INFINITE HORIZON OPTIMAL CONTROL PROBLEMS WITH DISCOUNT FACTOR ON THE STATE. PART I: ANALYSIS OF THE CONTROLLED STATE EQUATION *

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Abstract. This is the first part of an investigation of infinite horizon optimal control problems subject to semi-linear parabolic equations. A discount fact on the state variable is introduced in the cost. This allows the treatment of infinite horizon problems without stabilizability assumptions. The nonlinearities can be of polynomial type thus covering reaction diffusion equations which are important for applications. The control to state mapping and its regularity are analysed in details. This involves the relation between the type of the nonlinearity and the discount factor.

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Key words. Semi-linear parabolic equations, infinite horizon, discount factor, differentiability of the control-to-state mapping.

1. Introduction. This is the first part of our work in which we continue our efforts on infinite horizon optimal control problems for semi-linear parabolic differential equations. The specificity of the present contribution lies in the introduction of a discount factor on the state variable in the cost functional. This leads to important differences to our earlier work [8] and [9] with respect to the nature of the control problems and from the analytical perspective. Semi-linear parabolic equations appear in a multitude of applications, frequently with nonlinearities of polynomial type. Cubic polynomials arise for example in the Allen Cahn equation, modelling phase separation in multi-component alloys, in the Schlögel model, arising in chemical reactions, or the Newell-Whitehead equation, describing the evolution of self-organising systems. A quadratic nonlinearity appears in the Fisher equation modelling population growth, for example. In all this cases, when formulating optimal control problems of tracking type, the choice a specific time horizon over which the optimization takes place can be delicate and is to some extent ad hoc. The introduction of an infinite time horizon then arises a natural alternative to formulate the optimal control problem under consideration, unless it is conceived as infinite horizon problem from the start.

Let us mention some of the literature on infinite horizon optimal control. In the monograph [7] the importance of the infinite time horizon for problems in mathematical biology and in economics is stressed and examples are provided. The mathematical analysis of infinite horizon optimal control problems was likely started with the work of Halkin, see [10]. We also point at recent contributions in [3, 1, 2, 4, 15]. Except for one chapter in [7], which is devoted to partial differential equations, all of these contributions are concerned with control problems for ordinary differential equations. The case of partial differential equations has received significantly less attention. We point, however, to one section in the classical monograph [12, Chapter III.6], which is dedicated to infinite horizon problems, and to [5], [6], where bilinear optimal control problems are investigated. In these papers, just as in our previous papers [8], [9] no

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discount factors are utilized. Thus the infinite horizon formulation relates to optimal stabilization and with optimal trajectories typically asymptotically converging to steady states. This is not the case once a discount factor is introduced in the cost functional.

Our goal is the analysis of the optimal control problem

$$\text{(P)}\quad \min_{u\in\mathcal{U}_{ad}} J(u)=\frac{1}{2}\int_0^\infty \mathrm{e}^{-\sigma t}\|y_u-y_d\|^2_{L^2(\Omega)}\,dt+\frac{\nu}{2}\int_0^\infty \|u\|^2_{L^2(\omega)}\,dt+\gamma\int_0^\infty \|u\|_{L^2(\omega)}\,dt,$$

where $\mathcal{U}_{ad}=\{u\in L^2(0,\infty;L^2(\omega)):u_a\le u(x,t)\le u_b\ \text{ for a.a. } (x,t)\in\omega\times(0,\infty)\}$, $-\infty\le u_a\le 0\le u_b\le+\infty$, $\sigma>0$, $\nu>0$, and $\gamma\ge 0$. Here y_u denotes the solution of the following parabolic equation:

$$\left\{\begin{array}{l}\dfrac{\partial y}{\partial t}-\Delta y+ay+f(y)=g+u\chi_\omega\ \text{ in } Q=\Omega\times(0,\infty),\\ \partial_n y=0\ \text{ on } \Sigma=\Gamma\times(0,\infty),\ y(0)=y_0\ \text{ in } \Omega,\end{array}\right. \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $1\le n\le 3$, with a Lipschitz boundary Γ, ω is a subdomain of Ω, $g\in L^\infty(0,\infty;L^2(\Omega))$, χ_ω denotes the characteristic function of ω, $a\in L^\infty(\Omega)$, $0\le a\not\equiv 0$, and $y_0\in H^1(\Omega)$. The symbol $u\chi_\omega$ is defined as follows:

$$(u\chi_\omega)(x,t)=\left\{\begin{array}{ll}u(x,t) & \text{if } (x,t)\in Q_\omega=\omega\times(0,\infty),\\ 0 & \text{otherwise.}\end{array}\right.$$

The parameter σ is known as the discount factor. The last term in the cost functional is included to promote sparsity in time of the optimal controls.

In Part I, we mainly concentrate on the analysis of the control to state mapping $u\to y_u$. The fact that we need to consider this mapping over the half axis, leads to many technical challenges for which we cannot refer back to the finite horizon case, which has been intensely analyzed in the past, see for instance [16]. It is also different from the analysis in [9] which did not involve a discount factor, and which, as a consequence, did not allow us to analyze the differentiability properties of the control to state mapping. In Part II the optimization theoretic aspects of (P) will be investigated.

For the nonlinear term $f:\mathbb{R}\longrightarrow\mathbb{R}$ in state equation we assume that $f=f_1+f_2$, such that f_1 is a polynomial of odd degree $2m+1$ with a positive leading coefficient, $0\le m\le 1$ if $n=3$, and $m\ge 0$ is an arbitrary integer if $n\le 2$, and $f_2:\mathbb{R}\longrightarrow\mathbb{R}$ is a C^2 function satisfying

$$f_1(0)=f_2(0)=0\ \text{ and }\ \exists L_f>0:|f_2'(s)|+|f_2''(s)|\le L_f\ \forall s\in\mathbb{R}. \tag{1.2}$$

Since f_1 is a polynomial of odd order with positive leading coefficient we infer

$$\exists\Lambda_1\ge 0\ \text{ such that }\ f_1'(s)\ge-\Lambda_1\ \forall s\in\mathbb{R}. \tag{1.3}$$

From (1.2) and (1.3) we deduce

$$f'(s)\ge-\Lambda_f=-(\Lambda_1+L_f)\ \ \forall s\in\mathbb{R}, \tag{1.4}$$

$$\text{If } m>0,\text{then } \exists M_f \text{ such that } f'(s)>0 \text{ and } f(s)s\ge 0\ \ \forall|s|\ge M_f. \tag{1.5}$$

Remark 1.1. The assumption $a\not\equiv 0$ has been introduced for simplicity of the presentation, but it is not necessary. All the results of this paper remain valid if we take $a \equiv 0$. Indeed, if $a \equiv 0$ we redefine $f_2(s)$ as $f_2(s) - s$ and put $a \equiv 1$, and all the above assumptions are fulfilled. Analogously, if the condition $f_1(0) = f_2(0) = 0$ does not hold, we can replace f_i by $f_i(s) - f_i(0)$, $i = 1, 2$, and g by $g - f_1(0) - f_2(0)$.

Of course f can be reduced to a polynomial if we take $f_2 = 0$. Moreover, the case $f = f_2$ is included in the previous formulation. Indeed, it is enough to take $f_1(s) = s$ and redefine again $f_2(s)$ as $f_2(s) - s$. Then f satisfies the above assumptions.

This paper is structured as follows. Section 2 contains the existence theory for (1.1) and apriori estimates in appropriately weighted functions spaces. The differentiability properties of the control to state mapping are investigated in Section 3. This involves a detailed analysis of the relations between the discount factor σ, the nature of the nonlinearity f, and the weights characterizing the spaces in the which the linearized state equation and the adjoint equation are well-posed.

We end the introduction by fixing some notation. Given real numbers $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$, $L^p_\alpha(Q)$ denotes the space of measurable functions $\phi : Q \longrightarrow \mathbb{R}$ satisfying

$$\|\phi\|_{L^p_\alpha(Q)} = \Big(\int_0^\infty \mathrm{e}^{-\alpha t}\|\phi(t)\|^p_{L^p(\Omega)}\,dt\Big)^{\frac{1}{p}} < \infty \quad \text{if } p < \infty,$$
$$\|\phi\|_{L^\infty_\alpha(Q)} = \operatorname*{ess\,sup}_{(x,t)\in Q} \mathrm{e}^{-\frac{\alpha}{2}t}|\phi(x,t)| < \infty.$$

Let us observe that $L^p_\alpha(Q)$ is continuously embedded in $L^q_\alpha(Q)$ for $1 \le q < p \le \infty$ and $\alpha > 0$. The following well known inequality will be useful all along this paper

$$C_a\|z\|_{H^1(\Omega)} \le \left(\int_\Omega (|\nabla z|^2 + az^2)\,dx\right)^{\frac{1}{2}} \quad \forall z \in H^1(\Omega); \tag{1.6}$$

see, for instance, [13, Theorem 2.7.1].

2. Analysis of the State Equation. We shall denote by $L^2_{loc}(0,\infty; H^1(\Omega))$ the space of functions y belonging to $L^2(0,T; H^1(\Omega))$ for every $0 < T < \infty$. Analogously we define $L^2_{loc}(0,\infty; L^2(\Omega))$, $H^1_{loc}(0,\infty; L^2(\Omega))$, and $C_{loc}([0,\infty); L^2(\Omega))$. Following [8] we define the following solution concept.

Definition 2.1. *We call y a solution to (1.1) if $y \in L^2_{loc}(0,\infty; H^1(\Omega)) \cap C_{loc}([0,\infty); L^2(\Omega))$, $f(y) \in L^2_{loc}(0,\infty; L^2(\Omega))$, and for every $T > 0$ the restriction of y to $Q_T = \Omega \times (0,T)$ satisfies in the usual variational sense the equation*

$$\left\{\begin{array}{l} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) = g + u\chi_\omega \quad \text{in } Q_T, \\ \partial_n y = 0 \quad \text{on } \Sigma_T = \Gamma \times (0,T), \quad y(0) = y_0 \quad \text{in } \Omega; \end{array}\right. \tag{2.1}$$

see, for instance, [11, pages 136–137] or [14, page 108] for the definition of a variational solution (or generalized solution) of (2.1).

The following existence and uniqueness result can be proved as in [8, Theorem 2.2].

