INFINITE HORIZON OPTIMAL CONTROL PROBLEMS WITH DISCOUNT FACTOR ON THE STATE. PART II: ANALYSIS OF THE CONTROL PROBLEM *

EDUARDO CASAS† AND KARL KUNISCH‡

Abstract. This is the continuation of our work on infinite horizon optimal control problems with a discount factor on the state variable and nonlinear partial differential equations as constraints. Existence of a solution is proven, and first as well as second order optimality conditions are derived. They are used to analyze the approximation of the infinite horizon problem by finite horizon problems.

AMS subject classifications. 35K58, 49J20, 49J52, 49K20.

Key words. Semilinear parabolic equations, infinite horizon, discount factor, optimal control, sparse controls

1. Introduction. In this second part of our work on optimal control problems with discount factor on the state in the cost functional we focus on optimization theoretic aspects. In the first part [5] we analyzed the well-posedness of the controlled equation and differentiability properties of the control-to-state mapping. Concretely we investigate the problem

$$\text{(P)}\quad \min_{u\in\mathcal{U}_{ad}} J(u) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-\sigma t}\|y_u - y_d\|^2_{L^2(\Omega)}\,dt + \frac{\nu}{2}\int_0^\infty \|u\|^2_{L^2(\omega)}\,dt + \gamma\int_0^\infty \|u\|_{L^2(\omega)}\,dt,$$

where

$$\mathcal{U}_{ad} = \{u \in L^2(0,\infty;L^2(\omega)) : u_a \le u(x,t) \le u_b \ \text{ for a.a. } (x,t) \in \omega\times(0,\infty)\},$$

$-\infty \le u_a \le 0 \le u_b \le +\infty$ with $u_a < u_b$, $\sigma > 0$, $\nu > 0$, and $\gamma \ge 0$. Here y_u denotes the solution of the following parabolic equation:

$$\left\{\begin{array}{l} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) = g + u\chi_\omega \ \text{ in } Q = \Omega\times(0,\infty),\\ \partial_n y = 0 \ \text{ on } \Sigma = \Gamma\times(0,\infty),\ y(0) = y_0 \ \text{ in } \Omega, \end{array}\right. \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $1 \le n \le 3$, with a Lipschitz boundary Γ $g \in L^\infty(0,\infty;L^2(\Omega))$, ω is a subdomain of Ω, χ_ω denotes the characteristic function of ω, $a \in L^\infty(\Omega)$, $0 \le a \not\equiv 0$, and $y_0 \in H^1(\Omega)$. The symbol $u\chi_\omega$ is defined as follows:

$$(u\chi_\omega)(x,t) = \left\{\begin{array}{ll} u(x,t) & \text{if } (x,t) \in Q_\omega = \omega\times(0,\infty),\\ 0 & \text{otherwise.}\end{array}\right.$$

The target y_d is assumed to belong to $L^\infty(0,\infty;L^2(\Omega))$. The exponent $\sigma > 0$ is known as the discount factor. The last term of the cost functional is included to promote sparsity in time of the optimal controls.

Remark 1.1. *The choice $u_a \le 0 \le u_b$ is needed because if $u_a > 0$ or $u_b < 0$, then $\mathcal{U}_{ad} = \emptyset$.*

*The first author was supported by MCIN/ AEI/10.13039/501100011033/ under research project PID2020-114837GB-I00.

†Departamento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39005 Santander, Spain (eduardo.casas@unican.es).

‡Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, A-8010 Graz, Austria (karl.kunisch@uni-graz.at).

As pointed out in [5] there are only very few papers in which infinite horizon optimal control problems are investigated systematically. This is particularly true for problems involving partial differential equations. For references concerned with ordinary differential equations we refer to our references in [5]. Our own investigations started with [3], where infinite horizon problems with L^1 sparsity enhancing terms are investigated for stabilization problems, i.e. $y_d = 0$. The nonlinearities considered in that paper are of polynomial type and it is verified that for sufficiently large t, once the trajectory reaches a sufficiently small neighborhood of a stable equilibrium, the associated optimal control switches off, as expected due to the sparsifying term in the cost-functional. In [4] infinite horizon problems of tracking type are considered, under quite general assumptions on the nonlinearity f. The optimization problem is investigated under the assumption of existence of at least one optimal control, which is guaranteed, for instance, for sufficiently small initial conditions. Optimality conditions are derived without recourse to the regularity of the control to state mapping. The optimal states themselves are at least in $L^2(0,\infty;L^2(\Omega))$. In the present paper, on the contrary, due to the discounted term, the optimal states are allowed to be much more general, they need not lie in $L^2(0,\infty;L^2(\Omega))$. The nonlinearities are of polynomial nature or are globally Lipschitz continuous. The control to state mapping is well-defined and C^2 regular on all of the control space.

In [4] the nucleus of the proof technique, for the optimality conditions for instance, rested on the approximation of the infinite horizon by finite horizon problems. In the present paper we need not rely on this rather technical approach, rather the first and second order optimality conditions can be proved directly for the infinite horizon problem. However, we still address the approximation of (P) by means of finite horizon problems, and even derive a convergence rate estimate with respect to the time horizon, by exploiting sufficient second order optimality conditions. The importance of such an estimate, besides intrinsic interest, lies in the fact that numerical approaches many times rely on computations carried out for 'sufficiently' large time horizons. This suggests to investigate the error which is made by cutting off the time interval.

The paper is structured as follows. In Section 2 selected results from [5] are recalled and existence of a solution to (P) is verified. Differentiability properties of the cost functional on the basis of an appropriately defined adjoint equation are investigated in Section 3. In our work the transversality condition, known from Pontryagin's maximum principle, corresponds to the behavior of the adjoint state at ∞ here. For ordinary differential equations it has been analyzed in detail in [1]. Necessary and sufficient optimality conditions are contained in Section 4. The last section is devoted to the approximation of (P) by means of finite horizon problems.

Assumptions on f and notation.

For the nonlinear term in state equation $f : \mathbb{R} \longrightarrow \mathbb{R}$ we assume that $f = f_1 + f_2$ such that f_1 is a polynomial of odd degree $2m+1$ with a positive leading coefficient, $0 \le m \le 1$ if $n = 3$, and $m \ge 0$ arbitrary integer if $n = 2$, and $f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2 function satisfying

$$f_1(0) = f_2(0) = 0 \text{ and } \exists L_f > 0 : |f_2'(s)| + |f_2''(s)| \le L_f \ \forall s \in \mathbb{R}. \tag{1.2}$$

As established in [5], the assumptions on f imply that

$$\exists \Lambda_f \ge 0 \text{ such that } f'(s) \ge -\Lambda_f \quad \forall s \in \mathbb{R}, \tag{1.3}$$

$$\exists M_f \text{ such that } f'(s) > 0 \text{ and } f(s)s \ge 0 \quad \forall |s| \ge M_f. \tag{1.4}$$

Given a real number $\alpha \in \mathbb{R}$ and $p \in [1,\infty]$, $L^p_\alpha(Q)$ denotes the space of measurable functions $\phi : Q \longrightarrow \mathbb{R}$ satisfying

$$\|\phi\|_{L^p_\alpha(Q)} = \Big(\int_0^\infty \mathrm{e}^{-\alpha t}\|\phi(t)\|^p_{L^p(\Omega)}\,dt\Big)^{\frac{1}{p}} < \infty \quad \text{if } p < \infty,$$
$$\|\phi\|_{L^\infty_\alpha(Q)} = \operatorname*{ess\,sup}_{(x,t)\in Q} \mathrm{e}^{-\frac{\alpha}{2}t}|\phi(x,t)| < \infty.$$

With $L^2_\alpha(0,\infty;H^1(\Omega))$ and $C_\alpha([0,\infty);H^1(\Omega))$ we denote the Hilbert and Banach spaces of measurable, respectively continuous, functions $y : [0,\infty) \longrightarrow H^1(\Omega)$ endowed with the norms

$$\|y\|_{L^2_\alpha(0,\infty;H^1(\Omega))} = \Big(\int_0^\infty \mathrm{e}^{-\alpha t}\|y(t)\|^2_{H^1(\Omega)}\,dt\Big)^{\frac{1}{2}},$$
$$\|y\|_{C_\alpha([0,\infty);H^1(\Omega))} = \sup_{t\in[0,\infty)} \mathrm{e}^{-\frac{\alpha}{2}t}\|y(t)\|_{H^1(\Omega))}.$$

We also define $H^1_\alpha(Q)$ as the space of functions $y \in L^2_\alpha(0,\infty;H^1(\Omega))$ such that $\frac{\partial y}{\partial t} \in L^2_\alpha(Q)$. This is a Hilbert space for the norm

$$\|y\|_{H^1_\alpha(Q)} = \Big(\|y\|^2_{L^2_\alpha(0,\infty;H^1(\Omega))} + \Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2_\alpha(Q)}\Big)^{\frac{1}{2}}.$$

Finally, we set $Y_\alpha = H^1_\alpha(Q) \cap C_\alpha([0,\infty);H^1(\Omega))$. The next estimate was proved in [5]

$$\|y\|_{L^{4m+2}_\alpha(Q)} \le \begin{cases} C_m\|y\|_{Y_{\frac{\alpha}{4m}}} & \text{if } \alpha \ge 0,\\ C_m\|y\|_{Y_{\frac{\alpha}{2}}} & \text{if } \alpha < 0, \end{cases} \tag{1.5}$$

for a constant C_m. The following well known inequality will be useful all along this paper

$$C_a\|z\|_{H^1(\Omega)} \le \Big(\int_\Omega (|\nabla z|^2 + az^2)\,dx\Big)^{\frac{1}{2}} \quad \forall z \in H^1(\Omega). \tag{1.6}$$

2. Existence of a Solution for (P)**.** In this section, we will prove the existence of at least one solution to problem (P). Before we summarize some results concerning the state equation. Following [5, Definition 2.1], a function y is called a solution of (1.1) if it belongs to $L^2_{loc}(0,\infty;H^1(\Omega)) \cap C_{loc}([0,\infty);L^2(\Omega))$, $f(y) \in L^2_{loc}(0,\infty;L^2(\Omega))$, and it satisfies

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) = g + u\chi_\omega & \text{in } Q_T = \Omega \times (0,T),\\ \partial_n y = 0 \ \text{ on } \Sigma_T = \Gamma \times (0,T), \quad y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

for every $0 < T < \infty$.

THEOREM 2.1. *For every $u \in L^2(Q_\omega)$ equation* (1.1) *has a unique solution $y_u \in Y_\alpha$ for every $\alpha > 0$. Moreover, the following properties hold*

$$\lim_{T\to\infty} \mathrm{e}^{-\alpha T}\|y_u(T)\|_{H^1(\Omega)} = 0, \tag{2.2}$$

$$\|f(y_u)\|_{L^2_\alpha(Q)} + \|y_u{}^{2m+1}\|_{L^2_\alpha(Q)} + \|y_u\|_{Y_\alpha} \le C\Big(\|g\|_{L^\infty(0,\infty;L^2(\Omega))} + \|u\|_{L^2(Q_\omega)} + \|y_0\|^{m+1}_{H^1(\Omega)} + 1\Big), \tag{2.3}$$

where C is independent of g, u, and y_0.

The reader is referred to [5, Theorem 2.4] for the proof of this theorem. The continuous dependence of the state with respect to the control is established in the next lemma.

LEMMA 2.2. *Let $\{u_k\}_{k=1}^\infty$ be a sequence in $L^2(Q_\omega)$ with associated states $\{y_k\}_{k=1}^\infty$. If $u_k \rightharpoonup u$ in $L^2(Q_\omega)$, then for every $\alpha > 0$ the convergences $y_k \rightharpoonup y_u$ in $H^1_\alpha(Q)$ and $f(y_k) \rightharpoonup f(y)$ in $L^2_\alpha(Q)$ hold.*

Proof. From the boundedness of $\{u_k\}_{k=1}^\infty$ and (2.3) we deduce the existence of a subsequence, denoted in the same way, such that $y_k \rightharpoonup y$ in Y_α and $f(y_k) \rightharpoonup \phi$ in $L^2_\alpha(Q)$ for every $\alpha > 0$. Let $T > 0$ be arbitrary. From the compactness of the embedding $H^1(Q_T) \subset L^2(Q_T)$ we infer the existence of a further subsequence such that

$$y_k \to y \ \text{ in } L^2(Q_T) \ \text{ and } \ y_k(x,t) \to y(x,t) \ \text{ a.e. in } \ Q_T.$$

Using the above pointwise convergence we deduce that $\phi = f(y)$ and, hence, $f(y_k) \rightharpoonup f(y)$ in $L^2_\alpha(Q)$. Now, we prove that $y = y_u$. It is easy to pass to the limit weakly in the state equation (2.1) satisfied by (y_k, u_k) and to deduce that (y, u) satisfies the equation in the variational sense in Q_T for every $T > 0$. Moreover, from the continuity of the embedding $Y_\alpha \subset C_\alpha([0,\infty); H^1(\Omega))$ we have that $y_0 = y_k(0) \to y(0)$ in $L^2(\Omega)$, hence $y = y_u$. From the uniqueness of the solution of (1.1) we deduce that the whole sequence $\{y_k\}_{k=1}^\infty$ converges to y_u. ☐

