On the convergence and mesh-independent property of the Barzilai-Borwein method for PDE-constrained optimization

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Aiming at optimization problems governed by partial differential equations (PDE), local R-linear convergence of the Barzilai and Borwein (BB) method for a class of twice continuously Fréchet-differentiable functions is proven. Relying on this result, the mesh-independent principle for the BB-method is investigated. The applicability of the theoretical results is demonstrated for two different types of PDE-constrained optimization problems. Numerical experiments are given, which illustrate the theoretical results.

Keywords: Barzilai-Borwein method ; R-linear rate of convergence ; mesh independence ; PDE-constrained optimization.

1. Introduction

Since the pioneering work by Barzilai & Borwein (1988), the attention of many researchers has been turned again to gradient methods. In that work, by incorporating the quasi-Newton property, Barzilai and Borwein introduced new step-sizes for the negative gradient search direction, which leads to a significant acceleration over the steepest descent method. These step-sizes are obtained through approximating the Hessian matrix by a scalar times the identity which satisfies the secant condition in the sense of least squares. Later, global convergence and the R-linear convergence rate of the BB-method for finite-dimensional, strictly convex, and quadratic problems were established by Raydan (1993) and Dai & Liao (2002), respectively. Thereafter, Dai & Fletcher (2005a) provided a deep analysis concerning the asymptotic behaviour of the BB-method and the surprising computational efficiency of this algorithm in relation to its nonmonotonicity. In Fletcher (2005), several cases were discussed for which the BB-method is comparable, or can even be considered as an effective alternative to the conjugate gradient methods.

Due to their simplicity, efficiency and low memory requirements, the BB step-sizes have been widely

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used in various fields of mathematical optimization and applications. Recently, some researchers have employed the BB-method for optimal control problems with PDEs (see e.g., Azmi & Kunisch, 2020; Lemoine *et al.*, 2019; Dunst *et al.*, 2015; Peralta & Kunisch, 2020; Azmi *et al.*, 2018). In all of these works the BB-method appears to be very efficient and, competitive with the conjugate gradient method (Azmi & Kunisch, 2020; Lemoine *et al.*, 2019).

As suggested by Fletcher (2005), the BB-method can be advantageous in cases where the objective function consists of a quadratic function plus a small non-quadratic term (near quadratic), or if the gradient involves numerical error. To some degree these observations apply in PDE-constrained optimization as well. Indeed. due to numerical discretization, numerical error is inevitable. Moreover, optimal control problems with quadratic cost functionals subject to semi-linear PDEs lead to infinite-dimensional "near-quadratic" unconstrained optimization problems when considered in their reduced form.

In view of the above discussion, we are motivated to study the BB-method for unconstrained problems posed in infinite-dimensional Hilbert spaces. Here we focus on the following unconstrained optimization problem

$$\min_{u \in \mathscr{H}} \mathscr{F}(u), \tag{1.1}$$

where $\mathscr{F} : \mathscr{H} \to \mathbb{R}$ is a twice continuously Fréchet differentiable function defined on an abstract Hilbert space $\mathscr{H}$ endowed with the inner product $(\cdot,\cdot)$ and its associated norm $\|\cdot\|$. The Barzilai-Borwein iterations for solving (1.1) are defined by

$$u_{k+1} = u_k - \frac{1}{\alpha_k}\mathscr{G}_k, \tag{1.2}$$

where $\mathscr{G}_k := \mathscr{G}(u_k)$ and $\mathscr{G} : \mathscr{H} \to \mathscr{H}$ stands for the gradient of $\mathscr{F}$. This gradient is defined by $\mathscr{G} := \mathscr{R} \circ \mathscr{F}'$, where $\mathscr{F}' : \mathscr{H} \to \mathscr{H}'$ is the first derivative of $\mathscr{F}$, and $\mathscr{R} : \mathscr{H}' \to \mathscr{H}$ is the Riesz isomorphism, with $\mathscr{H}'$ denoting the dual space of $\mathscr{H}$. Therefore, for every $\delta u \in \mathscr{H}$, we obtain $\mathscr{F}'(u)\delta u = (\mathscr{G}(u), \delta u)$. Furthermore, the step-size $\alpha_k > 0$ is chosen according to either

$$\alpha_k^{BB1} := \frac{(\mathscr{S}_{k-1}, \mathscr{Y}_{k-1})}{(\mathscr{S}_{k-1}, \mathscr{S}_{k-1})} \quad \text{or} \quad \alpha_k^{BB2} := \frac{(\mathscr{Y}_{k-1}, \mathscr{Y}_{k-1})}{(\mathscr{S}_{k-1}, \mathscr{Y}_{k-1})}, \tag{1.3}$$

where $\mathscr{S}_{k-1} := u_k - u_{k-1}$ and $\mathscr{Y}_{k-1} := \mathscr{G}_k - \mathscr{G}_{k-1}$. With these specifications we are prepared to specify Algorithm 1 which will be investigated in this paper.

Algorithm 1 BB-method

Require: Let initial iterates $u_{-1}, u_0 \in \mathscr{H}$ with $u_{-1} \neq u_0$ be given.

1: Set $k = 0$.
2: If $\|\mathscr{G}_k\| = 0$ stop.
3: Choose α_k equal to either α_k^{BB1} or α_k^{BB2}.
4: Set $u_{k+1} = u_k - \frac{1}{\alpha_k}\mathscr{G}_k$, $k = k+1$, and go to Step 2.

In this work, we continue our investigation initiated in Azmi & Kunisch (2020) on the BB-method for optimization problems governed by partial differential equations. First, we establish the local R-linear convergence of Algorithm 1 for twice continuously Fréchet differentiable functions. Subsequently we analyse the mesh independence principle (MIP) for Algorithm 1. This important property states that the algorithm roughly requires the same number of iterations to reach the termination requirement for the infinite-dimensional problem as well as for the finite-dimensional approximations. This concept of MIP was initially introduced by Allgower *et al.* (1986) for Newton's method. Since then, MIP was studied for many different optimization algorithms and problem formulations. From these, we can mention generalized equations (Alt, 2001; Argyros, 1990, 1992), Newton methods (Karátson, 2012; Weiser *et al.*, 2005), SQP methods (Volkwein, 2000), shape design problems (Laumen, 1999), constrained Gauss-Newton methods (Heinkenschloss, 1993), gradient projection methods (Kelley & Sachs, 1992), quasi-Newton methods (Kelley & Sachs, 1987, 1990, 1991), and semi-smooth Newton methods (Hintermüller *et al.*, 2008; Hintermüller & Ulbrich, 2004). The convergence analysis of Algorithm 1 will show that, depending on the spectrum of the Hessian, the sequence $\{\|\mathscr{G}_k\|\}_k$ can be nonmonotone. This is different from the situation for quasi-Newton methods for which mesh-independence was analyzed in Kelley & Sachs (1987, 1990, 1991).

Our theoretical framework is supported by two optimizations problems with partial differential equations as constraints. The corresponding numerical experiments support the theoretical results, and illustrate the qualitative influence of the spectral condition number of the Hessian on the convergence, and the MIP property of Algorithm 1.

The rest of paper is organized as follows: In Section 2, we prove the local R-linear convergence of Algorithm 1 for a class of twice continuously Fréchet-differentiable functions. Section 3 is devoted to developing the mesh-independent principle for Algorithm 1. In Section 4, PDE-constrained optimal control problems are investigated. Finally, Section 5 presents the numerical experiments.

2. Convergence Analysis

In this section, we will prove the local *R*-linear convergence of Algorithm 1 for twice continuously Fréchet-differentiable functions with Lipschitz continuous second derivatives. This proof is based the global R-linear convergence of Algorithm 1 for the quadratic model around the strong minima, and comparing the sequences generated by Algorithm 1 applied to the original problem and the quadratic model. Beforehand, we recall the following convergence result from Azmi & Kunisch (2020) for strictly convex quadratic problems:

PROPOSITION 2.1 Assume that the objective function $\mathscr{F}$ is a strictly convex quadratic function, that is

$$\min_{u\in\mathscr{H}} \mathscr{F}(u) := \frac{1}{2}(\mathscr{A}u,u)-(b,u), \tag{QP}$$

where $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ is a bounded, self-adjoint, and uniformly positive operator and $b\in\mathscr{H}$. Moreover, let $\{u_k\}_k$ be the sequence generated by Algorithm 1 for (QP). Then there exists a positive integer m depending on the spectrum of $\mathscr{A}$ such that we have

$$\|\mathscr{G}_{k+m}\| \leqslant \frac{1}{2}\|\mathscr{G}_k\| \quad \text{for all } k\geqslant 0, \tag{2.1}$$

or equivalently,

$$\|u_{k+m}-u^*\| \leqslant \frac{1}{2}\|u_k-u^*\| \quad \text{for all } k\geqslant 0, \tag{2.2}$$

for all initial iterates $u_{-1},u_0\in\mathscr{H}$ with $u_{-1}\neq u_0$. Here $\mathscr{G}_k=\mathscr{G}(u_k):=\mathscr{A}u_k-b$ and u^* stands for the global minimum u^* of (QP).

Based on the convergence results and the discussion concerning the influence of the spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$ in Azmi & Kunisch (2020, Theorem 3.1, Remark 3.2), we can formulate the following remark:

REMARK 2.1 For strictly convex and quadratic functions of the from (QP), we have the following cases depending on the value of the spectral condition number $\kappa(\mathscr{A}) := \|\mathscr{A}\|\|\mathscr{A}^{-1}\|$:

1. $\kappa(\mathscr{A}) < 2$: In this case, Algorithm 1 is Q-linearly convergent with a rate $\gamma_{\mathscr{A}} < 1$. In other words, the sequence $\{\|\mathscr{G}_k\|\}_k$ is monotone decreasing with
$$\|\mathscr{G}_{k+1}\| \leqslant \gamma_{\mathscr{A}} \|\mathscr{G}_k\| \quad \text{for all } k \geqslant 0. \tag{2.3}$$
Moreover, as $\kappa(\mathscr{A})$ is getting smaller, $\gamma_{\mathscr{A}}$ becomes smaller and, as consequence, the convergence is getting faster.

2. $\kappa(\mathscr{A}) \geqslant 2$: In this case, there is a potential nonmonotonic behaviour of the sequence $\{\|\mathscr{G}_k\|\}_k$. As $\kappa(\mathscr{A})$ is getting larger, the nonmonotonic behaviour in the sequence $\{\|\mathscr{G}_k\|\}_k$ becomes stronger.

Now we are in the position to investigate the convergence of Algorithm 1 for a more general class of problems. Let $u^* \in \mathscr{H}$ be a local minimum of $\mathscr{F} : \mathscr{H} \to \mathbb{R}$, where $\mathscr{F}$ is twice continuously Fréchet-differentiable at u^* with Lipschitz continuous second derivative $\mathscr{F}''$ in a neighbourhood of u^*. If we identify the first derivative $\mathscr{F}'$ by its corresponding representation $\mathscr{G}$, we have the following first-order optimality condition
$$\mathscr{G}(u^*) = 0 \quad \text{in } \mathscr{H}. \tag{EP}$$
Due to the continuity of the bilinear map $\mathscr{F}''(u^*)$, there exists a positive constant δ_{sup} such that
$$\mathscr{F}''(u^*)(v,u) \leqslant \delta_{\text{sup}}\|v\|\|u\| \quad \text{for all } u, v \in \mathscr{H}. \tag{2.4}$$
Moreover, we assume that the continuous bilinear map $\mathscr{F}''(u^*)$ is uniformly positive, that is
$$\delta_{\text{inf}}\|v\|^2 \leqslant \mathscr{F}''(u^*)(v,v) \leqslant \delta_{\text{sup}}\|v\|^2 \quad \text{for all } v \in \mathscr{H}, \tag{2.5}$$
where $\delta_{\text{sup}} \geqslant \delta_{\text{inf}} > 0$. Then, due to the Riesz representation theorem, there exists a unique self-adjoint bounded operator $\mathscr{A}_{u^*}^{\mathscr{F}}$ (see Conway, 1990, Theorem. 2.2, p. 31) such that
$$\mathscr{F}''(u^*)(v,u) = (\mathscr{A}_{u^*}^{\mathscr{F}} v, u) \quad \text{for all } v, u \in \mathscr{H}.$$

Similarly to the analysis of Dai *et al.* (2006) and Liu & Dai (2001), the *R*-linearly convergence result is proven by comparing the sequences $\{u_k\}_k$ and $\{\hat{u}_k\}_k$ which are generated by Algorithm 1 applied to, respectively, $\mathscr{F}$ and its second-order Taylor approximation $\hat{\mathscr{F}}$ defined by
$$\hat{\mathscr{F}}(u) = \mathscr{F}(u^*) + \frac{1}{2}(\mathscr{A}_{u^*}^{\mathscr{F}}(u-u^*), u-u^*). \tag{2.6}$$

Throughout this section, all notations with the accent " ˆ " are related to the quadratic approximation (2.6). For instance with $\hat{\mathscr{G}}(\cdot)$ and $\hat{\alpha}_k$, we denote the gradient and the step-sizes of Algorithm 1 applied to $\hat{\mathscr{F}}$, respectively.