Theorem 2.2. *For every $u \in L^2(Q_\omega)$ equation (1.1) has a unique solution y_u. Moreover $y_u \in H^1_{loc}(0,\infty; L^2(\Omega)) \cap L^2_{loc}(0,\infty; H^1(\Omega))$ holds. Further, there exists a constant $K_f > 0$ independent of u, g, y_0, and $T > 0$ such that*

$$\begin{aligned} &\|y_u\|_{C([0,T];L^2(\Omega))} + \|y_u\|_{L^2(0,T;H^1(\Omega))} \\ &\le K_f\Big(\|y_0\|_{L^2(\Omega)} + \big[\|g\|_{L^\infty(0,\infty;L^2(\Omega))} + 1\big]\sqrt{T} + \|u\|_{L^2(Q_\omega)}\Big). \end{aligned} \tag{2.2}$$

Additionally, there exists a constant C_T independent of u, g, and y_0 such that

$$\|y_u\|_{H^1(Q_T)} + \|y_u\|_{C([0,T];H^1(\Omega))} + \|f(y_u)\|_{L^2(Q_T)} \\ \le C_T\Big(\|y_0\|^{m+1}_{H^1(\Omega)} + \|g\|_{L^\infty(0,\infty;L^2(\Omega))} + \|u\chi_\omega\|_{L^2(Q)} + 1\Big). \tag{2.3}$$

Proof. For this proof we cannot rely on the usual techniques because u is in L^2 rather than in L^p with p large enough and y_0 is not assumed to be in $L^\infty(\Omega)$. As a consequence, the corresponding state does not belong to $L^\infty(Q_T)$. Rather we follow the proof of [8, Theorem 2.2] and provide the estimates (2.2) and (2.3). Introducing $z(x,t) = \mathrm{e}^{-\Lambda_f t} y_u(x,t)$, where Λ_f was defined in (1.4), (1.1) is transformed to

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + \tilde f(t,z) = \mathrm{e}^{-\Lambda_f t}(g + u\chi_\omega) \text{ in } Q_T, \\ \partial_n z = 0 \text{ on } \Sigma_T,\ z(0) = y_0 \text{ in } \Omega, \end{cases} \tag{2.4}$$

where $\tilde f(t,s) = \mathrm{e}^{-\Lambda_f t} f(\mathrm{e}^{\Lambda_f t}s) + \Lambda_f s\ \forall (t,s) \in \mathbb{R}^2$. For any positive integer k setting $\tilde f_k(t,s) = \tilde f(t, \mathrm{Proj}_{[-k,+k]}(s))$ we consider the equation

$$\begin{cases} \dfrac{\partial z_k}{\partial t} - \Delta z_k + az_k + \tilde f_k(t,z_k) = \mathrm{e}^{-\Lambda_f t}(g + u\chi_\omega) \text{ in } Q_T, \\ \partial_n z_k = 0 \text{ on } \Sigma_T,\ z_k(0) = y_0 \text{ in } \Omega. \end{cases} \tag{2.5}$$

As a consequence of (1.2) and (1.4), we get $\tilde f_k(t,0) = 0$ and $\partial_s \tilde f_k(t,s) = f'(\mathrm{e}^{\Lambda t}s) + \Lambda_f \ge 0$ if $|s| < k$ and $\partial_s \tilde f_k(t,s) = 0$ if $|s| > k$. By an application of Schauder's fixed point theorem we obtain the existence of a solution $z_k \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ of (2.5). The uniqueness of z_k is a consequence of the monotonicity of $\tilde f_k$. Using that $\tilde f_k(t,z_k)z_k \ge 0$ and testing (2.5) with z_k we deduce

$$\|z_k\|_{L^\infty(0,T;L^2(\Omega))} + \|z_k\|_{L^2(0,T;H^1(\Omega))} \\ \le \frac{\sqrt{2}}{C_a}\big(\|g\|_{L^2_{\Lambda_f}(Q)} + \|u\|_{L^2(Q_\omega)}\big) + \sqrt{2}\|y_0\|_{L^2(\Omega)}. \tag{2.6}$$

Next we prove that $\{\tilde f_k(\cdot,z_k)\}_{k=1}^\infty$ is a bounded sequence in $L^2(Q)$. Since f_1 is a polynomial of degree $2m+1$ and leading positive coefficient, and $f_2^2(s) \le L_f s^2$ due to (1.2), elementary calculus leads to the existence of constants $C_1 > 0$, $C_2 \ge 0$, $C_3 > 0$, and $C_4 \le 0$ such that

$$\tilde f(t,s)^2 \le C_1 \tilde f(t,s)s^{2m+1} + C_2 \text{ and } \tilde f(t,s)s^{2m+1} \ge C_3 s^{4m+2} + C_4\ \forall (t,s) \in [0,T]\times\mathbb{R}.$$

Using these inequalities, the desired boundedness of $\{\tilde f_k(\cdot,z_k)\}_{k=1}^\infty$ is obtained as in the proof of [8, Theorem 2.2]. Following that proof we get the existence and uniqueness of a solution z of (2.4) which, in addition, belongs to $H^1(Q_T)$ with $\tilde f(\cdot,z) \in L^2(Q_T)$. Therefore, $y_u = \mathrm{e}^{\lambda_f t}z \in H^1(Q_T)$ is a solution of (2.1) for every $T > 0$. Hence y_u is the unique solution of (1.1). Additionally we have that $f(y_u) \in L^2(Q_T)$ and the estimate (2.3) for $f(y_u)$ is satisfied. The estimate (2.3) for y_u in $H^1(Q_T) \cap C([0,T];H^1(\Omega))$ is a well known consequence of the equation

$$\begin{cases} \dfrac{\partial y_u}{\partial t} - \Delta y_u + ay_u = g + u\chi_\omega - f(y_u) \text{ in } Q_T \\ \partial_n y_u = 0 \text{ on } \Sigma_T,\ y_u(0) = y_0 \text{ in } \Omega. \end{cases}$$

Now, we prove (2.2). For every $t \in (0,T)$ we set $\Omega_t = \{x \in \Omega : |y(x,t)| \le M_f\}$ with M_f given by (1.5). Let us set $C_{M_f} = \sup_{|s|\le M_f} |f(s)|$. Multiplying (2.1) by y and using (1.5) we infer

$$\frac{1}{2}\frac{d}{dt}\|y\|^2_{L^2(\Omega)} + \int_\Omega [|\nabla y|^2 + ay^2]\,dx \le \int_\Omega (g + \chi_\omega u)y\,dx + C_{M_f} M_f |\Omega_t|$$

Integrating in $(0,t)$ and using (1.6) we deduce

$$\begin{aligned}
&\frac{1}{2}\|y(t)\|^2_{L^2(\Omega)} + C_a \int_0^t \|y(s)\|^2_{H^1(\Omega)} dt \\
&\qquad \le \frac{1}{2}\|y_0\|^2_{L^2(\Omega)} + \frac{1}{C_a}\Big(\|g\|^2_{L^\infty(0,\infty;L^2(\Omega))} T + \|u\|^2_{L^2(0,\infty;L^2(\omega))}\Big) \\
&\qquad + \frac{C_a}{2}\int_0^t \|y(s)\|^2_{H^1(\Omega)}\,dt + C_{M_f} M_f |\Omega| T.
\end{aligned}$$

This yields (2.2). □

In the next theorem we establish some infinite horizon regularity properties of the solution of (1.1). First we introduce the following notation: for every $\alpha \in \mathbb{R}$ $L^2_\alpha(0,\infty;H^1(\Omega))$ and $C_\alpha([0,\infty);H^1(\Omega))$ denote the Hilbert and Banach spaces of measurable functions $y : [0,\infty) \longrightarrow H^1(\Omega)$ endowed with the norms

$$\begin{aligned}
\|y\|_{L^2_\alpha(0,\infty;H^1(\Omega))} &= \Big(\int_0^\infty \mathrm{e}^{-\alpha t}\|y(t)\|^2_{H^1(\Omega)}\,dt\Big)^{\frac{1}{2}}, \\
\|y\|_{C_\alpha([0,\infty);H^1(\Omega))} &= \sup_{t\in[0,\infty)} \mathrm{e}^{-\frac{\alpha}{2}t}\|y(t)\|_{H^1(\Omega)}.
\end{aligned}$$

We also define $H^1_\alpha(Q)$ as the space of functions $y \in L^2_\alpha(0,\infty;H^1(\Omega))$ such that $\frac{\partial y}{\partial t} \in L^2_\alpha(Q)$. This is a Hilbert space for the norm

$$\|y\|_{H^1_\alpha(Q)} = \Big(\|y\|^2_{L^2_\alpha(0,\infty;H^1(\Omega))} + \Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2_\alpha(Q)}\Big)^{\frac{1}{2}}.$$

The next corollary is an immediate consequence of (2.2).

Corollary 2.3. *For every $\alpha > 0$ and all $u \in L^2(Q_\omega)$ the solution y_u of* (1.1) *belongs to $L^2_\alpha(Q)$ and*

$$\|y_u\|_{L^2_\alpha(Q)} \le K_f \frac{1}{\sqrt{\alpha}}\Big(\|y_0\|_{L^2(\Omega)} + \frac{1}{\sqrt{\alpha}}\big[\|g\|_{L^\infty(0,\infty;L^2(\Omega))} + 1\big] + \|u\|_{L^2(Q_\omega)}\Big),$$

where K_f is the constant introduced in (2.2)

Theorem 2.4. *Let $u \in L^2(Q_\omega)$ and let y be the solution of* (1.1) *corresponding to u. Then the following properties hold for all $\alpha > 0$:*

$$f(y), y^{2m+1} \in L^2_\alpha(Q), \tag{2.7}$$

$$y \in H^1_\alpha(Q) \cap C_\alpha([0,\infty);H^1(\Omega)), \tag{2.8}$$

$$\lim_{T\to\infty} \mathrm{e}^{-\alpha T}\|y(T)\|_{H^1(\Omega)} = 0. \tag{2.9}$$

Moreover, there exists a constant C independent of α, u, g, and y_0 such that

$$\begin{aligned}
&\|f(y)\|_{L^2_\alpha(Q)} + \|y^{2m+1}\|_{L^2_\alpha(Q)} + \|y\|_{H^1_\alpha(Q)} + \|y\|_{C_\alpha([0,\infty);H^1(\Omega))} \\
&\quad \le \frac{C}{\min\{1,\alpha\}}\Big(\|g\|_{L^\infty(0,\infty;L^2(\Omega))} + \|u\|_{L^2(Q_\omega)} + \|y_0\|^{m+1}_{H^1(\Omega)} + 1\Big).
\end{aligned} \tag{2.10}$$

Proof. By Corollary 2.3, we know that $y \in L^2_\alpha(Q)$. We divide the proof into three parts.

Proof of (2.7). If $m = 0$, (2.7) is an immediate consequence of (1.2) and Corollary 2.3. Suppose $m > 0$. First we demonstrate that $\mathrm{e}^{-\alpha t} f(y) y^{2m+1} \in L^1(Q)$. Let us write

$$f_1(s) = \sum_{j=1}^{2m+1} a_j s^j \quad \text{and} \quad C_f = \sum_{j=1}^{2m+1} |a_j| + L_f. \tag{2.11}$$

Observe that $f_1(0) = 0$ implies that $a_0 = 0$. From here we infer

$$|f(s)s^{2m+1}| \le C_f s^{4m+2} \quad \forall |s| \ge 1 \quad \text{and} \quad |f(s)s^{2m+1}| \le C_f s^2 \quad \forall |s| \le 1. \tag{2.12}$$

We set

$$M = \max\Big\{1, M_f, \frac{2}{a_{2m+1}}\Big(\sum_{j=1}^{2m} |a_j| + L_f\Big)\Big\},$$

where M_f was introduced in (1.5). Let us denote $Q^M = \{(x,t) \in Q : |y(x,t)| > M\}$ and $C_M = \max_{1 \le |s| \le M} |f(s)|$. Then, with (1.5) we get for every $T > 0$

$$\begin{aligned}\int_Q \mathrm{e}^{-\alpha t} |f(y) y^{2m+1}| \, dx\, dt &\le \int_{Q \setminus Q^M} \mathrm{e}^{-\alpha t} |f(y) y^{2m+1}| \, dx\, dt \\ &\quad + \int_{Q^M} \mathrm{e}^{-\alpha t} |f(y) y^{2m+1}| \, dx\, dt \\ &\le \frac{C_M M^{2m+1} |\Omega|}{\alpha} + \int_{Q^M} \mathrm{e}^{-\alpha t} f(y) y^{2m+1} \, dx\, dt. \end{aligned} \tag{2.13}$$