THEOREM 2.3. *Problem* (P) *admits at least one solution.*

Proof. Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence for (P). Since $J(u_k) \le J(0)$ for every k large enough (unless $u \equiv 0$ is already an optimal control), the boundedness of $\{u_k\}_{k=1}^\infty$ in $L^2(Q_\omega)$ follows. Hence, there exists a subsequence, denoted in the same way, such that $u_k \rightharpoonup \bar u$ in $L^2(Q_\omega)$. Let us denote by $\{y_k\}_{k=1}^\infty$ the states associated with $\{u_k\}_{k=1}^\infty$. Lemma 2.2 implies that $y_k \rightharpoonup \bar y$ in $L^2_\sigma(Q)$, where $\bar y$ is the solution of (1.1) corresponding to $\bar u$. To prove that $\bar u$ is a solution to (P), we consider the following inequality for arbitrary $T > 0$:

$$\begin{aligned}&\frac12\int_Q \mathrm{e}^{-\sigma t}(\bar y - y_d)^2\,dx\,dt + \frac{\nu}{2}\int_0^\infty\int_\omega \bar u^2\,dx\,dt + \gamma\int_0^T\Big(\int_\omega \bar u^2\,dx\Big)^{1/2}dt\\ &\le \liminf_{k\to\infty} J(u_k) = \inf\,(\mathrm{P}),\end{aligned}$$

which follows from the convexity of the objective functional with respect to pair (y, u) and the continuity of the embedding $L^2(0,T; L^2(\omega)) \subset L^1(0,T; L^2(\omega))$. Now we have

$$\begin{aligned}J(\bar u) &= \sup_{T>0}\Big\{\frac12\int_Q \mathrm{e}^{-\sigma t}(\bar y - y_d)^2\,dx\,dt + \frac{\nu}{2}\int_0^\infty \|\bar u(t)\|^2_{L^2(\omega)}\,dt + \gamma\int_0^T \|\bar u(t)\|_{L^2(\omega)}\,dt\Big\}\\ &\le \inf\,(\mathrm{P}),\end{aligned}$$

which concludes the proof. ☐

3. Differentiability of the Cost Functional. The cost functional J is decomposed in two parts: $J(u) = F(u) + \gamma j(u)$ with

$$F(u) = \frac12\int_Q e^{-\sigma t}(y_u - y_d)^2\,dx\,dt + \frac{\nu}{2}\int_{Q_\omega} u^2\,dx\,dt, \tag{3.1}$$

$$j(u) = \int_0^\infty \|u(t)\|_{L^2(\omega)}\,dt. \tag{3.2}$$

Regarding the functional j we have the following properties, which can be obtained from [2] by reversing the role of the variables x and t.

LEMMA 3.1. *The functional* $j : L^1(0,\infty;L^2(\omega)) \longrightarrow \mathbb{R}$ *is Lipschitz and convex and the following relations hold*
(1) The subdifferential $\partial j(u)$ *is the set of functions* $\lambda \in L^\infty(0,\infty;L^2(\omega))$ *satisfying*

$$\left\{\begin{array}{ll} \|\lambda(t)\|_{L^2(\Omega)} \le 1 & \text{for a.a. } t \in I_u^0, \\ \lambda(x,t) = \dfrac{u(x,t)}{\|u(t)\|_{L^2(\omega)}} & \text{for a.a. } t \in I_u \text{ and } x \in \omega, \end{array}\right. \tag{3.3}$$

where $I_u = \{t \in (0,\infty) : \|u(t)\|_{L^2(\omega)} \neq 0\}$ *and* $I_u^0 = (0,\infty) \setminus I_u$.
(2) For every $u, v \in L^1(0,\infty;L^2(\omega))$ *the directional derivative is given by*

$$j'(u;v) = \int_{I_u^0} \|v(t)\|_{L^2(\omega)}\,dt + \int_{I_u} \frac{1}{\|u(t)\|_{L^2(\omega)}} \int_\omega u(x,t)v(x,t)\,dx\,dt. \tag{3.4}$$

As we will see later, F is differentiable. As usual, to represent its derivative we introduce the adjoint state. The next theorem establishes the existence and uniqueness of an adjoint state as well as its continuous dependence with respect to u.

THEOREM 3.2. *Let us assume that* $\sigma > \Lambda_f$, $h \in L^2_\beta(0,\infty;L^2(\Omega))$, *and* $y \in Y_\beta$ *for every* $\beta > 0$. *Then the problem*

$$\left\{\begin{array}{l} -\dfrac{\partial\varphi}{\partial t} - \Delta\varphi + a\varphi + f'(y)\varphi = \mathrm{e}^{-\sigma t}h \text{ in } Q, \\ \partial_n\varphi = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \mathrm{e}^{\Lambda_f t}\|\varphi(t)\|_{L^2(\Omega)} = 0 \end{array}\right. \tag{3.5}$$

has a unique solution $\varphi \in L^2(0,\infty;H^1(\Omega)) \cap C([0,\infty);L^2(\Omega))$. *Moreover, the regularity* $\varphi \in Y_{-\alpha}$ *holds for every* $\alpha < 2\sigma$. *Further, if* $\{(y_k,h_k)\}_{k=1}^\infty \subset Y_\beta \times L^2_\beta(0,\infty;L^2(\Omega))$ *and* $(y_k,h_k) \to (y,h)$ *in* $Y_\beta \times L^2_\beta(0,\infty;L^2(\Omega))$ *for every* $\beta > 0$, *then*

$$\lim_{k\to\infty} \|\varphi_k - \varphi\|_{Y_{-\alpha}} = 0 \quad \forall \alpha < 2\sigma, \tag{3.6}$$

where φ_k *is the solution of* (3.5) *with* (y,h) *replaced by* (y_k,h_k).

Proof. The proof is split in several steps.

Step 1 - Uniqueness of a solution. Since (3.5) is linear it is enough to prove that the only solution with a zero right hand side is $\varphi \equiv 0$. Multiplying (3.5) by φ, integrating by parts in $\Omega \times (t,T)$ for $0 < t < T < \infty$ with T arbitrarily large, and using that $f'(s) \ge -\Lambda_f$, we get

$$\begin{aligned} &\frac{1}{2}\Big\|\varphi(t)\Big\|^2_{L^2(\Omega)} \le \frac{1}{2}\Big\|\varphi(t)\Big\|^2_{L^2(\Omega)} + \int_t^T \int_\Omega [|\nabla\varphi|^2 + a\varphi^2]\,dx\,ds \\ &\le -\int_t^T \int_\Omega f'(y)\varphi^2\,dx\,ds + \frac{1}{2}\Big\|\varphi(T)\Big\|^2_{L^2(\Omega)} \le \Lambda_f \int_t^T \|\varphi(s)\|^2_{L^2(\Omega)}\,ds + \frac{1}{2}\Big\|\varphi(T)\Big\|^2_{L^2(\Omega)}. \end{aligned}$$

The inequality in the above expression holds because $f'(s) \ge -\Lambda_f$. By Gronwall's inequality we infer

$$\|\varphi(t)\|^2_{L^2(\Omega)} \le \mathrm{e}^{2\Lambda_f(T-t)}\|\varphi(T)\|^2_{L^2(\Omega)} \le [\mathrm{e}^{\Lambda_f T}\|\varphi(T)\|_{L^2(\Omega)}]^2 \stackrel{T\to\infty}{\longrightarrow} 0,$$

which proves that $\varphi = 0$.

Step 2 - Existence of a solution. Let $\{T_k\}_{k=1}^\infty$ be an increasing sequence converging to ∞. For every k we consider the equations

$$\begin{cases} -\dfrac{\partial \varphi_{T_k}}{\partial t} - \Delta\varphi_{T_k} + a\varphi_{T_k} + f'(y)\varphi_{T_k} = \mathrm{e}^{-\sigma t}h \text{ in } Q_{T_k}, \\ \partial_n\varphi_{T_k} = 0 \text{ on } \Sigma_{T_k},\ \varphi_{T_k}(T_k) = 0. \end{cases} \tag{3.7}$$

Since $\mathrm{e}^{-\sigma t}h \in L^2(Q_{T_k})$, the existence and uniqueness of a solution $\varphi_{T_k} \in H^1(Q_{T_k}) \cap C([0,T_k]; H^1(\Omega))$ is well known. We prove the convergence of $\{\varphi_{T_k}\}_{k=1}^\infty$ to the solution of (3.5). For this purpose we test the equation (3.7) with $\mathrm{e}^{\alpha t}\varphi_{T_k}$ for $\alpha \ge 2\Lambda_f$ and use that $\varphi_{T_k}(T_k) = 0$ we get for $t \in (0, T_k)$

$$\frac{\mathrm{e}^{\alpha t}}{2}\Big\|\varphi_{T_k}(t)\Big\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_t^{T_k}\mathrm{e}^{\alpha s}\|\varphi_{T_k}\|^2_{L^2(\Omega)}\,ds + \int_t^{T_k}\mathrm{e}^{\alpha s}\int_\Omega[|\nabla\varphi_{T_k}|^2 + a\varphi^2_{T_k}]\,dx\,ds$$
$$+\int_t^{T_k}\mathrm{e}^{\alpha s}\int_\Omega f'(y)\varphi^2_{T_k}\,dx\,ds = \int_t^{T_k}\mathrm{e}^{(\alpha-\sigma)s}\int_\Omega h\varphi_{T_k}\,dx\,ds.$$

Denoting by $\hat\varphi_{T_k}$ the extension of φ_{T_k} by zero for $t > T_k$ we deduce from the above equality with (1.3) and Young's inequality

$$\begin{aligned}
&\frac{\mathrm{e}^{\alpha t}}{2}\|\hat\varphi_{T_k}(t)\|^2_{L^2(\Omega)} + C_a^2\int_t^\infty \mathrm{e}^{\alpha s}\|\hat\varphi_{T_k}(s)\|^2_{H^1(\Omega)}\,ds\\
&\le \int_t^\infty \mathrm{e}^{(\alpha-\sigma)s}\|h\|_{L^2(\Omega)}\|\hat\varphi_{T_k}(s)\|_{L^2(\Omega)}\,ds\\
&\le \frac{1}{2C_a^2}\int_t^\infty \mathrm{e}^{(\alpha-2\sigma)s}\|h\|^2_{L^2(\Omega)}\,ds + \frac{C_a^2}{2}\int_t^\infty \mathrm{e}^{\alpha s}\|\hat\varphi_{T_k}(s)\|^2_{H^1(\Omega)}\,ds.
\end{aligned}$$

This yields for some constant C_1 independent of $\alpha \in [2\Lambda_f, 2\sigma)$ and k

$$\operatorname*{ess\,sup}_{t>0}\mathrm{e}^{\frac{\alpha}{2}t}\|\hat\varphi_{T_k}(t)\|_{L^2(\Omega)} + \Big(\int_0^\infty \mathrm{e}^{\alpha s}\|\hat\varphi_{T_k}\|^2_{H^1(\Omega)}\,ds\Big)^{\frac12}$$
$$\le C_1\|h\|_{L^2_{2\sigma-\alpha}(Q)} = K_\alpha < \infty. \tag{3.8}$$

Therefore, taking a subsequence, denoted in the same way, we have $\mathrm{e}^{\frac{\alpha}{2}t}\hat\varphi_{T_k} \rightharpoonup \mathrm{e}^{\frac{\alpha}{2}t}\varphi$ in $L^2(0,\infty;H^1(\Omega))$ and $\mathrm{e}^{\frac{\alpha}{2}t}\hat\varphi_{T_k} \stackrel{*}{\rightharpoonup} \mathrm{e}^{\frac{\alpha}{2}t}\varphi$ in $L^\infty(0,\infty;L^2(\Omega))$ for some function $\varphi \in L^2(0,\infty;H^1(\Omega)) \cap L^\infty(0,\infty;L^2(\Omega))$. The first convergence implies that φ satisfies the partial differential equation part of (3.5) and the second convergence yields

$$\|\mathrm{e}^{\alpha t}\varphi(t)\|_{L^2(\Omega)} \le K_\alpha \quad \text{for a.a. } t \ge 0 \text{ and } \alpha < 2\sigma. \tag{3.9}$$

Let us prove that $f'(y)\varphi \in L^2(Q)$. From our assumptions on f we deduce the existence of a constant C_2 such that $|f'(s)| \le C_2(s^{2m}+1)$ for all $s \in \mathbb{R}$. Then, using Hölder's inequality with $\frac{2m+1}{2m}$ and $2m+1$ we obtain

$$\begin{aligned}
\int_Q f'(y)^2\varphi^2\,dx\,dt &\le C_2'\int_0^\infty \big(\|y\|^{4m}_{L^{4m+2}(\Omega)} + 1\big)\|\varphi\|^2_{L^{4m+2}(\Omega)}\,dt\\
&\le C_2''\int_0^\infty \big([\mathrm{e}^{-\frac{\alpha}{4m}t}\|y\|_{H^1(\Omega)}]^{4m} + 1\big)\mathrm{e}^{\alpha t}\|\varphi\|^2_{H^1(\Omega)}\,dt\\
&\le C_2''\big(\|y\|^{4m}_{C_{\frac{\alpha}{2m}}(0,\infty;H^1(\Omega))} + 1\big)\|\varphi\|^2_{L^2_{-\alpha}(0,\infty;H^1(\Omega))} < \infty.
\end{aligned}$$

Hence we have $\varphi \in W(0,T) = H^1(0,T;H^1(\Omega)^*) \cap L^2(0,T;H^1(\Omega)) \subset C([0,T];L^2(\Omega))$ for every $T < \infty$ and, consequently, $\varphi \in C([0,\infty);L^2(\Omega))$. Furthermore, (3.9) implies that $\lim_{t\to\infty} \mathrm{e}^{\Lambda_f t}\|\varphi(t)\|_{L^2(\Omega)} = 0$. Hence, we have that φ is a solution of (3.5). Further, due to the uniqueness of a solution we deduce that the whole sequence $\{\varphi_{T_k}\}_{k=1}^\infty$ converges to φ in the sense specified above.