Since $\mathscr{F}'' : \mathscr{H} \to \mathscr{L}(\mathscr{H}, \mathscr{L}(\mathscr{H}, \mathbb{R}))$ is locally Lipschitz continuous and $\mathscr{F}''(u^*) : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is continuous and uniformly positive definite, there exist a ball $\mathscr{B}_\tau(u^*)$ centred at u^* with radius $\tau > 0$, constants $\alpha_{\text{sup}} \geqslant \alpha_{\text{inf}} > 0$ depending on τ, and L such that
$$\|\mathscr{G}(u) - \mathscr{A}_{u^*}^{\mathscr{F}}(u-u^*)\| \leqslant L\|u-u^*\|^2 \quad \text{for all } u \in \mathscr{B}_\tau(u^*), \tag{$L1$}$$
$$\mathscr{F}''(u)(v,w) \leqslant \alpha_{\text{sup}}\|v\|\|w\| \quad \text{for all } v, w \in \mathscr{H} \text{ and } u \in \mathscr{B}_\tau(u^*), \tag{$L2$}$$

and

$$\alpha_{\inf}\|v\|^2 \leqslant \mathscr{F}''(u)(v,v) \leqslant \alpha_{\sup}\|v\|^2 \quad \text{for all } v \in \mathscr{H} \text{ and } u \in \mathscr{B}_\tau(u^*). \tag{L3}$$

With these specifications, we show in the following lemma that all step-sizes α_k^{BB1} and α_k^{BB2} with $k \geqslant 0$ lie in the interval $[\alpha_{\inf}, \alpha_{\sup}]$.

LEMMA 2.1 Assume that for a twice continuously Fréchet-differentiable $\mathscr{F}$ in $\mathscr{B}_\tau(u^*)$ properties $L2$ and $L3$ hold. Then we have

$$\alpha_{\inf} \leqslant \alpha_k^{BB1}, \alpha_k^{BB2} \leqslant \alpha_{\sup} \quad \text{for all } u_k, u_{k-1} \in \mathscr{B}_\tau(u^*). \tag{2.7}$$

Proof. First we show that for arbitrary given $u_1, u_2 \in \mathscr{B}_\tau(u^*)$ it holds

$$\alpha_{\inf}\|u_1 - u_2\|^2 \leqslant (\mathscr{G}(u_1) - \mathscr{G}(u_2), u_1 - u_2) \leqslant \alpha_{\sup}\|u_1 - u_2\|^2. \tag{2.8}$$

Since $u_1, u_2 \in \mathscr{B}_\tau(u^*)$, by defining $s(t) := u_1 + t(u_2 - u_1)$, we have $s(t) \in \mathscr{B}_\tau(u^*)$ for every $t \in [0,1]$. Therefore, due to $L3$, we obtain

$$\begin{aligned}\alpha_{\inf}\|u_1 - u_2\|^2 &\leqslant \int_0^1 \mathscr{F}''(s(t))(u_1 - u_2, u_1 - u_2)dt = (\mathscr{G}(u_1) - \mathscr{G}(u_2), u_1 - u_2) \\ &\leqslant \alpha_{\sup}\|u_1 - u_2\|^2,\end{aligned} \tag{2.9}$$

and this completes the verification of (2.8). Now, using (2.8) for $u_1 = u_k$ and $u_2 = u_{k-1}$ and a short computation, it follows that $\alpha_{\inf} \leqslant \alpha_k^{BB1} \leqslant \alpha_{\sup}$.

Next we deal with α_k^{BB2}. Due to $L3$ and $L2$, the mapping $\mathscr{F}$ is convex on $\mathscr{B}_\tau(u^*)$ and its gradient is Lipschitz continuous with constant $\alpha_{\sup}$, respectively. Therefore, using Baillon-Haddad Theorem (Bauschke & Combettes, 2017, Cor. 18.17, p. 323), we conclude that the gradient of $\mathscr{F}$ is cocoercive with constant $\frac{1}{\alpha_{\sup}}$ and thus

$$(\mathscr{Y}_{k-1}, \mathscr{S}_{k-1}) \geqslant \frac{1}{\alpha_{\sup}}\|\mathscr{Y}_{k-1}\|^2. \tag{2.10}$$

Further using (2.8) we obtain that $\alpha_{\inf}\|\mathscr{S}_{k-1}\|^2 \leqslant (\mathscr{Y}_{k-1}, \mathscr{S}_{k-1}) \leqslant \|\mathscr{S}_{k-1}\|\|\mathscr{Y}_{k-1}\|$ and, thus, we have

$$(\mathscr{Y}_{k-1}, \mathscr{S}_{k-1}) \leqslant \|\mathscr{S}_{k-1}\|\|\mathscr{Y}_{k-1}\| \leqslant \frac{1}{\alpha_{\inf}}\|\mathscr{Y}_{k-1}\|^2. \tag{2.11}$$

From (2.10), (2.11), and the definition of α_k^{BB2} it follows that $\alpha_{\inf} \leqslant \alpha_k^{BB2} \leqslant \alpha_{\sup}$ and, thus (2.7) holds. □

As a consequence of Lemma 2.1, if the iterations of Algorithm 1 applied to $\hat{\mathscr{F}}$ lie in $\mathscr{B}_\tau(u^*)$, we have

$$\alpha_{\inf} \leqslant \delta_{\inf} \leqslant \hat{\alpha}_k^{BB1}, \hat{\alpha}_k^{BB2} \leqslant \delta_{\sup} \leqslant \alpha_{\sup}. \tag{2.12}$$

where the fact that $\hat{\alpha}_k^{BB1}, \hat{\alpha}_k^{BB2} \in [\delta_{\inf}, \delta_{\sup}]$ for $k \geqslant 0$ is justified as in Azmi & Kunisch (2020, p. 4). Further, using the fundamental theorem of calculus (see (2.9)), $L2$, and $L3$, we infer that

$$\alpha_{\inf}\|u - u^*\| \leqslant \|\mathscr{G}(u)\| = \|\mathscr{G}(u) - \mathscr{G}(u^*)\| \leqslant \alpha_{\sup}\|u - u^*\| \quad \text{for all } u \in \mathscr{B}_\tau(u^*). \tag{2.13}$$

In the next lemma we study the distance of the sequences $\{u_k\}_k$ and $\{\hat{u}_k\}_k$.

LEMMA 2.2 Let u^* be a local minimizer of $\mathscr{F}$ with $\mathscr{F} \in C^2(\mathscr{H}, \mathbb{R})$ and assume that $L1$-$L3$ hold for a radius τ and constants $\alpha_{\inf}$ and $\alpha_{\sup}$, and for the bilinear form $\mathscr{F}''(u^*)$ estimate (2.5) holds with the constants $\delta_{\sup}$ and $\delta_{\inf}$. Further, let $\{u_j\}_j$ be a sequence generated by Algorithm 1 applied to $\mathscr{F}$, and $\{\hat{u}_j^k\}_j$ be the sequence generated by Algorithm 1 applied to the quadratic approximation (2.6) of $\mathscr{F}$ at u^* with initial iterates $\hat{u}_0^k = u_k$ and $\hat{u}_{-1}^k = u_{k-1}$ for $k \geqslant 0$. Then for any fixed positive integer m, there exist positive constants $\eta \leqslant \tau$ and λ such that the following property holds:

If $u_{k-1} \in \mathscr{B}_\tau(u^*)$, $u_k \in \mathscr{B}_\eta(u^*)$, and if for some $\ell \in \{0, \ldots, m\}$, the following condition holds

$$\|\hat{u}_j^k - u^*\| \geqslant \frac{1}{2}\|u_k - u^*\| \quad \text{for all } j \in \{0, \ldots, \max\{0, \ell - 1\}\}, \tag{2.14}$$

then we have

$$u_{k+j} \in \mathscr{B}_\tau(u^*) \quad \text{and} \quad \|u_{k+j} - \hat{u}_j^k\| \leqslant \lambda \|u_k - u^*\|^2 \tag{2.15}$$

for all $j \in \{0, \ldots, \ell\}$.

Proof. The proof is given in Appendix A. □

In the next theorem, we present the main result of this section which is the local R-linearly convergence of Algorithm 1 applied to twice continuously Fréchet differentiable objective functions. The proof is inspired by the one given in Dai *et al.* (2006, Theorem. 2.3) for the finite-dimensional case. But, since some of its arguments are used later, and for the sake of completeness, we provide it here.

THEOREM 2.2 Let u^* be a local minimizer of a twice continuously Fréchet differentiable function $\mathscr{F}$, with a locally Lipschitz continuous second-derivative. Further suppose that the bilinear mapping $\mathscr{F}''(u^*)$ satisfies estimate (2.5) for constants $\delta_{\sup}$ and $\delta_{\inf}$. Then there exist positive constants ζ, λ_1, λ_2, and $\theta < 1$ such that the sequence $\{u_k\}_k$, generated by Algorithm 1, satisfies

$$\|u_k - u^*\| \leqslant \lambda_1 \theta^k \|u_0 - u^*\| \quad \text{for all } k \geqslant 0, \tag{2.16}$$

and

$$\|\mathscr{G}_k\| \leqslant \lambda_2 \theta^k \|\mathscr{G}_0\| \quad \text{for all } k \geqslant 0, \tag{2.17}$$

for all initial iterates $u_{-1}, u_0 \in \mathscr{B}_\zeta(u^*) \in \mathscr{H}$ with $u_{-1} \neq u_0$.

Proof. The assumptions on $\mathscr{F}$ imply that $L1$-$L3$ are satisfied for a radius τ and constants $\alpha_{\inf}$ and $\alpha_{\sup}$. The proof relies on Proposition 2.1 and Lemma 2.2 in an essential manner. Due to Proposition 2.1, there exists an integer m depending on $\sigma(\mathscr{A}_{u^*}^{\mathscr{F}})$ such that for every pair of initial iterates $(\hat{u}_{-1}^k, \hat{u}_0^k) = (u_{k-1}, u_k)$, we have for the sequences $\{\hat{u}_j^k\}_j$ that

$$\exists s \in \{1, \ldots, m\} \quad \text{such that} \quad \|\hat{u}_s^k - u^*\| \leqslant \frac{1}{2}\|u_k - u^*\|. \tag{2.18}$$

Given the constants $\eta \leqslant \tau$ and λ from Lemma 2.2 we define $\zeta := \min\{\eta, \tau_1\}$, where τ_1 is chosen such that $c_2 := \frac{1}{2} + \lambda\tau_1 < 1$. Then, due to Lemma 2.2, for the fixed integer m, if $\hat{u}_0^k = u_k \in \mathscr{B}_\zeta(u^*)$, $\hat{u}_{-1}^k = u_{k-1} \in \mathscr{B}_\tau(u^*)$, and if

$$\|\hat{u}_j^k - u^*\| \geqslant \frac{1}{2}\|u_k - u^*\| \quad \text{for all } j \in \{0, \ldots, \max\{0, \ell - 1\}\} \text{ with } \ell \leqslant m, \tag{2.19}$$

then we have

$$u_{k+j} \in \mathscr{B}_\tau(u^*) \text{ and } \|u_{k+j} - \hat{u}_j^k\| \leqslant \lambda \|u_k - u^*\|^2 \quad \text{for all } j \in \{0, 1, \ldots, \ell\}. \tag{2.20}$$

Next we show by induction that there exists a subsequence of indices $\{k_i\}_i$ with $k_0 = 0$, for which we have

$$k_{i+1} - k_i \leqslant m \quad \text{and} \quad \|u_{k_{i+1}} - u^*\| \leqslant c_2 \|u_{k_i} - u^*\|, \tag{2.21}$$

for all $i = 0, 1, \ldots$.

Due to Proposition 2.1 and (2.18), there exists a smallest integer $j_0 \leqslant m$ such that

$$\|\hat{u}^{k_0}_{j_0} - u^*\| \leqslant \frac{1}{2}\|\hat{u}^{k_0}_0 - u^*\| = \frac{1}{2}\|u_{k_0} - u^*\|. \tag{2.22}$$

Defining $k_1 := k_0 + j_0 > k_0$, and using (2.20) and (2.22), we have

$$\begin{aligned} \|u_{k_1} - u^*\| &= \|u_{k_0+j_0} - u^*\| \leqslant \|u_{k_0+j_0} - \hat{u}^{k_0}_{j_0}\| + \|\hat{u}^{k_0}_{j_0} - u^*\| \\ &\leqslant \lambda \|u_{k_0} - u^*\|^2 + \frac{1}{2}\|\hat{u}^{k_0}_0 - u^*\| \leqslant \lambda \tau_1 \|u_{k_0} - u^*\| + \frac{1}{2}\|u_{k_0} - u^*\| \leqslant c_2 \|u_{k_0} - u^*\|, \end{aligned} \tag{2.23}$$

and hence (2.21) follows for $i = 0$. By (2.23) and the fact that $u_{k_0} = u_0 \in \mathscr{B}_\zeta(u^*)$, it follows that $u_{k_1} \in \mathscr{B}_\zeta(u^*) \subset \mathscr{B}_\tau(u^*)$. Together with the inclusion in (2.20) we obtain that $u_k \in \mathscr{B}_\tau(u^*)$ for all $k \in \{0, 1, \ldots, k_1\}$.