Thus we only need to prove the integrability of $\mathrm{e}^{-\alpha t} f(y) y^{2m+1}$ in Q^M. To this end, for every integer $k > M$ we define the projection $y_k = \mathrm{Proj}_{[-k,+k]}(y) \in H^1_{loc}(Q)$ and we multiply (2.1) by $\mathrm{e}^{-\alpha t} y_k^{2m+1}$:

$$\begin{aligned}\int_{Q_T} \frac{\partial y}{\partial t} \mathrm{e}^{-\alpha t} y_k^{2m+1} \, dx\, dt &+ \int_{Q_T} \mathrm{e}^{-\alpha t} [\nabla y \nabla y_k^{2m+1} + a y y_k^{2m+1}] \, dx\, dt \\ &+ \int_{Q_T \cap Q^M} \mathrm{e}^{-\alpha t} f(y) y_k^{2m+1} \, dx\, dt \\ \le \frac{C_M M^{2m+1} |\Omega|}{\alpha} &+ \int_{Q_T} \mathrm{e}^{-\alpha t} (g + u\chi_\omega) y_k^{2m+1} \, dx\, dt. \end{aligned} \tag{2.14}$$

Using that $y y_k^{2m+1} \ge y_k^{2m+2}$ and $y y_k^{2m} \frac{\partial y_k}{\partial t} = \frac{1}{2m+2} \frac{\partial y_k^{2m+2}}{\partial t}$, and integrating by parts twice we obtain

$$\begin{aligned}&\int_0^T \int_\Omega \frac{\partial y}{\partial t} \mathrm{e}^{-\alpha t} y_k^{2m+1} \, dx\, dt \\ &\ge \frac{1}{2m+2}\Big(\mathrm{e}^{-\alpha T} \int_\Omega y_k^{2m+2}(T) \, dx + \alpha \int_{Q_T} \mathrm{e}^{-\alpha t} y_k^{2m+2} \, dx\, dt\Big) - \int_\Omega y_0^{2m+2} \, dx.\end{aligned}$$

Moreover, we have $\nabla y \nabla y_k^{2m+1} = (2m+1) y_k^{2m} \nabla y \nabla y_k = (2m+1) y_k^{2m} |\nabla y_k|^2$. Using this in (2.14), and taking into account that (1.5) implies that $f(y(x,t)) y_k(x,t) \geq 0$ for every $(x,t) \in Q^M$, we obtain

$$\begin{aligned}
&\int_{Q_T \cap Q^M} \mathrm{e}^{-\alpha t} f(y) y_k^{2m+1} \, dx \, dt \leq \int_\Omega y_0^{2m+2} \, dx + \frac{C_M M^{2m+1} |\Omega|}{\alpha} \\
&+ \int_{Q_T} \mathrm{e}^{-\alpha t} (g + u \chi_\Omega) y_k^{2m+1} \, dx \, dt \leq C \|y_0\|_{H^1(\Omega)}^{2m+2} + \frac{C_M M^{2m+1} |\Omega|}{\alpha} \\
&+ \big(\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)} \big) \Big(\int_{Q_T \cap Q^M} \mathrm{e}^{-\alpha t} y_k^{4m+2} \, dx \, dt \Big)^{1/2} \\
&\leq C \|y_0\|_{H^1(\Omega)}^{2m+2} + \frac{C_M M^{2m+1} |\Omega|}{\alpha} + \frac{1}{a_{2m+1}} \big(\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)} \big)^2 \\
&\qquad\qquad + \frac{a_{2m+1}}{4} \int_{Q \cap Q^M} \mathrm{e}^{-\alpha t} y_k^{4m+2} \, dx \, dt.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{Q_T \cap Q^M} \mathrm{e}^{-\alpha t} |f(y) y_k^{2m+1}| \, dx \, dt \\
&\leq C \big(\|y_0\|_{H^1(\Omega)}^{2m+2} + \|y\|_{L^2_\alpha(Q)}^2 + \big[\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)} \big]^2 + \frac{1}{\alpha} \big) \quad \forall T > 0 \text{ and } \forall k > M,
\end{aligned}$$

where C only depends on f and M. Since $y_k(x,t) \to y(x,t)$ a.e. in Q, we deduce from the above inequality, (2.13), and Fatou's lemma that

$$\begin{aligned}
&\int_Q \mathrm{e}^{-\alpha t} |f(y)| |y^{2m+1}| \, dx \, dt \\
&\leq C \big(\|y_0\|_{H^1(\Omega)}^{2m+2} + \|y\|_{L^2_\alpha(Q)}^2 + \big[\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)} \big]^2 + \frac{1}{\alpha} \big)
\end{aligned} \tag{2.15}$$

for a new C only depending on f and M. Due to the choice of M we have for $|s| \geq M$

$$\begin{aligned}
f(s) s^{2m+1} &\geq s^{4m+2} \Big(a_{2m+1} - \sum_{j=1}^{2m} |a_j| \frac{1}{|s|^{2m+1-j}} - L_f \frac{1}{s^{2m}} \Big) \\
&\geq s^{4m+2} \Big(a_{2m+1} - \frac{1}{M} \Big[\sum_{j=1}^{2m} |a_j| + L_f \Big] \Big) \geq \frac{a_{2m+1}}{2} s^{4m+2}.
\end{aligned}$$

Since $f'(s) > 0$ for $|s| \geq M$, we get

$$f(y(x,t)) y^{2m+1}(x,t) \geq f(y_k(x,t)) y^{2m+1}(x,t) \geq \frac{a_{2m+1}}{2} y^{4m+2}(x,t), \quad (x,t) \in Q^M.$$

Inserting this inequality in the left hand side of (2.15) we conclude that. Now we have

$$\begin{aligned}
&\int_Q \mathrm{e}^{-\alpha t} y^{4m+2} \, dx \, dt \leq M^{4m} \int_{Q \setminus Q^M} \mathrm{e}^{-\alpha t} y^2 \, dx \, dt + \int_{Q^M} \mathrm{e}^{-\alpha t} y^{4m+2} \, dx \, dt \\
&\leq M^{4m} \|y\|_{L^2_\alpha(Q)}^2 + \frac{2}{a_{2m+1}} \int_{Q^M} \mathrm{e}^{-\alpha t} f(y) y^{2m+1} \, dx \, dt < \infty,
\end{aligned}$$

which proves that $y^{2m+1} \in L^2_\alpha(Q)$. Moreover, since $|f(s)| \le C_f|s|^{2m+1}$ $\forall |s| \ge 1$ and $|f(s)| \le C_f|s|$ $\forall |s| \le 1$, we deduce that

$$f(s)^2 \le C_f^2(s^2 + s^{4m+2}) \quad \forall s \in \mathbb{R}.$$

Therefore, the fact that y and y^{2m+1} belong to $L^2_\alpha(Q)$ implies that $f(y) \in L^2_\alpha(Q)$ and the proof of (2.7) is complete. Additionally, these arguments and Corollary 2.3 lead to the estimates for the first two terms of (2.10).

Proof of (2.8). First we observe that $y \in C([0,T]; H^1(\Omega))$ for every $T > 0$. Indeed, this is a consequence of the fact that $f(y) \in L^2_\alpha(Q)$ and $y_0 \in H^1(\Omega)$; see [14, Proposition III-2.5]. Hence $y : [0,\infty) \to H^1(\Omega)$ is continuous. To prove that $y \in L^2_\alpha(0,\infty; H^1(\Omega))$ it is enough to multiply (2.1) by $\mathrm{e}^{-\alpha t}y$ and integrate in Q_T, $T > 0$ arbitrary, to get

$$\begin{aligned}
&\frac{\mathrm{e}^{-\alpha T}}{2}\|y(T)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_{Q_T} \mathrm{e}^{-\alpha t} y^2\,dx\,dt + \int_{Q_T} \mathrm{e}^{-\alpha t}(|\nabla y|^2 + ay^2)\,dx\,dt \\
&= \int_{Q_T} \mathrm{e}^{-\alpha t}(g + u\chi_\omega)y\,dx\,dt + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)} - \int_{Q_T} \mathrm{e}^{-\alpha t} f(y)y\,dx\,dt \\
&\le \big(\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)}\big)\|y\|_{L^2_\alpha(Q)} + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)} \\
&+ \|f(y)\|_{L^2_\alpha(Q)}\|y\|_{L^2_\alpha(Q)} < \infty.
\end{aligned}$$

Above we have used that $y \in L^2_\alpha(Q)$; see Corollary 2.3. Now it is enough to take $T \to \infty$ to deduce that $y \in L^2_\alpha(0,\infty; H^1(\Omega))$.

To prove that $y \in C_\alpha([0,\infty); H^1(\Omega))$ we take into account that by Theorem 2.2 $y \in H^1(Q_T)$ for every $T > 0$. We can multiply (2.1) by $\mathrm{e}^{-\alpha t}\frac{\partial y}{\partial t}$ and integrate in Q_T to get

$$\begin{aligned}
&\int_0^T \mathrm{e}^{-\alpha t}\Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2(\Omega)}\,dt + \int_0^T \mathrm{e}^{-\alpha t}\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla y|^2 + ay^2)\,dx\,dt + \int_{Q_T} \mathrm{e}^{-\alpha t} f(y)\frac{\partial y}{\partial t}\,dx\,dt \\
&= \int_{Q_T} \mathrm{e}^{-\alpha t}(g + u\chi_\omega)\frac{\partial y}{\partial t}\,dx\,dt. \qquad (2.16)
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_0^T \mathrm{e}^{-\alpha t}\Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2(\Omega)}\,dt \\
&+ \frac{1}{2}\mathrm{e}^{-\alpha T}\int_\Omega (|\nabla y(T)|^2 + a_0 y^2(T))\,dx + \frac{\alpha}{2}\int_0^T\int_\Omega \mathrm{e}^{-\alpha t}[|\nabla y|^2 + ay^2]\,dx\,dt \\
&\le \big(\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)} + \|f(y)\|_{L^2_\alpha(Q)}\big)\Big\|\frac{\partial y}{\partial t}\Big\|_{L^2_\alpha(Q)} + \frac{1}{2}\int_\Omega (|\nabla y_0|^2 + a_0 y_0^2)\,dx \\
&\le \frac{1}{2}\big(\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)} + \|f(y)\|_{L^2_\alpha(Q)}\big)^2 + \frac{1}{2}\int_0^T \mathrm{e}^{-\alpha t}\Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2(\Omega)}\,dt \\
&+ \frac{C}{2}\|y_0\|^2_{H^1(\Omega)},
\end{aligned}$$

and hence,

$$\int_0^T \mathrm{e}^{-\alpha t}\Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2(\Omega)}\,dt + \mathrm{e}^{-\alpha T}\int_\Omega (|\nabla y(T)|^2 + a_0 y^2(T))\,dx$$
$$+\frac{\alpha}{2}\int_0^T \mathrm{e}^{-\alpha t}[|\nabla y|^2 + ay^2]\,dx\,dt$$
$$\le \big(\|g\|_{L^2_\alpha(Q)} + \|u\|_{L^2(Q_\omega)} + \|f(y)\|_{L^2_\alpha(Q)}\big)^2 + C\|y_0\|^2_{H^1(\Omega)}\big). \tag{2.17}$$

Since $T > 0$ is arbitrary, the above inequality concludes the proof of (2.8). Moreover, from the obtained estimates and Corollary 2.3 the bounds for the last two terms in (2.10) follow.

