\mathcal{E}')^\star(s,u_0)$ *is bounded: there exist* $\overline{a}^\star > 0$ *such that*

$$\|(\mathcal{E}')^\star(s,u_0)\| \le \overline{a}^\star \quad \text{for } s \in I;$$

(E⋆5) $s \mapsto (\mathcal{E}')^\star(s, u_0)$ *is continuous as* $s \to 0^+$*;*

Then there exist $s_k \to 0^+$*, a subsequence of saddle points* $(u_{s_k}, v_{s_k}) \in K^{s_k} \times K_{s_k}$ *and* $(u_0, v_0) \in K^0 \times K_0$ *such that*

$$(u_{s_k}, v_{s_k}) \to (u_0, v_0) \quad \text{strongly in } V^2 \text{ as } k \to \infty. \tag{2.15}$$

The proof of Lemma 2.4 is technical and is presented in Appendix B. In the last lemma of this section directional differentiability of Lagrangians [3, 4] is recalled.

Lemma 2.5. *Let the set of saddle points* $(u_s, v_s) \in K^s \times K_s$ *satisfying* (2.14) *be nonempty for each* $s \in I$*; assume that a subsequence* $(u_{s_k}, v_{s_k}) \in K^{s_k} \times K_{s_k}$ *and an accumulation point* $(u_0, v_0) \in K^0 \times K_0$ *exist satisfying strong convergence* (2.15) *as* $s_k \to 0^+$*. If the following holds:*

(L1) There exists a partial derivative $\partial\mathcal{L}/\partial s : I \times V^3 \mapsto \overline{\mathbb{R}}$ *of the Lagrangian* $\mathcal{L}$ *with respect to the first argument at* $r \in I$ *such that*

$$\liminf_{r, s_k \to 0^+} \frac{\partial \mathcal{L}}{\partial s}(r, u_0, u_{s_k}, v_0) \geq \frac{\partial \mathcal{L}}{\partial s}(0, u_0, u_0, v_0) \quad \text{for all } v_0 \in K_0,$$
$$\limsup_{r, s_k \to 0^+} \frac{\partial \mathcal{L}}{\partial s}(r, u_0, u_0, v_{s_k}) \leq \frac{\partial \mathcal{L}}{\partial s}(0, u_0, u_0, v_0) \quad \text{for all } u_0 \in K^0,$$

then for the optimal value objective function $j : I \mapsto \overline{\mathbb{R}}$ *defined as*

$$j(s) := \mathcal{J}(s, Mu_s) \quad \text{for } u_s \in V \text{ solving (2.12)}, \tag{2.16}$$

the directional derivative $\partial_+ j(0)$ *in* (2.5) *exists. It is equal to a directional derivative* $\partial_+ l(0)$ *for the optimal value Lagrangian* $l : I \mapsto \overline{\mathbb{R}}$ *from* (2.14) *and expressed by the partial derivative* $\partial\mathcal{L}/\partial s$ *as follows*

$$\partial_+ j(0) = \partial_+ l(0) := \lim_{s_k \to 0^+} \frac{l(s_k) - l(0)}{s_k} = \frac{\partial \mathcal{L}}{\partial s}(0, u_0, u_0, v_0). \tag{2.17}$$

The proof of Lemma 2.5 is standard and given in Appendix C.

Based on Lemmas 2.1–2.5 we state the main theorem of this section.

Theorem 2.1. *Under assumptions (E1)–(E5), (J1)–(J4), (E⋆1)–(E⋆5), (L1) there exists the directional derivative* $\partial_+ j(0) = \partial_+ l(0)$ *in* (2.17)*, where* $(u_0, v_0) \in K^0 \times K_0 \in V^2$ *is a saddle point solving the reference variational equation* (2.2) *and the adjoint* (2.13) *at* $s = 0$*:*

$$\langle (\mathcal{E}')^\star(0, u_0) v_0, u \rangle = \langle \mathcal{J}'(0, Mu_0), Mu \rangle_X \quad \text{for all } u \in V. \tag{2.18}$$

Proof. From assumptions (E1)–(E3), (J1), (J2), $(E^\star1)$–$(E^\star3)$ and Lemmas 2.1–2.3 it follows that the set of saddle points $(u_s, v_s) \in K^s \times K_s$ satisfying (2.14) is nonempty. Together with assumptions (E4), (E5), (J3), (J4), ($E^\star 4$), ($E^\star 5$) Lemma 2.4 guarantees the existence of a subsequence $(u_{s_k}, v_{s_k}) \in K^{s_k} \times K_{s_k}$ and an accumulation point $(u_0, v_0) \in K^0 \times K_0$ satisfying the strong convergence (2.15) as $s_k \to 0^+$. Utilizing (L1) Lemma 2.5 implies the assertion of the theorem. □

In the following section we extend the directional differentiability result of Theorem 2.1 to a penalty-constrained optimization motivated by variational inequalities.

3. Directional differentiability of Lagrangians due to penalty constraints

Let H be another Banach space with an order relation denoted by $\geq$. We introduce a parameter-dependent family of linear operators $B(s) \in \mathcal{L}(V, H)$, with $s \in I$, and the associated *inequality constraints*

$$B(s)u \geq 0. \tag{3.1}$$

As a canonical example we may consider a trace operator. Using the decomposition $\zeta = [\zeta]^+ - [\zeta]^-$ into positive $[\zeta]^+ = \max(0, \zeta)$ and negative $[\zeta]^- = -\min(0, \zeta)$ parts, inequality (3.1) is equivalent to

$$[B(s)u]^- = 0. \tag{3.2}$$

Compared to (2.3), the constrained problem at $s = 0$:

$$\mathcal{E}(0, u_0) = \min_{u \in V,\, [B(0)u]^- = 0} \mathcal{E}(0, u) \tag{3.3}$$

leads to the *variational inequality*: find $u_0 \in V$, $[B(0)u_0]^- = 0$ such that

$$\langle \mathcal{E}'(0, u_0), v - u_0 \rangle \geq 0 \quad \text{for all } v \in V,\ [B(0)v]^- = 0. \tag{3.4}$$

In order to bring (3.4) in equality form akin (2.2), we regularize it by a penalty approximation.

For a small penalization parameter $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$, we define the *penalty* as a map $\beta_\varepsilon(s, \zeta) : I \times H \mapsto H^\star$ into the dual space $H^\star$. For the constraint $[\zeta]^- = 0$ according to (3.2), the standard penalty function $\beta_\varepsilon(0, \zeta) = -[\zeta]^-/\varepsilon \leq 0$ forces the compliance condition $\langle \beta_\varepsilon(0, \zeta), \zeta \rangle = ([\zeta]^-)^2/\varepsilon$. However, the min-based penalty function is not differentiable (see assumption (L3)). Therefore, within a Lavrentiev relaxation [25] satisfying: (B1) there exist $\underline{\beta}, \underline{\beta}^1 \geq 0$ such that for $\zeta \in H$, $\varepsilon \in (0, \varepsilon_0)$:

$$\frac{\|[\zeta]^-\|_H^2}{\varepsilon} - \varepsilon \underline{\beta} \leq \langle \beta_\varepsilon(0, \zeta), \zeta \rangle_{H^*, H}, \quad \beta_\varepsilon(0, \zeta) \leq \varepsilon \underline{\beta}^1, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle_{H^*, H}$ denotes the duality pairing between H and $H^\star$.

For example, a smooth ε-mollification of the minimum function

$$\beta_\varepsilon(0,\zeta) = \begin{cases} \zeta/\varepsilon & \text{for } \zeta < -\varepsilon \\ -\exp\big(2(\zeta+\varepsilon)/(\zeta-\varepsilon)\big) & \text{for } -\varepsilon \le \zeta < \varepsilon \\ 0 & \text{for } \zeta \ge \varepsilon \end{cases} \tag{3.6}$$

is depicted in Figure 2 together with its derivative. It satisfies (B1) with $\underline{\beta} = -\beta_\varepsilon(0,0) = \exp(-2)$ and $\underline{\beta}^1 = 0$.

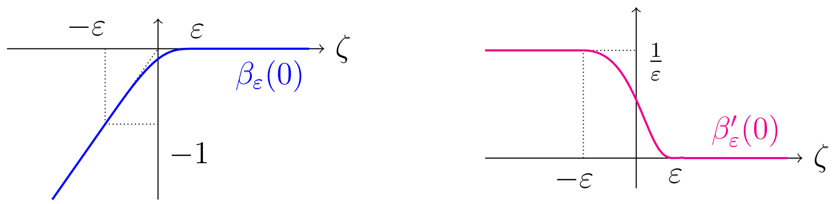

FIGURE 1. Example graphics of $\zeta \mapsto \beta_\varepsilon, \beta'_\varepsilon$ for fixed ε.

This leads to the *penalized problem*: find $u_0^\varepsilon \in V$ such that

$$\langle \mathcal{E}'(0,u_0^\varepsilon), v\rangle + \langle \beta_\varepsilon(0, B(0)u_0^\varepsilon), B(0)v\rangle_H = 0 \quad \text{for all } v \in V. \tag{3.7}$$

Lemma 3.1. *Let the asymptotic condition (B1) hold. If*

(B2) $[\zeta \mapsto \beta_\varepsilon(0,\zeta)] : H \mapsto H^\star$ *is sequentially weak-to-weak continuous,*

then there exists a solution $u_0^\varepsilon \in V$ *to* (3.7).

Proof. The operator of problem (3.7) is coercive due to assumption (E2) and the lower bound in (3.5). It is weakly continuous due to (E3) and (B2). The proof of Lemma 2.1 can be adapted to guarantee existence of a solution. □

Following Lemma 2.2 we assume that

(B*1) the Gateaux derivative $\beta'_\varepsilon(s, rB(0)u_0^\varepsilon) \in \mathscr{L}(H, H^\star)$ at $B(0)u_0^\varepsilon$ exists:

$$\langle \beta'_\varepsilon(s, rB(0)u_0^\varepsilon)\eta, \zeta\rangle_{H^*,H} = \lim_{\xi\to 0} \Big\langle \frac{\beta_\varepsilon(s, rB(0)u_0^\varepsilon + \xi\eta) - \beta_\varepsilon(s, rB(0)u_0^\varepsilon)}{\xi}, \zeta\Big\rangle_{H^*,H}$$

for $\zeta, \eta \in H$, and the mapping $r \mapsto \beta'_\varepsilon(s, rB(0)u_0^\varepsilon)$ is continuous for $r \in [0,1]$, where $s \in I$.

Then the adjoint $\beta^\star_\varepsilon(s, B(0)u_0^\varepsilon) \in \mathscr{L}(H, H^\star)$ exists, it is given by

$$\langle \beta^\star_\varepsilon(s, B(0)u_0^\varepsilon)\zeta, \eta\rangle_{H^*,H} := \int_0^1 \langle \beta'_\varepsilon(s, rB(0)u_0^\varepsilon)\eta, \zeta\rangle_{H^*,H}\, dr, \tag{3.8}$$

and satisfies the Lagrange identity for $\zeta \in H$, $s \in I$:

$$\langle \beta^\star_\varepsilon(s, B(0)u_0^\varepsilon)\zeta, B(0)u_0^\varepsilon\rangle_{H^*,H} = \langle \beta_\varepsilon(s, B(0)u_0^\varepsilon) - \beta_\varepsilon(s,0), \zeta\rangle_{H^*,H}. \tag{3.9}$$

Using (3.7) and (3.9) we modify (2.9) with a *penalized Lagrange function* $\mathcal{L}^\varepsilon : I \times V^3 \mapsto \overline{\mathbb{R}}$ expressed by

$$\text{(3.10)} \quad \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u, v) := \mathcal{L}(s, u_0^\varepsilon, u, v) - \langle \beta_\varepsilon(s, 0), B(s)v \rangle_{H^*,H} \\ - \langle \beta_\varepsilon^\star(s, B(0)u_0^\varepsilon) B(s)v, B(s)u \rangle_{H^*,H} \quad \text{for } u, v \in V.$$

The *penalized saddle-point problem* reads: find $(u_s^\varepsilon, v_s^\varepsilon) \in V^2$ such that

$$\text{(3.11)} \qquad \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u_s^\varepsilon, v) \le \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u_s^\varepsilon, v_s^\varepsilon) \le \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u, v_s^\varepsilon)$$

for all test functions $(u, v) \in V^2$. The optimal values and solution sets in (2.11) are

$$l_s^\varepsilon := \sup_{v \in V} \inf_{u \in V} \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u, v) \le \inf_{u \in V} \sup_{v \in V} \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u, v) =: l_\varepsilon^s,$$

$$\text{(3.12)} \qquad K_\varepsilon^s := \{u \in V |\ \sup_{v \in V} \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u, v) = l_\varepsilon^s\}, \\ K_s^\varepsilon := \{v \in V |\ \inf_{u \in V} \mathcal{L}^\varepsilon(s, u_0^\varepsilon, u, v) = l_s^\varepsilon\}.$$

We establish results for (3.12) analogous to those of Lemmas 2.3 and 2.4 .