To carry out the induction step we assume that for an index k_i we have $u_{k_i} \in \mathscr{B}_\zeta(u^*)$ and, $u_k \in \mathscr{B}_\tau(u^*)$ for all $k \in \{0, 1, \ldots, k_i\}$. We will show that there exists an index $k_{i+1} > k_i$ with $k_{i+1} - k_i \leqslant m$ such that $u_{k_{i+1}} \in \mathscr{B}_\zeta(u^*)$, $u_k \in \mathscr{B}_\tau(u^*)$ for all $k \in \{0, 1, \ldots, k_{i+1}\}$, and (2.21) holds. Using the fact that $u_{k_i}, u_{k_i-1} \in \mathscr{B}_\tau(u^*)$ and (2.18), there is an integer $j_i \leqslant m$ with the property that

$$\|\hat{u}^{k_i}_{j_i} - u^*\| \leqslant \frac{1}{2}\|\hat{u}^{k_i}_0 - u^*\| = \frac{1}{2}\|u_{k_i} - u^*\|.$$

Due to (2.20), by defining $k_{i+1} = k_i + j_i > k_i$ and using the similar argument as in (2.23), we can show that (2.21) holds and, consequently, we have $u_{k_{i+1}} \in \mathscr{B}_\zeta(u^*)$, and $u_k \in \mathscr{B}_\tau(u^*)$ for all $k \in \{0, 1, \ldots, k_{i+1}\}$.

Now, due to (A.11), there is a positive constant c_1 such that

$$\|u_{k+j} - u^*\| \leqslant c_1 \|u_k - u^*\| \quad \text{for all } j \in \{1, \ldots, m\}, \tag{2.24}$$

where c_1 depends only on m and the constants $\alpha_{\sup}$ and $\alpha_{\inf}$ which have been defined in $L3$. Further, for every $k \geqslant 0$, there exists an integer $i \geqslant 0$ such that $k_i \leqslant k < k_{i+1}$ with $k_{i+1} \leqslant m(i+1)$. Therefore, $i \geqslant \frac{k}{m}$ and also by (2.24), we obtain

$$\|u_k - u^*\| \leqslant c_1 \|u_{k_i} - u^*\| \leqslant c_1 (c_2)^i \|u_{k_0} - u^*\| \leqslant c_1 (c_2)^{\frac{k}{m}} \|u_{k_0} - u^*\|.$$

By setting $\theta := (c_2)^{\frac{1}{m}} < 1$, and $\lambda_1 := c_1$, we can conclude (2.16).

We turn to verification of (2.17). By using the fact that for every $k \geqslant 0$ the sequence $\{u_k\}_k$ lies in $\mathscr{B}_\tau(u^*)$, the property (2.13), and (2.16), we obtain

$$\|\mathscr{G}_k\| \leqslant \alpha_{\sup} \|u_k - u^*\| \leqslant \alpha_{\sup} \lambda_1 \theta^k \|u_0 - u^*\| \leqslant \frac{\alpha_{\sup} \lambda_1}{\alpha_{\inf}} \theta^k \|\mathscr{G}_0\|.$$

By setting $\lambda_2 := \frac{\alpha_{\sup} \lambda_1}{\alpha_{\inf}}$ we complete the proof. □

3. Mesh Independence Principle

In this section, we investigate finite-dimensional approximations of Algorithm 1. More specifically we investigate the dependence of the iteration count of the algorithm to achieve a desired accuracy of the residue under finite-dimensional approximations. We note that our objective here is not to estimate the error between the solutions of the discretized problem and continuous one.

Thus let $\{\mathscr{H}^h\}_h$ be a family of finite-dimensional Hilbert spaces indexed by some real number $h > 0$, and endowed with inner products and their associated norms denoted by $(\cdot,\cdot)_h$ and $\|\cdot\|_h$, respectively. Let $\mathscr{G}^h : \mathscr{H}^h \to \mathscr{H}^h$ denote continuous nonlinear mappings which will be required to approximate $\mathscr{G}$ in a sense to be made precise in Assumption A2 below. We then consider the family of problems:

$$\text{Find } u^{*h} \in \mathscr{H}^h \text{ such that } \quad \mathscr{G}^h(u^{*h}) = 0. \tag{EP^h}$$

Throughout this section we pose the following assumption:

A0: The assumptions of Theorem 2.2 in Section 2 hold and we denote by $\{u_k\}_k$ the sequence generated by Algorithm 1 which enjoys the properties asserted in Theorem 2.2.

In particular, it is assumed that $\|u_{-1} - u^*\|$ and $\|u_0 - u^*\|$ are sufficiently small ($< \zeta$ with ζ defined in Theorem 2.2) unless $\mathscr{F}$ is a strictly convex quadratic function. For the case of strictly convex quadratic functions, u_{-1} and u_0 can be chosen from the whole of $\mathscr{H}$.

To describe the family of approximating sequences, we choose $u^h_{-1}, u^h_0 \in \mathscr{H}^h$, and update u^h_k for $k \geqslant 0$ by

$$u^h_{k+1} = u^h_k - \frac{1}{\alpha^h_k}\mathscr{G}^h_k, \tag{3.1}$$

where $\mathscr{G}^h_k := \mathscr{G}^h(u^h_k)$ and the step-size α^h_k is chosen according to either

$$\alpha^{BB1,h}_k := \frac{(\mathscr{S}^h_{k-1}, \mathscr{Y}^h_{k-1})_h}{(\mathscr{S}^h_{k-1}, \mathscr{S}^h_{k-1})_h}, \quad \text{or} \quad \alpha^{BB2,h}_k := \frac{(\mathscr{Y}^h_{k-1}, \mathscr{Y}^h_{k-1})_h}{(\mathscr{S}^h_{k-1}, \mathscr{Y}^h_{k-1})_h}. \tag{3.2}$$

Here we have set $\mathscr{S}^h_{k-1} := u^h_k - u^h_{k-1}$ and $\mathscr{Y}^h_{k-1} := \mathscr{G}^h_k - \mathscr{G}^h_{k-1}$. We should point out that the inner product on $\mathscr{H}^h$ will typically reflect the norm on $\mathscr{H}$. It should not be thought of as the canonical inner product in $\mathbb{R}^{N(h)}$.

Let us now formulate some additional notation and assumptions that we require for the main result of this section. Suppose that $\{\mathbb{P}^h\}_h$ is a family of linear "prolongation" operators

$$\mathbb{P}^h : \mathscr{H}^h \to \mathscr{H}.$$

We use the following notion of convergence in the space $\mathscr{H}$. A sequence $u^h \in \mathscr{H}^h$ is $\mathscr{H}$-convergent to $u \in \mathscr{H}$ if

$$\lim_{h\downarrow 0} \|\mathbb{P}^h u^h - u\| = 0.$$

We have to assume that the discrete inner products approximate the original one in the following sense:

A1: If $u^h \overset{\mathscr{H}}{\to} u$ and $z^h \overset{\mathscr{H}}{\to} z$ for $u, z \in \mathscr{H}$, then

$$\lim_{h\downarrow 0} (u^h, z^h)_h = (u, z). \tag{3.3}$$

Moreover we need the following approximation property of $\mathscr{G}$ by the family $\mathscr{G}^h$.

A2: Suppose that $\mathscr{G}(u^*) = 0$. Then, if $u^h \stackrel{\mathscr{H}}{\to} u$ with u in a neighborhood of u^*, then

$$\mathscr{G}^h(u^h) \stackrel{\mathscr{H}}{\to} \mathscr{G}(u). \tag{3.4}$$

REMARK 3.1 In applications it can occur that the convergence specified in (3.4) requires additional regularity of u and $\mathscr{G}(u)$. In this case one assumes the existence of a subspace $\mathscr{W}$ in $\mathscr{H}$ of more regular functions, and one needs to assure that the limit of the iterations remains in $\mathscr{W}$. In this case Assumption A2 is replaced by A2' below. For details we refer to Kelley & Sachs (1987), for instance.

A2': There exists $u^* \in \mathscr{W}$ with $\mathscr{G}(u^*) = 0$, such that $\mathscr{G}$ is well-defined for all $u \in \mathscr{W}$ sufficiently near u^* with respect to the $\mathscr{H}$-norm. Moreover, if $u \in \mathscr{W}$ with $\|u - u^*\|$ sufficiently small and $u^h \stackrel{\mathscr{H}}{\to} u$, then $\mathscr{G}(u) \in \mathscr{W}$ and

$$\mathscr{G}^h(u^h) \stackrel{\mathscr{H}}{\to} \mathscr{G}(u).$$

THEOREM 3.1 Suppose that Assumptions A0-A2 hold. Moreover, let $u_i^h \stackrel{\mathscr{H}}{\to} u_i$ for $i = -1, 0$ with $u_{-1} \neq u_0$ and $u_{-1}^h \neq u_0^h$. Then for any $k' \geqslant 0$, we have

$$\lim_{h \downarrow 0} \max_{0 \leqslant k \leqslant k'} \|\mathbb{P}^h u_k^h - u_k\| = 0. \tag{3.5}$$

Proof. Using (3.1) and the triangle inequality we obtain

$$\|\mathbb{P}^h u_{k+1}^h - u_{k+1}\| \leqslant \|\mathbb{P}^h u_k^h - u_k\| + \left| \frac{1}{\alpha_k^h} - \frac{1}{\alpha_k} \right| \|\mathbb{P}^h \mathscr{G}_k^h\| + \left| \frac{1}{\alpha_k} \right| \|\mathbb{P}^h \mathscr{G}_k^h - \mathscr{G}_k\|, \tag{3.6}$$

for every $k \geqslant 0$. Then, proceeding by induction, using (3.3) and (3.4), and passing the limit in (3.2) and (3.6), it can be shown that (3.5) is true for every $k' \geqslant 0$. □

The termination condition for EP^h is based on the norm of the gradients for the approximated and the original problem. Thus for $\varepsilon > 0$ the iteration is terminated according to

$$\|\mathscr{G}_k^h\|_h < \varepsilon, \quad \text{and} \quad \|\mathscr{G}_k\| < \varepsilon, \tag{3.7}$$

where ε is a sufficiently small positive number. In order to investigate the behaviour of convergence of the approximated problem with respect to the original problem, we consider the following quantities:

$$k^*(\varepsilon) := \min\{k \in \mathbb{N} : \|\mathscr{G}_k\| < \varepsilon\}, \qquad k_h^*(\varepsilon) := \min\{k \in \mathbb{N} : \|\mathscr{G}_k^h\|_h < \varepsilon\},$$

where $k^*(\varepsilon)$ and $k_h^*(\varepsilon)$ are the smallest iteration numbers for which the norm of corresponding gradients is less than ε. In the following we study the relation between $k^*(\varepsilon)$ and $k_h^*(\varepsilon)$.

THEOREM 3.2 Suppose that Assumptions A0-A2 hold. Further, let $u_i^h \stackrel{\mathscr{H}}{\to} u_i$ for $i = -1, 0$ with $u_{-1} \neq u_0$ and $u_{-1}^h \neq u_0^h$. Then for each $\varepsilon > 0$ and $\delta > 0$, there exists a number $h_{\delta,\varepsilon} > 0$ such that

$$k^*(\varepsilon + \delta) \leqslant k_h^*(\varepsilon) \leqslant k^*(\varepsilon) \tag{3.8}$$

for every $h \in (0, h_{\delta,\varepsilon}]$.