Step 3 - $\varphi \in Y_{-\alpha}$. We test equation (3.7) with $-\mathrm{e}^{\alpha s}\frac{\partial \varphi_{T_k}(x,s)}{\partial s}$, $\alpha \in [2\Lambda_f, 2\sigma)$, and get for every $t \in (0,T_k)$

$$\int_t^{T_k} \mathrm{e}^{\alpha s}\Big\|\frac{\partial \varphi_{T_k}}{\partial s}\Big\|^2_{L^2(\Omega)} ds + \frac{\mathrm{e}^{\alpha t}}{2}\int_\Omega [|\nabla \varphi_{T_k}(t)|^2 + a\varphi_{T_k}^2(t)]\,dx$$
$$+ \frac{\alpha}{2}\int_t^{T_k} \mathrm{e}^{\alpha s}\int_\Omega [|\nabla\varphi_{T_k}|^2 + a\varphi_{T_k}^2]\,dx\,ds - \int_t^{T_k}\mathrm{e}^{\alpha s}\int_\Omega f'(y)\varphi_{T_k}\frac{\partial\varphi_{T_k}}{\partial s}\,dx\,ds$$
$$= -\int_t^{T_k}\mathrm{e}^{(\alpha-\sigma)s}\int_\Omega h\frac{\partial \varphi_{T_k}}{\partial s}\,dx\,ds.$$

Take $\varepsilon > 0$ such that $\alpha + \varepsilon < 2\sigma$. Using (1.5), $|f'(s)| \le C_2(s^{2m}+1)$, Hölder's inequality with $\frac{4m+2}{2m}$, $4m+2$ and 2 for $m > 0$ or Schwarz's inequality for $m = 0$, the fact that $y \in C_{\frac{\varepsilon}{2m}}([0,\infty);H^1(\Omega))$, and Young's inequality we infer from the above equality

$$\int_t^{T_k}\mathrm{e}^{\alpha s}\Big\|\frac{\partial\varphi_{T_k}}{\partial s}\Big\|^2_{L^2(\Omega)} ds + \frac{C_a^2\mathrm{e}^{\alpha t}}{2}\|\varphi_{T_k}(t)\|^2_{H^1(\Omega)}$$
$$\le C_3\int_t^{T_k}\mathrm{e}^{\alpha s}\big(\|y\|^{2m}_{L^{4m+2}(\Omega)}+1\big)\|\varphi_{T_k}\|_{L^{4m+2}(\Omega)}\Big\|\frac{\partial\varphi_{T_k}}{\partial s}\Big\|_{L^2(\Omega)} ds$$
$$+ \int_t^{T_k}\mathrm{e}^{(\alpha-\sigma)s}\|h\|_{L^2(\Omega)}\Big\|\frac{\partial\varphi_{T_k}}{\partial s}\Big\|_{L^2(\Omega)} ds$$
$$\le C_4\int_t^{T_k}\mathrm{e}^{(\alpha+\frac{\varepsilon}{2})s}\big([\mathrm{e}^{-\frac{\varepsilon}{4m}s}\|y\|_{H^1(\Omega)}]^{2m}+1\big)\|\varphi_{T_k}\|_{L^{4m+2}(\Omega)}\Big\|\frac{\partial\varphi_{T_k}}{\partial s}\Big\|_{L^2(\Omega)} ds$$
$$+ \int_t^{T_k}\mathrm{e}^{(\alpha-\sigma)s}\|h\|_{L^2(\Omega)}\Big\|\frac{\partial\varphi_{T_k}}{\partial s}\Big\|_{L^2(\Omega)} ds \le C_{5,\varepsilon}\int_t^{T_k}\mathrm{e}^{(\alpha+\varepsilon)s}\|\varphi_{T_k}\|^2_{H^1(\Omega)} ds$$
$$+ \int_t^{T_k}\mathrm{e}^{(\alpha-2\sigma)s}\|h\|^2_{L^2(\Omega)} ds + \frac{1}{2}\int_t^{T_k}\mathrm{e}^{\alpha s}\Big\|\frac{\partial\varphi_{T_k}}{\partial s}\Big\|^2_{L^2(\Omega)} ds.$$

Taking into account the definition of $\hat\varphi_{T_k}$, the above inequality leads to

$$\int_t^\infty \mathrm{e}^{\alpha s}\Big\|\frac{\partial\hat\varphi_{T_k}}{\partial s}\Big\|^2_{L^2(\Omega)} ds + C_a^2\mathrm{e}^{\alpha t}\|\hat\varphi_{T_k}(t)\|^2_{H^1(\Omega)}$$
$$\le 2C_{5,\varepsilon}\int_t^\infty \mathrm{e}^{(\alpha+\varepsilon)s}\|\hat\varphi_{T_k}\|^2_{H^1(\Omega)} ds + 2\int_t^\infty \mathrm{e}^{(\alpha-2\sigma)s}\|h\|^2_{L^2(\Omega)} ds.$$

Since $\alpha + \varepsilon < 2\sigma$, we deduce from the above inequality and (3.8)

$$\operatorname*{ess\,sup}_{t>0} \mathrm{e}^{\frac{\alpha}{2}t}\|\hat\varphi_{T_k}(t)\|_{H^1(\Omega)} + \Big(\int_0^\infty \mathrm{e}^{\alpha s}\Big\|\frac{\partial\hat\varphi_{T_k}}{\partial s}\Big\|^2_{L^2(\Omega)} ds\Big)^{\frac{1}{2}} \le C_{6,\varepsilon}\|h\|_{L^2_{2\sigma-(\alpha+\varepsilon)}(Q)}$$

for all $k \ge 1$. As a consequence we get $\mathrm{e}^{\frac{\alpha}{2}t}\hat\varphi_{T_k} \overset{*}{\rightharpoonup} \mathrm{e}^{\frac{\alpha}{2}t}\varphi$ in $L^\infty(0,\infty;H^1(\Omega))$ and $\mathrm{e}^{\frac{\alpha}{2}t}\frac{\partial\hat\varphi_{T_k}}{\partial t} \rightharpoonup \mathrm{e}^{\frac{\alpha}{2}t}\frac{\partial\varphi}{\partial t}$ in $L^2(Q)$. Then, the above inequality implies

$$\operatorname*{ess\,sup}_{t>0} \mathrm{e}^{\frac{\alpha}{2}t}\|\varphi(t)\|_{H^1(\Omega)} + \Big(\int_0^\infty \mathrm{e}^{\alpha s}\Big\|\frac{\partial\varphi}{\partial s}\Big\|^2_{L^2(\Omega)} ds\Big)^{\frac{1}{2}} \le C_{6,\varepsilon}\|h\|_{L^2_{2\sigma-(\alpha+\varepsilon)}(Q)}.$$

Hence we have that $\varphi \in H^1_{-\alpha}(Q)$. Moreover, since $\varphi(T) \in H^1(\Omega)$ for almost every $T > 0$ and $\frac{\partial \varphi}{\partial t} + \Delta\varphi \in L^2(Q)$, we infer that $\varphi \in C([0,T]; H^1(\Omega))$; see [7, Proposition III-2.5]. Combining this with the fact that $\varphi \in L^\infty_{-\alpha}(0,\infty; H^1(\Omega))$ we conclude that $\varphi \in C_{-\alpha}([0,\infty); H^1(\Omega))$. All together we obtain that $\varphi \in Y_{-\alpha}$ and

$$\|\varphi\|_{Y_{-\alpha}} \le M_\varepsilon \|h\|_{L^2_{2\sigma-(\alpha+\varepsilon)}(Q)} \quad \forall \alpha \in [2\Lambda_f, 2\sigma - \varepsilon). \tag{3.10}$$

Step 4 - Proof of (3.6). Take $\alpha < 2\sigma$ and $\varepsilon \in (0, 2\sigma - \alpha)$. We set $\phi_k = \varphi - \varphi_k$. Then, we have

$$\begin{cases} -\dfrac{\partial \phi_k}{\partial t} - \Delta\phi_k + a\phi_k + f'(y)\phi_k = \mathrm{e}^{-\sigma t}[(h - h_k) + \mathrm{e}^{\sigma t}(f'(y_k) - f'(y))\varphi_k] \text{ in } Q, \\ \partial_n \phi_k = 0 \text{ on } \Sigma, \quad \lim_{t\to\infty} \|\phi_k(t)\|_{L^2(\Omega)} = 0. \end{cases}$$

Analogously to (3.10) we have

$$\|\phi_k\|_{Y_{-\alpha}} \le M_\varepsilon \|(h - h_k) + \mathrm{e}^{\sigma t}(f'(y_k) - f'(y))\varphi_k\|_{L^2_{2\sigma-(\alpha+\varepsilon)}(Q)} \quad \forall \alpha \in [2\Lambda_f, 2\sigma - \varepsilon). \tag{3.11}$$

Observe that $h - h_k \to 0$ in $L^2_{2\sigma-(\alpha+\varepsilon)}(Q)$ due to $h_k \to h$ in Y_β for all $\beta > 0$. Let us prove that $\mathrm{e}^{\sigma t}[f'(y_k) - f'(y)]\varphi_k \to 0$ in $L^2_{2\sigma-\alpha}(Q)$ as $k \to \infty$ for $\alpha < 2\sigma$ arbitrary. We make the proof for $m \ge 1$, being simpler for $m = 0$. Using Hölder's inequality with $1 + \frac{1}{2m}$ and $2m + 1$ and taking $\varepsilon \in (0, \sigma)$ we get

$$\begin{aligned} &\|\mathrm{e}^{\sigma t}[f'(y_k) - f'(y)]\varphi_k\|^2_{L^2_{2\sigma-\alpha}(Q)} \\ &\le \int_0^\infty \mathrm{e}^{(\alpha-\sigma)t} \|f'(y_k) - f'(y)\|^2_{L^{2+\frac{1}{m}}(\Omega)} \|\varphi_k\|^2_{L^{4m+2}(\Omega)}\, dt \\ &\le C \int_0^\infty \mathrm{e}^{(\alpha+\varepsilon-3\sigma)t} \|f'(y_k) - f'(y)\|^2_{L^{2+\frac{1}{m}}(\Omega)} \big[\mathrm{e}^{(\sigma - \frac{\varepsilon}{2})t} \|\varphi_k\|_{H^1(\Omega)}\big]^2\, dt \\ &\le C \|\varphi_k\|^2_{C_{\varepsilon-2\sigma}([0,\infty); H^1(\Omega))} \|f'(y_k) - f'(y)\|_{L^2_{3\sigma-(\alpha+\varepsilon)}(Q)} \end{aligned}$$

From (3.10) the boundedness of $\{\varphi_k\}_{k=1}^\infty$ in $Y_{\varepsilon-2\sigma}$ follows. This combined with the above estimate and the fact that $\|f'(y_k) - f'(y)\|_{L^{2+\frac{1}{m}}_{3\sigma-(\alpha+\varepsilon)}(Q)} \to 0$ as $k \to \infty$ [5, Theorem 2.7] leads to

$$\lim_{k\to\infty} \|(h - h_k) + \mathrm{e}^{\sigma t}(f'(y_k) - f'(y))\varphi_k\|_{L^2_{2\sigma-\alpha}(Q)} = 0 \quad \forall \alpha < 2\sigma.$$

Hence, (3.11) yields $\lim_{k\to\infty} \|\phi_k\|_{Y_{-\alpha}} = 0$ for all $\alpha \in [2\Lambda_f, 2\sigma)$. Since the norm $\|\cdot\|_{Y_{-\alpha}}$ is monotonically increasing with respect to α, equality (3.6) holds. □

Remark 3.3. *We observe that $\Lambda_f = 0$ if f is a nondecreasing monotone function. Hence, existence and uniqueness of a solution for equation* (3.5) *holds for all $\sigma > 0$.*

The next theorem establishes the differentiability of the functional F.