Lemma 3.2. *Let (E1)–(E5), (J1)–(J4), (E***1)–(E***5), (B1), (B2), (B***1) with* u_0^ε *replacing* u_0*, and the following assumptions hold true:*

(B3) $B(s)$ *is bounded:* $0 < \underline{b} \le \|B(s)\| \le \overline{b}$ *for* $s \in I$*,*

(B4) $s \mapsto B(s)$ *is continuous for* $s \in I$*;*

(B5) $\beta_\varepsilon(s, 0)$ *is bounded: there exist* $\overline{b}_\varepsilon > 0$ *such that*

$$\|\beta_\varepsilon(s, 0)\|_{H^\star} \le \overline{b}_\varepsilon \quad \textit{for } s \in I;$$

(B6) $s \mapsto \beta_\varepsilon(s, 0)$ *is continuous as* $s \to 0^+$*;*

(B**2)* $\beta_\varepsilon^\star(s, B(0)u_0^\varepsilon)$ *is symmetric:*

$$\langle \beta_\varepsilon^\star(s, B(0)u_0^\varepsilon)\zeta, \eta \rangle_{H^*,H} = \langle \beta_\varepsilon^\star(s, B(0)u_0^\varepsilon)\eta, \zeta \rangle_{H^*,H} \quad \textit{for } \zeta, \eta \in H,\ s \in I;$$

(B**3) there exist* $\underline{b}^\star > 0$ *and* $f_b^\star \in H^\star$ *such that with* $\underline{a}^\star$ *from (E***3):*

$$\langle \beta_\varepsilon^\star(s, B(0)u_0^\varepsilon)\zeta, \zeta \rangle_{H^*,H} \ge \frac{\underline{b}^\star - \underline{a}^\star}{\underline{b}^2} \|\zeta\|_H^2 - \langle f_b^\star, \zeta \rangle_{H^*,H} \quad \textit{for } \zeta \in H,\ s \in I.$$

(B**4)* $\beta_\varepsilon^\star(s, B(0)u_0^\varepsilon)$ *is bounded: there exist* $\overline{b}_\varepsilon^\star \ge 0$ *such that*

$$\|\beta_\varepsilon^\star(s, B(0)u_0^\varepsilon)\| \le \overline{b}_\varepsilon^\star \quad \textit{for } s \in I;$$

(B**5)* $s \mapsto \beta_\varepsilon^\star(s, B(0)u_0^\varepsilon)$ *is continuous for* $s \in I$*.*

Then for every $s \in I$ there exist a state $u^\varepsilon_s \in V$ solving the equation:

$$\begin{aligned} \langle (\mathcal{E}')^\star(s,u^\varepsilon_0)v, u^\varepsilon_s\rangle + \langle \beta^\star_\varepsilon\big(s, B(0)u^\varepsilon_0\big)B(s)v, B(s)u^\varepsilon_s\rangle_{H^*,H} \\ + \langle \mathcal{E}'(s,0), v\rangle + \langle \beta_\varepsilon(s,0), B(s)v\rangle_{H^*,H} = 0 \quad \textit{for all } v \in V, \end{aligned} \tag{3.13}$$

and an adjoint state $v^\varepsilon_s \in V$ satisfying the adjoint equation:

$$\begin{aligned} \langle (\mathcal{E}')^\star(s,u^\varepsilon_0)v^\varepsilon_s, u\rangle + \langle \beta^\star_\varepsilon\big(s, B(0)u^\varepsilon_0\big)B(s)v^\varepsilon_s, B(s)u\rangle_{H^*,H} \\ = \langle \mathcal{J}'(s, Mu^\varepsilon_s), Mu\rangle_{X^*,X} \quad \textit{for all } u \in V. \end{aligned} \tag{3.14}$$

The pair $(u^\varepsilon_s, v^\varepsilon_s) \in K^s_\varepsilon \times K^\varepsilon_s$ is a saddle point satisfying

$$l(\varepsilon,s) := l^\varepsilon_s = \mathcal{L}^\varepsilon(s, u^\varepsilon_0, u^\varepsilon_s, v^\varepsilon_s) = l^s_\varepsilon, \quad s \in I. \tag{3.15}$$

If $f^\star = 0$ in (E*3) and $f^\star_b = 0$ in (B*3), then the saddle-point is unique.

Moreover, there exists a subsequence $s_k \to 0^+$ with associated saddle points $(u^\varepsilon_{s_k}, v^\varepsilon_{s_k}) \in K^{s_k}_\varepsilon \times K^\varepsilon_{s_k}$, and $(u^\varepsilon_0, v^\varepsilon_0) \in K^0 \times K_0$ such that

$$(u^\varepsilon_{s_k}, v^\varepsilon_{s_k}) \to (u^\varepsilon_0, v^\varepsilon_0) \quad \textit{strongly in } V^2 \textit{ as } k \to \infty. \tag{3.16}$$

The proof of Lemma 3.2 is technical and presented in Appendix D.

For illustration, we note that the derivative $\beta'_\varepsilon(0,\zeta)$ of the mollified minimum function from (3.6) satisfies (3.8). It fulfills the symmetry assumption ($B^\star 2$). Since $\beta^\star_\varepsilon(0,\zeta) \geq 0$, the lower bound in ($B^\star 3$) holds trivially with $\underline{b}^\star = \underline{a}^\star$ and $f^\star_b = 0$. The upper bound $\overline{b}^\star_\varepsilon$ in ($B^\star 4$) has the order $1/\varepsilon$ in this case.

Below we state a theorem on differentiability of $\mathcal{L}^\varepsilon$.

Theorem 3.1. *Let (E1)–(E5), (J1)–(J4), (E*1)–(E*5), (B1)–B(6), (B*1)–(B*5), (L1), and the two following assumptions hold:*

(L2) B is differentiable such that $\frac{d}{ds}B \in C(I, \mathcal{L}(V,H))$,

(L3) there exists the derivative $\frac{d}{ds}\beta^\star_\varepsilon(s, B(0)u^\varepsilon_0) \in C(I, \mathcal{L}(H,H^*))$.

The directional derivative of the optimal value function $j : (0,\varepsilon_0) \times I \mapsto \overline{\mathbb{R}}$ defined by

$$j(\varepsilon, s) := \mathcal{J}(s, Mu^\varepsilon_s) \quad \textit{for } u^\varepsilon_s \in V \textit{ solving } (3.13), \tag{3.17}$$

and the associated Lagrangian function $l : (0,\varepsilon_0) \times I \mapsto \overline{\mathbb{R}}$ from (3.15) satisfy

$$\partial_+ j(\varepsilon, 0) = \partial_+ l(\varepsilon, 0) = \frac{\partial \mathcal{L}^\varepsilon}{\partial s}(0, u^\varepsilon_0, u^\varepsilon_0, v^\varepsilon_0). \tag{3.18}$$

Here the partial derivative is given by

$$\begin{aligned}(3.19)\quad &\frac{\partial\mathcal{L}^\varepsilon}{\partial s}(s,u_0^\varepsilon,u,v)\\ &:= \frac{\partial\mathcal{L}}{\partial s}(s,u_0^\varepsilon,u,v) - \big\langle \frac{d}{ds}\beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)B(s)v, B(s)u\big\rangle_{H^*,H}\\ &-\big\langle \beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)\frac{d}{ds}B(s)u, B(s)v\big\rangle_{H^*,H} - \big\langle \beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)B(s)u, \frac{d}{ds}B(s)v\big\rangle_{H^*,H}.\end{aligned}$$

The saddle point $(u_0^\varepsilon, v_0^\varepsilon) \in K_\varepsilon^0 \times K_0^\varepsilon$ *solves the penalty problem* (3.7) *and the adjoint equation* (3.14) *at* $s=0$*:*

$$\begin{aligned}(3.20)\quad &\langle(\mathcal{E}')^\star(0,u_0^\varepsilon)v_0^\varepsilon, u\rangle + \langle\beta_\varepsilon^\star(0,B(0)u_0^\varepsilon)B(0)v_0^\varepsilon, B(0)u\rangle_{H^*,H}\\ &\qquad = \langle\mathcal{J}'(0,Mu_0^\varepsilon), Mu\rangle_{X^*,X} \quad \textit{for all } u\in V.\end{aligned}$$

Proof. The differentiability assumptions (L1)–(L3) together with the continuity in (B4), ($B^\star$5) imply the existence of the partial derivative of $\mathcal{L}^\varepsilon$ in (3.19) with respect to $s\in I$ and its semi-continuity properties:

$$\liminf_{r,s_k\to 0^+}\frac{\partial\mathcal{L}^\varepsilon}{\partial s}(r,u_0^\varepsilon,u_{s_k}^\varepsilon,v_0^\varepsilon) \ge \frac{\partial\mathcal{L}^\varepsilon}{\partial s}(0,u_0^\varepsilon,u_0^\varepsilon,v_0^\varepsilon) \quad \text{for all } v_0^\varepsilon\in K_0^\varepsilon,$$

$$\limsup_{r,s_k\to 0^+}\frac{\partial\mathcal{L}^\varepsilon}{\partial s}(r,u_0^\varepsilon,u_0^\varepsilon,v_{s_k}^\varepsilon) \le \frac{\partial\mathcal{L}^\varepsilon}{\partial s}(0,u_0^\varepsilon,u_0^\varepsilon,v_0^\varepsilon) \quad \text{for all } u_0^\varepsilon\in K_\varepsilon^0.$$

Therefore, utilizing Lemma 3.2 and proceeding as in Lemma 2.5, we obtain formula (3.18) for the directional derivative. Taking the limit $s\to 0^+$ in (3.13) and using (3.9) we arrived at (3.7). The adjoint equation (3.20) follows from (3.14). The proof is complete. □

Next we analyze the limit as $\varepsilon\to 0^+$. For this task we employ the Lagrangian $\mathcal{L}$ from (2.9) at $s=0$.

Theorem 3.2. *Let (E1)–(E5), (J1)–(J4), ($E^\star$1)–($E^\star$5), (B1)–(B6), ($B^\star$1)–($B^\star$5) and the following assumptions hold:*

(B7) $B(0)$ *is a compact operator;*

(B8) there exits a Banach space $\tilde H\subset H$ *exists such that* $B(0): V\mapsto \tilde H$ *is surjective: for each* $\zeta\in\tilde H$ *there exists* $u\in V$ *with* $B(0)u=\zeta$*;*

(J5) $u\mapsto \mathcal{J}'(0,Mu)$ *is sequentially weak-to-weak continuous from* V *to* X^**;*

($E^\star$6) $u\mapsto(\mathcal{E}')^\star(0,u): V\mapsto \mathscr{L}(V,V^\star)$ *is sequentially weak-to-weak continuous.*

Then there exists a quadruple $(u_0, \lambda_0, v_0, \mu_0) \in (V \times \tilde{H}^\star)^2$*, where* $\tilde{H}^\star$ *is the dual space to* $\tilde{H}$ *from (B8) with the duality pairing* $\langle \cdot\,, \cdot \rangle_{\tilde{H}^*,\tilde{H}}$*, which satisfies the primal problem:*

$$(3.21)\quad \mathcal{L}(0, u_0, u_0, v) - \langle \lambda_0, B(0)v \rangle_{\tilde{H}^*,\tilde{H}} \\ \le \mathcal{L}(0, u_0, u_0, v_0) - \langle \lambda_0, B(0)v_0 \rangle_{\tilde{H}^*,\tilde{H}} \quad \textit{for all } v \in V,$$

the adjoint problem:

$$(3.22)\quad \mathcal{L}(0, u_0, u_0, v_0) - \langle \mu_0, B(0)u_0 \rangle_{\tilde{H}^*,\tilde{H}} \\ \le \mathcal{L}(0, u_0, u, v_0) - \langle \mu_0, B(0)u \rangle_{\tilde{H}^*,\tilde{H}} \quad \textit{for all } u \in V,$$

the complementarity relations:

$$(3.23)\qquad [B(0)u_0]^- = 0, \quad [\lambda_0]^+ = 0, \quad \langle \lambda_0, B(0)u_0 \rangle_{\tilde{H}^*,\tilde{H}} = 0,$$

and the compatibility condition

$$(3.24)\qquad \langle \lambda_0 - \beta_\varepsilon(0,0), B(0)v_0 \rangle_{\tilde{H}^*,\tilde{H}} = \langle \mu_0, B(0)u_0 \rangle_{\tilde{H}^*,\tilde{H}},$$

where $\beta_\varepsilon(0,0) = -\exp(-2)$ *in* (3.6).

Moreover, u_0 *satisfies* $[B(0)u_0]^- = 0$ *and the variational inequality* (3.4)*. Together with the Lagrange multiplier* λ_0 *it solves*

$$(3.25)\qquad \langle \mathcal{E}'(0, u_0), v \rangle + \langle \lambda_0, B(0)v \rangle_{\tilde{H}^*,\tilde{H}} = 0 \quad \textit{for all } v \in V.$$

The adjoint v_0 *solves the variational equation for all* $u \in V$*:*

$$(3.26)\quad \langle (\mathcal{E}')^\star(0, u_0)v_0, u \rangle + \langle \mu_0, B(0)u \rangle_{\tilde{H}^*,\tilde{H}} = \langle \mathcal{J}'(0, Mu_0), Mu \rangle_{X^*,X}$$

for μ_0 *obtained as an accumulation point in the following limit:*

$$(3.27)\qquad \beta^\star_{\varepsilon_k}(0, B(0)u_0^{\varepsilon_k})B(0)v_0^{\varepsilon_k} \rightharpoonup \mu_0 \ \star\textit{-weakly in } \tilde{H}^\star \textit{ as } k \to \infty.$$

According to (3.21)–(3.24)*, the optimal value functions in* (3.17) *and* (3.15) *at* $\varepsilon = 0$ *are*

$$(3.28)\qquad j(0,0) = l(0,0) = \mathcal{L}(0, u_0, u_0, v_0) - \langle \lambda_0, B(0)v_0 \rangle_{\tilde{H}^*,\tilde{H}}.$$

The proof of Theorem 3.2 is technical and it is presented in Appendix E.

It is worth noting that we cannot pass to the limit as $\varepsilon \to 0^+$ in the derivative β'_ε of the penalty, since it is unbounded in general, see Figure 2. This would be needed for $\beta^\star_\varepsilon$ which enters into $\partial\mathcal{L}^\varepsilon/\partial s$ in (3.18).

4. Shape derivative for breaking-line identification

Now we turn to a model problem for a nonlinear Poisson equation. We derive a shape derivative suitable for shape optimization in the problem of breaking-line identification from a boundary measurement.

Let

$$[t \mapsto \Omega_t] : \ (t_0, t_1) \mapsto D \subset \mathbb{R}^2 \tag{4.1}$$

be a parameter dependent family of domains contained in the hold-all domain D. For some fixed $t \in (t_0, t_1)$ we refer to Ω_t as the reference domain. We assume that $\Omega_t = \Omega_t^+ \cup \Omega_t^- \cup \Sigma_t$ is split into two variable sub-domains $\Omega_t^\pm$ with Lipschitz boundaries $\partial\Omega_t^\pm$ and outward normal vectors $n_t^\pm$. The sub-domains are separated by a one-dimensional *breaking line*

$$[t \mapsto \Sigma_t] : \ (t_0, t_1) \mapsto D_\Sigma \subset D \tag{4.2}$$

with the chosen normal direction $\nu_t = n_t^- = -n_t^+$ (see Figure 2).

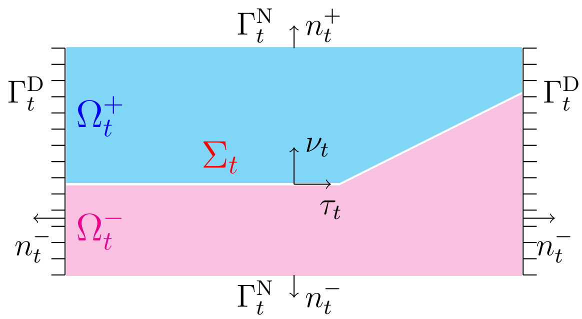

Figure 2. An example geometry Ω_t in 2D.