Proof. Due to (3.5) and A2, we have for every k that

$$\mathscr{G}_k^h \stackrel{\mathscr{H}}{\to} \mathscr{G}_k \tag{3.9}$$

and by A1, we obtain

$$\lim_{h\downarrow 0} \|\mathscr{G}_k^h\|_h = \|\mathscr{G}_k\|. \tag{3.10}$$

Now, we show that $\|\mathscr{G}_k^h\|_h < \varepsilon$ for a sufficiently small $h > 0$, provided that $\|\mathscr{G}_k\| < \varepsilon$ holds for an iterate k. Since $\|\mathscr{G}_{k^*(\varepsilon)}\| < \varepsilon$, there exists a positive number $\zeta := \zeta(\varepsilon)$ such that $\|\mathscr{G}_{k^*(\varepsilon)}\| + \zeta < \varepsilon$. Moreover, due to (3.10), there exists a positive number $h_\varepsilon > 0$ such that for every $h \in (0, h_\varepsilon]$ we have

$$\Big| \|\mathscr{G}_{k^*(\varepsilon)}^h\|_h - \|\mathscr{G}_{k^*(\varepsilon)}\| \Big| \leqslant \zeta. \tag{3.11}$$

Hence, for every $h \in (0, h_\varepsilon]$, we obtain

$$\|\mathscr{G}_{k^*(\varepsilon)}^h\|_h = \|\mathscr{G}_{k^*(\varepsilon)}\| + \|\mathscr{G}_{k^*(\varepsilon)}^h\|_h - \|\mathscr{G}_{k^*(\varepsilon)}\| \leqslant \|\mathscr{G}_{k^*(\varepsilon)}\| + \zeta < \varepsilon,$$

and, thus, we have

$$k_h^*(\varepsilon) \leqslant k^*(\varepsilon) \quad \text{for every } h \in (0, h_\varepsilon],$$

which implies the second inequality in (3.8). Now assume that $\delta > 0$ be given. Then due to (3.10) we have

$$\lim_{h\downarrow 0} \max_{0 \leqslant k < k^*(\delta+\varepsilon)} \Big| \|\mathscr{G}_k^h\|_h - \|\mathscr{G}_k\| \Big| = 0. \tag{3.12}$$

By the definition of $k^*(\delta + \varepsilon)$, we have

$$\|\mathscr{G}_k\| \geqslant \delta + \varepsilon \quad \text{for all } k < k^*(\delta + \varepsilon). \tag{3.13}$$

Moreover due to (3.12), there exists a positive number h_δ such that

$$\Big| \|\mathscr{G}_k^h\|_h - \|\mathscr{G}_k\| \Big| \leqslant \max_{0 \leqslant k' < k^*(\delta+\varepsilon)} \Big| \|\mathscr{G}_{k'}^h\|_h - \|\mathscr{G}_{k'}\| \Big| \leqslant \delta \quad \text{for all } h \in (0, h_\delta] \text{ and } k < k^*(\delta + \varepsilon). \tag{3.14}$$

Using (3.13) and (3.14) we infer for every $h \in (0, h_\delta]$ and $k < k^*(\delta + \varepsilon)$ that

$$\|\mathscr{G}_k^h\|_h \geqslant \|\mathscr{G}_k\| - \delta \geqslant \delta + \varepsilon - \delta = \varepsilon,$$

and, thus, $k_h^*(\varepsilon) \geqslant k^*(\delta + \varepsilon)$ for every $h \in (0, h_\delta]$. Now for the choice of $h_{\delta,\varepsilon} := \min\{h_\delta, h_\varepsilon\}$, the relation (3.8) holds for every $h \in (0, h_{\delta,\varepsilon}]$ and we are finished with the proof. □

THEOREM 3.3 Suppose that Assumptions A0-A2 hold. Further assume that $u_i^h \stackrel{\mathscr{H}}{\to} u_i$ for $i = -1, 0$ with $u_{-1} \neq u_0$ and $u_{-1}^h \neq u_0^h$. Then for each $\varepsilon > 0$ there exists $h_\varepsilon > 0$ such that

$$k^*(\varepsilon) - \ell \leqslant k_h^*(\varepsilon) \leqslant k^*(\varepsilon) \quad \text{for every } h \in (0, h_\varepsilon],$$

where the integer $\ell > 0$ is independent of h and ε.

Proof. Theorem 2.2 implies R-linear convergence of $u_k \to u^*$. It can be shown as in the proof of Theorem 2.2 that there exist a positive integer m, positive numbers $c_2 < 1$ and $\zeta \leqslant \tau$, and a subsequence of indices $\{k_i\}_i \in \mathbb{N}$ with $k_0 = 0$, for which we have

$$u_k \in \mathscr{B}_\zeta(u^*) \quad \text{for every } k \geqslant k_0, \tag{3.15}$$

and

$$k_{i+1} - k_i \leqslant m \quad \text{and} \quad \|u_{k_{i+1}} - u^*\| \leqslant c_2 \|u_{k_i} - u^*\|, \text{ for all } i \geqslant 0. \tag{3.16}$$

Moreover, as mentioned in the proof of Theorem 2.2, there exists a number $c_1 > 0$ such that

$$\|u_{k+j} - u^*\| \leqslant c_1 \|u_k - u^*\| \quad \text{for all } j \in \{1, \ldots, m\} \text{ and any } k \geqslant k_0. \tag{3.17}$$

Let us first denote the integer q^* as the smallest integer for which $c_2^{q^*} < \frac{\alpha_{\inf}}{c_1 \alpha_{\sup}}$ holds. The existence of such q^* is guaranteed since $c_2 < 1$. Next, we show for every $k \geqslant k_0$ that there exists a positive integer $i^+(k) \leqslant m(q^*+1) - 1 =: \ell$ such that

$$\|\mathscr{G}_{k+i^+(k)}\| < \|\mathscr{G}_k\|. \tag{3.18}$$

For every $k \geqslant k_0$, the exists an index i such that $k_i \leqslant k < k_{i+1}$. Due to (2.13), (3.16), (3.17), and the definition of q^*, we obtain

$$\begin{aligned} \|\mathscr{G}_{k_{i+q^*+1}}\| &\leqslant \alpha_{\sup} \|u_{k_{i+q^*+1}} - u^*\| \leqslant \alpha_{\sup} c_2^{q^*} \|u_{k_{i+1}} - u^*\| \leqslant \alpha_{\sup} c_1 c_2^{q^*} \|u_k - u^*\| \\ &\leqslant \frac{\alpha_{\sup} c_1 c_2^{q^*}}{\alpha_{\inf}} \|\mathscr{G}_k\| < \|\mathscr{G}_k\|. \end{aligned}$$

By setting $i^+(k) := k_{i+q^*+1} - k$, we have $i^+(k) \leqslant \ell$ and we are finished with the verification of (3.18).

Now, due to the definition of $k^*(\varepsilon)$, we have $\|\mathscr{G}_k\| \geqslant \varepsilon$ for every $k < k^*(\varepsilon)$. We will next show that for every $k^*(\varepsilon) \geqslant \ell$ that

$$\|\mathscr{G}_k\| > \varepsilon \quad \text{for every } k < k^*(\varepsilon) - \ell. \tag{3.19}$$

Suppose on contrary that there exists an index $\bar{k} < k^*(\varepsilon) - \ell$ with $\|\mathscr{G}_{\bar{k}}\| = \varepsilon$. Then due to (3.18) there exists an integer $i^+(\bar{k}) \leqslant \ell$ such that we have $\|\mathscr{G}_{\bar{k}+i^+(\bar{k})}\| < \|\mathscr{G}_{\bar{k}}\| = \varepsilon$ with $\bar{k} + i^+(\bar{k}) \leqslant \bar{k} + \ell < k^*(\varepsilon)$, and this contradicts the definition of $k^*(\varepsilon)$. Hence, (3.19) holds.

Due to (3.19), for $k < k^*(\varepsilon) - \ell$ there exist strictly positive numbers $\{\delta_k\}_k$ such that $\|\mathscr{G}_k\| = \varepsilon + \delta_k$ for $k < k^*(\varepsilon) - \ell$. By setting $0 < \delta := \min\{\delta_k : k < k^*(\varepsilon) - \ell\}$, we obtain

$$\|\mathscr{G}_k\| \geqslant \varepsilon + \delta \quad \text{for every } k < k^*(\varepsilon) - \ell.$$

Therefore we conclude that $k^*(\varepsilon) - \ell \leqslant k^*(\varepsilon + \delta)$. Due to Theorem 3.2, for $\varepsilon > 0$ and $\delta > 0$, there exists a number $h_\varepsilon > 0$ such that we have

$$k^*(\varepsilon) - \ell \leqslant k^*(\varepsilon + \delta) \leqslant k_h^*(\varepsilon) \leqslant k^*(\varepsilon) \quad \text{for every } h \in (0, h_\varepsilon]. \tag{3.20}$$

This concludes the proof. □

REMARK 3.2 In the case of quadratic functions (QP), due to Proposition 2.1, inequality (3.20) holds for $\ell = m$ and all initial iterates $u_{-1}, u_0 \in \mathscr{H}$ with $u_{-1} \neq u_0$. In particular, due to Remark 2.1,if also $\kappa(\mathscr{A}) < 2$, then (3.20) holds for $\ell = 1$. Further as $\kappa(\mathscr{A})$ increases, ℓ becomes larger.

REMARK 3.3 In general, the sequence $\{\|\mathscr{G}_k\|\}_k$ corresponding to Algorithm 1 is not monotonically decreasing. This is the reason why we have to introduce ℓ in Theorem 3.3 which can possibly be larger than 1. For the case that $\{\|\mathscr{G}_k\|\}_k$ is monotone decreasing, we have $\ell = 1$. Moreover, clearly ℓ depends on $\kappa(\mathscr{A}_{u^*}^{\mathscr{F}})$.

4. Application to Optimal Control Problems with PDEs

In this section we apply Algorithm 1 to optimal control problems which are governed by partial differential equations. For the sake of brevity, finite-dimensional approximation is only discussed for the first one.

4.1 *Dirichlet Optimal Control for the Poisson Equation*

4.1.1 *Continuous Problem.* We consider the following elliptic Dirichlet boundary control problem

$$\min_{u\in L^2(\Gamma)} J(u,y) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L^2(\Gamma)}, \tag{4.1}$$

$$\text{subject to } \begin{cases} -\Delta y = f & \text{in } \Omega, \\ y = u & \text{on } \Gamma, \end{cases} \tag{4.2}$$

on an open convex bounded polygonal set $\Omega \subset \mathbb{R}^2$ with boundary denoted by $\Gamma := \partial\Omega$. We assume that $f, y_d \in L^2(\Omega)$ and $\beta > 0$. Then, for a given $(u,f) \in L^2(\Gamma)\times L^2(\Omega)$, the solution $y(u,f) \in L^2(\Omega)$ of (4.2) exists in a very weak sense and it satisfies the following variational equation

$$(y, -\Delta\varphi)_{L^2(\Omega)} + (u, \partial_\nu \varphi)_{L^2(\Gamma)} = (\varphi, f)_{L^2(\Omega)} \quad \text{for all } \varphi \in H^2(\Omega)\cup H^1_0(\Omega).$$

The corresponding solution operator defined by $(u,f)\mapsto y(u,f)$ is a continuous operator from $L^2(\Gamma)\times L^2(\Omega)$ to $L^2(\Omega)$(see e.g., Grisvard, 1985, 1992). Moreover, the linear operators $\mathscr{L} : L^2(\Gamma)\to L^2(\Omega)$ defined by $u\mapsto y(u,0)$, and $\Pi : L^2(\Omega)\to L^2(\Omega)$ defined by $f\mapsto y(f,0)$ are continuous. Then by defining $\mathscr{X} := L^2(\Omega)$, $\mathscr{H} := L^2(\Gamma)$ and $\psi := -\Pi f + y_d$, we can express the optimal control problem (4.1)-(4.2) as the following linear least squares problem

$$\min_{u\in\mathscr{H}} \mathscr{F}(u) := \frac{1}{2}\|\mathscr{L}u-\psi\|^2_{\mathscr{X}} + \frac{\beta}{2}\|u\|^2_{\mathscr{H}}. \tag{LS}$$

Thus problem (4.1)-(4.2) can be written in the form of (QP) with

$$\mathscr{A} := \mathscr{L}^*\mathscr{L} + \beta I \quad \text{and} \quad b := \mathscr{L}^*\psi, \tag{4.3}$$

where $\mathscr{L}^* : \mathscr{X}\to\mathscr{H}$ defined as the adjoint operator of $\mathscr{L}$ and I is the identity mapping. The operator $\mathscr{A}$ is uniformly positive, bounded, and self-adjoint on $\mathscr{H}$, and thus, the existence and uniqueness of the solution to (4.1)-(4.2) can be obtained due the fact that $\mathscr{A}$ has a bounded inverse.