THEOREM 3.4. *Assume that $\sigma > 8\Lambda_f$. Then, the functional $F : L^2(Q_\omega) \longrightarrow \mathbb{R}$ is of class C^1 and the following expression for its derivative holds*

$$F'(u)v = \int_{Q_\omega} (\varphi_u + \nu u)v\, dx\, dt, \tag{3.12}$$

where φ_u is the solution of the adjoint equation (3.5) *with $y = y_u$ and $h = y_u - y_d$. If in addition $\sigma > 4\Lambda_f$, then F is of class C^2 and*

$$F''(u)(v_1, v_2) = \int_Q [\mathrm{e}^{-\sigma t} - \varphi_u f''(y_u)] z_{v_1} z_{v_2}\, dx\, dt + \nu \int_{Q_\omega} v_1 v_2\, dx\, dt. \tag{3.13}$$

Proof. Choose $\alpha \in (2\Lambda_f, \sigma)$, and define by $F_0 : L^2_\sigma(Q) \longrightarrow \mathbb{R}$ the quadratic form $F_0(y) = \frac{1}{2}\|y - y_d\|^2_{L^2_\sigma(Q)}$. Let $G_\alpha : L^2(Q_\omega) \longrightarrow Y_\alpha$ with $G_\alpha(u) = y_u$ denote the control to state mapping. From Theorem [5, Theorem 3.3] and the fact that $\alpha > 4\Lambda_f$ we get that G_α is of class C^1. Moreover, $z_v = G'(u)v$ is the solution of the equation

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z = v\chi_\omega \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega. \end{array}\right. \tag{3.14}$$

Further, since $\alpha < \sigma$, Y_α is continuously embedded in $L^2_\sigma(Q)$. Therefore, the mapping $F = F_0 \circ G_\alpha + \frac{\nu}{2}\|\cdot\|^2_{L^2(Q_\omega)}$ is of class C^1 and

$$\begin{aligned} F'(u)v &= \int_Q \mathrm{e}^{-\sigma t}(G_\alpha(u) - y_d)G'_\alpha(u)v\,dx\,dt + \nu\int_{Q_\omega} uv\,dx\,dt \\ &= \int_Q \mathrm{e}^{-\sigma t}(y_u - y_d)z_v\,dx\,dt + \nu\int_{Q_\omega} uv\,dx\,dt. \end{aligned}$$

Testing (3.14) with φ_u and integrating by parts, (3.12) follows from the above identity. To justify this testing the integrability in Q of every term of equation (3.14) multiplied by φ_u needs to be verified. Let us consider the integrability of the first term. Given $\beta \in (2\Lambda_f, 2\sigma)$ we get

$$\int_Q \Big|\frac{\partial z_v}{\partial t}\Big||\varphi_u|\,dx\,dt = \int_Q \mathrm{e}^{-\frac{\beta}{2}t}\Big|\frac{\partial z_v}{\partial t}\Big|\mathrm{e}^{\frac{\beta}{2}t}|\varphi_u|\,dx\,dt \le \Big\|\frac{\partial z_v}{\partial t}\Big\|_{L^2_\beta(Q)}\|\varphi_u\|_{L^2_{-\beta}(Q)} < \infty$$

due to the fact that $z_v \in Y_\beta$ for every $\beta > 2\Lambda_f$ and $\varphi_u \in Y_{-\beta}$ for every $\beta < 2\sigma$; see [5, Lemma 3.1] and Theorem 3.2. The other terms can be analyzed in a similar way except the one containing $f'(y_u)$. To deal with this term we consider the case $m \ge 1$, for $m = 0$ the proof is easier. We select $\beta_3 \in (2\Lambda_f, \frac{\sigma}{1+\frac{1}{2m}})$. This is possible because $\sigma > 4\Lambda_f$. Now we take $\beta_1 \in (0, \frac{\beta_3}{2m+1})$ and $\beta_2 = \beta_3 - \beta_1$. We define $\beta_{1m} = (2 + \frac{1}{m})\beta_1$, $\beta_{2m} = (2 + \frac{2m}{m+1})\beta_2$, and $\beta_{3m} = (2 + \frac{2m}{m+1})\beta_3$. Then, applying Hölder's inequality with $2 + \frac{1}{m}$, $2 + \frac{2m}{m+1}$, and $2 + \frac{2m}{m+1}$, and using that $H^1(\Omega) \subset L^{2+\frac{2m}{m+1}}(\Omega)$ we obtain

$$\begin{aligned} &\int_Q |f'(y_u)z_v\varphi_u|\,dx\,dt = \int_Q \mathrm{e}^{-\beta_1 t}|f'(y_u)|\mathrm{e}^{-\beta_2 t}|z_v|\mathrm{e}^{\beta_3 t}|\varphi_u|\,dx\,dt \\ &\le \|f'(y_u)\|_{L^{2+\frac{1}{m}}_{\beta_{1m}}(Q)}\|z_v\|_{L^{2+\frac{2m}{m+1}}_{\beta_{2m}}(Q)}\|\varphi_u\|_{L^{2+\frac{2m}{m+1}}_{-\beta_{3m}}(Q)} \\ &\le c\|y_u^{2m} + 1\|_{L^{2+\frac{1}{m}}_{\beta_{1m}}(Q)}\|z_v\|_{L^{2+\frac{2m}{m+1}}_{\beta_{2m}}(0,\infty;H^1(\Omega))}\|\varphi_u\|_{L^{2+\frac{2m}{m+1}}_{-\beta_{3m}}(0,\infty;H^1(\Omega))} \\ &\le c'(\|y_u\|_{L^{4m+2}_{\beta_{1m}}(Q)} + 1)\|z_v\|^{\frac{m}{2m+1}}_{C_{\frac{\beta_{2m}(m+1)}{2m}}(0,\infty;H^1(\Omega))}\|z_v\|^{\frac{m+1}{2m+1}}_{L^2_{\frac{\beta_{2m}}{2}}(0,\infty;H^1(\Omega))} \\ &\times \|\varphi_u\|^{\frac{m}{2m+1}}_{C_{-\frac{\beta_{3m}(m+1)}{2m}}(0,\infty;H^1(\Omega))}\|\varphi_u\|^{\frac{m+1}{2m+1}}_{L^2_{-\frac{\beta_{3m}}{2}}(0,\infty;H^1(\Omega))} < \infty. \end{aligned}$$

For the last inequality we have used that $\beta_{1m} > 0$ and (2.3), and the fact that

$z_v \in Y_{\frac{\beta_{2m}}{2}}$ and $\varphi_u \in Y_{-\frac{\beta_{3m}(m+1)}{2m}}$ due to the following properties:

$$\frac{\beta_{2m}(m+1)}{2m} = \frac{2m+1}{m}(\beta_3-\beta_1) > \frac{2m+1}{m}\Big(1-\frac{1}{2m+1}\Big)\beta_3 = 2\beta_3 > 4\Lambda_f,$$
$$\frac{\beta_{3m}}{2} \le \frac{2m+1}{m+1}\beta_3 < \frac{2m+1}{m+1}\frac{\sigma}{1+\frac{1}{2m}} < 2\sigma.$$

Now we turn to the second derivative and assume that $\sigma > 8\Lambda_f$. We select $\alpha \in (8\Lambda_f, \sigma)$. Then, Theorem [5, Theorem 3.3] implies that G_α is of class C^2 and $z_{v_1,v_2} = G''_\alpha(u)(v_1,v_2)$ is the solution of the equation

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z + f''(y_u)z_{v_1}z_{v_2} = 0 \text{ in } Q,\\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega, \end{cases} \tag{3.15}$$

where $z_{v_i} = G'_\alpha(u)v_i$, $i = 1,2$. Hence, by the chain rule we get that the function $F = F_0 \circ G_\alpha + \frac{\nu}{2}\|\cdot\|^2_{L^2(Q_\omega)}$ is of class C^2 and

$$\begin{aligned} &F''(u)(v_1,v_2)\\ &= \int_Q \mathrm{e}^{-\sigma t}\left\{(G_\alpha(u)-y_d)G''_\alpha(u)(v_1,v_2) + G'_\alpha(u)v_1G'_\alpha(u)v_2\right\}dx\,dt + \nu\int_{Q_\omega} v_1v_2\,dx\,dt\\ &= \int_Q \mathrm{e}^{-\sigma t}\left\{(y_u-y_d)z_{v_1v_2} + z_{v_1}z_{v_2}\right\}dx\,dt + \nu\int_{Q_\omega} v_1v_2\,dx\,dt, \end{aligned}$$

where $z_{v_i} = G'_\alpha(u)v_1$ and $z_{v_1v_2} = G''_\alpha(u)(v_1,v_2)$. Now, testing equation (3.15) with φ_u and integrating by parts we obtain (3.13). To check that the testing and integration by parts are justified, the most delicate issue is the one involving $f''(y_u)z_{v_1}z_{v_2}\varphi_u$. To prove its integrability we proceed as follows. As above we are going to consider the case $m \ge 1$, the case $m = 0$ being easier. First, we select $\beta \in (2\Lambda_f, \frac{2m+1}{4m}\sigma)$, $\beta_3 \in (\frac{4m}{2m+1}\beta, \sigma)$, $\beta_2 = \frac{2m}{2m+1}\beta$, and $\beta_1 = \beta_3 - 2\beta_2$. Let us denote $\beta_{1m} = \frac{4m+2}{2m-1}\beta_1$ and $\beta_{2m} = (4m+2)\beta_2$. Then we apply Hölder's inequality with $\frac{4m+2}{2m-1}$, $4m+2$, $4m+2$, and 2 to derive

$$\begin{aligned} &\int_Q |f''(y_u)z_{v_1}z_{v_2}\varphi_u|\,dx\,dt = \int_Q e^{-\beta_1 t}|f''(y_u)|\mathrm{e}^{-\beta_2 t}|z_{v_1}|\mathrm{e}^{-\beta_2 t}|z_{v_2}|\mathrm{e}^{\beta_3 t}|\varphi_u|\,dx\,dt\\ &\le \|f''(y_u)\|_{L^{2+\frac{4}{2m-1}}_{\beta_{1m}}(Q)}\|z_{v_1}\|_{L^{4m+2}_{\beta_{2m}}(Q)}\|z_{v_2}\|_{L^{4m+2}_{\beta_{2m}}(Q)}\|\varphi_u\|_{L^2_{-2\beta_3}(Q)}\\ &\le c(\|y_u\|_{L^{4m+2}_{\beta_{1m}}(Q)} + 1)\|z_{v_1}\|_{Y_{\frac{\beta_{2m}}{4m}}}\|z_{v_2}\|_{Y_{\frac{\beta_{2m}}{4m}}}\|\varphi_u\|_{Y_{-2\beta_3}} < \infty. \end{aligned}$$

In the last inequality we used that $|f''(s)| \le C(|s|^{2m-1}+1)$ and (2.3), $\beta_{1m} > 0$, $\frac{\beta_{2m}}{4m} = \beta > 2\Lambda_f$, and $\beta_3 < \sigma$ along with [5, Lemma 3.1] and Theorem 3.2. □

4. First and second order optimality conditions. The aim of this section is to establish the necessary and sufficient conditions for local optimality. Since problem (P) is not convex we will consider local minimizers in this section. We say that $\bar u$ is a local minimizer of (P) if $\bar u \in \mathcal{U}_{ad}$ and there exists a ball $B_\varepsilon(\bar u) \subset L^2(Q_\omega)$ such that $J(\bar u) \le J(u)$ for every $u \in B_\varepsilon(\bar u) \cap \mathcal{U}_{ad}$. If the inequality is strict for every $u \ne \bar u$, we call $\bar u$ a strict local minimizer. This definition implies that a local minimizer $\bar u$ satisfies $J(\bar u) < \infty$ and, consequently, it belongs to $L^1(0,\infty;L^2(\omega))$ if $\gamma > 0$. Indeed, given $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that for $u_\varepsilon(x,t) = \bar u(x,t)\chi_{[0,T_\varepsilon]}(t)$ the inequality $\|\bar{u} - \bar{u}_\varepsilon\|_{L^2(Q_\omega)} < \varepsilon$ holds. Moreover, we have that $u_\varepsilon \in \mathcal{U}_{ad}$ and then $J(\bar{u}) \leq J(u_\varepsilon) < \infty$.

Following the approach of [2] and using Theorem 3.4 the optimality conditions are deduced next.

THEOREM 4.1. *Let us assume that $\sigma > 4\Lambda_f$. If $\bar{u}$ is a local minimizer of* (P), *then there exists $\bar{y} \in Y_\alpha$ for every $\alpha > 0$, $\bar{\varphi} \in Y_{-\beta}$ for every $\beta < 2\sigma$, and $\bar{\lambda} \in \partial j(\bar{u})$ if $\gamma > 0$, such that*

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} - \Delta \bar{y} + a\bar{y} + f(\bar{y}) = g + \bar{u}\chi_\omega \quad \text{in } Q, \\ \partial_n \bar{y} = 0 \quad \text{on } \Sigma, \ \bar{y}(0) = y_0 \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

$$\begin{cases} -\dfrac{\partial \bar{\varphi}}{\partial t} - \Delta \bar{\varphi} + a\bar{\varphi} + f'(\bar{y})\bar{\varphi} = \mathrm{e}^{-\sigma t}(\bar{y} - y_d) \text{ in } Q, \\ \partial_n \bar{\varphi} = 0 \text{ on } \Sigma, \ \lim\limits_{t \to \infty} \mathrm{e}^{\Lambda_f t} \|\bar{\varphi}(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{4.2}$$

$$\int_{Q_\omega} [\bar{\varphi}_{|Q_\omega} + \nu \bar{u} + \gamma \bar{\lambda}](u - \bar{u}) \geq 0 \quad \forall u \in \mathcal{U}_{ad}. \tag{4.3}$$

PROPOSITION 4.2. *Let $(\bar{u}, \bar{\varphi}, \bar{\lambda})$ satisfy the optimality conditions* (4.1)–(4.3) *with $\bar{u} \in \mathcal{U}_{ad}$ and also $\bar{u} \in L^1(0,\infty; L^2(\omega))$ if $\gamma > 0$. Then $\bar{u} \in L^\infty(0,\infty; L^2(\omega))$ and the following identity holds for almost all $(x,t) \in Q_\omega$*

$$\bar{u}(x,t) = \mathrm{Proj}_{[u_a,u_b]}\Big(-\frac{1}{\nu}[\bar{\varphi}(x,t) + \gamma\,\bar{\lambda}(x,t)]\Big). \tag{4.4}$$

In addition, if $\gamma > 0$ and $u_a < 0 < u_b$, then we have for almost all $(x,t) \in Q_\omega$

$$\begin{cases} \|\bar{u}(t)\|_{L^2(\omega)} = 0 \Leftrightarrow \|\bar{\varphi}(t)\|_{L^2(\omega)} \leq \gamma, \\ \bar{\lambda}(x,t) = \begin{cases} -\dfrac{1}{\gamma}\bar{\varphi}(x,t) & \text{if } t \in I^0_{\bar{u}}, \\ \dfrac{\bar{u}(x,t)}{\|\bar{u}(t)\|_{L^2(\omega)}} & \text{if } t \in I_{\bar{u}}, \end{cases} \end{cases} \tag{4.5}$$

where $I_{\bar{u}} = \{t \in (0,\infty) : \|\bar{u}(t)\|_{L^2(\omega)} \neq 0\}$ and $I^0_{\bar{u}} = (0,\infty) \setminus I_{\bar{u}}$. Moreover, there exists $T^* < \infty$ such that $\|\bar{u}(t)\|_{L^2(\omega)} = 0$ for all $t \geq T^*$.