Let the outer boundary be split into two variable parts without intersection $\partial\Omega_t = \overline{\Gamma_t^{\rm D}} \cup \overline{\Gamma_t^{\rm N}}$, and the outward normal vector n_t be such that $n_t^\pm = n_t$ at $\partial\Omega_t$. The condition $\Gamma_t^{\rm D} \cap \partial\Omega_t^\pm \neq \emptyset$ on the Dirichlet boundary is assumed to guarantee the Poincare inequality in $\Omega_t^\pm$. A part of the Neumann boundary $\Gamma_t^{\rm O} \subset \Gamma_t^{\rm N}$ builds the *observation boundary*. Further we introduce

$$[t \mapsto (\Gamma_t^{\rm D}, \Gamma_t^{\rm N}, \Gamma_t^{\rm O})] : \ (t_0, t_1) \mapsto D_{\rm D} \times D_{\rm N} \times D_{\rm O} \subset D^3. \tag{4.3}$$

We adopt the formalism from Sections 2 and 3 to the geometry-dependent spaces of functions

$$\begin{aligned} &V(\Omega_t) := \{u \in H^1(\Omega_t^\pm) |\ u = 0 \text{ on } \Gamma_t^{\rm D}\}, \\ &X(\Omega_t) := L^2(\Gamma_t^{\rm O}), \quad H(\Omega_t) := L^2(\Sigma_t), \quad \tilde{H}(\Omega_t) := H^{1/2}(\Sigma_t). \end{aligned} \tag{4.4}$$

The observation operator $M : V(\Omega_t) \mapsto L^2(\Gamma_t^{\mathrm{O}})$ maps to the boundary traces on Γ_t^{O}. The restriction operator $B : V(\Omega_t) \mapsto L^2(\Sigma_t)$ is independent of s and describes a jump across the breaking line Σ_t subject to the *non-penetration condition* (see motivation in [10]):

$$u|_{\Sigma_t \cap \partial\Omega_t^+} - u|_{\Sigma_t \cap \partial\Omega_t^-} =: [\![u]\!] \geq 0. \tag{4.5}$$

This allows possible *contact* between the faces when $[\![u]\!] = 0$ in (4.5).

Here we take into account the dissipative interaction phenomenon of *cohesion* (see [2, 16]) described by a surface energy density $\alpha(s, \zeta)$. The following conditions are imposed:

$$\Big[(s,\zeta) \mapsto \alpha, \alpha', \alpha'', \frac{\partial \alpha'}{\partial s}, \frac{\partial \alpha''}{\partial s}\Big] \in C(I \times \mathbb{R}), \tag{4.6}$$

and the existence of $K_{\alpha 1} > 0$, $K_{\alpha 2} > 0$ such that:

$$|\alpha'(s,\zeta)| \leq K_{\alpha 1}, \quad |\alpha''(s,\zeta)| \leq K_{\alpha 2}. \tag{4.7}$$

For example, a mollification of the function $(K_{\mathrm{c}}/\kappa)\min(\kappa, |\zeta|)$ as

$$\alpha(0,\zeta) = K_{\mathrm{c}} \begin{cases} -1 & \text{for } \zeta < -\kappa - \delta \\ \frac{\delta}{\kappa}\exp\big(2\frac{\zeta+\kappa-\delta}{\zeta+\kappa+\delta}\big) - 1 & \text{for } -\kappa - \delta \leq \zeta < -\kappa + \delta \\ \zeta/\kappa & \text{for } -\kappa + \delta \leq \zeta < \kappa - \delta \\ 1 - \frac{\delta}{\kappa}\exp\big(2\frac{\zeta-\kappa+\delta}{\zeta-\kappa-\delta}\big) & \text{for } \kappa - \delta \leq \zeta < \kappa + \delta \\ 1 & \text{for } \zeta \geq \kappa + \delta \end{cases} \tag{4.8}$$

where $0 < \delta < \kappa$, $\kappa > 0$, and $K_{\mathrm{c}} > 0$ is the fracture toughness parameter. The function from (4.8) is depicted in Figure 3.

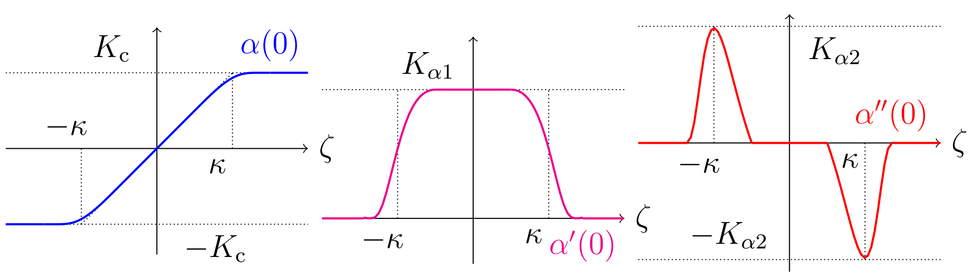

FIGURE 3. Example graphics of $\alpha, \alpha', \alpha''$ as $\delta = \kappa/2$.

Let the Lame parameter $\mu_{\mathrm{L}} > 0$ and the traction force $g \in H^1(D_{\mathrm{N}})$, ensuring that $g \in L^2(\Gamma_t^{\mathrm{N}})$ on Lipschitz curves $\Gamma_t^{\mathrm{N}} \subset D_{\mathrm{N}}$, be given. The bulk and the surface energies together constitute the *total potential energy* $\mathcal{E}(0) : V(\Omega_t) \mapsto \overline{\mathbb{R}}$:

$$\mathcal{E}(0, u; \Omega_t) := \frac{\mu_{\mathrm{L}}}{2} \int_{\Omega_t^{\pm}} |\nabla u|^2 \, dx - \int_{\Gamma_t^{\mathrm{N}}} g u \, dS_x + \int_{\Sigma_t} \alpha(0, [\![u]\!]) \, dS_x. \tag{4.9}$$

We calculate the Gateaux derivative $\mathcal{E}'(0) : V(\Omega_t) \mapsto V(\Omega_t)^\star$ at u:

$$\begin{aligned}\langle \mathcal{E}'(0,u;\Omega_t), v\rangle \\ = \mu_{\rm L}\int_{\Omega_t^\pm} \nabla u^\top \nabla v\, dx - \int_{\Gamma_t^{\rm N}} gv\, dS_x + \int_{\Sigma_t} \alpha'(0,[\![u]\!])[\![v]\!]\, dS_x,\end{aligned} \tag{4.10}$$

where $^\top$ denotes the transpose. The constrained optimization (3.3) leads to the variational inequality (3.4), which takes the form: find $u_t \in V(\Omega_t)$, $[\![u_t]\!]^- = 0$ on Σ_t, such that

$$\begin{aligned}\mu_{\rm L}\int_{\Omega_t^\pm} \nabla u_t^\top \nabla (v-u_t)\, dx + \int_{\Sigma_t} \alpha'(0,[\![u_t]\!])[\![v-u_t]\!]\, dS_x \\ \geq \int_{\Gamma_t^{\rm N}} g(v-u_t)\, dS_x \quad \text{for all } v\in V(\Omega_t),\ [\![v]\!]^- = 0 \text{ on } \Sigma_t.\end{aligned} \tag{4.11}$$

Lemma 4.1. *There exists a solution to the variational inequality* (4.11). *It satisfies the linear complementarity problem:*

$$\begin{aligned}-\mu_{\rm L}\Delta u_t = 0 \ \textit{in}\ \Omega_t^\pm; \quad u_t = 0 \ \textit{on}\ \Gamma_t^{\rm D}; \quad \mu_{\rm L} n_t^\top \nabla u_t = g \ \textit{on}\ \Gamma_t^{\rm N}; \\ \nu_t^\top [\![\nabla u_t]\!] = 0, \quad \big[\mu_{\rm L}\nu_t^\top \nabla u_t - \alpha'(0,[\![u_t]\!])\big]^+ = 0, \\ [\![u_t]\!]^- = 0, \quad [\![u_t]\!]\big(\mu_{\rm L}\nu_t^\top\nabla u_t - \alpha'(0,[\![u_t]\!])\big) = 0 \ \textit{on}\ \Sigma_t.\end{aligned} \tag{4.12}$$

The solution is unique for convex α (hence, monotone α').

Proof. For $u \in V(\Omega_t)$ we recall the Poincare inequality:

$$\int_{\Omega_t^\pm} |\nabla u|^2\, dx \geq K_{\rm P}\|u\|^2_{H^1(\Omega_t^\pm)}, \quad K_{\rm P} > 0, \tag{4.13}$$

and the trace inequality:

$$\|u\|_{L^2(\partial\Omega_t^\pm)} \leq \|u\|_{H^{1/2}(\partial\Omega_t^\pm)} \leq K_{\rm tr}\|u\|_{H^1(\Omega_t^\pm)}, \quad K_{\rm tr} > 0, \tag{4.14}$$

both uniform in $t \in (t_0, t_1)$. Using the bound $K_{\alpha 1} > 0$ in (4.7) and (4.13), (4.14) we can estimate $\langle \mathcal{E}'(0,u;\Omega_t), u\rangle$ in (4.10) from below and conclude the coercivity property (E2). The weak-to-weak continuity (E3) for $\mathcal{E}'(0,u)$ holds due to the continuity of α' assumed in (4.6).

Therefore, by Lemma 3.1 there exists a solution $u_t^\varepsilon \in V(\Omega_t)$ to the penalty equation (see (3.7)) in the form:

$$\mu_{\rm L}\int_{\Omega_t^\pm} \nabla (u_t^\varepsilon)^\top \nabla v\, dx + \int_{\Sigma_t} [\alpha' + \beta_\varepsilon](0,[\![u_t^\varepsilon]\!])[\![v]\!]\, dS_x = \int_{\Gamma_t^{\rm N}} gv\, dS_x \tag{4.15}$$

for all $v \in V(\Omega_t)$. It satisfies the mixed boundary value problem:

$$\begin{aligned}-\mu_{\rm L}\Delta u_t^\varepsilon = 0 \text{ in } \Omega_t^\pm; \quad u_t^\varepsilon = 0 \text{ on } \Gamma_t^{\rm D}; \quad \mu_{\rm L} n_t^\top \nabla u_t^\varepsilon = g \text{ on } \Gamma_t^{\rm N}; \\ \nu_t^\top [\![\nabla u_t^\varepsilon]\!] = 0, \quad \mu_{\rm L}\nu_t^\top \nabla u_t^\varepsilon = [\alpha' + \beta_\varepsilon](0,[\![u_t^\varepsilon]\!]) \text{ on } \Sigma_t.\end{aligned} \tag{4.16}$$

By the compactness argument used in the proof of Theorem 3.2 we get an accumulation point such that $u_t^{\varepsilon_k} \rightharpoonup u_t$ weakly in $V(\Omega_t)$ as $\varepsilon_k \to 0$, which solves the variational inequality (4.11). The derivation of relations (4.12) is standard, see e.g [14, Chapter 1]. $\square$

Let $z \in H^1(D_{\mathrm{O}})$ be given, providing an observation $z \in L^2(\Gamma_t^{\mathrm{O}})$ on Lipschitz curves $\Gamma_t^{\mathrm{O}} \subset D_{\mathrm{O}}$ from (4.3). We aim at the *shape optimization* problem for identification of an unknown breaking line from the observation: find Σ_* as the solution to

$$\text{(4.17)}\quad \min_{\Sigma_t \subset D_\Sigma} \Big\{ j(0,0) = J(0,u_t;\Omega_t) := \frac{1}{2}\int_{\Gamma_t^{\mathrm{O}}} (u_t - z)^2 \, dS_x + \rho|\Sigma_t| \\ \text{with } u_t \text{ satisfying (4.11)} \Big\},$$

where J represents $\mathcal{J}$ from (2.4), and $\rho \geq 0$ stands for the reason of perimeter regularization.

Lemma 4.2. *Let the observation z be feasible this means:*

$$\Omega_* = \Omega_*^{\pm} \cup \Sigma_* \subset D, \quad \Sigma_* \subset D_\Sigma, \quad (\Gamma_*^{\mathrm{D}}, \Gamma_*^{\mathrm{N}}, \Gamma_*^{\mathrm{O}}) \in D_{\mathrm{D}} \times D_{\mathrm{N}} \times D_{\mathrm{O}},$$

and $z \in V(\Omega_*)$, $[\![z]\!]^- = 0$ on Σ_* are such that

$$\text{(4.18)}\quad \mu_{\mathrm{L}} \int_{\Omega_*^{\pm}} \nabla z^\top \nabla(v - z)\, dx + \int_{\Sigma_*} \alpha'(0, [\![z]\!])[\![v - z]\!]\, dS_x \\ \geq \int_{\Gamma_*^{\mathrm{N}}} g(v - z)\, dS_x \quad \textit{for all } v \in V(\Omega_*),\ [\![v]\!]^- = 0 \textit{ on } \Sigma_*.$$

If $\rho = 0$, then there exists a solution to the shape optimization problem (4.17)*. In general, the solution is non-unique.*

Proof. The trivial minimum in (4.17) is evidently attained at the argument $\Sigma_t = \Sigma_*$ when $u_t = z$ and $\rho = 0$.

We construct a counter-example to uniqueness. Assume Σ_* solves (4.17) and z satisfies (4.18). Let the active part of the breaking line $\Sigma_*^a \subset \Sigma_*$, where the equality $[\![z]\!] = 0$ holds (i.e. contact happens), be nonempty. Then $z \in V(\tilde{\Omega}_*)$ satisfies (4.18) in $\tilde{\Omega}_* = \tilde{\Omega}_*^{\pm} \cup \tilde{\Sigma}_*$ for an arbitrary regular interface $\tilde{\Sigma}_* \subset D_\Sigma$ that coincides with Σ_* along $\Sigma_* \setminus \Sigma_*^a$. In this case, both $\tilde{\Sigma}_*$ and Σ_* solve (4.17). This situation is observed in the numerical experiment. $\square$

Under the penalty approach from Section 3 we approximate (4.17) by a differentiable constraint following Theorem 3.1: for $\varepsilon \in (0, \varepsilon_0)$ find $\Sigma_* \subset D_\Sigma$ such that

$$\text{(4.19)}\qquad \min_{\Sigma_t \subset D_\Sigma} \Big\{ j(\varepsilon, 0) = J(0, u_t^\varepsilon; \Omega_t) \quad \text{with } u_t^\varepsilon \text{ solving (4.15)} \Big\}.$$

Aiming to solve (4.19) by a gradient method, we look for a descent direction $\partial_+ j(\varepsilon, 0) < 0$ from Theorem 3.1. This requires to express the perturbation $j(\varepsilon, s)$ for $s \in I$ in a geometry-independent form.