For every $u\in\mathscr{H}$, the derivative of $\mathscr{F}$ at u in direction $\delta u\in\mathscr{H}$ can be expressed by

$$\mathscr{F}'(u)\delta u = (\mathscr{L}^*(\mathscr{L}u-\psi)+\beta u, \delta u), \tag{4.4}$$

and the gradient of $\mathscr{F}$ at u is identified by $\mathscr{G}(u) = \mathscr{L}^*(\mathscr{L}u-\psi)+\beta u$. Alternatively, if we consider the solution $p(u)\in H^2(\Omega)\cap H^1_0(\Omega)$ of the adjoint equation

$$\begin{cases} -\Delta p = y(u,f) - y_d & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma, \end{cases} \tag{4.5}$$

with the solution $y(u,f)\in L^2(\Omega)$ to (4.2), then the directional derivative (4.4) and the corresponding gradient $\mathscr{G}$ at point u can be rewritten as

$$\mathscr{F}'(u)\delta u = (\partial_\nu p(u)+\beta u, \delta u) \text{ for all } \delta u\in\mathscr{H}, \text{ and } \quad \mathscr{G}(u) = \partial_\nu p(u)+\beta u \text{ in } \mathscr{H}. \tag{4.6}$$

For the global minimizer $u^* \in \mathscr{H}$ to (LS), the first-order optimality condition can be expressed as

$$(\mathscr{L}^*\mathscr{L} + \beta I)u^* = \mathscr{L}^*\psi, \tag{4.7}$$

which can be rewritten, equivalently, as the following system of equations

$$\begin{cases} y^* = y(u^*, f) & \text{in } L^2(\Omega), \\ \partial_\nu p^* = -\beta u^* & \text{in } L^2(\Gamma), \\ -\Delta p^* = y^* - y_d & \text{in } L^2(\Omega), \text{with } p^* = 0 \text{ on } \Gamma. \end{cases}$$

4.1.2 *Discretized Problem* . For the discretization of (4.1)-(4.2), we use finite elements. Let us consider the regular family of triangulations $\{\mathscr{T}_h\}_{h>0}$ of $\overline{\Omega}$ with $\overline{\Omega} = \cup_{T\in\mathscr{T}_h} T$ and the mesh-size defined by $h := \max\{\text{diam}(T) : T \in \mathscr{T}_h\}$. Let $\{x_j\}_{1\leqslant j\leqslant N(h)}$ be the nodes which lies on the boundary with the counterclockwise numbering and $x_{N(h)+1} = x_1$. Then we define the space of discretized control by

$$\mathscr{H}^h := \{u^h \in C(\Gamma) : u^h|_{[x_j, x_{j+1}]} \in \mathscr{P}^1 \text{ for } j = 1, \dots, N(h)\},$$

and, we consider the space $V^h \subset H^1(\Omega)$ defined by

$$V^h := \{y^h \in C(\overline{\Omega}) : y^h|_T \in \mathscr{P}^1 \text{ for every } T \in \mathscr{T}_h\},$$

where $\mathscr{P}^1$ is the space of polynomials of degree less than or equal to 1. Further we set $V_0^h := V^h \cap H_0^1(\Omega)$. The space $\mathscr{H}^h$ is formed by the restriction of the functions of V^h to $\partial\Omega$. Clearly, we have $\mathscr{H}^h \subset \mathscr{H}$ and, as a result, the finite-dimensional space $\mathscr{H}^h$ is endowed with the inner product and the norm introduced by the space $\mathscr{H} = L^2(\Gamma)$. Then, naturally, the prolongation operator $\mathbb{P}^h : \mathscr{H}^h \to \mathscr{H}$ is defined to be the canonical injection operator i.e., $\mathbb{P}^h(u^h) = u^h$ for every $u^h \in \mathscr{H}^h$. Let us consider the orthogonal projection operator $\Pi^h : \mathscr{H} \to \mathscr{H}^h$ defined by

$$(\Pi^h v, u^h)_{\mathscr{H}} = (v, u^h)_{\mathscr{H}} \quad \text{for all } u^h \in \mathscr{H}^h.$$

It satisfies the following estimate

$$\|u - \Pi^h u\|_{\mathscr{H}} \leqslant ch^{\frac{1}{2}} \|u\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{4.8}$$

for every $u \in H^{\frac{1}{2}}(\Gamma)$, see, e.g., Berggren (2004) and Casas & Raymond (2006). For every $u \in \mathscr{H}$ we consider the unique discrete solution $y^h(u) \in V^h$ satisfying

$$\begin{cases} (\nabla y^h, \nabla \phi^h) = (f, \phi^h) & \text{for all } \phi^h \in V^h, \\ y^h|_\Gamma = \Pi^h u. \end{cases} \tag{4.9}$$

Then we can define the discrete objective function in $\mathscr{H}$ by

$$J^h(u, y^h(u)) := \frac{1}{2}\|y^h(u) - y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L^2(\Gamma)}. \tag{4.10}$$

The finite-dimensional approximation of (4.1)-(4.2) can be expressed as

$$\min_{u^h \in \mathscr{H}^h} \mathscr{F}^h(u^h) = \min_{u^h \in \mathscr{H}^h} J^h(u^h, y^h(u^h)). \tag{4.11}$$

Existence of a solution to (4.11) follows by similar arguments as for the continuous problem. Given $u \in \mathscr{H}$ we consider the adjoint state $p^h(u) \in V_0^h$ as the solution of

$$(\nabla p^h(u), \nabla \psi^h)_{L^2(\Omega)} = (y^h(u) - y_d, \psi^h)_{L^2(\Omega)} \quad \text{for all } \psi^h \in V_0^h. \tag{4.12}$$

In order to compute the gradient of $\mathscr{F}^h$, analogously to the expression (4.6), we need to characterise a discrete normal derivative $\partial_\nu^h p^h(u)$. For every $u \in \mathscr{H}$, similarly to Casas & Raymond (2006, Proposition 4.2), $\partial_\nu^h p^h(u) \in \mathscr{H}^h$ is characterized as the unique solution of the following variational problem

$$(\partial_\nu^h p^h(u), \varphi^h)_{\mathscr{H}} = (\nabla p^h(u), \nabla \varphi^h)_{L^2(\Omega)} - (y^h(u) - y_d, \varphi^h)_{L^2(\Omega)} \quad \text{for all } \varphi^h \in V^h,$$

where $p^h(u) \in V_0^h$ is the solution of (4.12). Next, we prove the following useful estimate.

LEMMA 4.1 There exists a constant c depending on f and y_d, and independent of h such that

$$\|\partial_\nu p(u) - \partial_\nu^h p^h(v)\|_{\mathscr{H}} \leqslant c\left(\|u - v\|_{\mathscr{H}} + h^{\frac{1}{2}}(1 + \|v\|_{\mathscr{H}})\right) \quad \text{for all } u, v \in \mathscr{H}. \tag{4.13}$$

Proof. This proof is based on the results from Casas & Raymond (2006), where $u \in L^\infty(\Omega)$ was used in the context of semilinear elliptic equation. Throughout the proof, $c > 0$ is a generic constant which is independent of h. First, using a similar argument as in Berggren (2004) and Casas & Raymond (2006), one can show that

$$\|y(u) - y^h(u)\|_{\mathscr{H}} \leqslant c(1 + \|u\|_{\mathscr{H}})^2 h^{\frac{1}{2}}, \tag{4.14}$$

where the constant c depends on f. From (4.14), it follows that

$$\|y(u) - y^h(v)\|_{\mathscr{H}} \leqslant c\left(\|u - v\|_{\mathscr{H}} + h^{\frac{1}{2}}(1 + \|u\|_{\mathscr{H}})\right) \quad \text{for all } u, v \in \mathscr{H},$$

Next, we show that

$$\|\partial_\nu p(u) - \partial_\nu^h p^h(u)\|_{\mathscr{H}} \leqslant c h^{\frac{1}{2}}(1 + \|u\|_{\mathscr{H}}) \quad \text{for all } u \in \mathscr{H}. \tag{4.15}$$

Recall that $p(u) \in H^2(\Omega) \cap H_0^1(\Omega)$ and therefore $\partial_\nu p(u) \in H^{\frac{1}{2}}(\Gamma)$. For the left hand-side of (4.15) we obtain

$$\|\partial_\nu p(u) - \partial_\nu^h p^h(u)\|_{\mathscr{H}}^2 = \int_\Gamma \left|\partial_\nu p(u) - \Pi^h \partial_\nu p(u)\right|^2 d\mathscr{S} + \int_\Gamma \left|\Pi^h \partial_\nu p(u) - \partial_\nu^h p^h(u)\right|^2 d\mathscr{S} =: I_1 + I_2. \tag{4.16}$$

The last term can be equivalently be expressed as

$$I_2 = \int_\Gamma (\partial_\nu p(u) - \partial_\nu^h p^h(u))(\Pi^h \partial_\nu p(u) - \partial_\nu^h p^h(u)) d\mathscr{S}. \tag{4.17}$$

Let $w^h \in V^h$ be the solution of the following variational equation

$$\begin{cases} (\nabla w^h, \nabla \phi^h) = 0 \quad \text{for all } \phi^h \in V^h, \\ w^h|_\Gamma = \Pi^h \partial_\nu p(u) - \partial_\nu^h p^h(u). \end{cases} \tag{4.18}$$

Then, by referring to Bramble *et al.* (1986, Lemma 3.2), we have the following estimate for (4.18)

$$\|w^h\|_{H^1(\Omega)} \leqslant c\|\Pi^h \partial_\nu p(u) - \partial_\nu^h p^h(u)\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{4.19}$$

with a constant c independent of h. Using the definition of $\partial_\nu^h p^h(u)$ and Green formula for $\partial_\nu p(u)$, we obtain

$$(\partial_\nu p(u) - \partial_\nu^h p^h(u), \phi^h)_{\mathscr{H}} = (\nabla(p(u) - p^h(u)), \nabla\phi^h)_{L^2(\Omega)} + (y^h(u) - y(u), \phi^h)_{L^2(\Omega)} \tag{4.20}$$

for every $\phi^h \in V^h$. Using (4.17), (4.18), and (4.20), we find

$$I_2 = (\nabla(p(u) - p^h(u)), \nabla w^h)_{L^2(\Omega)} + (y^h(u) - y(u), w^h)_{L^2(\Omega)}.$$

Moreover, we have

$$(\nabla p^h(u), \nabla w^h)_{L^2(\Omega)} = (\nabla I_h p(u), \nabla w^h)_{L^2(\Omega)} = 0, \tag{4.21}$$

where $I_h \in \mathscr{L}(C(\overline{\Omega}), V_0^h)$ stands for the classical interpolation operator, see e.g., Brenner & Scott (1994). Due to (4.21) and the definition of w^h from (4.18), we obtain

$$I_2 = (\nabla(p(u) - I_h p(u)), \nabla w^h)_{L^2(\Omega)} + (y^h(u) - y(u), w^h)_{L^2(\Omega)}. \tag{4.22}$$

Using (4.19), the interpolation estimate, and the following inverse estimate (see e.g., Berggren, 2004)

$$\|u^h\|_{H^{\frac{1}{2}}(\Gamma)} \leqslant \bar{C} h^{-\frac{1}{2}} \|u^h\|_{\mathscr{H}} \quad \text{for all } u^h \in \mathscr{H}^h,$$

for a constant $\bar{C} > 0$, we infer that

$$\begin{aligned}|(\nabla(p(u) - I_h p(u)), \nabla w^h)_{L^2(\Omega)}| &\leqslant \|\nabla(p(u) - I_h p(u))\|_{L^2(\Omega)} \|w^h\|_{H^1(\Omega)} \leqslant ch\|p(u)\|_{H^2(\Omega)} \|w^h|_\Gamma\|_{H^{\frac{1}{2}}(\Gamma)} \\ &\leqslant ch^{\frac{1}{2}}(1 + \|u\|_{\mathscr{H}})\|w^h|_\Gamma\|_{\mathscr{H}} \leqslant ch^{\frac{1}{2}}(1 + \|u\|_{\mathscr{H}})\sqrt{I_2},\end{aligned} \tag{4.23}$$

where the constant c from the second line of (4.23) depends also on y_d. Moreover, due to (4.14), we can write

$$|(y^h(u) - y(u), w^h)_{L^2(\Omega)}| \leqslant \|y^h(u) - y(u)\|_{L^2(\Omega)} \|w^h\|_{L^2(\Omega)} \leqslant ch^{\frac{1}{2}}(1 + \|u\|_{\mathscr{H}})\sqrt{I_2}. \tag{4.24}$$

From (4.22), (4.23), and (4.24), it follows that

$$I_2 \leqslant ch(1 + \|u\|_{\mathscr{H}})^2. \tag{4.25}$$

Further, using (4.8) we obtain

$$I_1 \leqslant ch\|\partial_\nu p(u)\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \leqslant ch\|p(u)\|^2_{H^2(\Omega)} \leqslant ch(1 + \|u\|_{\mathscr{H}})^2. \tag{4.26}$$

Now, from (4.16), (4.25), and (4.26), we conclude (4.15). Finally, using (4.15) we can write that

$$\begin{aligned}&\|\partial_\nu p(u) - \partial_\nu^h p^h(v)\|_{\mathscr{H}} \leqslant \|\partial_\nu p(u) - \partial_\nu p(v)\|_{\mathscr{H}} + \|\partial_\nu p(v) - \partial_\nu^h p^h(v)\|_{\mathscr{H}} \\ &\leqslant c\|p(u) - p(v)\|_{H^2(\Omega)} + ch^{\frac{1}{2}}(1 + \|v\|_{\mathscr{H}}) \leqslant c\left(\|u - v\|_{\mathscr{H}} + h^{\frac{1}{2}}(1 + \|v\|_{\mathscr{H}})\right),\end{aligned}$$

for every $u, v \in \mathscr{H}$ and we are finished with the verification of (4.13). □

Now we are in the position in which we can verify the assumptions A1-A2 of Section 3.A1 follows from the definition of $\mathscr{H}^h$ and $\mathbb{P}^h$. To verify A2, assume that $u^h \overset{\mathscr{H}}{\to} u$ with $u^h \in \mathscr{H}^h$. Similarly to (4.6), the directional derivative and its corresponding gradient of $\mathscr{F}^h$ of the discretized problem (4.11) at point u^h can be rewritten as

$$\mathscr{F}^{h'}(u^h)\delta u^h = (\partial_\nu^h p^h(u^h) + \beta u^h, \delta u^h) \text{ for all } \delta u^h \in \mathscr{H}^h, \text{ and } \quad \mathscr{G}^h(u^h) = \partial_\nu^h p^h(u^h) + \beta u^h \text{ in } \mathscr{H}^h. \tag{4.27}$$

Then by (4.6), (4.27), and (4.13), we obtain

$$\begin{aligned} \|\mathscr{G}(u) - \mathbb{P}^h\mathscr{G}^h(u^h)\|_{\mathscr{H}} &\leqslant \|\partial_\nu p(u) - \partial_\nu^h p^h(u^h)\|_{\mathscr{H}} + \beta\|u-u^h\|_{\mathscr{H}} \\ &\leqslant (c+\beta)\|u-u^h\|_{\mathscr{H}} + ch^{\frac{1}{2}}(1+\|u^h\|_{\mathscr{H}}). \end{aligned} \tag{4.28}$$

Hence, $\mathscr{G}^h(u^h) \overset{\mathscr{H}}{\to} \mathscr{G}(u)$ follows by sending h to zero in (4.28).