Identity (4.4) is standard and the regularity $\bar{u} \in L^\infty(0,\infty; L^2(\omega))$ is a consequence of it. The reader is referred to [2, Corollary 3.9] for the proof of (4.5) just by reversing the roles of x and t. The existence of T^* follows from the property $\lim_{t\to\infty} \|\bar{\varphi}(t)\|_{L^2(\Omega)} = 0$ and (4.5).

Now, we formulate the necessary second order optimality conditions for (P). Once again, following [2] we introduce the cone of critical directions associated with a control $\bar{u} \in \mathcal{U}_{ad}$ satisfying the first order optimality conditions (4.1)–(4.3) as follows

$$C_{\bar{u}} = \{v \in \mathcal{U} : v \text{ satisfies (4.6) below and } F'(\bar{u})v + \gamma j'(\bar{u}; v) = 0\},$$

where $\mathcal{U} = L^2(Q_\omega) \cap L^1(0,\infty; L^2(\omega))$ if $\gamma > 0$ and $\mathcal{U} = L^2(Q_\omega)$ if $\gamma = 0$, and

$$v(x,t) \begin{cases} \geq 0 & \text{if } \bar{u}(x,t) = u_a, \\ \leq 0 & \text{if } \bar{u}(x,t) = u_b, \end{cases} \quad \text{a.e. in } Q_\omega. \tag{4.6}$$

We also define $j''(u) : L^2(Q_\omega) \longrightarrow [0,\infty]$ by

$$j''(u; v^2) = \begin{cases} \displaystyle\int_{I_u} \frac{1}{\|u(t)\|_{L^2(\omega)}} \Big[\int_\omega v^2(x,t)\,dx - \Big(\int_\omega \frac{u(x,t)v(x,t)}{\|u(t)\|_{L^2(\omega)}}\,dx\Big)^2\Big]dt \text{ if } u \not\equiv 0, \\ 0 \text{ if } u \equiv 0. \end{cases}$$

The expression for $j''(u;v^2)$ is just a definition, it does not represent the second directional derivative, except for special cases. Actually, the expression $j''(u;v^2)$ can be ∞ for some functions u and v. Nevertheless, the integral is always well defined because the integrand is non negative. The following necessary second order optimality conditions can be established as in [2] with the help of Theorem 3.4.

THEOREM 4.3. *Assume that $\sigma > 8\Lambda_f$ and $u_a < 0 < u_b$ if $\gamma > 0$. Let $\bar u$ be a local minimizer of* (P). *Then, we have that $F''(\bar u)v^2 + \gamma j''(\bar u; v^2) \geq 0$ for every $v \in C_{\bar u}$.*

Proof. If $\gamma = 0$, the proof is well known. Assume that $\gamma > 0$ and $j''(\bar u; v^2) < \infty$, otherwise the inequality is obvious. Given $v \in C_{\bar u}$ and $T > 0$, we define $v_T(x,t) = v(x,t)$ if $t \leq T$ and zero otherwise. Following the steps of [2, Proof of Theorem 4.3, Case III], once again reversing the roles of x and t, and using that $\bar\varphi \in L^\infty(Q_T)$ due to $\bar y, y_d \in L^\infty(0,T;L^2(\Omega))$, we obtain that $F''(\bar u)v_T^2 + \gamma j''(\bar u; v_T^2) \geq 0$. We can pass to the limit in this inequality to conclude that $F''(\bar u)v^2 + \gamma j''(\bar u; v^2) \geq 0$. Indeed, using that $|v_T(x,t)| \leq |v(x,t)|$ and $v_T(x,t) \to v(x,t)$ when $T \to \infty$, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned}\lim_{T\to\infty} g_T(t) &:= \lim_{T\to\infty}\Big[\int_\omega v_T^2(x,t)\,dx - \Big(\int_\omega \frac{u(x,t)v_T(x,t)}{\|u(t)\|_{L^2(\omega)}}\,dx\Big)^2\Big]\\ &= g(t) := \Big[\int_\omega v^2(x,t)\,dx - \Big(\int_\omega \frac{u(x,t)v(x,t)}{\|u(t)\|_{L^2(\omega)}}\,dx\Big)^2\Big].\end{aligned}$$

Moreover, we have that $0 \leq g(t)$ for every t, $g_T(t) = g(t)$ if $t \leq T$, and $g_T(t) = 0$ if $t > T$. Hence, the monotone convergence theorem implies that

$$\lim_{T\to\infty}\int_{I_{\bar u}} \frac{1}{\|\bar u(t)\|_{L^2(\omega)}} g_T(t)\,dt = \int_{I_{\bar u}} \frac{1}{\|\bar u(t)\|_{L^2(\omega)}} g(t)\,dt.$$

This implies that $\lim_{T\to\infty} j''(\bar u; v_T^2) = j''(\bar u; v^2)$. Finally, as a consequence of the convergence $v_T \to v$ in $L^2(Q_\omega)$ and Theorem 3.4 we infer that $F''(\bar u)v_T^2 \to F''(\bar u)v^2$. ∎

In the next theorem the sufficient second order conditions for local optimality are established.

THEOREM 4.4. *Assume that $u_a < 0 < u_b$ if $\gamma > 0$ and $\sigma > 8\Lambda_f$. Let $\bar u \in \mathcal{U}\cap\mathcal{U}_{ad}$ satisfy the first order optimality conditions given by Theorem 4.1 and the second order condition $F''(\bar u)v^2 + \gamma j''(\bar u; v^2) > 0$ $\forall v \in C_{\bar u}\setminus\{0\}$. Then, there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(Q_\omega)} \leq J(u) \quad \forall u \in \mathcal{U}_{ad}\cap \bar B_\varepsilon(\bar u), \tag{4.7}$$

where $\bar B_\varepsilon(\bar u) = \{u \in L^2(Q_\omega) : \|u - \bar u\|_{L^2(Q_\omega)} \leq \varepsilon\}$.

LEMMA 4.5. *Under the assumptions of Theorem 4.4, if there are no $\delta > 0$ and $\varepsilon > 0$ such that* (4.7) *holds for every $u \in \mathcal{U}_{ad}\cap\bar B_\varepsilon(\bar u)\cap L^\infty(0,\infty;L^2(\omega))$, then there exists a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{ad}\cap L^\infty(0,\infty;L^2(\omega))$ such that for every $k \geq 1$*

$$\|u_k - \bar u\|_{L^2(Q_\omega)} < \frac{1}{k}, \tag{4.8}$$

$$J(u_k) < J(\bar u) + \frac{1}{2k}\|u_k - \bar u\|^2_{L^2(Q_\omega)}. \tag{4.9}$$

Furthermore, if $\gamma > 0$ and T^* is as introduced in Proposition 4.2, then there exists a sequence of measurable sets $I_k \subset (0,T^*)$ with $T^* - |I_k| < \frac{1}{k}$ such that $\{u_k\}_{k=1}^\infty$ satisfies *additionally*

$$\|u_k - \bar u\|_{L^\infty(I_k;L^2(\omega))} < \frac{1}{k}. \tag{4.10}$$

Proof. If (4.7) does not hold, then for any integer $k \ge 1$ there exists an element $w_k \in \mathcal{U}_{ad} \cap L^\infty(0,\infty;L^2(\omega))$ such that

$$\|w_k - \bar u\|_{L^2(Q_\omega)} < \frac{1}{k} \text{ and } J(w_k) < J(\bar u) + \frac{1}{2k}\|w_k - \bar u\|^2_{L^2(Q_\omega)}. \tag{4.11}$$

If $\gamma = 0$, then we take $u_k = w_k$. Otherwise, we proceed as follows. From the convergence $\|w_k(t) - \bar u(t)\|_{L^2(\omega)} \to 0$ in $L^2(0,\infty)$, we deduce the existence of a subsequence, denoted in the same way, such that $\|w_k(t) - \bar u(t)\|_{L^2(\omega)} \to 0$ for almost all $t \in (0,\infty)$. Then, from Egorov's theorem we deduce the existence of a subsequence $\{w_{j_k}\}_{k=1}^\infty$ and a sequence $\{I_k\}_{k=1}^\infty$ of measurable subsets of $(0,T^*)$ such that $T^* - |I_k| < \frac{1}{k}$ holds and

$$\|w_{j_k} - \bar u\|_{L^\infty(I_k;L^2(\omega))} = \operatorname*{ess\,sup}_{t\in I_k} \|w_{j_k}(t) - \bar u(t)\|_{L^2(\omega)} < \frac{1}{2k}.$$

Moreover, j_k can be chosen so that $j_k > 2k$. Then setting $u_k = w_{j_k}$ we get with (4.11)

$$\begin{aligned}&\|u_k - \bar u\|_{L^\infty(I_k;L^2(\omega))} + \|u_k - \bar u\|_{L^2(Q_\omega)}\\ &= \|w_{j_k} - \bar u\|_{L^\infty(I_k;L^2(\omega))} + \|w_{j_k} - \bar u\|_{L^2(Q_\omega)} < \frac{1}{2k} + \frac{1}{j_k} < \frac{1}{k}\end{aligned}$$

and

$$J(u_k) = J(w_{j_k}) < J(\bar u) + \frac{1}{j_k}\|w_{j_k} - \bar u\|^2_{L^2(Q_\omega)} < J(\bar u) + \frac{1}{2k}\|u_k - \bar u\|^2_{L^2(Q_\omega)},$$

which proves (4.8)–(4.10). □

Proof of Theorem 4.4. First we prove that (4.7) holds for every $u \in \mathcal{U}_{ad} \cap \bar B_\varepsilon(\bar u) \cap L^\infty(0,\infty;L^2(\omega))$. We argue by contradiction. Let us consider the case $\gamma > 0$. If $\gamma = 0$, the proof follows the same steps with the obvious simplifications. If (4.7) does not hold in $\mathcal{U}_{ad} \cap \bar B_\varepsilon(\bar u) \cap L^\infty(0,\infty;L^2(\omega))$, then we get from Lemma 4.5 a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{ad} \cap L^\infty(0,\infty;L^2(\omega))$ satisfying (4.8)-(4.10). Let us define $\rho_k = \|u_k - \bar u\|_{L^2(Q_\omega)} < 1/k$ and $v_k = (u_k - \bar u)/\rho_k$. Since, $\|v_k\|_{L^2(Q)} = 1$ for every k, we can extract a subsequence denoted in the same way so that $v_k \rightharpoonup v$ in $L^2(Q_\omega)$. The proof is split into four steps.

Step I. $v \in C_{\bar u}$. For all $T < \infty$ we define the functional $j_T : L^1(0,T;L^2(\omega)) \longrightarrow R$ by

$$j_T(u) = \int_0^T \|u(t)\|_{L^2(\omega)}\,dt.$$

Then, we have

$$j'_T(u;v) = \int_{(0,T)\cap I_u^0} \|v(t)\|_{L^2(\omega)}\,dt + \int_{(0,T)\cap I_u}\int_\omega \frac{u(t)v(t)}{\|u(t)\|_{L^2(\omega)}}\,dx\,dt.$$

For every $T \geq T^*$ the following identity holds

$$j'_T(\bar u; v) = j'_{T^*}(\bar u; v) + \int_{T^*}^T \|v(t)\|_{L^2(\omega)}\, dt.$$

The convergence $v_k \rightharpoonup v$ in $L^2(Q_\omega)$ implies that $v_k \rightharpoonup v$ in $L^1(0,T;L^2(\omega))$. Then, using that $v \to j'_T(\bar u; v)$ is convex and continuous, we have for every $T \geq T^*$

$$\begin{aligned}
j'_{T^*}(\bar u; v) &+ \int_{T^*}^T \|v(t)\|_{L^2(\omega)}\, dt = j'_T(\bar u; v) \leq \liminf_{k\to\infty} j'_T(\bar u; v_k)\\
&\leq \liminf_{k\to\infty} \frac{j_T(\bar u + \rho_k v_k) - j_T(\bar u)}{\rho_k} = \liminf_{k\to\infty} \frac{j_T(u_k) - j_T(\bar u)}{\rho_k} \leq \liminf_{k\to\infty} \frac{j(u_k) - j(\bar u)}{\rho_k}.
\end{aligned}$$

Taking the supremum with respect to T we infer

$$j'(\bar u; v) = j'_{T^*}(\bar u; v) + \int_{T^*}^\infty \|v(t)\|_{L^2(\omega)}\, dt \leq \liminf_{k\to\infty} \frac{j(u_k) - j(\bar u)}{\rho_k}.$$

From this inequality and (4.8)–(4.9) we get

$$\begin{aligned}
F'(\bar u)\, v &+ \gamma\, j'(\bar u; v)\\
&\leq \liminf_{k\to\infty} \frac{1}{\rho_k}\{[F(\bar u + \rho_k v_k) - F(\bar u)] + \gamma[j(\bar u + \rho_k v_k) - j(\bar u)]\}\\
&= \liminf_{k\to\infty} \frac{1}{\rho_k}[J(u_k) - J(\bar u)] \leq \liminf_{k\to\infty} \frac{1}{2k\rho_k}\|u_k - \bar u\|^2_{L^2(Q_\omega)} = \liminf_{k\to\infty} \frac{\rho_k}{2k} = 0.
\end{aligned}$$

To prove the converse inequality we use that

$$\max_{\lambda \in \partial j(\bar u)} \int_{Q_\omega} \lambda v\, dx\, dt = j'(\bar u; v).$$

Then, with inequality (4.3) we get

$$\begin{aligned}
F'(\bar u)v + \gamma j'(\bar u; v) &\geq F'(\bar u)v + \gamma \int_{Q_\omega} \bar\lambda v\, dx\, dt\\
&= \lim_{k\to\infty} \frac{1}{\rho_k} \int_{Q_\omega} (\bar\varphi + \nu \bar u + \gamma\bar\lambda)(u_k - \bar u)\, dx\, dt \geq 0. \qquad (4.12)
\end{aligned}$$

The last two inequalities imply that $F'(\bar u)\, v + \gamma\, j'(\bar u; v) = 0$. Moreover, from this identity we infer

$$\int_{T^*}^\infty \|v(t)\|_{L^2(\omega)}\, dt = \frac{1}{\gamma}F'(\bar u)v - j'_{T*}(\bar u; v) < \infty.$$

Consequently, $v \in \mathcal{U}$ holds. In addition, since v_k satisfies the sign conditions (4.6) for every k and the set of elements of $L^2(Q_\omega)$ satisfying (4.6) is convex and closed, we deduce that v satisfies (4.6) as well. Hence, we have that $v \in C_{\bar u}$.