For this task we employ the velocity method based on coordinate transformations. Let I have the end-point $s_0 \leq t_1 - t$, and let us fix a kinematic flow and its inverse

$$(4.20)\quad [(s,x) \mapsto \phi_s], [(s,y) \mapsto \phi_s^{-1}] \in C^1(t_0 - t_1, t_1 - t_0; W^{1,\infty}(\overline{D})^2)^2.$$

This defines an associateed *coordinate transformation* $y = \phi_s(x)$ and its inverse $x = \phi_s^{-1}(y)$. We suppose that the mapping introduced in (4.1)–(4.3) forms a diffeomorphism:

$$(4.21)\quad x \mapsto \phi_s : (\Omega_t, \Sigma_t, \Gamma_t^{\mathrm{D}}, \Gamma_t^{\mathrm{N}}, \Gamma_t^{\mathrm{O}}) \mapsto (\Omega_{t+s}, \Sigma_{t+s}, \Gamma_{t+s}^{\mathrm{D}}, \Gamma_{t+s}^{\mathrm{N}}, \Gamma_{t+s}^{\mathrm{O}}).$$

Then the *kinematic velocity* $\Lambda(t,x) \in C([t_0, t_1]; W^{1,\infty}(\overline{D})^2)$ can be defined from (4.20) by the formula

$$(4.22)\quad \Lambda(t+s, y) := \tfrac{d\phi_s}{ds}(\phi_s^{-1}(y)).$$

If a velocity vector is given explicitly

$$(4.23)\quad \Lambda = (\Lambda_1, \Lambda_2)(t,x) \in C([t_0, t_1]; W^{1,\infty}(\overline{D}))^2, \quad \Lambda|_{\partial D} = 0,$$

preserving the hold-all domain D, it determines the flows in (4.20) as solution vector $\phi_s = ((\phi_s)_1, (\phi_s)_2)$ to the non-autonomous ODE system

$$(4.24)\quad \frac{d}{ds}\phi_s = \Lambda(t+s, \phi_s) \text{ for } s \in I, \quad \phi_s = x \text{ as } s = 0,$$

and $\phi_s^{-1}(y) = ((\phi_s^{-1})_1, (\phi_s^{-1})_2)$ to the transport equation

$$(4.25)\quad \frac{\partial}{\partial s}\phi_s^{-1} + (\nabla_y \phi_s^{-1})\Lambda|_{t+s} = 0 \text{ in } I \times D, \quad \phi_s^{-1} = y \text{ as } s = 0.$$

In (4.25) we utilize the second order tensor $\nabla_y \phi_s^{-1} = (\partial(\phi_s^{-1})_i / \partial y_j)_{i,j=1}^2$, and $\Lambda|_{t+s} = \Lambda(t+s, y)$. For validation of (4.20)–(4.25) see [9, 21].

The diffeomorphism (4.21) preserves the bijectivity between the function spaces in (4.4):

$$(4.26)\quad [u \mapsto u \circ \phi_s^{-1}] : \big(V(\Omega_t), L^2(\Gamma_t^{\mathrm{O}}), L^2(\Sigma_t), H^{1/2}(\Sigma_t)\big) \\ \mapsto \big(V(\Omega_{t+s}), L^2(\Gamma_{t+s}^{\mathrm{O}}), L^2(\Sigma_{t+s}), H^{1/2}(\Sigma_{t+s})\big).$$

With the help of (4.26) we transform the perturbed objective $J(0, \tilde{u}; \Omega_{t+s})$ from (4.19) for $\tilde{u} \in V(\Omega_{t+s})$ such that

$$(4.27)\quad J(0, u \circ \phi_s^{-1}; \Omega_{t+s}) = J(s, u; \Omega_t) \\ := \frac{1}{2}\int_{\Gamma_t^{\mathrm{O}}} (u - z \circ \phi_s)^2\, \omega_s dS_x + \rho \int_{\Sigma_t} \omega_s dS_x,$$

where ω_s will be defined later. Based on the second derivative in the identity (see (2.6)):

$$\int_0^1 \alpha''(0, [\![ru_t^\varepsilon]\!])[\![u_t^\varepsilon]\!]\,dr = \alpha'(0, [\![u_t^\varepsilon]\!]) - \alpha'(0,0), \tag{4.28}$$

we linearize at the solution u_t^ε the perturbed state operator in (4.10):

$$\langle \mathcal{E}'(0, u\circ\phi_s^{-1}; \Omega_{t+s}), v\circ\phi_s^{-1}\rangle \sim \langle (\mathcal{E}')^\star(s, u_t^\varepsilon)v, u\rangle + \langle \mathcal{E}'(s,0), v\rangle, \tag{4.29}$$

where the terms are

$$\begin{aligned} \langle (\mathcal{E}')^\star(s, u_t^\varepsilon)v, u\rangle &:= \mu_{\mathrm{L}} \int_{\Omega_t^\pm} ([\nabla\phi_s^{-\top}\circ\phi_s]u)^\top [\nabla\phi_s^{-\top}\circ\phi_s]v\, J_s dx \\ &+ \int_{\Sigma_t}\int_0^1 \alpha''(0, [\![ru_t^\varepsilon]\!])\,dr [\![u]\!][\![v]\!]\,\omega_s dS_x, \\ \langle \mathcal{E}'(s,0), v\rangle &:= -\int_{\Gamma_t^{\mathrm{N}}} (g\circ\phi_s)v\,\omega_s dS_x + \int_{\Sigma_t} \alpha'(0,0)[\![v]\!]\,\omega_s dS_x. \end{aligned} \tag{4.30}$$

In (4.27) and (4.30) we use the chain rule

$$\nabla_y(u\circ\phi_s^{-1}) = (\nabla\phi_s^{-T}\circ\phi_s)\nabla u, \tag{4.31}$$

and the Jacobian in the domain and at the boundary:

$$J_s := \det(\nabla\phi_s) \text{ in } \Omega_t^\pm, \quad \omega_s := |(\nabla\phi_s^{-\top}\circ\phi_s)n_t^\pm| J_s \text{ at } \partial\Omega_t^\pm, \tag{4.32}$$

for more details, see e.g. [17, 18, 23].

Similarly, using the following identity analogous to (4.28):

$$\int_0^1 \beta_\varepsilon'(0, [\![ru_t^\varepsilon]\!])[\![u_t^\varepsilon]\!]\,dr = \beta_\varepsilon(0, [\![u_t^\varepsilon]\!]) - \beta_\varepsilon(0,0), \tag{4.33}$$

we perturb the penalty term in (4.15) linearized at u_t^ε such that

$$\begin{aligned} &\langle \beta_\varepsilon(s, [\![u\circ\phi_s^{-1}]\!]), [\![v\circ\phi_s^{-1}]\!]\rangle_{L^2(\Sigma_{t+s})} \sim \langle \beta_\varepsilon^\star(s, [\![u_t^\varepsilon]\!])[\![v]\!], [\![u]\!]\rangle_{L^2(\Sigma_t)} \\ &+\langle \beta_\varepsilon(s,0), [\![v]\!]\rangle_{L^2(\Sigma_t)} := \int_{\Sigma_t}\Big(\int_0^1 \beta_\varepsilon'(0, [\![ru_t^\varepsilon]\!])[\![u]\!]\,dr + \beta_\varepsilon(0,0)\Big)[\![v]\!]\,\omega_s dS_x. \end{aligned} \tag{4.34}$$

Combining formulas (4.27)–(4.34) we get a perturbed Lagrange function in (3.10) expressed by the integrals

$$\begin{aligned} \mathcal{L}^\varepsilon(s, u_t^\varepsilon, u, v) &= \frac{1}{2}\int_{\Gamma_t^{\mathrm{O}}} (u - z\circ\phi_s)^2\,\omega_s dS_x + \rho\int_{\Sigma_t} \omega_s dS_x \\ &- \mu_{\mathrm{L}}\int_{\Omega_t^\pm} ([\nabla\phi_s^{-\top}\circ\phi_s]u)^\top[\nabla\phi_s^{-\top}\circ\phi_s]v\,J_s dx + \int_{\Gamma_t^{\mathrm{N}}} (g\circ\phi_s)v\,\omega_s dS_x \\ &- \int_{\Sigma_t}\Big(\int_0^1 [\alpha'' + \beta_\varepsilon'](0, [\![ru_t^\varepsilon]\!])[\![u]\!]\,dr + [\alpha' + \beta_\varepsilon](0,0)\Big)[\![v]\!]\,\omega_s dS_x. \end{aligned} \tag{4.35}$$

Next we present a formula for the shape derivative.

Theorem 4.1. *Let the bound $K_{\alpha 2} > 0$ in (4.7) be sufficiently small such that*

$$\underline{a}^\star := \mu_{\rm L} K_{\rm P} - 2K_{\alpha 2} K_{\rm tr}^2 > 0, \tag{4.36}$$

where $K_{\rm P}$ and $K_{\rm tr}$ are the constants from the Poincare and the trace estimates (4.13) and (4.14). Then the directional derivative of $\mathcal{L}^\varepsilon$ exists and is given by the formula

$$\partial_+ j(\varepsilon, 0) = \frac{\partial \mathcal{L}^\varepsilon}{\partial s}(0, u_t^\varepsilon, u_t^\varepsilon, v_t^\varepsilon) = \int_{\Gamma_t^{\rm O}} \Big(\frac{1}{2} {\rm div}_{\tau_t} \Lambda \, (u_t^\varepsilon - z)^2 - \Lambda^\top \nabla z (u_t^\varepsilon - z) \Big) dS_x - \mu_{\rm L} \int_{\Omega_t^\pm} (\nabla u_t^\varepsilon)^\top ({\rm div}\Lambda - \nabla \Lambda - \nabla \Lambda^\top) \nabla v_t^\varepsilon \, dx + \int_{\Gamma_t^{\rm N}} ({\rm div}_{\tau_t} \Lambda \, g + \Lambda^\top \nabla g) v_t^\varepsilon \, dS_x + \int_{\Sigma_t} {\rm div}_{\tau_t} \Lambda \big(\rho - [\alpha' + \beta_\varepsilon](0, [\![u_t^\varepsilon]\!]) [\![v_t^\varepsilon]\!] \big) \, dS_x, \tag{4.37}$$

where the tangential divergence is defined as

$${\rm div}_{\tau_t} \Lambda := {\rm div}\Lambda - (n_t^\pm)^\top \nabla \Lambda \, n_t^\pm \ \text{ at } \partial\Omega_t^\pm. \tag{4.38}$$

The saddle point $(u_t^\varepsilon, v_t^\varepsilon) \in V(\Omega_t)^2$ solves the penalty equation (4.15) and the adjoint equation:

$$\mu_{\rm L} \int_{\Omega_t^\pm} \nabla u^\top \nabla v_t^\varepsilon \, dx + \int_{\Sigma_t} \int_0^1 [\alpha'' + \beta'_\varepsilon](0, [\![r u_t^\varepsilon]\!]) [\![v_t^\varepsilon]\!] [\![u]\!] \, dr dS_x = \int_{\Gamma_t^{\rm O}} (u_t^\varepsilon - z) u \, dS_x \quad \text{for all } u \in V(\Omega_t), \tag{4.39}$$

for which the mixed boundary value formulation is given by:

$$-\mu_{\rm L} \Delta v_t^\varepsilon = 0 \text{ in } \Omega_t^\pm; \quad v_t^\varepsilon = 0 \text{ on } \Gamma_t^{\rm D};$$
$$\mu_{\rm L} n_t^\top \nabla v_t^\varepsilon = u_t^\varepsilon - z \text{ on } \Gamma_t^{\rm O}; \quad \mu_{\rm L} n_t^\top \nabla v_t^\varepsilon = 0 \text{ on } \Gamma_t^{\rm N} \setminus \Gamma_t^{\rm O};$$
$$\nu_t^\top [\![\nabla v_t^\varepsilon]\!] = 0, \ \mu_{\rm L} \nu_t^\top \nabla v_t^\varepsilon = \int_0^1 [\alpha'' + \beta'_\varepsilon](0, [\![r u_t^\varepsilon]\!]) [\![v_t^\varepsilon]\!] \, dr \text{ on } \Sigma_t. \tag{4.40}$$

For the proof of Theorem 4.1 one checks the conditions of Theorem 3.1. It is given in Appendix F.

In the following we decompose the velocity into the normal and tangential vectors at the boundary:

$$\Lambda = ((n_t^\pm)^\top \Lambda) n_t^\pm + ((\tau_t^\pm)^\top \Lambda) \tau_t^\pm \quad \text{on } \partial\Omega_t^\pm, \tag{4.41}$$

where $\tau_t^\pm$ is the tangential vector positively oriented to $n_t^\pm$.