REMARK 4.1 Due the fact that $\partial_\nu p(u) \in H^{\frac{1}{2}}(\Gamma)$ for every $u \in \mathscr{H}$, using (4.6) and Step 4 in Algorithm 1, it is easy to see that for every $u_{-1}, u_0 \in H^{\frac{1}{2}}(\Gamma)$, the sequence $\{u_k\}_k$ stays in the space $H^{\frac{1}{2}}(\Gamma)$. Moreover, for given $y_d, f \in L^{p^*}(\Omega)$ with $p^* > 2$, we have $y \in W^{1,\overline{p}}(\Omega)$ and $p(y,y_d) \in W^{2,\overline{p}}(\Omega)$ for $\overline{p} \in (2, p^*]$ depending on Ω, see e.g., Casas & Raymond (2006, Theorem 3.4). Hence, $\{u_k\}_k \subset W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma) \subset C(\Gamma)$ provided that $u_{-1}, u_0 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma)$. In this case, u_0^h, u_1^h can be chosen as $I_h^\Gamma u_{-1}, I_h^\Gamma u_0 \in \mathscr{H}^h$ where $I_h^\Gamma \in \mathscr{L}(W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma), \mathscr{H}^h)$ is the standard interpolation operator.

4.2 *Distributed Optimal Control for the Burgers Equation*

We consider the following optimal control problem

$$\min_{u \in L^2(\hat{Q})} J(y,u) := \frac{\alpha_1}{2}\|y - y_d\|^2_{L^2(Q)} + \frac{\alpha_2}{2}\|y(T) - z_d\|^2_{L^2(0,1)} + \frac{\beta}{2}\|u\|^2_{L^2(\hat{Q})}, \tag{4.29}$$

$$\text{subject to } \begin{cases} y_t - \vartheta y_{xx} + y y_x = Bu + f, & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{4.30}$$

where ϑ, β α_1, α_2, and T are positive constants, $y(t) = y(t,x), u(t) = u(t,x)$, $Q := (0,T) \times (0,1)$, and $\hat{Q} := (0,T) \times \hat{\Omega}$ where $\hat{\Omega}$ is an open subset of $(0,1)$. Moreover, $y_0 \in L^2(0,1)$, $f \in L^2(0,T;H^{-1}(0,1))$, the desired states y_d and z_d are smooth enough, and the extension operator $B \in \mathscr{L}(L^2(\hat{\Omega}), L^2(0,1))$ is defined by

$$(Bu)(x) = \begin{cases} u(x), & x \in \hat{\Omega}, \\ 0 & x \in (0,1)\backslash\hat{\Omega}. \end{cases}$$

Considering $W(0,T) := \{\phi : \phi \in L^2(0,T;H_0^1(0,1)), \phi_t \in L^2(0,T;H^{-1}(0,1))\}$ as the space of solutions, we have the following notion of weak solution.

DEFINITION 4.1 Let $(y_0,u,f) \in L^2(0,1) \times L^2(\hat{Q}) \times L^2(0,T;H^{-1}(0,1))$ be given. Then, a function $y \in W(0,T)$ is referred as a weak solution to (4.30) if $y(0) = y_0$ is satisfied in $L^2(0,1)$ and for almost every $t \in (0,T)$, the following equality

$$\langle y_t(t), \varphi \rangle_{H^{-1},H_0^1} + \vartheta(y(t),\varphi)_{H_0^1} + b(y(t),y(t),\varphi) = \langle Bu(t) + f, \varphi \rangle_{H^{-1},H_0^1}$$

for all $\varphi \in H_0^1(0,1)$ holds, where the continuous trilinear form $b : H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1) \to \mathbb{R}_+$ is defined as

$$b(\varphi,\psi,\phi) = \int_0^1 \varphi\psi_x\phi dx \quad \text{for all } \varphi,\psi,\phi \in H_0^1(0,1).$$

It is known that, for every triple $(y_0,u,f) \in L^2(0,1)\times L^2(\hat{Q})\times L^2(0,T;H^{-1}(0,1))$, equation (4.30) admits a unique weak solution $y(y_0,u,f) \in W(0,T)$ and for this weak solution we have the following estimate

$$\|y\|_{W(0,T)} \leqslant C\left(\|y_0\|_{L^2(\Omega)} + \|u\|_{L^2(\hat{Q})} + \|f\|_{L^2(0,T;H^{-1}(0,1))}\right)^2, \tag{4.31}$$

where the constant C depends only on T and ϑ. Now, by setting $X = W(0,T)\times\mathscr{H}$ with $\mathscr{H} := L^2(\hat{Q})$, and $Y := L^2(0,T;H^{-1}(0,1))\times L^2(0,1)$, we define $e : X \to Y$ by

$$e(y,u) := \begin{pmatrix} y_t - \vartheta y_{xx} + yy_x - Bu - f \\ y(0) - y_0 \end{pmatrix}.$$

The mapping $e : X \to Y$ consists of a sum of continuous linear terms and a continuous bilinear term. Hence, it can be shown that it is infinitely Fréchet differentiable. Moreover due to the unique solvability of (4.30), for every $u \in \mathscr{H}$ there exists a unique element $y = y(u) \in W(0,T)$ satisfying $e(y(u),u) = 0$ and estimate (4.31) holds. Therefore the control-to-state $u \in \mathscr{H} \mapsto y(u) \in W(0,T)$ is well-defined. Then we can rewrite the optimal control problem (4.29)-(4.30) in the following form

$$\min_{u\in\mathscr{H}} \mathscr{F}(u) = \min_{u\in\mathscr{H}} J(y(u),u) = \min_{(y,u)\in X}\{J(y,u) : \text{ subject to } e(y,u) = 0\}. \tag{4.32}$$

Further, due to estimate (4.31) and the compact embedding from the space $W(0,T)$ to the space $L^2(Q)$, it follows from standard subsequential limit arguments that the problem (4.29)-(4.30) admits a solution (see e.g., Tröltzsch & Volkwein, 2001; Volkwein, 2001). Before dealing with the optimality conditions, we refer to the following linearized Burgers equation at $y \in W(0,T)$ and its corresponding backward in time adjoint equation

$$\begin{cases} q_t - \vartheta q_{xx} + (yq)_x = \phi & (t,x)\in Q, \\ q(t,0) = q(t,1) = 0 & t \in (0,T), \\ q(0) = q_0 & x \in (0,1), \end{cases} \tag{4.33}$$

$$\begin{cases} -p_t - \vartheta p_{xx} - yp_x = \psi & (t,x)\in Q, \\ p(t,0) = p(t,1) = 0 & t \in (0,T), \\ p(T,x) = p_T & x \in (0,1). \end{cases} \tag{4.34}$$

It can be shown that for all pairs (ϕ,q_0) and (ψ,p_T) in the space Y, the solution operators $\mathscr{S}_{lin}^y : Y \to W(0,T)$ of (4.33), and $\mathscr{S}_{adj}^y : Y \to W(0,T)$ of (4.34) defined by $(\phi,q_0) \mapsto v$ and $(\psi,p_T) \mapsto p$, respectively, are well-defined and continuous (see e.g., Tröltzsch & Volkwein, 2001; Volkwein, 2001).

Due to the definitions of $\mathscr{S}_{lin}^{y(u)}$ and $e_y(y(u),u)$, we can infer that $e_y^{-1}(y(u),u) = \mathscr{S}_{lin}^{y(u)}$ and, as consequence, $e_y(y,u)$ is continuously invertible. In addition, since e is inifinitely often continuously differentiable, the implicit function theorem (Hinze *et al.*, 2009, Theorem. 1.41) implies that the control-to-state operator $u \to y(u)$ is infinitely continuously differentiable from $\mathscr{H}$ to $W(0,T)$ as well. Due to the compact embedding $W(0,T) \hookrightarrow L^2(Q)$ (see Temam, 1984, Theorem. 2.3), one obtains that all Fréchet derivatives of this mapping from $\mathscr{H}$ to $L^2(Q)$ are Lipschitz continuous on bounded sets. Now we are in the position to derive the first-order optimality conditions. First, by using the implicit function theorem, the first derivative of the mapping $u \mapsto y(u)$ at u in direction of an arbitrary $\delta u \in \mathscr{H}$ is given by

$$y'(u)\delta u = -e_y^{-1}(x)e_u(x)\delta u, \tag{4.35}$$

where $x := (y(u), u) \in X$. Then, by the chain rule we obtain

$$\mathscr{F}'(u)\delta u = (\mathscr{G}(u), \delta u) = ((y'(u))^* J_y(x) + J_u(x), \delta u),$$

where $(y'(u))^*$ stands for the adjoint operator of $y'(u)$. Since δu is arbitrary, the first-order optimality condition (EP) can be written as

$$\mathscr{G}(u^*) = J_u(x^*) - e_u^*(x) e_y^{-*}(x) J_y(x^*) = 0. \tag{4.36}$$

where $x^* := (y(u^*), u^*)$. Moreover, by setting $(p^*, \bar{p}) := -e_y^{-*}(x) J_y(x^*)$ with $(p^*, \bar{p}) \in Y^*$ and $\bar{p} = p^*(0)$, the first-order optimality condition (4.36) can be expressed as the following system of differential equations

$$\begin{cases} B^* p^* = \beta u^* \quad \text{in } L^2(\hat{Q}), \\ p^* = \mathscr{S}_{adj}^{y(u^*)}(-\alpha_1(y(u^*) - y_d), -\alpha_2((y(u^*))(T) - z_d)). \end{cases}$$

Next, we compute the second derivative of $\mathscr{F}$. Let $(\delta u, \delta v) \in \mathscr{H} \times \mathscr{H}$ be arbitrary, then using the implicit functions theorem, the second derivative of the operator $u \mapsto y(u)$ from $\mathscr{H}$ to $W(0,T)$ can be written as

$$y''(u)(\delta u, \delta v) = -e_y^{-1}(x) e_{yy}(x)(y'(u)\delta u, y'(u)\delta v). \tag{4.37}$$

Now, by using the chain rule and (4.37) as in Hinze & Kunisch (2001) and Hinze *et al.* (2009), we obtain

$$\begin{aligned} \mathscr{F}''(u)(\delta u, \delta v) &= \langle J_{yy}(x) y'(u)\delta u, y'(u)\delta v\rangle \\ &\quad + \langle -e_y^{-*}(x) J_y(x), e_{yy}(x)(y'(u)\delta u, y'(u)\delta v)\rangle + \langle J_{uu}(x)\delta u, \delta v\rangle_{\mathscr{H}}. \end{aligned} \tag{4.38}$$

Furthermore, due to the first estimate in (4.38), and the fact that $J : \mathscr{H} \times W(0,T) \to \mathbb{R}$ and the control-to-state operator are infinitely Fréchet differentiable, it follows that $\mathscr{F}'' : \mathscr{H} \to \mathscr{L}(\mathscr{H}, \mathscr{L}(\mathscr{H}, \mathbb{R}))$ is locally Lipschitz continuous. Then, the uniformly positiveness of $\mathscr{F}''(u^*)$ can be expressed as

$$\begin{aligned} \mathscr{F}''(u^*)(v,v) :&= \alpha_1 \|q^*\|^2_{L^2(Q)} + \alpha_2 \|q^*(T)\|^2_{L^2(0,1)} + (p^*, 2q^* q^*_x)_{L^2(Q)} + \beta\|v\|^2_{\mathscr{H}} \\ &\geqslant \delta_{\inf}\|v\|^2_{\mathscr{H}} \quad \text{for all } v \in \mathscr{H}, \end{aligned} \tag{4.39}$$

where $\delta_{\inf} > 0$, $p^* := \mathscr{S}_{adj}^{y(u^*)}(-\alpha_1(y(u^*) - y_d), -\alpha_2((y(u^*))(T) - z_d)$, and $q^* := \mathscr{S}_{lin}^{y(u^*)}(v, 0)$.