Step II. $v = 0$. For $\beta > 0$ small we define

$$I_\beta = \{t \in (0,\infty) : \|\bar u(t)\|_{L^2(\omega)} \geq \beta\} \;\text{ and }\; j_\beta(u) = \int_{I_\beta} \|u(t)\|_{L^2(\omega)}\, dt,$$

and with Lemma 4.5

$$I_{\beta,k} = I_\beta \cap I_k \quad \text{and} \quad j_{\beta,k}(u) = \int_{I_{\beta,k}} \|u(t)\|_{L^2(\omega)}\, dt.$$

Since $\|\bar u(t)\|_{L^2(\omega)} \geq \beta > 0$ for every $t \in I_{\beta,k}$, $j_{\beta,k} : L^\infty(0,\infty; L^2(\omega)) \longrightarrow \mathbb{R}$ is infinitely differentiable in the $L^\infty(0,\infty;L^2(\omega))$ open ball $B_{\frac{\beta}{2}}(\bar u)$. Hence, if $k > \frac{2}{\beta}$, (4.10) implies that $\bar u + \rho_k v_k = u_k \in B_{\frac{\beta}{2}}(\bar u)$. Then, by a Taylor expansion we get

$$\begin{aligned}
&j_{\beta,k}(\bar u + \rho_k\, v_k) - j_{\beta,k}(\bar u) = \rho_k j'_{\beta,k}(\bar u; v_k) + \frac{\rho_k^2}{2} j''_{\beta,k}(\bar u; v_k^2) + \frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)\\
&= \rho_k \int_{I_{\beta,k}} \frac{1}{\|\bar u(t)\|_{L^2(\omega)}} \int_\omega \bar u(x,t)\, v_k(x,t)\, dx\, dt\\
&\quad + \frac{\rho_k^2}{2} \int_{I_{\beta,k}} \frac{1}{\|\bar u(t)\|_{L^2(\omega)}} \left\{ \int_\omega v_k^2(x,t)\, dx - \left(\int_\omega \frac{\bar u(x,t)}{\|\bar u(t)\|_{L^2(\omega)}} v_k(x,t)\, dx \right)^2 \right\} dt\\
&\quad + \frac{\rho_k^3}{6} \int_{I_{\beta,k}} \frac{3}{\|u_{\vartheta_k}(t)\|^3_{L^2(\omega)}} \left\{ \frac{1}{\|u_{\vartheta_k}(t)\|^2_{L^2(\omega)}} \left(\int_\omega u_{\vartheta_k}(x,t)\, v_k(x,t)\, dx \right)^3 \right.\\
&\quad \left. - \left(\int_\omega v_k(x,t)^2\, dx \right) \left(\int_\omega u_{\vartheta_k}(x,t)\, v_k(x,t)\, dx \right) \right\} dt,
\end{aligned}$$

where $u_{\vartheta_k} = \bar u + \vartheta_k \rho_k\, v_k \in B_{\frac{\beta}{2}}(\bar u)$ with $0 \leq \vartheta_k(t) \leq 1$. Now, using the convexity of the mapping $w \to \|w\|_{L^2(\omega)}$, we get for every t

$$\|(\bar u + \rho_k\, v_k)(t)\|_{L^2(\omega)} - \|\bar u(t)\|_{L^2(\omega)} \geq \frac{\rho_k}{\|\bar u(t)\|_{L^2(\omega)}} \int_\omega \bar u(x,t) v_k(x,t)\, dx.$$

Using this inequality we obtain

$$\begin{aligned}
&j(\bar u + \rho_k\, v_k) - j(\bar u) = \rho_k \int_{I^0_{\bar u}} \|v_k(t)\|_{L^2(\omega)}\\
&+ \int_{I_{\bar u} \setminus I_{\beta,k}} \left\{ \|(\bar u + \rho_k\, v_k)(t)\|_{L^2(\omega)} - \|\bar u(t)\|_{L^2(\omega)} \right\} dt + [j_{\beta,k}(\bar u + \rho_k\, v_k) - j_{\beta,k}(\bar u)]\\
&\geq \rho_k\, j'(\bar u; v_k) + \frac{\rho_k^2}{2} j''_{\beta,k}(\bar u; v_k^2) + \frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3).
\end{aligned}$$

From (4.9), the above inequality, and the fact that $F'(\bar u)\, v_k + \gamma\, j'(\bar u; v_k) \geq 0$, see (4.12), we get

$$\begin{aligned}
&\frac{\rho_k^2}{2k} > J(\bar u + \rho_k\, v_k) - J(\bar u) \geq \rho_k \{F'(\bar u)\, v_k + \gamma\, j'(\bar u; v_k)\}\\
&\quad + \frac{\rho_k^2}{2} \{F''(\bar u)\, v_k^2 + \gamma\, j''_{\beta,k}(\bar u; v_k^2)\} + \frac{\rho_k^2}{2} [F''(u_{\theta_k}) - F''(\bar u)] v_k^2 + \gamma \frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3),
\end{aligned}$$

where $u_{\theta_k} = \bar u + \theta_k \rho_k (u_k - \bar u)$ with $0 \leq \theta_k \leq 1$. We deduce from (4.3)

$$\frac{\rho_k^2}{2k} > \frac{\rho_k^2}{2} \{F''(\bar u)\, v_k^2 + \gamma j''_{\beta,k}(\bar u; v_k^2)\} + \frac{\rho_k^2}{2} [F''(u_{\theta_k}) - F''(\bar u)] v_k^2 + \gamma \frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3).$$

Dividing this expression by $\rho_k^2/2$ we obtain

$$F''(\bar u)\, v_k^2 + \gamma\, j''_{\beta,k}(\bar u; v_k^2) < |[F''(u_{\theta_k}) - F''(\bar u)]v_k^2| + \gamma \frac{\rho_k}{3} |j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)| + \frac{1}{k}. \tag{4.13}$$

Since $F : L^2(Q_\omega) \longrightarrow \mathbb{R}$ is of class C^2, $u_{\theta_k} \to \bar u$ in $L^2(Q_\omega)$, and $\|v_k\|_{L^2(Q_\omega)} = 1$ we have

$$\lim_{k\to\infty} |[F''(u_{\theta_k}) - F''(\bar u)]v_k^2| \le \lim_{k\to\infty} \|F''(u_{\theta_k}) - F''(\bar u)\| = 0, \tag{4.14}$$

where $\|\cdot\|$ denotes the norm in the space of bilinear forms on $L^2(Q_\omega)^2$. Let us estimate the second term of (4.13). Observe that every element $u \in B_{\frac{\beta}{2}}(\bar u)$ satisfies $\|u(t)\|_{L^2(\omega)} \ge \frac{\beta}{2}$. Using that $u_{\vartheta_k} \in B_{\frac{\beta}{2}}(\bar u)$ for $k > \frac{2}{\beta}$, Hölder's inequality, the expression of $j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)$, (4.10), and $\|v_k\|_{L^2(Q_\omega)} = 1$, we obtain

$$\begin{aligned} |j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)| \quad &\le 6 \int_{I_{\beta,k}} \frac{\|v_k(t)\|^3_{L^2(\omega)}}{\|u_{\vartheta_k}(t)\|^2_{L^2(\omega)}}\, dt \le \frac{24}{\beta^2} \int_{I_{\beta,k}} \|v_k(t)\|^3_{L^2(\omega)} dt \\ &\le \frac{24}{\beta^2} \|v_k\|_{L^\infty(I_{\beta,k};L^2(\omega))} \int_{I_{\beta,k}} \|v_k(x)\|^2_{L^2(\omega)} dt \le \frac{24}{\beta^2 \rho_k k}. \end{aligned}$$

So we get

$$\gamma \frac{\rho_k}{3} |j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)| \le \frac{8\gamma}{k\beta^2} \to 0 \ \text{ as } k \to \infty. \tag{4.15}$$

Since $I_\beta \subset (0, T^*)$, we have that $|I_\beta \setminus I_{\beta,k}| \to 0$ as $k \to \infty$. Hence, the convergence $\chi_{I_k} v_k \rightharpoonup v$ in $L^2(I_\beta, L^2(\omega))$ holds. Using this fact and the convexity and continuity of the quadratic form $j''_\beta(\bar u) : L^2(Q_\omega) \to \mathbb{R}$, (4.13), (4.14), and (4.15) we infer the following inequality

$$F''(\bar u)\, v^2 + \gamma\, j''_\beta(\bar u; v^2) \le \liminf_{k\to\infty} \{F''(\bar u)\, v_k^2 + \gamma\, j''_{\beta,k}(\bar u; v_k^2)\} \le 0 \ \ \forall \beta > 0. \tag{4.16}$$

Now, taking the limit as $\beta \to 0$ we conclude that $J''(\bar u; v^2) = F''(\bar u) v^2 + \gamma\, j''(\bar u; v^2) \le 0$. According to the assumption of the theorem, this is possible only if $v = 0$.

Step III. $\lim_{k\to\infty} F''(\bar u) v_k^2 = \nu$. Since $\|v_k\|_{L^2(Q_\omega)} = 1$, from the expression (3.13) we deduce

$$F''(\bar u) v_k^2 = \int_Q [\mathrm{e}^{-\sigma t} - \bar\varphi f''(\bar y)] z_{v_k}^2 \, dx\, dt + \nu.$$

Therefore, it is enough to prove that the integral converges to 0 as $k \to \infty$. Let us select α and β satisfying $2\Lambda_f < \alpha < \beta < \sigma$. Since $v_k \rightharpoonup v = 0$ in $L^2(Q)$ we get from [5, Lemma 3.1] that $z_{v_k} \rightharpoonup 0$ in $H^1_\alpha(Q)$ and there exists a constant C_1 such that

$$\|z_{v_k}\|_{C_\alpha([0,\infty);H^1(\Omega))} \le C_1 \quad \forall k \ge 1. \tag{4.17}$$

From the compactness of the embedding $H^1(Q_T) \subset L^r(Q_T)$ for every $T > 0$ and $r < 4$, we infer that $z_{v_k} \to 0$ as $k \to \infty$ in $L^r(Q_T)$ for every $r < 4$ and $T < \infty$. We also know that $|f''(s)| \le C_2(|s|^{2m-1} + 1)$ for every $s \in \mathbb{R}$ and some constant C_2. Moreover, since $y_d \in L^\infty(0, \infty; L^2(\Omega))$ and $\bar y \in C([0, T]; H^1(\Omega))$, see Theorem 2.1, we get that $\bar\varphi \in C(\bar Q_T)$ for every $T > 0$; see [6, Chapter 3]. All these facts imply with Hölder's inequality for $\frac{4m+2}{2m-1}$ and $\frac{4m+2}{2m+3}$

$$\int_{Q_T} |\mathrm{e}^{-\sigma t} - \bar\varphi f''(\bar y)| z_{v_k}^2\,dx\,dt \le \int_{Q_T} \big(1 + \|\bar\varphi\|_{C(\bar Q_T)} C_2[|\bar y|^{2m-1} + 1]\big) z_{v_k}^2\,dx\,dt$$

$$\le \|z_{v_k}\|^2_{L^2(Q_T)} + \|\bar\varphi\|_{C(\bar Q_T)} C_3\Big(\|\bar y\|^{2m-1}_{L^{4m+2}(Q_T)} + 1\Big)\|z_{v_k}\|^2_{L^{\frac{8m+4}{2m+3}}(Q_T)} \to 0 \qquad (4.18)$$

as $k \to \infty$. Above we have used that $\frac{8m+4}{2m+3} < 4$. Next we prove estimates in the intervals (T,∞). From (4.17) and Hölder's inequality with 2, $\frac{4m+2}{2m-1}$, and $2m+1$, we obtain

$$\int_T^\infty \int_\Omega |\mathrm{e}^{-\sigma t} - \bar\varphi f''(\bar y)| z_{v_k}^2\,dx\,dt = \int_T^\infty \int_\Omega |\mathrm{e}^{(\alpha-\sigma)t} - e^{\beta t}\bar\varphi \mathrm{e}^{(\alpha-\beta)t} f''(\bar y)|\mathrm{e}^{-\alpha t} z_{v_k}^2\,dx\,dt$$

$$\le \|z_{v_k}\|^2_{C_\alpha(0,\infty;L^2(\Omega))} \int_T^\infty \mathrm{e}^{(\alpha-\sigma)t}\,dt$$

$$+ C_4 \int_T^\infty \mathrm{e}^{\beta t}\|\bar\varphi\|_{L^2(\Omega)} \mathrm{e}^{(\alpha-\beta)t}\big(\|\bar y\|^{2m-1}_{L^{4m+2}(\Omega)} + 1\big)\mathrm{e}^{-\alpha t}\|z_{v_k}\|^2_{L^{4m+2}(\Omega)}\,dt$$

$$\le \frac{C_1^2}{\sigma-\alpha}\mathrm{e}^{(\alpha-\sigma)T} + C_5\big(\|\bar y\|^{2m-1}_{C_{\frac{2(\beta-\alpha)}{2m-1}}(0,\infty;H^1(\Omega))} + 1\big) C_1^2 \int_T^\infty \mathrm{e}^{\beta t}\|\varphi\|_{L^2(\Omega)}\,dt \qquad (4.19)$$

Using that $\varphi \in L^2_\alpha(Q)$ for every $\alpha < 2\sigma$ we get for $\rho \in (0, \sigma-\beta)$

$$\int_T^\infty \mathrm{e}^{\beta t}\|\varphi\|_{L^2(\Omega)}\,dt$$

$$\le \Big(\int_T^\infty \mathrm{e}^{-2\rho t}\,dt\Big)^{\frac12}\Big(\int_T^\infty \mathrm{e}^{2(\beta+\rho)t}\|\varphi\|_{L^2(\Omega)}\,dt\Big)^{\frac12} = \frac{1}{\sqrt{2\rho}}\mathrm{e}^{-\rho T}\|\varphi\|_{L^2_{2(\beta+\rho)}(Q)}.$$

From this inequality and (4.19) we infer that for every $\varepsilon > 0$ there exits $T_\varepsilon < \infty$ such that

$$\int_{T_\varepsilon}^\infty \int_\Omega |\mathrm{e}^{-\sigma t} - \bar\varphi f''(\bar y)| z_{v_k}^2\,dx\,dt < \frac{\varepsilon}{2}.$$

Moreover, from (4.18) we obtain the existence of k_ε such that

$$\int_0^{T_\varepsilon} \int_\Omega |\mathrm{e}^{-\sigma t} - \bar\varphi f''(\bar y)| z_{v_k}^2\,dx\,dt < \frac{\varepsilon}{2} \quad \forall k \ge k_\varepsilon.$$

The last two inequalities proves the convergence to zero of the integral.