Proof of (2.9). From (2.16) we get

$$\frac{1}{2}\mathrm{e}^{-\alpha T}\int_\Omega (|\nabla y(T)|^2 + a_0 y^2(T))\,dx = \int_0^T\int_\Omega \mathrm{e}^{-\alpha t}(g + u\chi_\omega)\frac{\partial y}{\partial t}\,dx\,dt$$
$$-\int_0^T \mathrm{e}^{-\alpha t}\Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2(\Omega)}\,dt - \frac{\alpha}{2}\int_0^T \mathrm{e}^{-\alpha t}[|\nabla y|^2 + ay^2]\,dx\,dt$$
$$-\int_Q \mathrm{e}^{-\alpha t} f(y)\frac{\partial y}{\partial t}\,dx\,dt + \frac{1}{2}\int_\Omega [|\nabla y_0|^2 + ay_0^2]\,dx.$$

Taking the limit in T we infer

$$\frac{1}{2}\lim_{T\to\infty}\mathrm{e}^{-\alpha T}\int_\Omega (|\nabla y(T)|^2 + a_0 y^2(T))\,dx = \int_0^\infty\int_\Omega \mathrm{e}^{-\alpha t}(g + u\chi_\omega)\frac{\partial y}{\partial t}\,dx\,dt$$
$$-\int_0^\infty \mathrm{e}^{-\alpha t}\Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2(\Omega)}\,dt - \frac{\alpha}{2}\int_0^\infty \mathrm{e}^{-\alpha t}[|\nabla y|^2 + ay^2]\,dx\,dt$$
$$-\int_Q \mathrm{e}^{-\alpha t} f(y)\frac{\partial y}{\partial t}\,dx\,dt + \frac{1}{2}\int_\Omega [|\nabla y_0|^2 + ay_0^2]\,dx.$$

We have proved that $\mathrm{e}^{-\alpha T}\int_\Omega (|\nabla y(T)|^2 + a_0 y^2(T))\,dx \to \beta$ as $T \to \infty$ for a certain real number β. But, we know that $y \in L^2_\alpha(0,\infty;H^1(\Omega))$, hence there exists a sequence $\{T_k\}_{k=1}^\infty$ converging to ∞ such that $\mathrm{e}^{-\alpha T_k}\int_\Omega (|\nabla y(T_k)|^2 + a_0 y^2(T_k))\,dx \to 0$. Therefore, $\beta = 0$ and, since $\alpha > 0$ is arbitrary, (2.9) holds with (1.6). □

Corollary 2.5. *For every $u \in L^2(Q_\omega)$ the following identities hold*

$$\int_Q e^{-\alpha t}\frac{\partial y_u}{\partial t}z\,dx\,dt + \int_Q e^{-\alpha t}[\nabla y_u \nabla z + ay_u z]\,dx\,dt + \int_Q e^{-\alpha t} f(y_u) z\,dx\,dt$$
$$= \int_Q e^{-\alpha t}(g + u\chi_\omega) z\,dx\,dt \qquad \forall z \in H^1_\alpha(Q), \tag{2.18}$$

$$\frac{\alpha}{2}\int_0^\infty e^{-\alpha t}\|y_u(t)\|^2_{L^2(\Omega)}\,dt + \int_Q e^{-\alpha t}[|\nabla y_u|^2 + ay_u^2]\,dx\,dt + \int_Q e^{-\alpha t} f(y_u) y_u\,dx\,dt$$
$$= \int_Q e^{-\alpha t}(g + u\chi_\omega) y_u\,dx\,dt + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)}, \tag{2.19}$$

$$\int_Q e^{-\alpha t}\Big\|\frac{\partial y_u}{\partial t}(t)\Big\|^2_{L^2(\Omega)}\,dt + \frac{\alpha}{2}\int_Q e^{-\alpha t}[|\nabla y_u|^2 + ay_u^2]\,dx\,dt + \int_Q e^{-\alpha t} f(y_u)\frac{\partial y_u}{\partial t}\,dx\,dt$$
$$\int_Q e^{-\alpha t}(g + u\chi_\omega)\frac{\partial y_u}{\partial t}\,dx\,dt + \frac{1}{2}\int_\Omega [|\nabla y_0|^2 + ay_0^2]\,dx. \tag{2.20}$$

The identity (2.18) is obtained multiplying (2.1) by $\mathrm{e}^{-\alpha t}z$, performing integration by parts, and passing to the limit as $T \to \infty$ with the help of (2.7) and (2.8). To prove (2.19) we set $z = y_u$ and integrate by parts in the first integral. The identity (2.20) was established in the last part of the proof of Theorem 2.4.

Remark 2.6. *In case of $g \in L^2(Q)$ and $f'(s) \geq 0$ for every $s \in \mathbb{R}$, we can take $M_f = 0$ in the last part of the proof of Theorem 2.2 and get*

$$\|y_u\|_{C([0,\infty);L^2(\Omega))} + \|y_u\|_{L^2(0,\infty;H^1(\Omega))} \leq K_f\Big(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} + \|u\|_{L^2(Q_\omega)}\Big).$$

Further, relations (2.7)–(2.10) hold with $\alpha = 0$, except on the right hand side of (2.10) where α and $\|g\|_{L^\infty(0,\infty;L^2(\Omega))}$ are replaced by 1 and $\|g\|_{L^2(Q)}$, respectively; see [8, Theorem 2.4].

THEOREM 2.7. *Let $\{u_k\}_{k=1}^\infty \subset L^2(Q_\omega)$ be a sequence converging to u. Then, the following convergences hold for every $\alpha > 0$*

$$\lim_{k\to\infty} \|y_k - y\|_{H^1_\alpha(Q)} = 0, \tag{2.21}$$

$$\lim_{k\to\infty} \|y_k - y\|_{L^{4m+2}_\alpha(Q)} = 0, \tag{2.22}$$

$$\lim_{k\to\infty} \|f(y_k) - f(y)\|_{L^2_\alpha(Q)} = 0, \tag{2.23}$$

$$\lim_{k\to\infty} \|y_k - y\|_{C_\alpha([0,\infty);H^1(\Omega))} = 0, \tag{2.24}$$

where $y = y_u$ and $y_k = y_{u_k}$. Moreover, we have for $m \geq 1$

$$\lim_{k\to\infty} \|f'(y_k) - f'(y)\|_{L_\alpha^{2+\frac{1}{m}}(Q)} = \lim_{k\to\infty} \|f''(y_k) - f''(y)\|_{L_\alpha^{2+\frac{4}{2m-1}}(Q)} = 0, \tag{2.25}$$

and for $m = 0$

$$\lim_{k\to\infty} \|f'(y_k) - f'(y)\|_{L^p_\alpha(Q)} = \lim_{k\to\infty} \|f''(y_k) - f''(y)\|_{L^p_\alpha(Q)} = 0 \ \forall p \in [1,\infty). \tag{2.26}$$

Proof. From (2.10) we deduce the existence of a subsequence, denoted in the same way, such that $y_k \rightharpoonup y$ in $H^1_\alpha(Q)$ and $f(y_k) \rightharpoonup \phi$ in $L^2_\alpha(Q)$. Let us prove that the convergence of $\{y_k\}_{k=1}^\infty$ to y is strong in $L^2_\alpha(Q)$. Given $\varepsilon > 0$, from (2.2) and the boundedness of $\{u_k\}_{k=1}^\infty$ in $L^2(Q_\omega)$ we infer the existence of $T_\varepsilon > 0$ such that

$$\int_{T_\varepsilon}^\infty \mathrm{e}^{-\alpha t}\|y_k(t) - y(t)\|^2_{L^2(\Omega)}\,dt \leq \int_{T_\varepsilon}^\infty \mathrm{e}^{-\alpha t}(C_1 + C_2 t)\,dt + 2\int_{T_\varepsilon}^\infty \mathrm{e}^{-\alpha t}\|y(t)\|^2_{L^2(\Omega)}\,dt < \varepsilon.$$

Moreover, the compactness of the embedding $H^1(Q_{T_\varepsilon}) \subset L^2(Q_{T_\varepsilon})$ implies that $y_k \to y$ in $L^2(Q_{T_\varepsilon})$. Combining these facts we deduce the strong convergence $y_k \to y$ in $L^2_\alpha(Q)$ as $k \to \infty$. Furthermore, taking a new subsequence we assume that $y_k(x,t) \to y(x,t)$ for almost every point $(x,t) \in Q$. Then, by the continuity of f we deduce that $\phi = f(y)$ and, hence, $f(y_k) \rightharpoonup f(y)$ in $L^2_\alpha(Q)$. Now, we prove that $y = y_u$. For this purpose we have to check Definition 2.1. It is easy to pass to the limit weakly in the state equation (2.1) satisfied by (y_k, u_k) and to deduce that (y, u) satisfies the equation in the variational sense in Q_T for every $T > 0$. Moreover, from the continuity of the embedding $H^1_\alpha(Q) \subset C_\alpha([0,\infty); L^2(\Omega))$ we have that $y_0 = y_k(0) \rightharpoonup y(0)$ in $L^2(\Omega)$, hence $y = y_u$. Now, the uniqueness of the solution of (1.1) implies that the whole sequence $\{y_k\}_{k=1}^\infty$ converges to $y = y_u$.

Taking $u = u_k$ in (2.19) we obtain obtain

$$\int_0^\infty e^{-\alpha t}\int_\Omega[|\nabla y_k|^2 + ay_k^2]\,dx\,dt = \int_0^\infty e^{-\alpha t}\int_\Omega(g + u_k\chi_\omega)y_k\,dx\,dt + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)}$$
$$-\frac{\alpha}{2}\int_0^\infty e^{-\alpha t}\|y_k\|^2_{L^2(\Omega)}\,dt - \int_0^\infty e^{-\alpha t}\int_\Omega f(y_k)y_k\,dx\,dt.$$

Using that $u_k \to u$ in $L^2(Q_\omega)$, $f(y_k) \rightharpoonup f(y)$ in $L^2_\alpha(Q)$, and $y_k \to y$ in $L^2_\alpha(Q)$, we can pass to the limit in the above identity and deduce from (2.19) with $u = \bar u$

$$\lim_{k\to\infty}\int_0^\infty e^{-\alpha t}\int_\Omega[|\nabla y_k|^2 + ay_k^2]\,dx\,dt = \int_0^\infty e^{-\alpha t}\int_\Omega(g + u\chi_\omega)y\,dx\,dt + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)}$$
$$-\frac{\alpha}{2}\int_0^\infty e^{-\alpha t}\|y\|^2_{L^2(\Omega)}\,dt - \int_0^\infty e^{-\alpha t}\int_\Omega f(y)y dx\,dt = \int_0^\infty e^{-\alpha t}\int_\Omega[|\nabla y|^2 + ay^2]dx\,dt.$$

This implies that $\lim_{k\to\infty}\|y_k\|_{L^2_\alpha(0,\infty;H^1(\Omega))} = \|y\|_{L^2_\alpha(0,\infty;H^1(\Omega))}$. Then, the continuous inclusion $H^1_\alpha(Q) \subset L^2_\alpha(0,\infty;H^1(\Omega))$ yields $y_k \rightharpoonup y$ in $L^2_\alpha(0,\infty;H^1(\Omega))$ for every $\alpha > 0$. Consequently, strong convergence $y_k \to y$ in $L^2_\alpha(0,\infty;H^1(\Omega))$ holds for every $\alpha > 0$.