REMARK 4.2 Clearly, the only term in (4.39) that can spoil the uniformly positiveness of $\mathscr{F}''(u^*)$ is the term involving p^*. This term originates from the nonlinear convection term in the state equation. Since

$$\left|(p^*, 2q^* q^*_x)_{L^2(Q)}\right| \leqslant c\|p^*\|_{L^2(0,T;L^\infty(0,1))} \|q^*\|^2_{W(0,T)}$$

for a constant $c > 0$, the uniform positiveness of $\mathscr{F}''(u^*)$ holds, provided that $\|p^*\|_{L^2(0,T;L^\infty(0,1))}$ is small enough. Indeed, for $p^* = 0$, inequality (4.39) holds for $\delta_{\inf} := \beta$. For instance, by setting $y_d = z_d = f = 0$, inequality (4.39) holds for every initial function y_0 with sufficiently small $\|y_0\|_{L^2(0,1)}$.

5. Numerical Experiments

In order to validate our theoretical findings in the previous sections, we report numerical results corresponding to the optimal control problems introduced in the previous section. We investigate the application of Algorithm 1 with respect to different strategies for selecting step-sizes and different choices of the discretization parameter h, the control cost parameter β, and the tolerance ε in the termination condition (3.7). For Algorithm 1, we consider the cases:

BB1: $\alpha_k := \alpha_k^{BB1}$ for every $k \geqslant 0$.

BB2: $\alpha_k := \alpha_k^{BB2}$ for every $k \geqslant 0$.

ABB: $\alpha_k := \alpha_k^{BB1}$ for even $k \geqslant 0$, and α_k^{BB2} for odd $k \geqslant 0$.

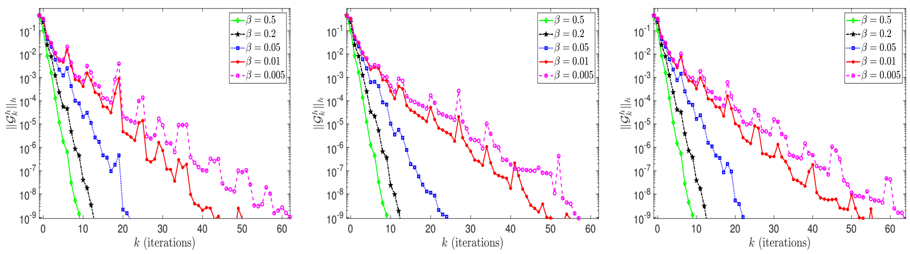
The last case, which is known as the alternating strategy, has already been introduced by e.g., Dai *et al.* (2002) and Grippo & Sciandrone (2002). Further, Dai & Fletcher (2005b) reported numerical results for the case of finite-dimensional bound-constrained optimization problems which show that projected ABB works somewhat better than projected BB1. According to (4.3) and (4.39), the value β in all the optimal control problems of the previous section has a direct influence on the spectral condition number of $\mathscr{A}_{u^*}^{\mathscr{F}}$ corresponding to $\mathscr{F}$. To be more precise, as the value of β increases, the value of $\kappa(\mathscr{A}_{u^*}^{\mathscr{F}})$ is getting smaller. Therefore, as its has been discussed in Remarks 2.1, one expects a larger total number of iterations for a smaller value of β and a fixed tolerance ε. Moreover, according to Remark 3.2, the number ℓ depends on the behaviour (monotonicity versus nonmonotonicity) of $\{\|\mathscr{G}_k\|\}_k$, and consequently also on $\kappa(\mathscr{A}_{u^*}^{\mathscr{F}})$. Hence, the smaller β is chosen, the larger the value of ℓ is expected to be. We report the total number of iterations of the optimization Algorithm for different levels of discretization, or equivalently, different values of mesh-sizes. Then, for every example and fixed tolerance ε, ℓ is reported as the maximum of the pairwise differences of $k_h(\varepsilon)$ for different choices of h. We have chosen $u_{-1} = 0$ and $u_0 := -\mathscr{G}(0)$ as the initial iterates. All computations were done on a MATLAB platform.

EXAMPLE 5.1 (Dirichlet optimal control for the Poisson equation) We consider the problem introduced in Subsection 4.1 which is posed on the domain $\Omega := (0,1)^2$. For the discretization a uniform mesh was generated by triangulation. Then over this mesh, the discretization was done by a conforming linear finite element scheme using continuous piecewise linear basis functions as described in Subsection 4.1.2. We set $f(x) = 10\sin(\pi(x_1+x_2))$ and $y_d(x) = (x_1^2+x_2^2)^{\frac{1}{3}}$ where $x := (x_1,x_2) \in \Omega$. Table 1 shows the number of required iterations $k_h^*(\varepsilon)$ for different step-size strategies, and different values of β, ε, and the mesh-size h. From Table 1, it can be observed that:

1. For every fixed h, ε, and choice of step-size, decreasing in the value of β implies that the number of required iterations $k_h^*(\varepsilon)$ becomes larger and, thus, the convergence is getting slower. This is in accordance with the fact that there is a trade-off between the magnitude of β and the value of $\kappa(\mathscr{A})$ where $\mathscr{A} = \mathscr{L}^*\mathscr{L} + \beta I$ with $\mathscr{L}$ specified in Subsection 4.1.1. More precisely, $\kappa(\mathscr{A}) = \frac{\beta+\delta_{\sup}}{\beta+\delta_{\inf}}$ with $\delta_{\inf} := \inf(\sigma(\mathscr{L}^*\mathscr{L}))$ and $\delta_{\sup} := \sup(\sigma(\mathscr{L}^*\mathscr{L}))$. Hence a larger value of β yields a smaller value of $\kappa(\mathscr{A})$. That is as expected from the theory, for a larger β Algorithm 1 requires fewer iterations $k_h^*(\varepsilon)$ for every fixed h and ε. This behaviour is clearly illustrated in Figure 1 which depicts the convergence of $\|\mathscr{G}_k^h\|_h$ for the choice $h = 2^{-9}\sqrt{2}$, and different step-sizes strategies and values of β. As can be seen from Figure 1, the convergence for the cases $\beta = 0.5$ and $\beta = 0.2$ is Q-linear. For these cases, based on the discussion given in Remark 2.1, we might conjecture that $\kappa(\mathscr{A}) < 2$ with a smaller value of convergence rate $\gamma_{\mathscr{A}}$ for $\beta = 0.5$ compared to $\beta = 0.2$. However, for the rest of the cases, nonmonotonic behaviour occurs, which corresponds to $\kappa(\mathscr{A}) \geqslant 2$. Apparently, as β decreases, the nonmonotonic behaviour in the sequences $\{\|\mathscr{G}_k\|\}_k$ and, consequently, in $\{\|\mathscr{G}_k^h\|_h\}_k$ becomes stronger.

2. Mesh-independence can be observed from Table 1.Indeed we can see that for every fixed β, ε, and step-size strategy, the iterations $k_h(\varepsilon)$ stay almost constant and do not change as the discretezation levels changes. Moreover, for $\beta = 0.2$, $\beta = 0.05$, and $\beta = 0.01$ we can state that $\ell \approx 1$, $\ell \approx 3$, and

$\ell \approx 6$, respectively. This is also due to the dependence of the spectrum of $\mathscr{A} = \mathscr{L}^*\mathscr{L} + \beta I$ on the magnitude of β (see Remark 3.2).



(a) BB1 (b) BB2 (c) ABB

FIG. 1. Example 5.1: Convergence of $\|\mathscr{G}_k^h\|_h$ with $h = 2^{-9}\sqrt{2}$ for different choices of β and step-size strategies

To further study the behaviour of Algorithms 1, we consider Table 2 which summarizes the values of $\|\mathscr{G}_k^h\|_h$ for the choice of $\beta = 0.01$, at the iterations $k = 36, \ldots, 44$, and different levels of discretization. It can be seen that in any case the sequence $\{\|\mathscr{G}_k^h\|_h\}_k$ has a nonmonotonic behaviour. For every case the members of $\{\|\mathscr{G}_k^h\|_h\}_k$ at which the monotonicity of the sequence is violated, are indicated by bold type. With the superscript star we denote the members corresponding to $k_h^*(\varepsilon)$ with $\varepsilon = 1e-8$.

EXAMPLE 5.2 (Distributed optimal control for the Burgers equation) The spatial discretization of (4.29)-(4.30) was done by the standard Galerkin method based on piecewise linear basis functions with mesh-size h. For temporal discretization, we used the implicit Euler method with a step-size denoted by Δt. The resulting nonlinear systems were solved by Newton's method with the tolerance $\varepsilon_n = 10^{-13}$. Here the control acts on the open interval $\hat{\Omega} = (0.1, 0.4)$. Moreover we set $\vartheta = 0.01$, $y_0(x) = 8\exp(-20(x-0.5)^2)$, and $y_d(t,x) = z_d(x) = f(t,x) = 0$. To illustrate the mesh-independence, we report the values of $k_{\Delta t,h}^*(\varepsilon)$ for different levels of temporal and spatial discretizations. These results are gathered in Table 3. Figure 2 shows the convergence of Algorithm 1 applied to Example 5.2, for $(\Delta t, h) = (2^{-7}, 2^{-8})$, and different step-size strategies and values of β. Similarly to the previous example, due to (4.39), there is a trade-off between the magnitude of β and the value of $\kappa(\mathscr{A}_{u^*}^{\mathscr{F}})$. Further, as can be seen from Table 3 and Figures 2, despite the nonlinearity the observations 1 and 2 from Example 5.1 hold also true for this example, with the difference that here for $\beta = 0.5$, and $\beta = 0.05$, we have $\ell \approx 1$ and $\ell \approx 5$, respectively.

6. Conclusions

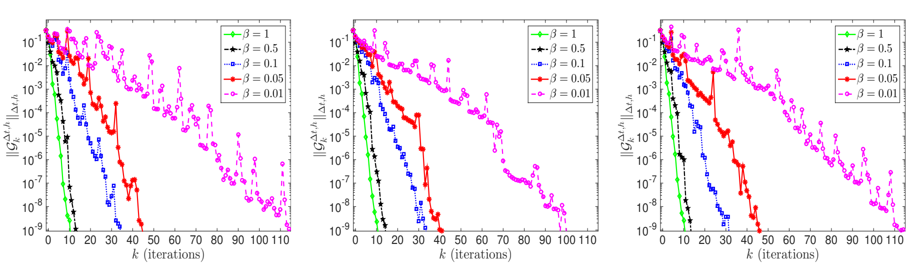
We have studied the performance of the BB-method in the context of PDE-contained optimization. Relying on the convergence results given in Azmi & Kunisch (2020) for infinite-dimensional, strictly convex, and quadratic functions, we have established the local R-linear convergence for a class of twice continuously Frećhet-differentiable functions. Based on this convergence result, the mesh-independent principle for the BB-method has been investigated. More precisely, we have proved that, for sufficiently small mesh-size, there is at most a difference of ℓ iterates between the number of iterations required for the infinite-dimensional problem and for its discretization to converge within a given tolerance. Here ℓ