Step IV. Contradiction. Using that $j''_{\beta,k}(\bar u; v_k^2) \ge 0$, and (4.16), we deduce that $\nu \le \liminf_{k\to\infty}\{F''(\bar u)\,v_k^2 + \gamma\, j''_{\beta,k}(\bar u; v_k^2)\} \le 0$, which contradicts the assumption $\nu > 0$.

Step V. Removing the assumption $u \in L^\infty(0,\infty;L^2(\omega))$. Given $u \in \mathcal{U}_{ad} \cap \bar B_\varepsilon(\bar u)$, we set $v = u - \bar u$, $v_k(x,t) = \mathrm{Proj}_{[-k,+k]}(v(x,t))$, and $u_k = \bar u + v_k$. From Proposition 4.2 we get that $\bar u \in L^\infty(0,\infty;L^2(\omega))$. Then, it is obvious that $\|u_k - \bar u\|_{L^2(Q_\omega)} \le \|u - \bar u\|_{L^2(Q_\omega)} \le \varepsilon$ and $u_k \in \mathcal{U}_{ad} \cap L^\infty(0,\infty;L^2(\omega))$. Hence, (4.7) implies

$$J(\bar u) + \frac{\delta}{2}\|u_k - \bar u\|^2_{L^2(Q_\omega)} \le J(u_k) \quad \forall k \ge 1.$$

Of course, we assume that $u \in L^1(0,\infty;L^2(\omega))$, otherwise the inequality (4.7) is obvious. By Lebesgue's dominated convergence theorem we obtain that $u_k \to u$ in $L^2(Q_\omega)\cap L^1(0,\infty;L^2(\omega))$ and, consequently, $y_{u_k} \to y_u$ in $L^2_\sigma(Q)$; see [5, Theorem 2.7]. Then, we can pass to the limit in the above inequality and deduce that u satisfies (4.7). ∎

5. Approximation of (P)**.** The aim of this section is to approximate the control problem (P) by a sequence of finite horizon optimal control problems and to obtain estimates on the error of these approximations. For every $0 < T < \infty$ we consider the control problem

$$(\mathrm{P}_T) \qquad \min_{u\in\mathcal{U}_{T,ad}} J_T(u) = F_T(u) + \gamma j_T(u),$$

where

$$\mathcal{U}_{T,ad} = \{u \in L^2(Q_{T,\omega}) : u_a \le u(x,t) \le u_b \text{ for a.a. } (x,t) \in Q_{T,\omega}\},$$

$F_T(u) = \frac{1}{2}\|y_{T,u} - y_d\|^2_{L^2_\sigma(Q_T)} + \frac{\nu}{2}\|u\|^2_{L^2(Q_{T,\omega})}$, $j_T(u) = \|u\|_{L^1(0,T;L^2(\omega))}$, $y_{T,u}$ is the solution of (2.1) corresponding to u, and $Q_{T,\omega} = \omega\times(0,T)$.

Given a solution (or a local minimizer) u_T of (P_T), we denote by y_T its associated state. For every control $u \in L^2(Q_{T,\omega})$ with associated state $y_{T,u}$ we consider extensions to Q_ω and Q, denoted by $\hat u_T$ and $\hat y_{T,u}$, by setting $\hat u_T(x,t) = 0$ if $t > T$ and $\hat y_{T,u}$ the corresponding solution of (1.1) associated with the extension $\hat u$. Let us observe that if $u \in \mathcal{U}_{T,ad}$, then $\hat u \in \mathcal{U}_{ad}$ due to our assumptions on u_a and u_b. Using this notation, we prove the following theorem.

THEOREM 5.1. *For every $T > 0$ the control problem* (P_T) *has at least one solution u_T. The extensions $\{\hat u_T\}_{T>0}$ of any family of solutions are bounded in $L^2(Q_\omega)$. Every weak limit $\bar u$ in $L^2(Q_\omega)$ of a sequence $\{\hat u_{T_k}\}_{k=1}^\infty$ with $T_k \to \infty$ as $k \to \infty$ is a solution of* (P)*. Moreover, the strong convergence $\hat u_{T_k} \to \bar u$ in $L^2(Q_\omega)$ holds.*

Proof. The existence proof of a solution is analogous to the one for (P). Let us denote by y^0 the solution of (1.1) corresponding to the control $u \equiv 0$. From Theorem 2.1 we know that $y^0 \in L^2_\sigma(Q)$. Then, using the optimality of u_T we get

$$\frac{\nu}{2}\|\hat u_T\|^2_{L^2(Q_\omega)} + \gamma\|\hat u_T\|_{L^1(0,\infty;L^2(\omega))} \le J_T(u_T) \le J(0).$$

This implies the boundedness of $\{\hat u_T\}_{T>0}$ in $\mathcal{U}$. Therefore, there exists a sequence $\{T_k\}_{k=1}^\infty$ converging to ∞ such that $\hat u_{T_k} \rightharpoonup \bar u$ in $L^2(Q_\omega)$. Then, using Lemma 2.2 we get that $\hat y_{T_k} \rightharpoonup \bar y$ in $H^1_\sigma(Q)$. These facts yield

$$\begin{cases} \|\bar u\|_{L^2(Q_\omega)} \le \liminf_{k\to\infty} \|\hat u_{T_k}\|_{L^2(Q_\omega)}, \\ \|\bar y - y_d\|_{L^2_\sigma(Q)} \le \liminf_{k\to\infty} \|\hat y_{T_k} - y_d\|_{L^2_\sigma(Q)}. \end{cases} \tag{5.1}$$

Moreover, the continuous inclusion $L^2(0,T;L^2(\omega)) \subset L^1(0,T;L^2(\omega))$ for all $T > 0$ implies for $\gamma > 0$

$$\|\bar u\|_{L^1(0,T;L^2(\omega))} \le \liminf_{k\to\infty} \|\hat u_{T_k}\|_{L^1(0,T;L^2(\omega))} \le \liminf_{k\to\infty} \|\hat u_{T_k}\|_{L^1(0,\infty;L^2(\omega))} \le \frac{1}{\gamma}J(0).$$

From here we get

$$\|\bar u\|_{L^1(0,\infty;L^2(\omega))} = \sup_{T>0} \|\bar u\|_{L^1(0,T;L^2(\omega))} \le \liminf_{k\to\infty} \|\hat u_{T_k}\|_{L^1(0,\infty;L^2(\omega))} < \infty. \tag{5.2}$$

From (5.1), (5.2), and the optimality of u_T we infer for every $u \in \mathcal{U}_{ad}$

$$J(\bar u) \le \liminf_{k\to\infty} J(\hat u_{T_k}) \le \liminf_{k\to\infty} J_{T_k}(u_{T_k}) + \limsup_{k\to\infty} \frac{1}{2}\int_{T_k}^{\infty} \mathrm{e}^{-\sigma t}\|\hat y_{T_k}(t) - y_d(t)\|^2_{L^2(\Omega)}\,dt$$
$$\le \liminf_{k\to\infty} J_{T_k}(u) = J(u).$$

Above we have used the inequality

$$\|y_u\|_{C([0,T];L^2(\Omega))} \le K_f\Big(\|y_0\|_{L^2(\Omega)} + \big[\|g\|_{L^\infty(0,\infty;L^2(\Omega))} + 1\big]\sqrt{T} + \|u\|_{L^2(Q_\omega)}\Big), \quad (5.3)$$

see [5, Theorem 2.2], and the fact that $y_d \in L^\infty(0,\infty;L^2(\Omega))$ to deduce

$$\limsup_{k\to\infty} \frac{1}{2}\int_{T_k}^{\infty} \mathrm{e}^{-\sigma t}\|\hat y_{T_k}(t) - y_d(t)\|^2_{L^2(\Omega)}\,dt \le 2\limsup_{k\to\infty}\int_{T_k}^{\infty} \mathrm{e}^{-\sigma t}\|\hat y_{T_k}(t)\|^2_{L^2(\Omega)}\,dt$$
$$+ 2\limsup_{k\to\infty}\int_{T_k}^{\infty} \mathrm{e}^{-\sigma t}\|y_d(t)\|^2_{L^2(\Omega)}\,dt \le C\limsup_{k\to\infty}\int_{T_k}^{\infty}(t+1)\mathrm{e}^{-\sigma t}\,dt = 0.$$

This proves that $\bar u$ is a solution of (P). Let us prove that the convergence of $\{\hat u_{T_k}\}_{k=1}^\infty$ to $\bar u$ is strong. First we observe that the above convergence to zero and the optimality of u_{T_k} imply

$$J(\bar u) \le \liminf_{k\to\infty} J(\hat u_{T_k}) \le \limsup_{k\to\infty} J(\hat u_{T_k}) = \limsup_{k\to\infty} J_{T_k}(u_{T_k}) \le \limsup_{k\to\infty} J_{T_k}(\bar u) = J(\bar u),$$

which implies that $J(\bar u) = \lim_{k\to\infty} J(\hat u_{T_k})$. This identity along with (5.1) and (5.2), and Lemma 5.2 below lead to $\lim_{k\to\infty}\|\hat u_{T_k}\|_{L^2(Q_\omega)} = \|\bar u\|_{L^2(Q_\omega)}$. Thus, $\hat u_{T_k} \to \bar u$ in $L^2(Q_\omega)$ holds. □

Lemma 5.2. *For every $j = 1,\dots,k$ with $k \ge 2$, let $\{\alpha_{j,T}\}_{T>0}$ be a family of real numbers satisfying*

$$\alpha_j \le \liminf_{T\to\infty} \alpha_{j,T} \text{ for } 1 \le j \le k \quad \text{and} \quad \lim_{T\to\infty}\sum_{j=1}^k \alpha_{j,T} = \sum_{j=1}^k \alpha_j,$$

where $\{\alpha_j\}_{j=1}^k \subset \mathbb{R}$. Then, the equalities $\lim_{T\to\infty}\alpha_{j,T} = \alpha_j$ hold for every $j = 1,\dots,k$.