Next we prove that $\|y_k - y\|_{L^p_\alpha(Q)} \to 0$ as $k \to \infty$ for every $p \in [1, 4m+2]$. Let us take a subsequence, denoted in the same way, such that $y_k(x,t) \to y(x,t)$ almost everywhere in Q. From (2.10) we get the boundedness of $\{y_k\}_{k=1}^\infty$ in $L^{4m+2}_\alpha(Q)$. Therefore $y_k \rightharpoonup y$ in $L^{4m+2}_\alpha(Q)$ holds. Due to the continuous embedding $L^{4m+2}_\alpha(Q) \subset L^p_\alpha(Q)$ for $p < 4m+2$, we only need to prove the convergence of $\{y_k\}_{k=1}^\infty$ to y in $L^{4m+2}_\alpha(Q)$. Since this convergence is obvious for $m = 0$, let us consider the case $m \geq 1$. Setting $\beta = \frac{\alpha}{4m}$, using the inclusion $H^1(\Omega) \subset L^{4m+2}(\Omega)$, (2.10), the boundedness of $\{u_k\}_{k=1}^\infty$ in $L^2(Q_\omega)$, and the convergence $y_k \to y$ in $L^2_{\frac{\alpha}{2}}(0,\infty;H^1(\Omega))$ we infer

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\alpha t}\int_\Omega(y_k - y)^{4m+2}\,dx\,dt \leq C\int_0^\infty \mathrm{e}^{-\alpha t}\|y_k(t) - y(t)\|^{4m+2}_{H^1(\Omega)}\,dt\\ &= C\int_0^\infty \mathrm{e}^{-\frac{\alpha}{2}t}\|y_k - y\|^2_{H^1(\Omega)}[\mathrm{e}^{-\frac{\beta}{2}t}\|y_k(t) - y(t)\|_{H^1(\Omega)}]^{4m}\,dt\\ &\leq C\|y_k - y\|^{4m}_{C_\beta([0,\infty);H^1(\Omega))}\|y_k - y\|^2_{L^2_{\frac{\alpha}{2}}(0,\infty;H^1(\Omega))} \to 0 \text{ as k } \to \infty.\end{aligned}$$

Since f_1 is a polynomial of degree $2m+1$, we conclude that $f_1(y_k) \to f_1(y)$ in $L^2_\alpha(Q)$, $f_1'(y_k) \to f_1'(y)$ in $L^{2+\frac{1}{m}}_\alpha(Q)$, and $f_1''(y_k) \to f_1''(y)$ in $L^{2+\frac{4}{2m-1}}_\alpha(Q)$. Moreover, the inequality $|f_2(y_k) - f_2(y)| \leq L_f|y_k - y|$ also yields the convergence $f_2(y_k) \to f_2(y)$ in $L^2_\alpha(Q)$. Hence, (2.23) holds. Using (1.2) and applying the Lebesgue's dominated convergence theorem we infer that $f_2'(y_k) \to f_2'(y)$ in $L^{2+\frac{1}{m}}_\alpha(Q)$ and $f_2''(y_k) \to f_2''(y)$ in $L^{2+\frac{4}{2m-1}}_\alpha(Q)$. Therefore, (2.25) follows. If $m = 0$, then (2.26) follows again by the Lebesgue's dominated convergence theorem and the fact that f' and f'' are bounded. Finally, let us prove the convergence of $\left\{\frac{\partial y_k}{\partial t}\right\}_{k=1}^\infty$ and (2.24). Setting $w_k = y - y_k$ and subtracting the equations satisfied by y and y_k we get

$$\left\{\begin{array}{l}\dfrac{\partial w_k}{\partial t} - \Delta w_k + aw_k = \chi_\omega(u - u_k) + [f(y_k) - f(y)] \text{ in } Q,\\ \partial_n w_k = 0 \text{ on } \Sigma,\ \ w_k(0) = 0 \text{ in } \Omega.\end{array}\right.$$

Testing this equation with $\mathrm{e}^{-\alpha t}\frac{\partial w_k}{\partial t}$ and integrating by parts in $\Omega \times (0,t)$ we obtain

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\alpha s}\|\frac{\partial w_k}{\partial t}\|^2_{L^2(\Omega)}\,ds + \frac{\mathrm{e}^{-\alpha t}}{2}\int_\Omega [|\nabla w_k(t)|^2 + a w_k(t)^2]\,dx \\
&+ \frac{\alpha}{2}\int_0^t \mathrm{e}^{-\alpha s}\int_\Omega [|\nabla w_k|^2 + a w_k^2]\,dx\,ds \\
&= \int_0^t \mathrm{e}^{-\alpha s}\int_\Omega [\chi_\omega(u-u_k) + f(y_k) - f(y)]\frac{\partial w_k}{\partial t}\,dx\,ds \\
&\le \Big(\|u-u_k\|_{L^2(0,\infty;L^2(\omega))} + \|f(y_k)-f(y)\|_{L^2_\alpha(Q)}\Big)\Big\|\frac{\partial w_k}{\partial t}\Big\|_{L^2_\alpha(Q_t)}.
\end{aligned}$$

Using $u_k \to u$ and (2.23) we deduce (2.21) and (2.24) from the above inequality. ◻

3. Differentiability of the Control-to-State Mapping. In this section, we prove that the mapping $u \to y_u$ is of class C^2 in appropriately chosen spaces. First we analyze the linearized state equation. For every $\beta \in \mathbb{R}$ let us define the space $Y_\beta = H^1_\beta(Q) \cap C_\beta([0,\infty); H^1(\Omega))$ endowed with the norm

$$\|y\|_{Y_\beta} = \|y\|_{H^1_\beta(Q)} + \|y\|_{C_\beta([0,\infty);H^1(\Omega))}.$$

If $m \ge 1$, using the inclusion $H^1(\Omega) \subset L^{4m+2}(\Omega)$ we infer with Young's inequality

$$\begin{aligned}
&\|y\|_{L^{4m+2}_\beta(Q)} = \Big(\int_Q \mathrm{e}^{-\beta t} y^{4m+2}\,dx\,dt\Big)^{\frac{1}{4m+2}} \le C\Big(\int_0^\infty \mathrm{e}^{-\beta t}\|y(t)\|^{4m+2}_{H^1(\Omega)}\,dt\Big)^{\frac{1}{4m+2}} \\
&\le C\|y\|^{\frac{4m}{4m+2}}_{C_{\frac{\beta}{4m}}([0,\infty);H^1(\Omega))}\|y\|^{\frac{1}{2m+1}}_{L^2_{\frac{\beta}{2}}(0,\infty;H^1(\Omega))} \\
&\le C\Big(\frac{2m}{2m+1}\|y\|_{C_{\frac{\beta}{4m}}([0,\infty);H^1(\Omega))} + \frac{1}{2m+1}\|y\|_{L^2_{\frac{\beta}{2}}(0,\infty;H^1(\Omega))}\Big).
\end{aligned}$$

Then we have

$$\|y\|_{L^{4m+2}_\beta(Q)} \le \begin{cases} \dfrac{2mC}{2m+1}\|y\|_{Y_{\frac{\beta}{4m}}} & \text{if } \beta \ge 0, \\ \dfrac{2mC}{2m+1}\|y\|_{Y_{\frac{\beta}{2}}} & \text{if } \beta < 0. \end{cases} \tag{3.1}$$

Hence, $Y_{\frac{\beta}{4m}}$ and $Y_{\frac{\beta}{2}}$ are continuously embedded in $L^{4m+2}_\beta(Q)$ if $\beta \ge 0$ respectively $\beta < 0$.

LEMMA 3.1. *Assume that $y \in Y_\beta$ for every $\beta > 0$ and take $\alpha_f = 2\Lambda_f$. Then, for every $h \in L^2_{\alpha_f}(Q)$ the linear equation*

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y)z = h & \text{in } Q, \\ \partial_n z = 0 \ \text{ on } \Sigma, \quad z(0) = 0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

has a unique solution $z \in H^1_{loc}(Q) \cap C_{loc}([0,\infty); H^1(\Omega))$. Further, $z \in Y_\alpha$ for all $\alpha > \alpha_f$ and the estimates

$$\|z\|_{L^2_\alpha(0,\infty;H^1(\Omega))} + \|z\|_{C_\alpha([0,\infty);L^2(\Omega))} \le K_1\|h\|_{L^2_\alpha(Q)} \quad \forall \alpha \ge \alpha_f, \tag{3.3}$$

$$\|z\|_{Y_\alpha} \le K_2\big(\|y\|^{2m}_{C_{\frac{\alpha-\alpha'}{2m}}([0,\infty);H^1(\Omega))} + 1\big)\|h\|_{L^2_{\alpha'}(Q)} \quad \forall \alpha > \alpha' > \alpha_f, \tag{3.4}$$

hold with constants independent K_i, $i = 1, 2$, independent of y and h.

Proof. From our assumptions on f we deduce the existence of a constant C_1 such that $|f'(s)| \le C_1(s^{2m} + 1)$. Then, from (2.7) we deduce that $f'(y) \in L_\beta^{2+\frac{1}{m}}(Q)$ for all $\beta > 0$. Therefore, from the classical theory of evolution partial differential equations and (1.4), the existence and uniqueness of a solution of (3.2), $z_T \in H^1(Q_T) \cap C([0,T]; H^1(\Omega))$, follows for every $0 < T < \infty$; see [11, Secs. III.1–III.4]. Hence, defining $z = z_T$ in Q_T for every T we infer that $z \in H^1_{loc}(Q) \cap C_{loc}([0,\infty); H^1(\Omega))$. Testing equation (3.2) with $\mathrm{e}^{-\alpha s} z$ with $\alpha \ge \alpha_f$ and integrating in $\Omega \times (0,t)$ we obtain after integration by parts

$$\begin{aligned}\frac{1}{2}\mathrm{e}^{-\alpha t}\|z(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^t \mathrm{e}^{-\alpha s}\|z(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \mathrm{e}^{-\alpha s}\int_\Omega [|\nabla z|^2 + az^2]\,dx\,ds \\ + \int_0^t \mathrm{e}^{-\alpha s}\int_\Omega f'(y) z^2\,dx\,ds = \int_0^t \mathrm{e}^{-\alpha s}\int_\Omega hz\,dx\,ds.\end{aligned}$$

Using (1.4), (1.6), and the fact that $\frac{\alpha}{2} \ge \Lambda_f$ we infer

$$\begin{aligned}&\frac{1}{2}\mathrm{e}^{-\alpha t}\|z(t)\|^2_{L^2(\Omega)} + C_a^2\int_0^t \mathrm{e}^{-\alpha s}\|z(s)\|^2_{H^1(\Omega)}\,ds \\ &\le \frac{1}{2C_a^2}\int_0^t \mathrm{e}^{-\alpha s}\|h(s)\|^2_{L^2(\Omega)}\,ds + \frac{C_a^2}{2}\int_0^t \mathrm{e}^{-\alpha s}\|z(s)\|^2_{H^1(\Omega)}\,ds.\end{aligned}$$

Since $t > 0$ is arbitrary, the above inequality implies (3.3). To prove (3.4) we take $\alpha > \alpha_f$, and test equation (3.2) with $\mathrm{e}^{-\alpha t}\frac{\partial z}{\partial t}$ and get

$$\begin{aligned}&\int_0^t \mathrm{e}^{-\alpha s}\Big\|\frac{\partial z}{\partial t}\Big\|^2_{L^2(\Omega)}\,ds + \frac{\mathrm{e}^{-\alpha t}}{2}\int_\Omega [|\nabla z(t)|^2 + az^2(t)]\,dx \\ &+ \frac{\alpha}{2}\int_0^t \mathrm{e}^{-\alpha s}\int_\Omega [|\nabla z|^2 + az^2]\,dx\,ds + \int_0^t \mathrm{e}^{-\alpha s}\int_\Omega f'(y) z\frac{\partial z}{\partial t}\,dx\,ds \\ &= \int_0^t \mathrm{e}^{-\alpha s}\int_\Omega h\frac{\partial z}{\partial t}\,dx\,ds.\end{aligned}$$