Theorem 4.2. *Let the solution of* (4.15), (4.39) *be smooth such that* $(u_t^\varepsilon, v_t^\varepsilon) \in H^2(\Omega_t^\pm)^2$*. Then the shape derivative in Theorem 4.1 satisfies an equivalent Hadamard's representation by the boundary integrals:*

$$\begin{aligned}(4.42)\quad \partial_+ j(\varepsilon,0) = &\int_{\Gamma_t^{\mathrm{D}}} (n_t^\top \Lambda)(n_t^\top \mathcal{D}_1)\, dS_x + (\tau_t^\top \Lambda)(\tau_t^\top [\![\mathcal{D}_1]\!])_{\partial\Gamma_t^{\mathrm{D}}\cap\Sigma_t} \\ &+ \int_{\Gamma_t^{\mathrm{N}}} (n_t^\top \Lambda)(\varkappa_t \mathcal{D}_2 + n_t^\top \nabla \mathcal{D}_2)\, dS_x + (\tau_t^\top \Lambda)[\![\mathcal{D}_2]\!]_{\partial\Gamma_t^{\mathrm{N}}\cap\Sigma_t} \\ &+ \int_{\Sigma_t} \big((\nu_t^\top \Lambda)\mathcal{D}_3^\varepsilon + (\tau_t^\top \Lambda)\mathcal{D}_4^\varepsilon\big)\, dS_x + (\tau_t^\top \Lambda)[\![\mathcal{D}_5^\varepsilon]\!]_{\partial\Sigma_t} \\ &+ \int_{\Gamma_t^{\mathrm{O}}} (n_t^\top \Lambda)(\varkappa_t \mathcal{D}_6 + n_t^\top \nabla \mathcal{D}_6)\, dS_x + (\tau_t^\top \Lambda)\mathcal{D}_6|_{\partial\Gamma_t^{\mathrm{O}}}.\end{aligned}$$

The terms in (4.42) *are given by*

$$\begin{aligned}(4.43)\quad &\mathcal{D}_1 := \mu_{\mathrm{L}}\big(\nabla u_t^\varepsilon(n_t^\top \nabla v_t^\varepsilon) + \nabla v_t^\varepsilon(n_t^\top \nabla u_t^\varepsilon)\big), \quad \mathcal{D}_2 := g v_t^\varepsilon, \\ &\mathcal{D}_3^\varepsilon := \varkappa_t \mathcal{D}_5^\varepsilon + \mu_{\mathrm{L}}[\![(\nabla u_t^\varepsilon)^\top \nabla v_t^\varepsilon]\!] - \nu_t^\top(\nabla p_\varepsilon + q_\varepsilon), \\ &\mathcal{D}_4^\varepsilon := -\tau_t^\top q_\varepsilon, \quad \mathcal{D}_5^\varepsilon := \rho - p_\varepsilon, \quad \mathcal{D}_6 := \frac{1}{2}(u_t^\varepsilon - z)^2,\end{aligned}$$

where $\varkappa_t^\pm := \mathrm{div}_{\tau_t} n_t^\pm$ *denotes the curvature at* $\partial\Omega_t^\pm$*, and we utilize the notation at* Σ_t*:*

$$\begin{aligned}(4.44)\quad &p_\varepsilon := [\alpha' + \beta_\varepsilon](0, [\![u_t^\varepsilon]\!])\, [\![v_t^\varepsilon]\!], \\ &q_\varepsilon := [\![\nabla u_t^\varepsilon]\!]\Big(\int_0^1 [\alpha'' + \beta_\varepsilon'](0, [\![r u_t^\varepsilon]\!])\, dr - [\alpha'' + \beta_\varepsilon'](0, [\![u_t^\varepsilon]\!])\Big)[\![v_t^\varepsilon]\!].\end{aligned}$$

A descent direction $\partial_+ j(\varepsilon, 0) < 0$ *in* (4.42) *is provided by the choice*

$$\begin{aligned}(4.45)\quad &n_t^\top \Lambda = -k_7(n_t^\top \mathcal{D}_1) \text{ at } \Gamma_t^{\mathrm{D}}, \quad \tau_t^\top \Lambda = -k_1(\tau_t^\top [\![\mathcal{D}_1]\!]) \text{ at } \partial\Gamma_t^{\mathrm{D}} \cap \Sigma_t, \\ &n_t^\top \Lambda = -k_2(\varkappa_t \mathcal{D}_2 + n_t^\top \nabla \mathcal{D}_2) \text{ at } \Gamma_t^{\mathrm{N}}, \quad \tau_t^\top \Lambda = -k_8 [\![\mathcal{D}_2]\!] \text{ at } \partial\Gamma_t^{\mathrm{N}} \cap \Sigma_t, \\ &\nu_t^\top \Lambda = -k_3 \mathcal{D}_3^\varepsilon \text{ and } \tau_t^\top \Lambda = -k_4 \mathcal{D}_4^\varepsilon \text{ at } \Sigma_t, \quad \tau_t^\top \Lambda = -k_5 [\![\mathcal{D}_5^\varepsilon]\!] \text{ at } \partial\Sigma_t, \\ &n_t^\top \Lambda = -k_6(\varkappa_t \mathcal{D}_6 + n_t^\top \nabla \mathcal{D}_6) \text{ at } \Gamma_t^{\mathrm{O}}, \quad \tau_t^\top \Lambda = -k_9 \mathcal{D}_6 \text{ at } \partial\Gamma_t^{\mathrm{O}},\end{aligned}$$

with $k_i \geq 0$*,* $i = 1, \dots, 9$*, and not all simultaneously equal to zero.*

The proof of Theorem 4.2 is based on integration by parts and is presented in Appendix G. The expression (4.42) is important for gradient-based iterative techniques.

5. Numerical simulation

We set a piecewise-linear breaking line $\Sigma_* \subset D_\Sigma$ to be identified:

$$D_\Sigma = \{x_1 \in (0,1),\ x_2 = \psi(x_1) \in (0, 0.5)\},\quad \Sigma_* := \{x_1 \in (0,1),\ \psi_*(x_1) = \max(0.2, (x_1-1)/3 + 0.4)\}, \tag{5.1}$$

which breaks the rectangle $\Omega = (0,1)\times(0,0.5)$ into two parts $\Omega_*^\pm$. Let the boundary $\partial\Omega$ be split into fixed Dirichlet and Neumann parts:

$$\Gamma^{\rm D}_* = \{x_1 \in \{0,1\},\ x_2 \in (0,0.5)\},\ \Gamma^{\rm N}_* = \{x_1 \in (0,1),\ x_2 \in \{0, 0.5\}\}, \tag{5.2}$$

see the illustration of the geometry in Figure 2. We choose for the Young's modulus $E_{\rm Y} = 73000$ (mPa) and Poisson's ratio $\nu_{\rm P} = 0.34$, the Lamé parameter $\mu_{\rm L} = E_{\rm Y}/(2(1+\nu_{\rm P})) \approx 27239$, and the linear traction force

$$g(x) = \mu_{\rm L}(1 - 1.68x_1)(4x_2 - 1). \tag{5.3}$$

Then there exists a solution $z \in H^1(\Omega_*^\pm)$ such that $z = 0$ on $\Gamma^{\rm D}_*$, $[\![z]\!]^- = 0$ on Σ_*, which satisfies the variational equation (4.18) according to Lemma 4.1. Let the observation boundary be $\Gamma^{\rm O}_* = \Gamma^{\rm N}_*$.

Now we discretize the problem. For $\Sigma_t \subset D_\Sigma$ breaking Ω into $\Omega_t^\pm$, let $\Omega^\pm_{t,h}$ be a triangulation of mesh size $h > 0$ of $\Omega_t^\pm$, which is compatible at the interface $\Sigma_{t,h} := \Sigma_t \cap \partial\Omega^1_{t,h} = \Sigma_t \cap \partial\Omega^2_{t,h}$. At $\Sigma_{t,h}$ the cohesion function $\alpha(0,\zeta)$ is set as in (4.8) with $K_{\rm c} = 10^{-3}$ (mPa·m), $\kappa = 10^{-2}$ (m). For small δ and h we rely on the discretization $\alpha_h(0,\zeta)$ such that

$$\alpha_h = \frac{K_{\rm c}}{\kappa}\min(\kappa, |\zeta|), \quad \alpha_h' = \frac{K_{\rm c}}{\kappa}\,{\rm ind}\{|\zeta| < \kappa\}. \tag{5.4}$$

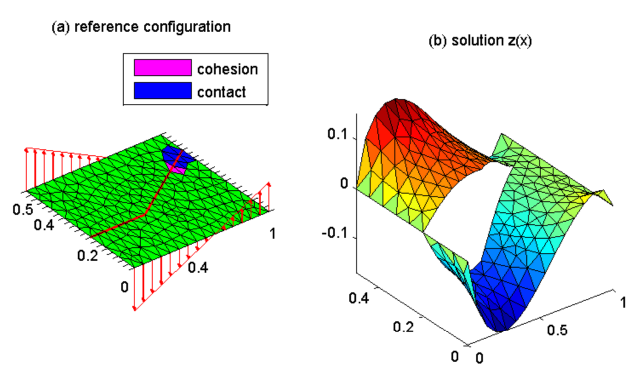

Figure 4. Reference configuration (a); true solution z_h (b).

After piecewise-linear FE discretization of the problem on a grid of mesh size $h = 10^{-2}$ according to (5.1)–(5.4), we solve the variational equation (4.18) by a primal-dual active set (PDAS) iterative algorithm developed in [11]. The numerical solution z_h obtained after 3 iterations with zero residual is plotted in Figure 4 (b). In plot (a) we depict the computational grid $\Omega^{\pm}_{t,h}$, the traction force g at Γ^{N}_*, the cohesion (where $[\![z_h]\!] < \kappa$) and contact (where $[\![z_h]\!] = 0$) parts of Σ_*, which are marked in the triangles adjacent to the interface.

According to the proof given in Lemma 4.1 we approximate the variational inequality (4.11) by the penalty equation (4.15). For small ε and h, the penalty operator from (3.6) is discretized as

$$\beta_{\varepsilon,h}(0,\zeta) = \frac{1}{\varepsilon}\min(0,\zeta), \quad \beta'_{\varepsilon,h}(0,\zeta) = \frac{1}{\varepsilon}\,\mathrm{ind}\{\zeta < 0\}. \tag{5.5}$$

Let $V_{t,h}(\Omega_{t,h})$ be a conforming piecewise-linear FE-space such that

$$V_{t,h}(\Omega_{t,h}) \subset V(\Omega_{t,h}) = \{u \in H^1(\Omega^{\pm}_{t,h}) | \quad u = 0 \text{ on } \Gamma^{\mathrm{D}}_*\}.$$

The discrete penalty equation (4.15) determines $u^{\varepsilon}_{t,h} \in V_{t,h}(\Omega_{t,h})$ such that

$$\int_{\Omega^{\pm}_{t,h}} (\nabla u^{\varepsilon}_{t,h})^{\top} \nabla v_h \, dx + \int_{\Sigma_{t,h}} [\alpha'_h + \beta_{\varepsilon,h}](0, [\![u^{\varepsilon}_{t,h}]\!]) [\![v_h]\!] \, dS_x = \int_{\Gamma^{\mathrm{N}}_*} g v_h \, dS_x, \tag{5.6}$$

and ignoring the singularity of α'_h the discrete adjoint equation (4.39) reads: find $v^{\varepsilon}_{t,h} \in V_{t,h}(\Omega_{t,h})$ such that

$$\begin{aligned} \int_{\Omega^{\pm}_{t,h}} (\nabla u_h)^{\top} \nabla v^{\varepsilon}_{t,h} \, dx + \int_{\Sigma_{t,h}} \beta'_{\varepsilon,h}(0, [\![u^{\varepsilon}_{t,h}]\!]) [\![u_h]\!] \, dS_x \\ = \int_{\Gamma^{\mathrm{N}}_*} (u^{\varepsilon}_{t,h} - z_h) u_h \, dS_x \quad \text{for all } u_h, v_h \in V_{t,h}(\Omega_{t,h}). \end{aligned} \tag{5.7}$$

After solving problems (5.6) and (5.7), according to Theorem 4.2 we calculate $\mathcal{D}^{\varepsilon}_3$ at the moving boundary $\Sigma_{t,h}$, and $\mathcal{D}_1$ at $\Sigma_{t,h} \cap \Gamma^{\mathrm{D}}_*$, where $\rho = 1/\mu_{\mathrm{L}}$ is set. By the virtue of (5.4), (5.5) here $q_{\varepsilon,h} = 0$ and

$$\begin{aligned} p_{\varepsilon,h} &= [\alpha'_h + \beta_{\varepsilon,h}](0, [\![u^{\varepsilon}_{t,h}]\!]) \, [\![v^{\varepsilon}_{t,h}]\!], \\ \nabla p_{\varepsilon,h} &= [\![\nabla v^{\varepsilon}_{t,h}]\!] [\alpha'_h + \beta_{\varepsilon,h}](0, [\![u^{\varepsilon}_{t,h}]\!]) + [\![\nabla u^{\varepsilon}_{t,h}]\!] \beta'_{\varepsilon,h}(0, [\![u^{\varepsilon}_{t,h}]\!]) [\![v^{\varepsilon}_{t,h}]\!]. \end{aligned} \tag{5.8}$$

Since Γ^{D}_* and $\Gamma^{\mathrm{N}}_* = \Gamma^{\mathrm{O}}_*$ are fixed in the identification problem, the normal velocity $n_t^{\top} \Lambda = 0$ at $\partial\Omega$ when $k_2 = k_6 = k_7 = 0$ in (4.45). The tangential velocity is set $\tau_t^{\top} \Lambda = 0$ at Σ_t by means of $k_4 = k_5 = k_8 =$

$k_9 = 0$. Therefore, we get a descent direction when $\Lambda_{1,H} = 0$ and

$$\Lambda_{2,H} = \frac{k_3}{\sqrt{h}}(2x_1 - 1)[\![\mathcal{D}_{1,h}]\!]_2 \text{ at } \Sigma_{t,h} \cap \Gamma_*^{\rm D}, \qquad \Lambda_{2,H} = -k_3 \mathcal{D}^\varepsilon_{3,h} \text{ at } \Sigma_{t,h} \setminus \Gamma_*^{\rm D}. \tag{5.9}$$

The scaling $k_3 = 0.1h/\|\Lambda_{2,H}\|_{C(\overline{\Sigma_{t,h}})}$ is chosen, and the weight $k_1 = k_3/\sqrt{h}$ at $\Gamma_*^{\rm D}$ was found empirically in [6]. We point out that the discrete velocity Λ_H at the interface Σ_t is defined on a coarser grid of size $H > 0$, compared to the mesh size h of the problem.

We summarize the optimization algorithm for breaking line identification.

Algorithm 1.

- **(0)** *Initialize constant grid function* $\psi_H^{(0)} = 0.25$ *at points* $s_H \in [0,1]$ *and the linear interpolate* $\Sigma^{(0)} = \{x_1 \in (0,1), x_2 = \psi_H^{(0)}(x_1)\}$; *set* $n = 0$.
- **(1)** *Set the interface* $\Sigma_{t,h} = \Sigma^{(n)}$ *and triangulate* $\Omega^\pm_{t,h}$; *find solutions* $u^\varepsilon_{t,h}$, $v^\varepsilon_{t,h}$ *to the discrete equations* (5.6), (5.7).
- **(2)** *Calculate a velocity* $\Lambda_{2,H}$ *from* (**??**)–(5.9); *update the values*

$$\psi_H^{(n+1)} = \psi_H^{(n)} + \Lambda_{2,H} \quad \text{at the points } s_H \in [0,1]; \tag{5.10}$$

from linear interpolant $\psi_H^{(n+1)}$ *determine the piecewise-linear segment* $\Sigma^{(n+1)} = \{x_1 \in (0,1),\, x_2 = \psi_H^{(n+1)}(x_1)\}$.
- **(3)** *Until a stopping rule is reached, set* $n = n+1$ *and go to Step* **(1)**.