Proof. We proceed by induction on k. First, we assume that $k = 2$. The convergence $\alpha_{1,T} \to \alpha_1$ is obtained as follows

$$\alpha_1 \le \liminf_{T\to\infty}\alpha_{1,T} \le \limsup_{T\to\infty}\alpha_{1,T}$$
$$\le \limsup_{T\to\infty}(\alpha_{1,T} + \alpha_{2,T}) - \liminf_{T\to\infty}\alpha_{2,T} \le (\alpha_1 + \alpha_2) - \alpha_2 = \alpha_1.$$

Now the convergence $\lim_{T\to\infty}\alpha_{2,T} = \alpha_2$ is immediate. Let us take $k > 2$ and assume that the statement is valid for $k-1$. Proceeding as above we get

$$\alpha_1 \le \liminf_{T\to\infty}\alpha_{1,T} \le \limsup_{T\to\infty}\alpha_{1,T}$$
$$\le \limsup_{T\to\infty}\sum_{j=1}^k \alpha_{j,k} - \liminf_{T\to\infty}\sum_{j=2}^k \alpha_{j,k} \le \sum_{j=1}^k \alpha_j - \sum_{j=2}^k \alpha_j = \alpha_1$$

Then, we have from the above inequalities and the assumption of the lemma

$$\lim_{T\to\infty} \alpha_{1,T} = \alpha_1 \quad \text{and} \quad \lim_{T\to\infty} \sum_{j=2}^{k} \alpha_{j,T} = \sum_{j=2}^{k} \alpha_j.$$

Now, the statement follows by the induction hypothesis. □

THEOREM 5.3. *Let $\bar u$ be a strict local minimizer of* (P)*. Then, there exist $T_0 > 0$ and a family $\{u_T\}_{T>T_0}$ of local minimizers to* (P_T) *such that $\hat u_T \to \bar u$ in $L^2(Q_\omega)$ as $T \to \infty$.*

Proof. Since $\bar u$ is a strict local minimizer of (P), there exists $\rho > 0$ such that $J(\bar u) < J(u)$ for every $u \in \mathcal{U}_{ad} \cap B_\rho(\bar u)$ with $u \neq \bar u$, where $B_\rho(\bar u)$ is the closed ball in $L^2(Q_\omega)$ centered at $\bar u$ and radius $\rho > 0$. We consider the control problems

$$(\mathrm{P}_\rho) \quad \min_{u\in B_\rho(\bar u)\cap \mathcal{U}_{ad}} J(u) \quad \text{and} \quad (\mathrm{P}_{T,\rho}) \quad \min_{u\in B_{T,\rho}(\bar u)\cap \mathcal{U}_{T,ad}} J_T(u),$$

where $B_{T,\rho}(\bar u) = \{u \in L^2(Q_{T,\omega}) : \|u - \bar u\|_{L^2(Q_{T,\omega})} \le \rho\}$. Obviously $\bar u$ is the unique solution of (P_ρ). Existence of a solution u_T of ($\mathrm{P}_{T,\rho}$) is straightforward. Then, arguing as in the proof of Theorem 5.1 and using the uniqueness of the solution of (P_ρ), we deduce the convergence $\hat u_T \to \bar u$ in $L^2(Q_\omega)$. This implies the existence of $T_0 > 0$ such that $\|u_T - \bar u\|_{L^2(Q_{T,\omega})} \le \|\hat u_T - \bar u\|_{L^2(Q_\omega)} < \rho$ for all $T > T_0$. Hence, u_T is also a local minimizer of (P_T) for $T > T_0$. □

THEOREM 5.4. *Assume that $\sigma > 8\Lambda_f$, $u_a < 0 < u_b$ if $\gamma > 0$, and let $\bar u$ be a local minimizer of* (P) *satisfying the sufficient second order optimality condition $F''(\bar u)v^2 + \gamma j''(\bar u; v^2) > 0$ for all $v \in C_{\bar u} \setminus \{0\}$. Let $\{u_T\}_{T>T_0}$ be a family of local minimizers of* (P_T) *as selected in Theorem 5.3. Then, there exists a constant C independent of T such that*

$$\|\hat u_T - \bar u\|_{L^2(Q_\omega)} \le \frac{C}{\sigma - \Lambda_f}\Big(T + 1\Big)\mathrm{e}^{-\sigma T} \quad \forall T > T_0. \tag{5.4}$$

In addition, for every $\alpha > 4\Lambda_f$ there exists a constant C_α such that

$$\|\hat y_T - \bar y\|_{Y_\alpha} \le \frac{C_\alpha}{\sigma - \Lambda_f}\Big(T + 1\Big)\mathrm{e}^{-\sigma T} \quad \forall T > T_0. \tag{5.5}$$

Proof. Under our hypotheses, Theorem 4.4 is applicable. Accordingly let $\varepsilon > 0$ and $\delta > 0$ be such that (4.7) holds. Following the proof of Theorem 5.3 with $\rho = \varepsilon$, we have that u_T is a solution of ($\mathrm{P}_{T,\rho}$). Using (4.7), the optimality of u_T and the fact that $\hat u(x,t) = 0$ for $t > T$ we get

$$\begin{aligned} &\frac{\delta}{2}\|\hat u_T - \bar u\|^2_{L^2(Q_\omega)} \le J(\hat u_T) - J(\bar u) \le J_T(u_T) - J_T(\bar u)\\ &+ \frac12\int_T^\infty \mathrm{e}^{-\sigma t}\|\hat y_T - y_d\|^2_{L^2(\Omega)}\,dt - \frac12\int_T^\infty \mathrm{e}^{-\sigma t}\|\bar y - y_d\|^2_{L^2(\Omega)}\,dt\\ &\le \frac12\int_0^\infty \mathrm{e}^{-\sigma t}\chi_{(T,\infty)}(t)\|\hat y_T - y_d\|^2_{L^2(\Omega)}\,dt - \frac12\int_0^\infty \mathrm{e}^{-\sigma t}\chi_{(T,\infty)}(t)\|\bar y - y_d\|^2_{L^2(\Omega)}\,dt, \end{aligned} \tag{5.6}$$

where $\chi_{(T,\infty)}$ denotes the characteristic function of the interval (T,∞).

Let $\mathcal{F} : L^2(Q_\omega) \longrightarrow \mathbb{R}$ be the function defined by

$$\mathcal{F}(u) = \frac12\int_0^\infty \mathrm{e}^{-\sigma t}\chi_{(T,\infty)}(t)\|y_u - y_d\|^2_{L^2(\Omega)}\,dt.$$

By [5, Theorem 3.3] and the chain rule we infer that $\mathcal{F}$ is of class C^2. Arguing as in the proof of Theorem 3.4 we get

$$\mathcal{F}'(u)v = \int_{Q_\omega} \varphi_u v \, dx \, dt,$$

where φ_u is the solution of the adjoint state equation

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} - \Delta\varphi + a\varphi + f'(y_u)\varphi = \mathrm{e}^{-\sigma t}\chi_{(T,\infty)}(y_u - y_d) \text{ in } Q, \\ \partial_n \varphi = 0 \text{ on } \Sigma, \ \lim_{t\to\infty} \mathrm{e}^{\Lambda_f t}\|\varphi(t)\|_{L^2(\Omega)} = 0. \end{cases} \tag{5.7}$$

Applying the mean value theorem in the right hand side of (5.6) we infer

$$\frac{\delta}{2}\|\hat{u}_T - \bar{u}\|^2_{L^2(Q_\omega)} \le \int_{Q_\omega} \varphi_{\theta_T}(\hat{u}_T - \bar{u}) \, dx \, dt \le \|\varphi_{\theta_T}\|_{L^2(Q_\omega)}\|\hat{u}_T - \bar{u}\|_{L^2(Q_\omega)},$$

which implies

$$\|\hat{u}_T - \bar{u}\|_{L^2(Q_\omega)} \le \frac{2}{\delta}\|\varphi_{\theta_T}\|_{L^2(Q_\omega)}, \tag{5.8}$$

where φ_{θ_T} is the solution of (5.7) with y_u replaced by y_{θ_T}, the state associated with $u_{\theta_T} = \bar{u} + \theta_T(\hat{u}_T - \bar{u})$, $\theta_T \in (0,1)$. We use that $\{u_{\theta_T}\}_{T>T_0}$ is bounded in $L^2(Q)$, $y_d \in L^\infty(0,\infty;L^2(\Omega))$ to estimate φ_{θ_T}. First, we test the equation satisfied by φ_{θ_T} with $\mathrm{e}^{2\Lambda_f s}\varphi_{\theta_T}$ and integrate in $(t,\hat{T})$ for $\hat{T} \ge T$ and $t \in (0,\hat{T})$

$$\begin{aligned} &\frac{\mathrm{e}^{2\Lambda_f t}}{2}\Big\|\varphi_{\theta_T}(t)\Big\|^2_{L^2(\Omega)} + \Lambda_f\int_t^{\hat{T}} \mathrm{e}^{2\Lambda_f s}\|\varphi_{\theta_T}(s)\|^2_{L^2(\Omega)}\, ds \\ &+ \Lambda_f\int_t^{\hat{T}} \mathrm{e}^{2\Lambda_f s}\int_\Omega [|\nabla\varphi_{\theta_T}|^2 + a\varphi^2_{\theta_T} + f'(y)\varphi^2_{\theta_T}]\, dx\, ds \\ &= \int_T^{\hat{T}} \mathrm{e}^{(2\Lambda_f - \sigma)s}\int_\Omega (y_{\theta_T} - y_d)\varphi_{\theta_T}\, dx\, ds + \frac{\mathrm{e}^{2\Lambda_f \hat{T}}}{2}\Big\|\varphi_{\theta_T}(\hat{T})\Big\|^2_{L^2(\Omega)}. \end{aligned}$$

Arguing as in the proof of (3.8), using that $\lim_{t\to\infty}\mathrm{e}^{\Lambda_f t}\|\varphi_{\theta_T}(t)\|_{L^2(\Omega)} = 0$ and taking the limit as $\hat{T} \to \infty$, we infer

$$\operatorname*{ess\,sup}_{t>0} \mathrm{e}^{\Lambda_f t}\|\varphi_{\theta_T}(t)\|_{L^2(\Omega)} + \|\varphi_{\theta_T}\|_{L^2_{-2\Lambda_f}(0,\infty;H^1(\Omega))} \le C_1\|\chi_{[T,\infty)}(y_{\theta_T} - y_d)\|_{L^2_{2(\sigma-\Lambda_f)}(Q)}.$$

Additionally, taking $\hat{T} = T$ in the above inequalities we get $\operatorname*{ess\,sup}_{t\in[0,T]}\|\varphi_{\theta_T}(t)\|_{L^2(\Omega)} \le \|\varphi_{\theta_T}(T)\|_{L^2(\Omega)}$. Then, we have

$$\begin{aligned} &\|\varphi_{\theta_T}\|_{L^2(Q_T)} \le \sqrt{T} \operatorname*{ess\,sup}_{t\in[0,T]}\|\varphi_{\theta_T}(t)\|_{L^2(\Omega)} \le \sqrt{T}\|\varphi_{\theta_T}(T)\|_{L^2(\Omega)} \\ &\le \mathrm{e}^{-\Lambda_f T}\sqrt{T}\mathrm{e}^{\Lambda_f T}\|\varphi_{\theta_T}(T)\|_{L^2(\Omega)} \le C_1\mathrm{e}^{-\Lambda_f T}\sqrt{T}\|\chi_{[T,\infty)}(y_{\theta_T} - y_d)\|_{L^2_{2(\sigma-\Lambda_f)}(Q)}. \end{aligned}$$

Finally, we get

$$\|\varphi_{\theta_T}\|_{L^2(Q)} \le \|\varphi_{\theta_T}\|_{L^2(Q_T)} + \Big(\int_T^\infty \|\varphi_{\theta_T}(t)\|^2_{L^2(\Omega)}\,dt\Big)^{\frac12} \le \|\varphi_{\theta_T}\|_{L^2(Q_T)}$$
$$+ \mathrm{e}^{-\Lambda_f T}\|\varphi_{\theta_T}\|_{L^2_{-2\Lambda_f}(Q)} \le C_1\mathrm{e}^{-\Lambda_f T}(\sqrt{T}+1)\|\chi_{[T,\infty)}(y_{\theta_T}-y_d)\|_{L^2_{2(\sigma-\Lambda_f)}(Q)}$$
$$= C_1\mathrm{e}^{-\Lambda_f T}(\sqrt{T}+1)\left(\int_T^\infty \mathrm{e}^{2(\Lambda_f-\sigma)t}\|y_{\theta_T}(t)-y_d(t)\|^2_{L^2(\Omega)}\,dt\right)^{\frac12}$$
$$\le C_2\mathrm{e}^{-\Lambda_f T}(\sqrt{T}+1)\left(\int_T^\infty \mathrm{e}^{2(\Lambda_f-\sigma)t}(t+1)\,dt\right)^{\frac12} \le \frac{C_3}{\sigma-\Lambda_f}\Big(T+1\Big)\mathrm{e}^{-\sigma T}.$$

This estimate along with (5.8) leads to (5.4).

The estimate (5.5) follows from (5.4) and the generalized mean value theorem

$$\|\hat{y}_T-\bar{y}\|_{Y_\alpha} \le \sup_{\theta\in[0,1]}\|G'_\alpha(\bar{u}+\theta(\hat{u}_T-\bar{u})\|\|\hat{u}_T-\bar{u}\|_{L^2(Q_\omega)},$$

where $\|G'_\alpha(\bar{u}+\theta(\hat{u}_T-\bar{u}))\|$ denotes the norm in $\mathcal{L}(L^2(Q_\omega),Y_\alpha)$. ☐

Remark 5.5. *In the formulation of the control problem* (P)*, the discount factor could be introduced after some period of time. This means that the weight* $\mathrm{e}^{-\sigma t}$ *could be replaced by weights of type*

$$w(t) = \begin{cases} 1 & \text{if } t \le \bar{T}, \\ \mathrm{e}^{-\sigma(t-\bar{T})} & \text{if } t > \bar{T}. \end{cases}$$

The results proved in this paper can be easily modified to include this type of weights.

REFERENCES

[1] S.M. Aseev and A.V. Kryazhimskiy. The Pontryagin maximum principle and transversality conditions for a class of optimal control problems with infinite time horizons. *SIAM J. Control Optim.*, 43(3):1094–1119, 2004.

[2] E. Casas, R. Herzog, and G. Wachsmuth. Analysis of spatio-temporally sparse optimal control problems of semilinear parabolic equations. *ESAIM Control Optim. Calc. Var.*, 23:263–295, 2017.

[3] E. Casas and K. Kunisch. Stabilization by sparse controls for a class of semilinear parabolic equations. *SIAM J. Control Optim.*, 55(1):512–532, 2017.

[4] E. Casas and K. Kunisch. Infinite horizon optimal control problems for a class of semilinear parabolic equations. *SIAM J. Control Optim.*, 60(4):2070–2094, 2022.

[5] E. Casas and K. Kunisch. Infinite horizon optimal control problems with discount factor on the state. Part I: Analysis of the controlled state equation. *Submitted*, 2022.

[6] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva. *Linear and Quasilinear Equations of Parabolic Type*. American Mathematical Society, Providence, 1988.

[7] R.E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, volume 49 of *Math. Surv. and Monogr.* American Mathematical Society, Providence, RI, 1997.