Using (1.6), Hölder's inequality with $\frac{4m+2}{2m}$, $4m+2$ and 2, and Schwarz's and Young's inequalities, we infer from the above equality for $\varepsilon = \alpha - \alpha'$ and constants $C_2 > 0$, $C_3 > 0$

$$\begin{aligned}&\int_0^t \mathrm{e}^{-\alpha s}\Big\|\frac{\partial z}{\partial t}\Big\|^2_{L^2(\Omega)}\,ds + \frac{C_a^2\mathrm{e}^{-\alpha t}}{2}\|z(t)\|^2_{H^1(\Omega)} \\ &\le C_2\int_0^t \mathrm{e}^{-\alpha s}\big(\|y\|^{2m}_{L^{4m+2}(\Omega)} + 1\big)\|z\|_{L^{4m+2}(\Omega)}\Big\|\frac{\partial z}{\partial t}\Big\|_{L^2(\Omega)}\,ds \\ &+ \int_0^t \mathrm{e}^{-\alpha s}\|h\|_{L^2(\Omega)}\Big\|\frac{\partial z}{\partial t}\Big\|_{L^2(\Omega)}\,ds \\ &\le C_3\big(\|y\|^{4m}_{C_{\frac{\varepsilon}{2m}}([0,\infty);H^1(\Omega))} + 1\big)\int_0^t \mathrm{e}^{(\varepsilon-\alpha)s}\|z\|^2_{H^1(\Omega)}\,ds \\ &+ \int_0^t \mathrm{e}^{-\alpha s}\|h\|^2_{L^2(\Omega)}\,ds + \frac{1}{2}\int_0^t \mathrm{e}^{-\alpha s}\Big\|\frac{\partial z}{\partial t}\Big\|^2_{L^2(\Omega)}\,ds.\end{aligned}$$

This implies with the identity $\alpha - \varepsilon = \alpha'$

$$\int_0^t \mathrm{e}^{-\alpha s}\Big\|\frac{\partial z}{\partial t}\Big\|^2_{L^2(\Omega)}\,ds + C_a^2\mathrm{e}^{-\alpha t}\|z(t)\|^2_{H^1(\Omega)}$$
$$\le 2C_3\big(\|y\|^{4m}_{C_{\frac{\alpha-\alpha'}{2m}}([0,\infty);H^1(\Omega))} + 1\big)\int_0^t \mathrm{e}^{-\alpha' s}\|z\|^2_{H^1(\Omega)}\,ds + 2\int_0^t \mathrm{e}^{-\alpha' s}\|h\|^2_{L^2(\Omega)}\,ds.$$

Finally, (3.4) follows from the above estimate and (3.3). □

LEMMA 3.2. *Let $y_k \to y$ in Y_β for every $\beta > 0$. Given $v \in L^2(0,\infty;L^2(\omega))$ with $v \not\equiv 0$, we denote by z_v and $z_{k,v}$ the solutions of the equations*

$$\left\{\begin{array}{l}\dfrac{\partial z}{\partial t} - \Delta z + az + f'(y)z = v\chi_\omega \text{ in } Q,\\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega,\end{array}\right. \tag{3.5}$$

$$\left\{\begin{array}{l}\dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_k)z = v\chi_\omega \text{ in } Q,\\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega.\end{array}\right. \tag{3.6}$$

Then, we have for every $\alpha > \alpha_f = 2\Lambda_f$

$$\lim_{k\to\infty}\frac{1}{\|v\|_{L^2(Q_\omega)}}\|z_v - z_{k,v}\|_{Y_\alpha} = 0. \tag{3.7}$$

Proof. Let us set $w_k = z_v - z_{k,v}$. Subtracting (3.5) and (3.6) we get

$$\left\{\begin{array}{l}\dfrac{\partial w_k}{\partial t} - \Delta w_k + aw_k + f'(y)w_k = [f'(y_k) - f'(y)]z_{k,v} \text{ in } Q,\\ \partial_n w_k = 0 \text{ on } \Sigma,\ w_k(0) = 0 \text{ in } \Omega.\end{array}\right.$$

The identity (3.7) follows from (3.4) if the right hand side of the above equation converges to zero in $L^2_{\alpha'}(Q)$ as $k \to \infty$ for $\alpha' \in (\alpha_f, \alpha)$. To prove this we first consider the case $m > 0$. Let us observe that due to the assumptions on f there exists a constant C_1 such that

$$(f'(s_2) - f'(s_1))^2 = |f''(s_1 + \theta(s_2 - s_1))|^2(s_2 - s_1)^2 \le C_1(s_1^{4m-2} + s_2^{4m-2} + 1)(s_2 - s_1)^2.$$

for some $\theta \in (0,1)$. We define $\beta = \alpha' - \alpha_f$. Then, applying Hölder's inequality in Ω with $\frac{4m+2}{4m-2}$, $\frac{4m+2}{2}$, and $\frac{4m+2}{2}$, we estimate

$$\|[f'(y_k) - f'(y)]z_{k,v}\|^2_{L^2_{\alpha'}(Q)}$$
$$\le C_1\int_Q \mathrm{e}^{-\alpha' t}(|y|^{4m-2} + |y_k|^{4m-2} + 1)(y - y_k)^2 z_{k,v}^2\,dx\,dt$$
$$\le C_2\int_0^\infty \mathrm{e}^{-\alpha' t}\Big(\|y\|^{4m-2}_{L^{4m+2}(\Omega)} + \|y_k\|^{4m-2}_{L^{4m+2}(\Omega)} + 1\Big)\|y - y_k\|^2_{L^{4m+2}(\Omega)}\|z_{k,v}\|^2_{L^{4m+2}(\Omega)}\,dt$$
$$\le C_3\int_0^\infty \mathrm{e}^{-\alpha' t}\Big(\|y\|^{4m-2}_{H^1(\Omega)} + \|y_k\|^{4m-2}_{H^1(\Omega)} + 1\Big)\|y - y_k\|^2_{H^1(\Omega)}\|z_{k,v}\|^2_{H^1(\Omega)}\,dt$$
$$\le C_4\Big(\|y\|^{4m-2}_{C_{\frac{\beta}{4m-2}}([0,\infty);H^1(\Omega))} + \|y_k\|^{4m-2}_{C_{\frac{\beta}{4m-2}}([0,\infty);H^1(\Omega))} + 1\Big)$$
$$\times\|y - y_k\|^2_{C_{\frac{\beta}{2}}([0,\infty);H^1(\Omega))}\|z_{k,v}\|^2_{L^2_{\alpha_f}(0,\infty;H^1(\Omega))}.$$

Taking into account that (3.3) yields the estimate $\|z_{k,v}\|_{L^2_{\alpha_f}(0,\infty;H^1(\Omega))} \le C\|v\|_{L^2(Q_\omega)}$, and using the assumption on $\{y_k\}_{k=1}^\infty$ the convergence (3.7) follows.

In the case $m = 0$, we have that $(f'(s_2) - f'(s_1))^2 \le L_f^2(s_2 - s_1)^2$. Then, we can argue similarly as above just dropping the term $|y|^{4m-2} + |y_k|^{4m-2} + 1$ and using Schwarz's inequality instead of Hölder's inequality. ∎

Next, given $\alpha > 0$ we denote by $G_\alpha : L^2(Q_\omega) \longrightarrow Y_\alpha$ the mapping $G_\alpha(u) = y_u$.

THEOREM 3.3. *Let us assume that $\alpha > 2\alpha_f$ with $\alpha_f = 2\Lambda_f$. Then, the mapping G_α is of class C^1 and for every $u, v \in L^2(Q_\omega)$, $z_v = G'_\alpha(u)v$ is the solution of the equation*

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z = v\chi_\omega \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega. \end{array}\right. \tag{3.8}$$

If in addition $\alpha > 4\alpha_f$, then G_α is of class C^2 and for every $u, v_1, v_2 \in L^2(Q_\omega)$, $z_{v_1,v_2} = G''_\alpha(u)(v_1, v_2)$ is the solution of the equation

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z + f''(y_u)z_{v_1}z_{v_2} = 0 \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega, \end{array}\right. \tag{3.9}$$

where $z_{v_i} = G'_\alpha(u)v_i$, $i = 1, 2$.

Proof. Given $u, v \in L^2_\alpha(Q_\omega)$ we set $w = G_\alpha(u+v) - G_\alpha(u) - z_v = y_{u+v} - y_u - z_v$, where z_v is the solution of (3.8). Then, w satisfies the equation

$$\left\{\begin{array}{l} \dfrac{\partial w}{\partial t} - \Delta w + aw + f'(y_u)w = [f'(y_u) - f'(y_u + \theta(y_{u+v} - y_u))](y_{u+v} - y_u) \text{ in } Q, \\ \partial_n w = 0 \text{ on } \Sigma,\ w(0) = 0 \text{ in } \Omega, \end{array}\right.$$

with $0 \le \theta(x,t) \le 1$. Assume that $m > 0$. Let us select $\alpha' \in (2\alpha_f, \alpha)$ and $\varepsilon > 0$ such that $\alpha' - \varepsilon > 2\alpha_f$. Set $\varepsilon_m = \varepsilon^{1+\frac{1}{2m}}$ and $\alpha_m = (2m+1)(\alpha' - \varepsilon)$. Utilizing Lemma 3.1 and Hölder's inequality with $1 + \frac{1}{2m}$ and $2m+1$, and (3.1) we infer

$$\begin{aligned} &\|w\|_{Y_\alpha} \le K_u\|[f'(y_u) - f'(y_u + \theta(y_{u+v} - y_u)](y_{u+v} - y_u)\|_{L^2_{\alpha'}(Q)} \\ &\le K_u\|f'(y_u) - f'(y_u + \theta(y_{u+v} - y_u)\|_{L^{2+\frac{1}{m}}_{\varepsilon_m}(Q)}\|y_{u+v} - y_u\|_{L^{4m+2}_{\alpha_m}(Q)} \\ &\le \frac{2mC}{2m+1}K_u\|f'(y_u) - f'(y_u + \theta(y_{u+v} - y_u)\|_{L^{2+\frac{1}{m}}_{\varepsilon_m}(Q)}\|y_{u+v} - y_u\|_{Y_{\frac{\alpha_m}{4m}}}, \end{aligned} \tag{3.10}$$

where $K_u = K_2\big(\|y\|^{2m}_{C_{\frac{\alpha-\alpha'}{2m}}([0,\infty);H^1(\Omega))} + 1\big)$.