For 11 equidistant points s_H with $H = 0.1$, the numerical result of Algorithm 1 after $\#n = 200$ iterations (the stopping rule) is depicted in Figure 5. The penalty parameter $\varepsilon = 10^{-10}$ was taken. In plot (a)

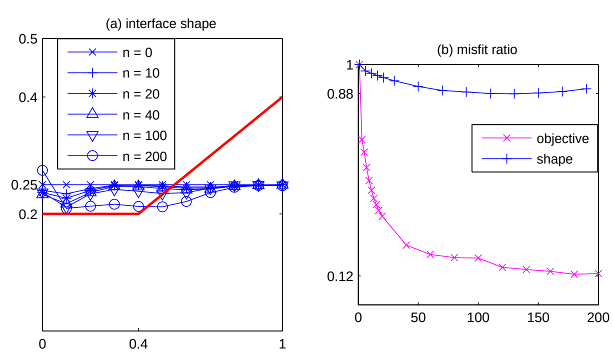

FIGURE 5. Iterations $\Sigma^{(n)}$ (a); misfit ratio (b).

the selected iterations $n = 0, 10, 20, 40, 100, 200$ of $\Sigma^{(n)}$ according to

(5.10) are drawn in Ω in comparison with the true interface Σ_* (the thick solid line). In plot (b) of Figure 5 we plot the ratio $J^{(n)}/J^{(0)}$ of the objective optimal values recalled here to be

$$J^{(n)}(u^\varepsilon_{t,h};\Omega^\pm_{t,h}) = \frac{1}{2}\int_{\Gamma^{\rm O}_*}(u^\varepsilon_{t,h}-z_h)^2\,dS_x + \rho|\Sigma^{(n)}|, \tag{5.11}$$

and the shape ratio $\|\psi^{(n)}-\psi_*\|_{C([0,1])}/\|\psi^{(0)}-\psi_*\|_{C([0,1])}$. The computed misfit ratios attain as minimum 12% and 88%, respectively.

From the simulation we conclude the following feature. In Figure 5 (a) it can be observed that the left part of curve Σ_*, where no contact occurs (see Figure 4 (a)), is recovered well by the identification Algorithm 1, whereas the right part of interface, where contact occurs, the initialization $\Sigma^{(0)}$ is almost unchanged during the iterations.

To remedy the hidden part of Σ_*, we apply to the same configuration a traction force which is more stretching than that in (5.3):

$$g(x) = \mu_{\rm L}(1-1.55x_1)(4x_2-1). \tag{5.12}$$

As the result, the whole Σ_* is open without contact, however, the cohesion occurs at the interface as shown in Figure 6.

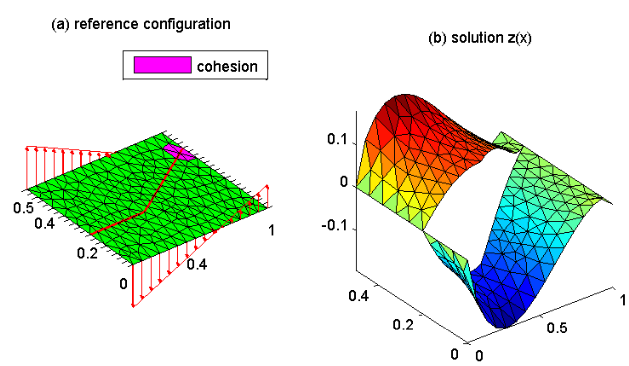

FIGURE 6. Reference configuration (a); true solution z_h (b).

In this case, the result of Algorithm 1 for $n \in \{0,\dots,400\}$ is depicted in Figure 7. The objective ratio attains the minimum $0,4\%$, and the shape error ratio 25%. We observe in Figure 7 (a) that the whole curve Σ_* is recovered well compared to the previous case of contacting faces.

On the basis of our numerical simulation, we conclude that the breaking line identification algorithm is consistent with the setup of destructive physical analysis (DPA).

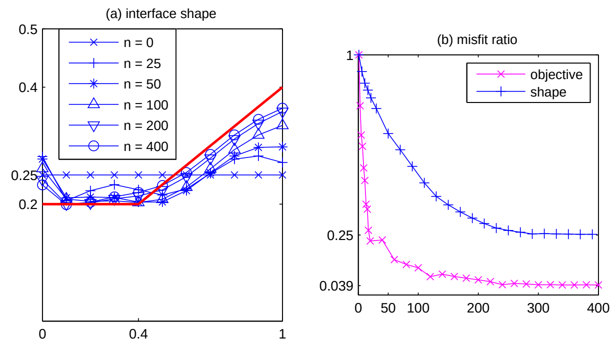

FIGURE 7. Iterations $\Sigma^{(n)}$ (a); misfit ratio (b).

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Appendix A. Proof of Lemma 2.3

Let us define the quadratic functional $\mathcal{E}^\star : I \times V^2 \mapsto \overline{\mathbb{R}}$ by

$$\mathcal{E}^\star(s, u_0, v) := \frac{1}{2}\langle(\mathcal{E}')^\star(s, u_0)v, v\rangle \quad \text{for } v \in V. \tag{A.1}$$

It is weakly lower semi-continuous and coercive due to $(E^\star 3)$, Gateaux-differentiable by $(E^\star 2)$, and $(\mathcal{E}^\star)'(s, u_0) = (\mathcal{E}')^\star(s, u_0)$. Adding to $\mathcal{E}^\star$ in (A.1) the linear term $\langle \mathcal{E}'(s, 0), v\rangle$, the above properties provide an argument $u_s \in V$ of the minimum:

$$\min_{v\in V}\big\{\mathcal{E}^\star(s, u_0, v) + \langle\mathcal{E}'(s, 0), v\rangle\big\}, \tag{A.2}$$

with an optimality condition in the form of the variational equation (2.12). Similarly, using (J1), there exists a minimizer $v_s \in V$ of the problem:

$$\min_{u\in V}\big\{\mathcal{E}^\star(s, u_0, u) - \langle\mathcal{J}'(s, Mu_s), Mu\rangle_{X^*,X}\big\}, \tag{A.3}$$

resulting in the adjoint equation (2.13). The uniqueness in (2.12) and (2.13) under the coercivity assumption $(E^\star 3)$ if $f^\star = 0$ follows in a standard way.

Indeed, inserting the explicit expression (2.9) of $\mathcal{L}$ into (2.10), we have the first inequality

$$\begin{aligned}\mathcal{J}(s, Mu_s) - \langle(\mathcal{E}')^\star(s, u_0)v, u_s\rangle - \langle\mathcal{E}'(s, 0), v\rangle \\ \le \mathcal{J}(s, Mu_s) - \langle(\mathcal{E}')^\star(s, u_0)v_s, u_s\rangle - \langle\mathcal{E}'(s, 0), v_s\rangle.\end{aligned}$$

After cancelling $\mathcal{J}(s, Mu_s)$ and testing with $v = v_s \pm w$ we obtain the variational equation (2.12). Conversely, (2.12) satisfies the first inequality of (2.10) as equality.

On the other side, the second inequality of (2.10) after cancelling the term $-\langle\mathcal{E}'(s, 0), v_s\rangle$ reads

$$\mathcal{J}(s, Mu_s) - \langle(\mathcal{E}')^\star(s, u_0)v_s, u_s\rangle \le \mathcal{J}(s, Mu) - \langle(\mathcal{E}')^\star(s, u_0)v_s, u\rangle.$$

Substituting here $u = u_s \pm rw$, dividing the results with r and passing $r \to 0$, by the virtue of differentiability of $\mathcal{J}$ assumed in (J1), this leads to the variational equation (2.13). Conversely, by the convexity assumption (J2) the necessary optimality condition (2.13) is sufficient for the minimum in the second inequality of (2.10) provided by u_s.

This proves that $(u_s, v_s) \in V^2$ is a saddle point to problem (2.10). The definition (2.11) of solution sets K^s, K_s implies that $(u_s, v_s) \in K^s \times K_s$ and satisfies the equality (2.14). This completes the proof of Lemma 2.3.

Appendix B. Proof of Lemma 2.4

We test the primal equation (2.12) with $v = u_s$, apply the Cauchy–Schwarz inequality, the coercivity ($E^\star 3$) with $u = u_s$, and the boundedness assumption (E4) to derive the upper bound

$$\text{(B.1)} \quad \underline{a}^\star \|u_s\|_V^2 \leq \langle (\mathcal{E}')^\star(s, u_0)u_s + f^\star, u_s\rangle = \langle f^\star - \mathcal{E}'(s, 0), u_s\rangle \leq (\overline{a} + \|f^\star\|_{V^\star})\|u_s\|_V.$$

Testing the adjoint equation (2.13) with $u = v_s$, from ($E^\star 3$) with $u = v_s$ and (J3) it follows similarly that

$$\text{(B.2)} \quad \underline{a}^\star \|v_s\|_V^2 \leq \langle (\mathcal{E}')^\star(s, u_0)v_s + f^\star, v_s\rangle = \langle \mathcal{J}'(s, Mu_s), Mv_s\rangle_{X^*,X} + \langle f^\star, v_s\rangle \leq \overline{a}_{\mathcal{J}} \|u_s\|_V \|Mv_s\|_X + \|f^\star\|_{V^\star}\|v_s\|_V.$$

We combine (B.1) and (B.2) together in the uniform in $s \in I$ estimate

$$\text{(B.3)} \quad \|u_s\|_V + \|v_s\|_V \leq \frac{1}{\underline{a}^\star}(\overline{a} + \|f^\star\|_{V^\star})\big(1 + \frac{\overline{a}_{\mathcal{J}}}{\underline{a}^\star}\|M\|\big) + \frac{1}{\underline{a}^\star}\|f^\star\|_{V^\star}.$$

Then there exist $s_k \to 0^+$, a subsequence of saddle points $(u_{s_k}, v_{s_k}) \in K^{s_k} \times K_{s_k}$ and an accumulation point $(u_0, v_0) \in V^2$ such that

$$\text{(B.4)} \quad (u_{s_k}, v_{s_k}) \rightharpoonup (u_0, v_0) \quad \text{weakly in } V^2 \text{ as } k \to \infty.$$

For $u = u_{s_k} - u_0$ in the coercivity inequality ($E^\star 3$) we have

$$\text{(B.5)} \quad \underline{a}^\star \|u_{s_k} - u_0\|_V^2 \leq \langle (\mathcal{E}')^\star(s_k, u_0)(u_{s_k} - u_0) + f^\star, u_{s_k} - u_0\rangle = \langle f^\star - \mathcal{E}'(s_k, 0) - (\mathcal{E}')^\star(s_k, u_0)u_0, u_{s_k} - u_0\rangle = \langle f^\star - \mathcal{E}'(s_k, u_0), u_{s_k} - u_0\rangle,$$

where (2.12) was tested with $v = u_{s_k} - u_0$, and (2.6) and property ($E^\star 2$) were used. Inserting $u = v_{s_k} - v_0$ into ($E^\star 3$) and using (2.13) with $u = v_{s_k} - v_0$ gives similarly

$$\text{(B.6)} \quad \underline{a}^\star \|v_{s_k} - v_0\|_V^2 \leq \langle \mathcal{J}'(s_k, Mu_{s_k}), M(v_{s_k} - v_0)\rangle_{X^*,X} + \langle f^\star - (\mathcal{E}')^\star(s_k, u_0)v_0, v_{s_k} - v_0\rangle.$$

Taking the limit as $k \to \infty$ in (B.5) and (B.6), we get (2.15) with the help of the weak convergence in (B.4) and the boundedness properties (E4), (J3), ($E^\star 4$) of $\mathcal{E}'$, $\mathcal{J}'$, $(\mathcal{E}')^\star$.

Finally, taking the limits in the primal (2.12) and adjoint (2.13) equations and using the strong convergence (2.15) and the continuity assumptions (E5), (J4) and ($E^\star 5$), this guarantees that the pair (u_0, v_0) solves (2.2) (due to identity (2.6)) and (2.13) at $s = 0$. Therefore, $(u_0, v_0) \in K^0 \times K_0$ which ends the proof of Lemma 2.4.

Appendix C. Proof of Lemma 2.5

From definition (2.9) of $\mathcal{L}$ and the variational equation (2.12), it follows straightforwardly that for $s \in I$:

$$\text{(C.1)} \quad l(s) = \mathcal{L}(s, u_0, u_s, v_s) = \mathcal{J}(s, Mu_s) - \langle (\mathcal{E}')^\star(s, u_0)v_s, u_s\rangle - \langle \mathcal{E}'(s, 0), v_s\rangle = \mathcal{J}(s, Mu_s) = j(s)$$

for the optimal values of the objective j in (2.16) and the Lagrange function l in (2.14). Next we prove the directional differentiability of the Lagrangian l at 0.. Then by (C.1) we have $\partial_+ j(0) = \partial_+ l(0)$ in (2.17).