In the case $m = 0$ we have that $f' = a_1 + f'_2$ is Lipschitz. The above estimate is replaced by the following

$$\begin{aligned} &\|w\|_{Y_\alpha} \le K_u\|[f'_2(y_u) - f'_2(y_u + \theta(y_{u+v} - y_u))](y_{u+v} - y_u)\|_{L^2_{\alpha'}(Q)} \\ &\le K_u\|f'_2(y_u) - f'_2(y_u + \theta(y_{u+v} - y_u)\|_{L^4_{\alpha'}(Q)}\|y_{u+v} - y_u\|_{L^4_{\alpha'}(Q)} \end{aligned}$$

Using that

$$\begin{aligned} &\|y\|_{L^4_{\alpha'}(Q)} \le C\Big(\int_0^\infty e^{-\alpha' t}\|y(t)\|^4_{H^1(\Omega)}\,dt\Big)^{\frac{1}{4}} \le C\|y\|^{\frac{1}{2}}_{C_{\frac{\alpha'}{2}}([0,\infty);H^1(\Omega))}\|y\|^{\frac{1}{2}}_{L^2_{\frac{\alpha'}{2}}(0,\infty;H^1(\Omega))} \\ &\le \frac{1}{2}C\Big(\|y\|_{C_{\frac{\alpha'}{2}}([0,\infty);H^1(\Omega))} + \|y\|_{L^2_{\frac{\alpha'}{2}}(0,\infty;H^1(\Omega))}\Big) \le \frac{1}{2}C\|y\|_{Y_{\frac{\alpha'}{2}}} \quad \forall y \in Y_{\frac{\alpha'}{2}}, \end{aligned} \tag{3.11}$$

we obtain for $m = 0$

$$\|w\|_{Y_\alpha} \le \frac{1}{2} C K_u \|f_2'(y_u) - f_2'(y_u + \theta(y_{u+v} - y_u)\|_{L^4_{\alpha'}(Q)} \|y_{u+v} - y_u\|_{Y_{\frac{\alpha'}{2}}}. \quad (3.12)$$

Let us estimate $\phi = y_{u+v} - y_u$, that satisfies the equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - \Delta\phi + a\phi + f'(y_u + \theta(y_{u+v} - y_u))\phi = v\chi_\omega & \text{in } Q, \\ \partial_n \phi = 0 \text{ on } \Sigma,\ \phi(0) = 0 \text{ in } \Omega. \end{cases}$$

From (3.4) we infer for $K_{u,v} = K_2\big(\|y_u + \theta(y_{u+v} - y_u)\|^m_{C_{\frac{\beta-\alpha_f}{m}}([0,\infty);H^1(\Omega))} + 1\big)$

$$\|y_{u+v} - y_u\|_{Y_\beta} = \|\phi\|_{Y_\beta} \le K_{u,v}\|v\|_{L^2(Q_\omega)} \quad \forall \beta > \alpha_f. \quad (3.13)$$

Then, applying (2.25) with $\beta = \frac{\alpha_m}{4m} > \alpha_f$, respectively (2.26) with $\beta = \frac{\alpha}{2} > \alpha_f$, and (3.13) in (3.10), respectively in (3.12), we deduce that

$$\lim_{\|v\|_{L^2(Q_\omega)} \to 0} \frac{\|G_\alpha(u+v) - G_\alpha(u) - z_v\|_{Y_\alpha}}{\|v\|_{L^2(Q_\omega)}} = \lim_{\|v\|_{L^2(Q_\omega)} \to 0} \frac{\|w\|_{Y_\alpha}}{\|v\|_{L^2(Q_\omega)}} = 0.$$

This proves that G_α is Fréchet differentiable and $G_\alpha'(u)v = z_v$. To prove the continuity of $G_\alpha' : L^2(Q_\omega) \longrightarrow \mathcal{L}(L^2(Q_\omega), Y_\alpha)$ we take a sequence $u_k \to u$ in $L^2(Q_\omega)$. Setting $z_{k,v} = G_\alpha'(u_k)v$ and $z_v = G_\alpha'(u)v$ for arbitrary $v \in L^2(Q_\omega)$ with $\|v\|_{L^2(Q_\omega)} = 1$ the convergence

$$\lim_{k\to\infty} \sup_{\|v\|_{L^2(Q_\omega)}=1} \|G_\alpha'(u_k)v - G_\alpha'(u)v\|_{Y_\alpha} = \lim_{k\to\infty} \sup_{\|v\|_{L^2(Q_\omega)}=1} \|z_{k,v} - z_v\|_{Y_\alpha} = 0.$$

follows from (3.7). Thus, G_α is of class C^1 for every $\alpha > 2\alpha_f$.

Now we consider the second derivative for $\alpha > 4\alpha_f$. Let $v_1, v_2 \in L^2(Q_\omega)$ with $\|v_2\|_{L^2(Q_\omega)} = 1$. We denote $y_{u+v_1} = G_\alpha(u+v_1)$, $y_u = G_\alpha(u)$, $z_{v_i} = G_\alpha'(u)v_i$, $i = 1, 2$, and $\eta_{v_1,v_2} = G_\alpha'(u+v_1)v_2$. Let z_{v_1,v_2} be the solution of (3.9). Then, we will prove

$$\lim_{\|v_1\|_{L^2(Q_\omega)}\to 0} \sup_{\|v_2\|_{L^2(Q_\omega)}=1} \frac{\|\eta_{v_1,v_2} - z_{v_2} - z_{v_1,v_2}\|_{Y_\alpha}}{\|v_1\|_{L^2(Q_\omega)}} = 0 \quad (3.14)$$

Taking $w = \eta_{v_1,v_2} - z_{v_2} - z_{v_1,v_2}$ we have

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + aw + f'(y_u)w \\ \qquad\qquad = f''(y_u + \theta(y_{u+v_1} - y_u))(y_u - y_{u+v_1})\eta_{v_1,v_2} + f''(y_u)z_{v_1}z_{v_2} \quad \text{in } Q, \\ \partial_n w = 0 \text{ on } \Sigma,\ w(0) = 0 \text{ in } \Omega. \end{cases}$$

Let us denote $y_\theta = y_u + \theta(y_{u+v_1} - y_u)$. Take $\alpha' \in (4\alpha_f, \alpha)$. From (3.4) we get for $K = K_2\big(\|y_u\|^{2m}_{C_{\frac{\alpha-\alpha'}{2m}}([0,\infty);H^1(\Omega))} + 1\big)$

$$\begin{aligned} \|w\|_{Y_\alpha} &\le K\|f''(y_\theta)(y_u - y_{u+v_1})\eta_{v_1,v_2} + f''(y_u)z_{v_1}z_{v_2}\|_{L^2_{\alpha'}(Q)} \\ &\le K\|f''(y_\theta)(y_u - y_{u+v_1})(\eta_{v_1,v_2} - z_{v_2})\|_{L^2_{\alpha'}(Q)} \\ &+ K\|f''(y_\theta)(y_u + z_{v_1} - y_{u+v_1})z_{v_2}\|_{L^2_{\alpha'}(Q)} \\ &+ K\|[f''(y_u) - f''(y_\theta)]z_{v_1}z_{v_2}\|_{L^2_{\alpha'}(Q)} = K(I_1 + I_2 + I_3). \end{aligned} \quad (3.15)$$

Let us consider the case $m \geq 1$. We select $\varepsilon > 0$ such that $\alpha' - \varepsilon > 4\alpha_f$. Denote $\varepsilon_m = \varepsilon^{\frac{2m+1}{2m-1}}$ and $\alpha_m = (2m+1)(\alpha' - \varepsilon)$. To estimate I_1 we use Hölder's inequality with $\frac{4m+2}{4m-2}$, $2m+1$ and $2m+1$, and (3.1)

$$\begin{aligned} I_1 &\leq \|f''(y_\theta)\|_{L_{\varepsilon_m}^{2+\frac{4}{2m-1}}(Q)} \|y_{u+v_1} - y_u\|_{L_{\alpha_m}^{4m+2}(Q)} \|\eta_{v_1,v_2} - z_{v_2}\|_{L_{\alpha_m}^{4m+2}(Q)} \\ &\leq \frac{4m^2C^2}{(2m+1)^2} \|f''(y_\theta)\|_{L_{\varepsilon_m}^{2+\frac{4}{2m-1}}(Q)} \|y_{u+v_1} - y_u\|_{Y_{\frac{\alpha_m}{4m}}} \|\eta_{v_1,v_2} - z_{v_2}\|_{Y_{\frac{\alpha_m}{4m}}}. \end{aligned} \tag{3.16}$$

The term $\|f''(y_\theta)\|_{L_{\varepsilon_m}^{2+\frac{4}{2m-1}}(Q)}$ is bounded; see (2.25). Regarding the term $\eta_{v_1,v_2} - z_{v_2}$ we have with (3.1)

$$\begin{aligned} &\|\eta_{v_1,v_2} - z_{v_2}\|_{L_{\alpha_m}^{4m+2}(Q)} \leq \frac{2mC}{2m+1} \|\eta_{v_1,v_2} - z_{v_2}\|_{Y_{\frac{\alpha_m}{4m}}} \\ &\leq \|G'_{\frac{\alpha_m}{4m}}(u+v_1) - G'_{\frac{\alpha_m}{4m}}(u)\|_{\mathcal{L}(L^2(Q_\omega),Y_{\frac{\alpha_m}{4m}})} \to 0 \quad \text{as } \|v_1\|_{L^2(Q_\omega)} \to 0. \end{aligned}$$

The convergence to 0 follows form the fact that $\frac{\alpha_m}{4m} > 2\alpha_f$ and, hence, the mapping $G_{\frac{\alpha_m}{4m}} : L^2(Q_\omega) \longrightarrow Y_{\frac{\alpha_m}{4m}}$ is of class C^1. Using this fact and the estimate (3.13) we deduce from (3.16) that $\lim_{\|v_1\|_{L^2(Q_\omega)}\to 0} \frac{I_1}{\|v_1\|_{L^2(Q_\omega)}} = 0$.

In the case $m = 0$, we have that $f'' = f_2''$ is bounded by L_f; see (1.2). Then, using Schwarz's inequality and (3.11) we obtain

$$I_1 \leq L_f \|y_{u+v_1} - y_u\|_{L_\alpha^4(Q)} \|\eta_{v_1,v_2} - z_{v_2}\|_{L_\alpha^4(Q)} \leq L_f \frac{C^2}{2} \|y_{u+v_1} - y_u\|_{Y_{\frac{\alpha}{2}}} \|\eta_{v_1,v_2} - z_{v_2}\|_{Y_{\frac{\alpha}{2}}}.$$

Now, we can continue similarly as above taking into account that $\frac{\alpha}{2} > 2\alpha_f$.

Let us estimate I_2, first for $m \geq 1$. We use Hölder's inequality as for I_1 and obtain

$$I_2 \leq \frac{4m^2C^2}{(2m+1)^2} \|f''(y_\theta)\|_{L_{\varepsilon_m}^{2+\frac{4}{2m-1}}(Q)} \|y_{u+v_1} - y_u - z_{v_1}\|_{Y_{\frac{\alpha_m}{4m}}} \|z_{v_2}\|_{Y_{\frac{\alpha_m}{4m}}}. \tag{3.17}$$

Inequality (3.4) leads to

$$\|z_{v_2}\|_{Y_{\frac{\alpha_m}{4m}}} \leq K_2\big(\|y_u\|^{2m}_{C_\delta([0,\infty);H^1(\Omega))} + 1\big)\|v_2\|_{L^2(Q_\omega)} = K_2\big(\|y_u\|^{2m}_{C_\delta([0,\infty);H^1(\Omega))} + 1\big)$$

with $\delta = \frac{\frac{\alpha_m}{4m} - \alpha_f}{2m}$. Since $\frac{\alpha_m}{4m} > 2\alpha_f$, the mapping $G_{\frac{\alpha_m}{4m}} : L^2(Q_\omega) \longrightarrow Y_{\frac{\alpha_m}{4m}}$ is Fréchet differentiable, and therefore

$$\begin{aligned} &\lim_{\|v_1\|_{L^2(Q_\omega)}\to 0} \frac{\|y_{u+v_1} - y_u - z_{v_1}\|_{Y_{\frac{\alpha_m}{4m}}}}{\|v_1\|_{L^2(Q_\omega)}} \\ &= \lim_{\|v_1\|_{L^2(Q_\omega)}\to 0} \frac{\|G_{\frac{\alpha_m}{4m}}(u+v_1) - G_{\frac{\alpha_m}{4m}}(u) - G'_{\frac{\alpha_m}{4m}}(u)v_1\|_{Y_{\frac{\alpha_m}{4m}}}}{\|v_1\|_{L^2(Q_\omega)}} = 0. \end{aligned}$$

Inserting these estimates in (3.17) we get that $\lim_{\|v_1\|_{L^2(Q_\omega)}\to 0} \frac{I_2}{\|v_1\|_{L^2(Q_\omega)}} = 0$. If $m = 0$ we proceed similarly as we did for I_1.