We sketch the proof following [4, Chapter 10, Theorem 5.1]. For a test function $(u, v) = (u_0, v_0) \in K^0 \times K_0$, the saddle-point inequalities (2.10) at $s = s_k$ give:

$$\mathcal{L}(s_k, u_0, u_{s_k}, v_0) \leq \mathcal{L}(s_k, u_0, u_{s_k}, v_{s_k}) \leq \mathcal{L}(s_k, u_0, u_0, v_{s_k}). \tag{C.2}$$

Also we insert $(u, v) = (u_{s_k}, v_{s_k}) \in K^{s_k} \times K_{t+s_k}$ into (2.10) at $s = 0$:

$$\mathcal{L}(0, u_0, u_0, v_{s_k}) \leq \mathcal{L}(0, u_0, u_0, v_0) \leq \mathcal{L}(0, u_0, u_{s_k}, v_0). \tag{C.3}$$

Subtracting $l(0) = \mathcal{L}(0, u_0, u_0, v_0)$ from the left inequality (C.2) and using the right inequality (C.3), after division with s_k and applying the mean value theorem with $\underline{\alpha}_k \in (0, 1)$ leads to the inequalities

$$\frac{\mathcal{L}(s_k, u_0, u_{s_k}, v_{s_k}) - \mathcal{L}(0, u_0, u_0, v_0)}{s_k} \geq \frac{\mathcal{L}(s_k, u_0, u_{s_k}, v_0) - \mathcal{L}(0, u_0, u_0, v_0)}{s_k}$$
$$\geq \frac{\mathcal{L}(s_k, u_0, u_{s_k}, v_0) - \mathcal{L}(0, u_0, u_{s_k}, v_0)}{s_k} = \frac{\partial \mathcal{L}}{\partial s}(\underline{\alpha}_k s_k, u_0, u_{s_k}, v_0).$$

Upon taking the limit as $s_k \to 0^+$ by the virtue of the lower bound in assumption (L1) we obtain

$$\liminf_{s_k \to 0^+} \frac{\mathcal{L}(s_k, u_0, u_{s_k}, v_{s_k}) - \mathcal{L}(0, u_0, u_0, v_0)}{s_k} \geq \frac{\partial \mathcal{L}}{\partial s}(0, u_0, u_0, v_0). \tag{C.4}$$

On the other hand, subtracting $\mathcal{L}(0, u_0, u_0, v_0)$ from the right inequality (C.2), using the left inequality (C.3) and the mean value theorem with weights $\overline{\alpha}_k \in (0, 1)$ provides the following relations:

$$\frac{\mathcal{L}(s_k, u_0, u_{s_k}, v_{s_k}) - \mathcal{L}(0, u_0, u_0, v_0)}{s_k} \leq \frac{\mathcal{L}(s_k, u_0, u_0, v_{s_k}) - \mathcal{L}(0, u_0, u_0, v_0)}{s_k}$$
$$\leq \frac{\mathcal{L}(s_k, u_0, u_0, v_{s_k}) - \mathcal{L}(0, u_0, u_0, v_{s_k})}{s_k} = \frac{\partial \mathcal{L}}{\partial s}(\overline{\alpha}_k s_k, u_0, u_0, v_{s_k}).$$

Together with the upper bound in (L1) this leads to the upper estimate

$$\limsup_{s_k \to 0^+} \frac{\mathcal{L}(s_k, u_0, u_{s_k}, v_{s_k}) - \mathcal{L}(0, u_0, u_0, v_0)}{s_k} \leq \frac{\partial \mathcal{L}}{\partial s}(0, u_0, u_0, v_0). \tag{C.5}$$

Inequalities (C.4) and (C.5) prove the limit in (2.17).

Appendix D. Proof of Lemma 3.2

The modified quadratic functional $\mathcal{E}^\star_\varepsilon : I \times V^2 \mapsto \overline{\mathbb{R}}$ defined for $v \in V$ by

$$\mathcal{E}^\star_\varepsilon(s, u^\varepsilon_0, v) := \frac{1}{2}\langle (\mathcal{E}')^\star(s, u^\varepsilon_0)v, v\rangle + \frac{1}{2}\langle \beta^\star_\varepsilon(s, B(0)u^\varepsilon_0)B(s)v, B(s)v\rangle_H \tag{D.1}$$

is weakly lower semi-continuous and coercive due to ($E^\star 3$), (B3), and ($B^\star 3$). Using ($E^\star 2$) and ($B^\star 2$) its Gateaux derivative is given by

$$\langle (\mathcal{E}^\star_\varepsilon)'(s, u^\varepsilon_0)u, v\rangle = \langle (\mathcal{E}')^\star(s, u^\varepsilon_0)v, u\rangle + \langle \beta^\star_\varepsilon(s, B(0)u^\varepsilon_0)B(s)u, B(s)v\rangle_H.$$

Consequently, the variational equation (3.13) is an optimality condition for the minimizer $u^\varepsilon_s \in V$ of the following problem:

$$\min_{v \in V}\big\{\mathcal{E}^\star_\varepsilon(s, u^\varepsilon_0, v) + \langle \mathcal{E}'(s, 0), v\rangle + \langle \beta_\varepsilon(s, 0), B(s)v\rangle_H\big\}, \tag{D.2}$$

and the adjoint equation (3.14) provides an argument $v_s^\varepsilon \in V$ for

$$\min_{u\in V}\big\{\mathcal{E}_\varepsilon^\star(s,u_0^\varepsilon,v) - \langle \mathcal{J}'(s,Mu_s^\varepsilon),Mu\rangle_{X^*,X}\big\}. \tag{D.3}$$

The uniqueness assertion is similar to Lemma 2.3 and done by coercivity.

The left-hand side of the saddle-point formulation (3.11) is equivalent to the primal problem:

$$\begin{aligned}&-\langle(\mathcal{E}')^\star(s,u_0^\varepsilon)v,u_s^\varepsilon\rangle - \langle\beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)B(s)v,B(s)u_s^\varepsilon\rangle_H\\ &\quad-\langle\mathcal{E}'(s,0),v\rangle - \langle\beta_\varepsilon(s,0),B(s)v\rangle_H \le -\langle(\mathcal{E}')^\star(s,u_0^\varepsilon)v_s^\varepsilon,u_s^\varepsilon\rangle - \langle\mathcal{E}'(s,0),v_s^\varepsilon\rangle\\ &\qquad\qquad-\langle\beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)B(s)v_s^\varepsilon,B(s)u_s^\varepsilon\rangle_H - \langle\beta_\varepsilon(s,0),B(s)v_s^\varepsilon\rangle_H,\end{aligned}$$

which implies equation (3.13). The right-hand side

$$\begin{aligned}&\mathcal{J}(s,Mu_s^\varepsilon) - \langle(\mathcal{E}')^\star(s,u_0^\varepsilon)v_s^\varepsilon,u_s^\varepsilon\rangle - \langle\beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)B(s)v_s^\varepsilon,B(s)u_s^\varepsilon\rangle_H\\ &\qquad\le \mathcal{J}(s,Mu) - \langle(\mathcal{E}')^\star(s,u_0^\varepsilon)v_s^\varepsilon,u\rangle - \langle\beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)B(s)v_s^\varepsilon,B(s)u\rangle_H\end{aligned}$$

is equivalent to the adjoint equation (3.14) due to the convexity (J2). Then $(u_s^\varepsilon,v_s^\varepsilon)\in K_\varepsilon^s\times K_s^\varepsilon$ satisfies the saddle-point condition (3.15).

The proof of (3.15) is analogous to that of Lemma 2.4. By the coercivity $(E^\star 3)$, $(B^\star 3)$ and boundedness assumptions (E4), (B3), (B5), $(B^\star 4)$ we derive from equation (3.13)

$$\begin{aligned}&\underline{b}^\star\|u_s^\varepsilon\|_V^2 \le \underline{a}^\star\|u_s^\varepsilon\|_V^2 + \frac{\underline{b}^\star-\underline{a}^\star}{\underline{b}^2}\|B(s)u_s^\varepsilon\|_H^2 \le \langle(\mathcal{E}')^\star(s,u_0^\varepsilon)u_s^\varepsilon + f^\star,u_s^\varepsilon\rangle \\ &+\langle\beta_\varepsilon^\star(s,B(0)u_0^\varepsilon)B(s)u_s^\varepsilon + f_b^\star,B(s)u_s^\varepsilon\rangle_H = \langle f^\star - \mathcal{E}'(s,0),u_s^\varepsilon\rangle + \langle f_b^\star - \beta_\varepsilon(s,0),B(s)u_s^\varepsilon\rangle_H\\ &\qquad\le \big(\|f^\star\|_{V^\star} + \overline{a} + \overline{b}(\|f_b^\star\|_{H^\star} + \overline{b}_\varepsilon)\big)\|u_s^\varepsilon\|_V,\end{aligned} \tag{D.4}$$

and from the adjoint equation (3.14) using (J3) we get the upper bound

$$\begin{aligned}\underline{b}^\star\|v_s^\varepsilon\|_V^2 &\le \langle\mathcal{J}'(s,Mu_s^\varepsilon),Mv_s^\varepsilon\rangle_{X^*,X}\langle f^\star,v_s^\varepsilon\rangle + \langle f_b^\star,B(s)v_s^\varepsilon\rangle_H\\ &\le \big(\overline{a}_{\mathcal{J}}\|M\|\|u_s^\varepsilon\|_V + \|f^\star\|_{V^\star} + \overline{b}\|f_b^\star\|_{H^\star}\big)\|v_s^\varepsilon\|_V.\end{aligned} \tag{D.5}$$

By the boundedness of $(u_s^\varepsilon,v_s^\varepsilon)$, there exists a subsequence $(u_{s_k}^\varepsilon,v_{s_k}^\varepsilon)\in K_\varepsilon^{s_k}\times K_{s_k}^\varepsilon$ and an accumulation point $(u_0^\varepsilon,v_0^\varepsilon)\in V^2$ such that

$$(u_{s_k}^\varepsilon,v_{s_k}^\varepsilon) \rightharpoonup (u_0^\varepsilon,v_0^\varepsilon)\quad\text{weakly in } V^2 \text{ as } s_k\to 0^+. \tag{D.6}$$

We test equation (3.13) with $v = u_{s_k}^\varepsilon - u_0^\varepsilon$ and in analogy to (D.4) we find using identity (2.6):

$$\begin{aligned}&\underline{b}^\star\|u_{s_k}^\varepsilon - u_0^\varepsilon\|_V^2 \le \langle(\mathcal{E}')^\star(s_k,u_0^\varepsilon)(u_{s_k}^\varepsilon - u_0^\varepsilon) + f^\star,u_{s_k}^\varepsilon - u_0^\varepsilon\rangle\\ &+\langle\beta_\varepsilon^\star(s_k,B(0)u_0^\varepsilon)B(s_k)(u_{s_k}^\varepsilon - u_0^\varepsilon) + f_b^\star,B(s_k)(u_{s_k}^\varepsilon - u_0^\varepsilon)\rangle_H = \langle f^\star - \mathcal{E}'(s_k,u_0^\varepsilon),u_{s_k}^\varepsilon - u_0^\varepsilon\rangle\\ &\qquad+\langle f_b^\star - \beta_\varepsilon(s_k,0) - \beta_\varepsilon^\star(s_k,B(0)u_0^\varepsilon)B(s_k)u_0^\varepsilon,B(s_k)(u_{s_k}^\varepsilon - u_0^\varepsilon)\rangle_H.\end{aligned} \tag{D.7}$$

The adjoint equation (3.14) for $u = v_{s_k}^\varepsilon - v_0^\varepsilon$ gives

$$\begin{aligned}&\underline{b}^\star\|v_{s_k}^\varepsilon - v_0^\varepsilon\|_V^2 \le \langle\mathcal{J}'(s_k,Mu_{s_k}^\varepsilon),M(v_{s_k}^\varepsilon - v_0^\varepsilon)\rangle_{X^*,X}\\ &+\langle f^\star - (\mathcal{E}')^\star(s_k,u_0^\varepsilon)v_0^\varepsilon,v_{s_k}^\varepsilon - v_0^\varepsilon\rangle + \langle f_b^\star - \beta_\varepsilon^\star(s_k,B(0)u_0^\varepsilon)B(s_k)v_0^\varepsilon,B(s_k)(v_{s_k}^\varepsilon - v_0^\varepsilon)\rangle_H.\end{aligned} \tag{D.8}$$

Passing $k\to\infty$ in (D.7) and (D.8) with the help of weak convergence in (D.6) and recalling boundedness of $B(s)$ (3.16) follows. The limit as $s\to 0^+$ in equations (D.7) and (D.8) due to strong convergence (3.16) and continuity properties (E5), (J4), $(E^\star 5)$, (B4), (B6) and $(B^\star 5)$ agrees with the solution $(u_0^\varepsilon, v_0^\varepsilon) \in K_\varepsilon^0 \times K_0^\varepsilon$ to (3.7) (due to identity (3.9)) and to (3.14) at $s = 0$. This proves Lemma 3.2.

Appendix E. Proof of Theorem 3.2

Passing $s_k \to 0^+$ due to the strong convergence (3.16) we refine the estimates (D.4) as follows. Using the lower bound in (3.5) and (E2), from (3.7) tested with $v = u_0^\varepsilon$ we get:

$$\text{(E.1)}\quad \underline{a}\|u_0^\varepsilon\|_V^2 + \frac{1}{\varepsilon}\|[B(0)u_0^\varepsilon]^-\|_H^2 \le \langle \beta_\varepsilon(0, B(0)u_0^\varepsilon), B(0)u_0^\varepsilon\rangle_H \\ + \langle \mathcal{E}'(0, u_0^\varepsilon), u_0^\varepsilon\rangle + \langle f, u_0^\varepsilon\rangle + \varepsilon\underline{\beta} \le \|f\|_{V^\star}\|u_0^\varepsilon\|_V + \varepsilon\underline{\beta},$$

which is uniform in $\varepsilon \in (0, \varepsilon_0)$. From (D.5) as $s_k \to 0^+$ it follows that

$$\text{(E.2)}\qquad \underline{b}^\star\|v_0^\varepsilon\|_V \le \overline{a}_{\mathcal{J}}\|M\|\|u_0^\varepsilon\|_V + \|f^\star\|_{V^\star} + \overline{b}\|f_b^\star\|_{H^\star}.$$

Hence, there exists a subsequence $\varepsilon_k \to 0$ and a weak accumulation point $(u_0, v_0) \in V^2$ such that $[B(0)u_0]^- = 0$ since $\|[B(0)u_0^{\varepsilon_k}]^-\|_H \to 0$, and

$$\text{(E.3)}\qquad (u_0^{\varepsilon_k}, v_0^{\varepsilon_k}) \rightharpoonup (u_0, v_0) \text{ weakly in } V^2 \text{ as } k \to \infty.$$

Taking the limit in (3.7) due to the convergence (E.3) and (E3), according to the surjectivity in (B8) we determine $\lambda_0 \in \tilde{H}^\star$ such that

$$\text{(E.4)}\quad \lim_{\varepsilon_k \to 0}\langle \beta_{\varepsilon_k}(0, B(0)u_0^{\varepsilon_k}), B(0)v\rangle_H = -\lim_{\varepsilon_k \to 0}\langle \mathcal{E}'(0, u_0^{\varepsilon_k}), v\rangle \\ = -\langle \mathcal{E}'(0, u_0), v\rangle =: \langle \lambda_0, B(0)v\rangle_{\tilde{H}^*,\tilde{H}} \quad \text{for } v \in V.$$

This implies that $u_0 \in V$ is a solution to the variational equation (3.25) and establishes the weak convergence

$$\text{(E.5)}\qquad \beta_{\varepsilon_k}(0, B(0)u_0^{\varepsilon_k}) \rightharpoonup \lambda_0 \text{ weakly in } \tilde{H}^\star \text{ as } k \to \infty.$$

The space $\tilde{H}$ has the order relation of H. Consequently $\lambda_0 \le 0$ because of (3.5). In particular, $\langle \lambda_0, B(0)u_0\rangle_{\tilde{H}^*,\tilde{H}} \le 0$ for $B(0)u_0 \ge 0$. On the other hand, by virtue of assumption (B7) and (E.5) the strong convergence holds:

$$\text{(E.6)}\qquad B(0)u_0^{\varepsilon_k} \to B(0)u_0 \text{ strongly in } H \text{ as } k \to \infty.$$

Using (3.5) and taking $\varepsilon_k \to 0$ in $\langle \beta_{\varepsilon_k}(0, B(0)u_0^{\varepsilon_k}), B(0)u_0^{\varepsilon_k}\rangle_H \ge -\varepsilon_k\underline{\beta}$ provides the opposite inequality $\langle \lambda_0, B(0)u_0\rangle_{\tilde{H}^*,\tilde{H}} \ge 0$, which together ensures the complementarity relations (3.23). The variational equation (3.25) together with (3.23) is equivalent to the variational inequality (3.4).