To estimate I_3 we use again Hölder's inequality as for I_1 and obtain for $m \geq 1$

$$\begin{aligned} I_3 &\leq \frac{4m^2C^2}{(2m+1)^2} \|f''(y_u) - f''(y_\theta)\|_{L_{\varepsilon_m}^{2+\frac{4}{2m-1}}(Q)} \|z_{v_1}\|_{Y_{\frac{\alpha_m}{4m}}} \|z_{v_2}\|_{Y_{\frac{\alpha_m}{4m}}} \\ &\leq K_u^2 \frac{4m^2C^2}{(2m+1)^2} \|f''(y_u) - f''(y_\theta)\|_{L_{\varepsilon_m}^{2+\frac{4}{2m-1}}(Q)} \|v_1\|_{L^2(Q_\omega)} \|v_2\|_{L^2(Q_\omega)}, \end{aligned}$$

where $K_u = K_2\big(\|y_u\|^{2m}_{C_\delta([0,\infty);H^1(\Omega))}+1\big)$ with δ as defined above. Since $\|v_2\|_{L^2(Q_\omega)} = 1$, using (2.25) we deduce that $\lim_{\|v_1\|_{L^2(Q_\omega)}\to 0}\frac{I_3}{\|v_1\|_{L^2(Q_\omega)}} = 0$. If $m = 0$ we proceed similarly as we did for I_1.

Finally, (3.14) follows from (3.15) and the established convergences for I_i, $1 \le i \le 3$.

It remains to prove that $G''_\alpha : L^2(Q_\omega) \longrightarrow \mathcal{B}(L^2(Q_\omega)^2, Y_\alpha)$ is continuous, where $\mathcal{B}(L^2(Q_\omega)^2, Y_\alpha)$ denotes the space of continuous bilinear mappings from $L^2(Q_\omega)^2$ to Y_α. If we take a sequence $u_k \to u$ in $L^2(Q_\omega)$ we have to prove that

$$\lim_{k\to\infty}\sup_{\|v_1\|_{L^2(Q_\omega)}=1,\|v_2\|_{L^2(Q_\omega)}=1}\|[G''_\alpha(u_k) - G''_\alpha(u)](v_1, v_2)\|_{Y_\alpha} = 0.$$

Denoting $z^k_{v_1,v_2} = G''_\alpha(u_k)(v_1, v_2)$ and $z_{v_1,v_2} = G''_\alpha(u)(v_1, v_2)$ and putting $w_k = z^k_{v_1,v_2} - z_{v_1,v_2}$ we obtain

$$\begin{cases} \dfrac{\partial w_k}{\partial t} - \Delta w_k + a w_k + f'(y_u)w_k \\ \qquad\qquad = [f'(y_u) - f'(y_{u_k})]z^k_{v_1,v_2} + [f''(y_u)z_{v_1}z_{v_2} - f''(y_{u_k})z^k_{v_1}z^k_{v_2}] \ \text{ in } Q,\\ \partial_n w_k = 0 \text{ on } \Sigma,\ w_k(0) = 0 \text{ in } \Omega, \end{cases}$$

where $z_{v_i} = G'_\alpha(u)v_i$ and $z_{k,v_i} = G'_\alpha(u_k)v_i$ for $i = 1, 2$. From (3.4) we infer for $\alpha' \in (4\alpha_f, \alpha)$

$$\begin{aligned} &\|w_k\|_{Y_\alpha} \le K_{\alpha-\alpha'}\Big(\|[f'(y_u) - f'(y_{u_k})]z^k_{v_1,v_2}\|_{L^2_{\alpha'}(Q)}\\ &+ \|f''(y_u)z_{v_1}z_{v_2} - f''(y_{u_k})z_{k,v_1}z_{k,v_2}\|_{L^2_{\alpha'}(Q)}\Big)\\ &\le K_{\alpha-\alpha'}\Big(\|[f'(y_u) - f'(y_{u_k})]z^k_{v_1,v_2}\|_{L^2_{\alpha'}(Q)} + \|[f''(y_u) - f''(y_{u_k})]z_{v_1}z_{v_2}\|_{L^2_{\alpha'}(Q)}\\ &+ \|f''(y_{u_k})(z_{v_1} - z_{k,v_1})z_{v_2}\|_{L^2_{\alpha'}(Q)} + \|f''(y_{u_k})z_{k,v_1}(z_{v_2} - z_{k,v_2})\|_{L^2_{\alpha'}(Q)}\Big)\\ &= K_{\alpha-\alpha'}\sum_{i=1}^4 I_{k,i}, \end{aligned}\tag{3.18}$$

where $K_{\alpha-\alpha'} = K_2\big(\|y_u\|^{2m}_{C_{\frac{\alpha-\alpha'}{2m}}([0,\infty);H^1(\Omega))} + 1\big)$. We estimate w_k for $m \ge 1$, the case $m = 0$ is obtained in a similar way just using the modifications considered above. We first estimate $z^k_{v_1,v_2}$. Take $\beta > \beta' > 2\alpha_f$ and $\varepsilon > 0$ such $\beta' - \varepsilon > 2\alpha_f$. We set $C_{\beta-\beta'} = \max_{k\ge1} K_2\big(\|y_k\|^{2m}_{C_{\frac{\beta-\beta'}{2m}}([0,\infty);H^1(\Omega))} + 1\big)$, $\varepsilon_m = \frac{4m+2}{2m-1}\varepsilon$ and $\beta_m = (2m+1)(\beta' - \varepsilon)$. Looking at the equation satisfied by $z^k_{v_1,v_2}$ and using (3.4) we get with Hölder's inequality for $\frac{2m+1}{2m-1}$, $2m+1$, and $2m+1$, and (3.1) for a constant $C_{\beta',m}$

$$\begin{aligned} &\|z^k_{v_1,v_2}\|_{Y_\beta} \le C_{\beta-\beta'}\|f''(y_{u_k})z_{k,v_1}z_{k,v_2}\|_{L^2_{\beta'}(Q)}\\ &\le C_{\beta-\beta'}\|f''(y_{u_k})\|_{L^{2+\frac{4}{2m-1}}_{\varepsilon_m}(Q)}\|z_{k,v_1}\|_{L^{4m+2}_{\beta_m}(Q)}\|z_{k,v_2}\|_{L^{4m+2}_{\beta_m}(Q)}\\ &\le \frac{4m^2C^2}{(2m+1)^2}C_{\beta-\beta'}\|f''(y_{u_k})\|_{L^{2+\frac{4}{2m-1}}_{\varepsilon_m}(Q)}\|z_{k,v_1}\|_{Y_{\frac{\beta_m}{4m}}}\|z_{k,v_2}\|_{Y_{\frac{\beta_m}{4m}}} \le C_{\beta',m} < \infty, \end{aligned}$$

where we have used (2.25), $\frac{\beta_m}{4m} > \alpha_f$, (3.4), and $\|v_i\|_{L^2(Q_\omega)} = 1$ for $i = 1, 2$. The same estimate is obtained for z_{v_1,v_2}.

Now, we estimate $I_{k,1}$. Selecting again $\varepsilon > 0$ such that $\alpha' - \varepsilon > 4\alpha_f$, putting $\varepsilon_m = \varepsilon^{1+\frac{1}{2m}}$ and $\alpha_m = (2m+1)(\alpha' - \varepsilon)$, and noting that $\beta = \frac{\alpha_m}{4m} > 2\alpha_f$ we get with the above estimate

$$I_{k,1} \leq \|f'(y_u) - f'(y_{u_k})\|_{L^{2+\frac{1}{m}}_{\varepsilon_m}(Q)} \|z^k_{v_1,v_2}\|_{L^{4m+2}_{\alpha_m}(Q)}$$
$$\leq \frac{2mC}{2m+1} \|f'(y_u) - f'(y_{u_k})\|_{L^{2+\frac{1}{m}}_{\varepsilon_m}(Q)} \|z^k_{v_1,v_2}\|_{Y_{\frac{\alpha_m}{4m}}} \leq C_1 \|f'(y_u) - f'(y_{u_k})\|_{L^{2+\frac{1}{m}}_{\varepsilon_m}(Q)}.$$

The convergence $I_{k,1} \to 0$ as $k \to \infty$ follows from (2.25).

To deal with $I_{k,2}$ we set $\varepsilon_m = \varepsilon^{1+\frac{2}{2m-1}}$ and $\alpha_m = (2m+1)(\alpha' - \varepsilon)$. Now, using Hölder's inequality with $\frac{2m+1}{2m-1}$, $2m+1$, and $2m+1$, (3.1), and (3.4) we obtain

$$I_{k,2} \leq \|f''(y_u) - f''(y_{u_k})\|_{L^{2+\frac{4}{2m-1}}_{\varepsilon_m}(Q)} \|z_{v_1}\|_{L^{4m+2}_{\alpha_m}(Q)} \|z_{v_2}\|_{L^{4m+2}_{\alpha_m}(Q)}$$
$$\leq \frac{4m^2C^2}{(2m+1)^2} \|f''(y_u) - f''(y_{u_k})\|_{L^{2+\frac{4}{2m-1}}_{\varepsilon_m}(Q)} \|z_{v_1}\|_{Y_{\frac{\alpha_m}{4m}}} \|z_{v_2}\|_{Y_{\frac{\alpha_m}{4m}}}$$
$$\leq C_2 \|f''(y_u) - f''(y_{u_k})\|_{L^{2+\frac{4}{2m-1}}_{\varepsilon_m}(Q)}.$$

Using again (2.25) the convergence $I_{k,2} \to 0$ as $k \to \infty$ follows.

Arguing as we did for $I_{k,2}$ we obtain

$$I_{k,3} \leq \frac{4m^2C^2}{(2m+1)^2} \|f''(y_{u_k})\|_{L^{2+\frac{4}{2m-1}}_{\varepsilon_m}(Q)} \|z_{v_1} - z_{k,v_1}\|_{Y_{\frac{\alpha_m}{4m}}} \|z_{v_2}\|_{Y_{\frac{\alpha_m}{4m}}}$$
$$\leq C_3 \|z_{v_1} - z_{k,v_1}\|_{Y_{\frac{\alpha_m}{4m}}}.$$

Taking into account that $\frac{\alpha_m}{4m} > 2\alpha_f$, we know that $G_{\frac{\alpha_m}{4m}} : L^2(Q_\omega) \longrightarrow Y_{\frac{\alpha_m}{4m}}$ is of class C^1 and, consequently,

$$\sup_{\|v_1\|_{L^2(Q_\omega)}=1} \|z_{v_1} - z_{k,v_1}\|_{Y_{\frac{\alpha_m}{4m}}} = \|G'_{\frac{\alpha_m}{4m}}(u) - G'_{\frac{\alpha_m}{4m}}(u_k)\|_{\mathcal{L}(L^2(Q_\omega), Y_{\frac{\alpha_m}{4m}})} \to 0 \quad \text{as } k \to \infty.$$

This proves the convergence to zero of $I_{k,3}$. The term $I_{k,4}$ is treated in an identical way. Therefore, with (3.18) we conclude that $w_k \to 0$ as $k \to \infty$ and, hence, G_α is of class C^2. ☐

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