By the identity (2.6) at $s = 0$ equation (3.25) is equivalent to

$$\langle (\mathcal{E}')^\star(0, u_0)v, u_0\rangle + \langle \mathcal{E}'(0, 0), v\rangle + \langle \lambda_0, B(0)v\rangle_{\tilde{H}^*,\tilde{H}} = 0 \quad \text{for all } v \in V,$$

which yields the first order necessary and sufficient optimality condition for the unconstrained, primal limit problem (3.21).

Applying (E.3) and assumptions (J5), $(E^\star 6)$, (B8), the limit of the adjoint equation (3.20) determines $\mu_0 \in \tilde{H}^\star$ such that

$$\text{(E.7)}\quad \lim_{\varepsilon_k \to 0}\langle \beta_{\varepsilon_k}^\star(0, B(0)u_0^{\varepsilon_k})B(0)v_0^{\varepsilon_k}, B(0)u\rangle_H \\ = \lim_{\varepsilon_k \to 0}\langle \mathcal{J}'(0, Mu_0^{\varepsilon_k}), Mu\rangle_{X^*,X} - \lim_{\varepsilon_k \to 0}\langle (\mathcal{E}')^\star(0, u_0^{\varepsilon_k})v_0^{\varepsilon_k}, u\rangle \\ = \langle \mathcal{J}'(0, Mu_0), Mu\rangle_{X^*,X} - \langle (\mathcal{E}')^\star(0, u_0)v_0, u\rangle =: \langle \mu_0, B(0)u\rangle_{\tilde{H}^*,\tilde{H}} \quad \text{for } u \in V.$$

From (E.7) we conclude the existence of a solution $v_0 \in V$ to the limit adjoint equation (3.26) and the $\star$-weak convergence (3.27). Applying the convergences (E.5) and (3.27) to the identity (3.9) at $s = 0$, in the limit the compatibility condition (3.24) follows. Equation (3.26) is the necessary and sufficient optimality condition for the adjoint limit problem (3.22). The proof of Theorem 3.2 is complete.

Appendix F. Proof of Theorem 4.1

Using inequalities $\|[\![u]\!]\|^2_{L^2(\Sigma_t)} \leq 2\|u\|^2_{L^2(\partial\Omega_t^\pm)}$ and (4.13), (4.14) we estimate from below $\langle(\mathcal{E}')^\star(0, u_t^\varepsilon)u, u\rangle$ in (4.30) as

$$\text{(F.1)}\quad \mu_{\rm L}\int_{\Omega_t^\pm} |\nabla u|^2\,dx + \int_{\Sigma_t}\int_0^1 \alpha''(0, [\![ru_t^\varepsilon]\!])[\![u]\!]^2\,drdS_x \\ \geq \mu_{\rm L}K_{\rm P}\|u\|^2_{H^1(\Omega_t^\pm)} - K_{\alpha 2}\|[\![u]\!]\|^2_{L^2(\Sigma_t)} \geq \underline{a}^\star\|u\|^2_{H^1(\Omega_t^\pm)}.$$

Then (4.36) provides the coercivity property $(E^\star 3)$ with u_t^ε replacing u_0.

As $s \to 0$, by the mean value theorem there exists $r(s) \in [0, 1]$ such that from (4.24), (4.25) it follows that $\phi_s = x + s\Lambda|_{t+rs}$ and the expansions (see e.g. [34, Chapter 2]):

$$\text{(F.2)}\quad z \circ \phi_s = z + s\Lambda|_{t+rs}^\top \nabla z, \quad \nabla\phi_s^{-1} \circ \phi_s = I - s\nabla\Lambda|_{t+rs}, \\ J_s = 1 + s\,\mathrm{div}\Lambda|_{t+rs}, \quad \omega_s = 1 + s\,\mathrm{div}_{\tau_t}\Lambda|_{t+rs}$$

for $u \in V(\Omega_t)$, and $\mathrm{div}_{\tau_t}\Lambda$ defined in (4.38). Inserting (F.2) into the perturbed Lagrangian (4.35) we derive its expansion in the first argument:

$$\text{(F.3)}\qquad \mathcal{L}^\varepsilon(s, u_t^\varepsilon, u, v; \Omega_t) = \mathcal{L}^\varepsilon(0, u_t^\varepsilon, u, v; \Omega_t) + s\frac{\partial\mathcal{L}^\varepsilon}{\partial s}(rs, u_t^\varepsilon, u, v; \Omega_t)$$

with the partial derivative $\partial\mathcal{L}^\varepsilon/\partial s : I \times V(\Omega_t)^3 \mapsto \mathbb{R}$ in (F.3), which is a continuous function and is given by

$$\text{(F.4)}\quad \frac{\partial\mathcal{L}^\varepsilon}{\partial s}(s, u_t^\varepsilon, u, v) := \int_{\Gamma_t^{\rm O}} \Big(\frac{1}{2}\mathrm{div}_{\tau_t}\Lambda|_{t+s}(u-z)^2 - \Lambda|_{t+s}^\top\nabla z(u-z)\Big)dS_x \\ -\mu_{\rm L}\int_{\Omega_t^\pm}(\nabla u)^\top(\mathrm{div}\Lambda|_{t+s} - \nabla\Lambda|_{t+s} - \nabla\Lambda|_{t+s}^\top)\nabla v\,dx + \int_{\Gamma_t^{\rm N}}(\mathrm{div}_{\tau_t}\Lambda|_{t+s}g + \Lambda|_{t+s}^\top\nabla g)v\,dS_x \\ + \int_{\Sigma_t}\mathrm{div}_{\tau_t}\Lambda|_{t+s}\big\{\rho - \Big(\int_0^1 [\alpha'' + \beta'_\varepsilon](0, [\![ru_t^\varepsilon]\!])[\![u]\!]\,dr + [\alpha' + \beta_\varepsilon](0,0)\Big)[\![v]\!]\big\}\,dS_x.$$

Here we recall the identity when $u = u_t^\varepsilon$:

$$\text{(F.5)}\qquad \int_0^1 [\alpha'' + \beta'_\varepsilon](0, [\![ru_t^\varepsilon]\!])[\![u_t^\varepsilon]\!]\,dr + [\alpha' + \beta_\varepsilon](0,0) = [\alpha' + \beta_\varepsilon](0, [\![u_t^\varepsilon]\!]).$$

With the help of (F.1), (F.3) we check properties (E1)–(E5), (J1)–(J4), $(E^\star 1)$–$(E^\star 5)$, (B1)–(B6), $(B^\star 1)$–$(B^\star 5)$, (L1)–(L3) with u_t^ε replacing u_0 in Theorem 3.1. This proves the assertion of Theorem 4.1.

Appendix G. Proof of Theorem 4.2

We integrate by parts the domain integral from (4.37):

$$I(\Omega_t^\pm) := -\mu_{\rm L} \int_{\Omega_t^\pm} (\nabla u_t^\varepsilon)^\top (\mathrm{div}\Lambda - \nabla\Lambda - \nabla\Lambda^\top) \nabla v_t^\varepsilon \, dx$$
$$= \mu_{\rm L} \int_{\Omega_t^\pm} \big((\Lambda^\top \nabla u_t^\varepsilon) \Delta v_t^\varepsilon + (\Lambda^\top \nabla v_t^\varepsilon) \Delta u_t^\varepsilon \big) dx$$
$$- \mu_{\rm L} \int_{\partial\Omega_t^\pm} \Lambda^\top \Big(n_t^\pm (\nabla u_t^\varepsilon)^\top \nabla v_t^\varepsilon - \nabla u_t^\varepsilon ((n_t^\pm)^\top \nabla v_t^\varepsilon - \nabla v_t^\varepsilon ((n_t^\pm)^\top \nabla u_t^\varepsilon \Big) \, dS_x$$

and since $\Delta u_t^\varepsilon = \Delta v_t^\varepsilon = 0$ in $\Omega_t^\pm$:

$$I(\Omega_t^\pm) = \mu_{\rm L} \int_{\Sigma_t} \Lambda^\top \Big(\nu_t [\![(\nabla u_t^\varepsilon)^\top \nabla v_t^\varepsilon]\!] - [\![\nabla u_t^\varepsilon (\nu^\top \nabla v_t^\varepsilon)]\!] - [\![\nabla v_t^\varepsilon (\nu^\top \nabla u_t^\varepsilon)]\!] \Big) \, dS_x$$
$$+ \mu_{\rm L} \int_{\Gamma_t^{\rm D} \cup \Gamma_t^{\rm N}} \Lambda^\top \big(\nabla u_t^\varepsilon (n_t^\top \nabla v_t^\varepsilon) + \nabla v_t^\varepsilon (n_t^\top \nabla u_t^\varepsilon) \big) \, dS_x.$$

Using the boundary conditions for $(u_t^\varepsilon, v_t^\varepsilon)$ from (4.16), (4.40), it follows that $\tau_t^\top \nabla u_t^\varepsilon = \tau_t^\top \nabla v_t^\varepsilon = 0$ at $\Gamma_t^{\rm D} \setminus \Sigma_t$. Decomposing $\mathcal{D}_1 = (n_t^\top \mathcal{D}_1) n_t + (\tau_t^\top \mathcal{D}_1) \tau_t$ in (4.43) gives

$$I(\Omega_t^\pm) = \int_{\Sigma_t} \Lambda^\top i_{\Sigma_t} \, dS_x + \int_{\Gamma_t^{\rm D}} (n_t^\top \Lambda)(n_t^\top \mathcal{D}_1) \, dS_x + (\tau_t^\top \Lambda)(\tau_t^\top [\![\mathcal{D}_1]\!])_{\partial\Gamma_t^{\rm D} \cap \Sigma_t}$$
$$+ \int_{\Gamma_t^{\rm N}} (\Lambda^\top \nabla v_t^\varepsilon) g \, dS_x + \int_{\Gamma_t^{\rm O}} (\Lambda^\top \nabla u_t^\varepsilon)(u_t^\varepsilon - z) \, dS_x, \tag{G.1}$$

where the integrand along Σ_t in (G.1) is expressed as

$$i_{\Sigma_t} := \nu_t \mu_{\rm L} [\![(\nabla u_t^\varepsilon)^\top \nabla v_t^\varepsilon]\!] - [\![\nabla v_t^\varepsilon]\!] \, [\alpha' + \beta_\varepsilon](0, [\![u_t^\varepsilon]\!])$$
$$- [\![\nabla u_t^\varepsilon]\!] \int_0^1 [\alpha'' + \beta_\varepsilon'](0, [\![r u_t^\varepsilon]\!]) [\![v_t^\varepsilon]\!] \, dr = \nu_t \mu_{\rm L} [\![(\nabla u_t^\varepsilon)^\top \nabla v_t^\varepsilon]\!] - \nabla p_\varepsilon - q_\varepsilon, \tag{G.2}$$

with the notation (4.44) for q_ε and p_ε. Here the gradient is given by

$$\nabla p_\varepsilon = [\![\nabla v_t^\varepsilon]\!] [\alpha' + \beta_\varepsilon](0, [\![u_t^\varepsilon]\!]) + [\![\nabla u_t^\varepsilon]\!] [\alpha'' + \beta_\varepsilon'](0, [\![u_t^\varepsilon]\!]) [\![v_t^\varepsilon]\!].$$

By the virtue of (4.45) and (G.2) and exploiting the calculus $\nabla(\xi\eta) = \nabla\xi^\top \eta + \nabla\eta^\top \xi$ we rearrange the terms in (4.37):

$$\partial_+ j(\varepsilon, 0) = \frac{1}{2} \int_{\Gamma_t^{\rm O}} \big(\mathrm{div}_{\tau_t}\Lambda \, (u_t^\varepsilon - z)^2 + \Lambda^\top \nabla((u_t^\varepsilon - z)^2) \big) dS_x$$
$$+ \int_{\Sigma_t} \big(\mathrm{div}_{\tau_t}\Lambda \, (\rho - p_\varepsilon) + \Lambda^\top \big(\nu_t \mu_{\rm L} [\![(\nabla u_t^\varepsilon)^\top \nabla v_t^\varepsilon]\!] - \nabla p_\varepsilon - q_\varepsilon \big) \big) \, dS_x$$
$$+ \int_{\Gamma_t^{\rm N}} \big(\mathrm{div}_{\tau_t}\Lambda (g v_t^\varepsilon) + \Lambda^\top \nabla (g v_t^\varepsilon) \big) \, dS_x + \int_{\Gamma_t^{\rm D}} (n_t^\top \Lambda)(n_t^\top \mathcal{D}_1) \, dS_x + (\tau_t^\top \Lambda)(\tau_t^\top [\![\mathcal{D}_1]\!])_{\partial\Gamma_t^{\rm D} \cap \Sigma_t}. \tag{G.3}$$

The integration along a boundary $\Gamma_t \subset \partial\Omega_t^\pm$ is given by the formula (see e.g. [34, (2.125)]) for smooth $p \in H^2(\Omega_t^\pm)$:

$$\int_{\Gamma_t} (\mathrm{div}_{\tau_t}\Lambda \, p + \Lambda^\top \nabla p) \, dS_x = \int_{\Gamma_t} (n_t^\top \Lambda)(\varkappa_t p + n_t^\top \nabla p) \, dS_x + (\tau_t^\top \Lambda) p|_{\partial\Gamma_t}, \tag{G.4}$$

where the curvature $\varkappa_t = \mathrm{div}_{\tau_t} n_t$, the normal n_t and tangential τ_t vectors at $\partial\Gamma_t$ are positively oriented. Applying (G.4) to (G.3) and decomposing the velocity (4.41), we conclude the Hadamard representation (4.42)–(4.44).

The substitution of (4.45) into (4.42) implies that $\partial_+ j(\varepsilon, 0) < 0$.