The number of required iteration $k_h^*(\varepsilon)$							
$\beta = 0.2$							
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
BB1	$1e-2$	2	2	2	2	2	2
	$1e-4$	5	5	5	5	5	5
	$1e-6$	8	8	8	8	8	8
	$1e-8$	11	12	12	12	12	12
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
BB2	$1e-2$	2	2	2	2	2	2
	$1e-4$	5	5	5	5	5	5
	$1e-6$	8	8	8	8	8	8
	$1e-8$	10	11	11	11	11	11
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
ABB	$1e-2$	2	2	2	2	2	2
	$1e-4$	5	5	5	5	5	5
	$1e-6$	8	8	8	8	8	8
	$1e-8$	11	11	12	12	12	12
$\beta = 0.05$							
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
BB1	$1e-2$	3	3	3	3	3	3
	$1e-4$	8	8	8	8	9	9
	$1e-6$	13	15	15	15	15	15
	$1e-8$	20	20	20	20	20	20
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
BB2	$1e-2$	3	3	3	3	3	3
	$1e-4$	8	8	8	8	8	8
	$1e-6$	13	14	14	14	14	14
	$1e-8$	18	20	20	20	21	21
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
ABB	$1e-2$	3	3	3	3	3	3
	$1e-4$	8	8	8	8	8	8
	$1e-6$	13	13	14	15	15	15
	$1e-8$	17	20	20	20	20	20
$\beta = 0.01$							
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
BB1	$1e-2$	3	3	3	4	4	4
	$1e-4$	15	15	15	15	15	15
	$1e-6$	23	27	26	26	26	26
	$1e-8$	37	38	37	39	37	38
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
BB2	$1e-2$	3	3	4	4	4	4
	$1e-4$	12	14	14	14	14	14
	$1e-6$	25	25	29	30	30	31
	$1e-8$	38	43	42	44	44	43
	ε \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
ABB	$1e-2$	3	3	4	4	4	4
	$1e-4$	14	14	15	15	15	15
	$1e-6$	23	23	27	27	27	27
	$1e-8$	42	37	42	42	42	39

Table 1. Example 5.1: Numerical results

	The value of $\|\mathscr{G}_k^h\|$ at an iteration k						
	k \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
BB1	36	$1.29e-8$	$1.59e-8$	$3.96e-8$	$5.25e-7$	$1.90e-7$	$1.98e-7$
	37	$9.22e-9^*$	$1.05e-8$	$4.26e-9^*$	$1.03e-7$	$9.69e-9^*$	$1.39e-8$
	38	$\mathbf{2.93e-8}$	$5.17e-9^*$	$3.78e-9$	$5.93e-8$	$3.88e-9$	$8.75e-9^*$
	39	$\mathbf{1.05e-7}$	$1.38e-9$	$2.38e-9$	$7.21e-9^*$	$3.25e-9$	$3.46e-9$
	40	$1.43e-8$	$1.08e-9$	$\mathbf{6.90e-9}$	$4.30e-9$	$2.02e-9$	$1.90e-9$
	41	$\mathbf{1.46e-8}$	$\mathbf{1.69e-9}$	$4.72e-9$	$\mathbf{5.76e-9}$	$2.66e-9$	$1.28e-9$
	42	$7.26e-10$	$\mathbf{1.26e-8}$	$3.74e-9$	$1.12e-9$	$2.47e-9$	$6.74e-9$
	43	$6.24e-10$	$1.55e-9$	$2.63e-10$	$8.49e-10$	$8.46e-10$	$\mathbf{1.84e-8}$
	44	$4.09e-10$	$1.43e-9$	$1.74e-10$	$7.33e-10$	$\mathbf{1.11e-9}$	$4.10e-10$
BB2	k \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
	36	$5.94e-8$	$2.91e-8$	$7.59e-8$	$3.38e-7$	$7.83e-8$	$2.40e-7$
	37	$2.52e-8$	$2.71e-8$	$4.59e-8$	$2.19e-7$	$6.21e-8$	$1.95e-7$
	38	$4.97e-9^*$	$1.78e-8$	$1.28e-8$	$2.08e-7$	$5.32e-8$	$3.46e-8$
	39	$4.06e-9$	$\mathbf{2.87e-8}$	$\mathbf{1.38e-7}$	$1.71e-7$	$4.72e-8$	$\mathbf{2.61e-7}$
	40	$3.52e-9$	$\mathbf{6.00e-8}$	$4.12e-8$	$6.06e-8$	$1.77e-8$	$4.37e-8$
	41	$\mathbf{4.08e-9}$	$2.13e-8$	$2.63e-8$	$\mathbf{1.13e-6}$	$\mathbf{2.12e-7}$	$1.40e-8$
	42	$3.36e-9$	$1.23e-8$	$1.02e-9^*$	$9.80e-8$	$6.14e-8$	$1.00e-8$
	43	$2.38e-9$	$1.88e-9^*$	$7.07e-10$	$4.43e-8$	$1.64e-8$	$2.19e-9^*$
	44	$2.67e-9$	$1.78e-9$	$4.10e-10$	$9.39e-9^*$	$7.97e-9^*$	$2.00e-9$
ABB	k \ h	$2^{-5}\sqrt{2}$	$2^{-6}\sqrt{2}$	$2^{-7}\sqrt{2}$	$2^{-8}\sqrt{2}$	$2^{-9}\sqrt{2}$	$2^{-10}\sqrt{2}$
	36	$5.10e-8$	$1.57e-8$	$2.93e-8$	$3.44e-8$	$3.83e-8$	$2.55e-8$
	37	$2.70e-8$	$7.44e-9^*$	$1.96e-8$	$2.49e-8$	$2.94e-8$	$1.55e-8$
	38	$2.37e-8$	$5.79e-9$	$\mathbf{3.70e-8}$	$\mathbf{4.67e-8}$	$2.20e-8$	$1.15e-8$
	39	$\mathbf{2.50e-8}$	$4.61e-9$	$\mathbf{7.85e-8}$	$\mathbf{1.50e-7}$	$\mathbf{9.20e-8}$	$2.21e-9^*$
	40	$\mathbf{1.50e-7}$	$\mathbf{1.25e-8}$	$\mathbf{8.04e-8}$	$9.71e-8$	$\mathbf{1.81e-7}$	$\mathbf{5.17e-8}$
	41	$1.31e-8$	$\mathbf{1.39e-8}$	$1.06e-8$	$1.08e-8$	$1.17e-8$	$3.26e-8$
	42	$9.26e-9^*$	$8.82e-9$	$2.66e-9^*$	$6.73e-9^*$	$6.90e-9^*$	$1.36e-8$
	43	$8.76e-9$	$1.06e-9$	$1.75e-9$	$1.90e-9$	$6.11e-9$	$2.33e-9$
	44	$4.43e-9$	$9.25e-10$	$7.39e-10$	$1.29e-9$	$5.62e-9$	$1.90e-9$

Table 2. Example 5.1: The values of $\|\mathscr{G}_k^h\|_h$ for $\beta = 0.01$ and iterations $k = 36,\dots,44$



(a) BB1 (b) BB2 (c) ABB

FIG. 2. Example 5.2: Convergence of $\|\mathscr{G}_k^{\Delta t,h}\|_{\Delta t,h}$ with $(\Delta t, h) = (2^{-7}, 2^{-8})$ for different choices of β and step-size strategies

The number of required iteration $k^*_{\Delta t,h}(\varepsilon)$						
$\beta = 0.5$						
	ε \ $(\Delta t,h)$	$(2^{-4},2^{-5})$	$(2^{-5},2^{-6})$	$(2^{-6},2^{-7})$	$(2^{-7},2^{-8})$	$(2^{-8},2^{-9})$
BB1	$1e-2$	4	4	4	3	3
	$1e-4$	7	7	7	7	7
	$1e-6$	10	10	10	10	10
	$1e-8$	12	13	12	12	12
	ε \ $(\Delta t,h)$	$(2^{-4},2^{-5})$	$(2^{-5},2^{-6})$	$(2^{-6},2^{-7})$	$(2^{-7},2^{-8})$	$(2^{-8},2^{-9})$
BB2	$1e-2$	3	3	3	3	3
	$1e-4$	7	8	8	8	8
	$1e-6$	9	10	10	10	10
	$1e-8$	12	13	13	13	13
	ε \ $(\Delta t,h)$	$(2^{-4},2^{-5})$	$(2^{-5},2^{-6})$	$(2^{-6},2^{-7})$	$(2^{-7},2^{-8})$	$(2^{-8},2^{-9})$
ABB	$1e-2$	3	3	3	3	3
	$1e-4$	7	7	7	7	7
	$1e-6$	9	10	11	11	11
	$1e-8$	12	12	12	12	12
$\beta = 0.05$						
	ε \ $(\Delta t,h)$	$(2^{-4},2^{-5})$	$(2^{-5},2^{-6})$	$(2^{-6},2^{-7})$	$(2^{-7},2^{-8})$	$(2^{-8},2^{-9})$
BB1	$1e-2$	14	15	15	15	15
	$1e-4$	25	24	24	26	26
	$1e-6$	34	36	31	34	37
	$1e-8$	47	42	42	43	45
	ε \ $(\Delta t,h)$	$(2^{-4},2^{-5})$	$(2^{-5},2^{-6})$	$(2^{-6},2^{-7})$	$(2^{-7},2^{-8})$	$(2^{-8},2^{-9})$
BB2	$1e-2$	8	8	8	8	8
	$1e-4$	24	23	23	23	23
	$1e-6$	31	30	32	33	33
	$1e-8$	37	35	40	36	36
	ε \ $(\Delta t,h)$	$(2^{-4},2^{-5})$	$(2^{-5},2^{-6})$	$(2^{-6},2^{-7})$	$(2^{-7},2^{-8})$	$(2^{-8},2^{-9})$
ABB	$1e-2$	11	11	12	12	12
	$1e-4$	26	24	25	25	23
	$1e-6$	30	28	34	34	32
	$1e-8$	39	38	37	42	38

Table 3. Example 5.2: Numerical results

is independent of the mesh and depends on the spectrum of the Hessian. These results were confirmed by the numerical experiments.

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A. Appendix: Proof of Lemma 2.2

For every $k \geqslant 0$, we consider the sequence $\{\hat{\alpha}_j^k\}_j$ associated to $\{\hat{u}_j^k\}_j$, which is defined by

$$\hat{\alpha}_j^k := \begin{cases} \hat{\alpha}_j^{BB1,k} & \text{if } \alpha_{k+j} = \alpha_{k+j}^{BB1}, \\ \hat{\alpha}_j^{BB2,k} & \text{if } \alpha_{k+j} = \alpha_{k+j}^{BB2}. \end{cases} \tag{A.1}$$

for all $j \geqslant 0$. We will show by induction that for every $q \in \{0, \ldots, m\}$, there exist positive constants λ_q and η_q such that

$$\begin{cases} \text{If } u_k \in \mathscr{B}_{\eta_q}(u^*),\ u_{k-1} \in \mathscr{B}_\tau(u^*), \text{ and if for some } \ell \in \{0, \ldots, q\}, \text{ property (2.14) holds,} \\ \text{then we have } u_{k+j} \in \mathscr{B}_\tau(u^*), \text{ and } \|u_{k+j} - \hat{u}_j^k\| \leqslant \lambda_q \|u_k - u^*\|^2 \text{ for all } j \in \{0, \ldots, \ell\}. \end{cases} \tag{P_q}$$

For the case that $q = \ell = 0$ and the choice of $\eta_0 = \tau$ and arbitrary $\lambda_0 > 0$, property (P_q) holds clearly since $\hat{u}_0^k = u_k$.

Induction step: Let p be an integer with $1 \leqslant p < m$ such that property (P_q) holds for $q = p$ and, constants λ_p and η_p. We will show that this property holds for $q = p+1$, a positive constant $\lambda_{p+1} \geqslant \lambda_p$, and for the choice of

$$\eta_{p+1} := \min\left\{\frac{1}{4\lambda_p}, \eta_p, \tau\left(1 + \frac{\alpha_{\sup}}{\alpha_{\inf}}\right)^{-(p+1)}\right\}, \tag{A.2}$$

where due to (A.2), we obtain $\eta_{p+1} \leqslant \eta_p$.

Now assume that $u_k \in \mathscr{B}_{\eta_{p+1}}(u^*)$ and $u_{k-1} \in \mathscr{B}_\tau(u^*)$. First we investigate property (P_q) for $q = p+1$ and $\ell \leqslant p$. That is, we assume that (2.14) holds for any given $\ell \leqslant p$ and we show that (2.15) holds. In this case, since $\eta_{p+1} \leqslant \eta_p$, we can use the induction hypothesis (property (P_q) for $q = p$) and conclude, for every $j \in \{0, \ldots, \ell\}$ and $\lambda_{p+1} \geqslant \lambda_p$, that

$$u_{k+j} \in \mathscr{B}_\tau(u^*) \quad \text{and} \quad \|u_{k+j} - \hat{u}_j^k\| \leqslant \lambda_p \|u_k - u^*\|^2 \leqslant \lambda_{p+1} \|u_k - u^*\|^2, \tag{A.3}$$

and, thus, (2.15) holds. In the remainder of the proof, we consider the case $\ell = p+1$. In this case $u_k \in \mathscr{B}_{\eta_{p+1}}(u^*)$, $u_{k-1} \in \mathscr{B}_\tau(u^*)$, and

$$\|\hat{u}_j^k - u^*\| \geqslant \frac{1}{2}\|u_k - u^*\| \quad \text{for all } j \in \{0, \ldots, p\}, \tag{A.4}$$

and we need to verify that $u_{k+j} \in \mathscr{B}_\tau(u^*)$ for $j = \{1, \ldots, p+1\}$ and

$$\|u_{k+j+1} - \hat{u}_{j+1}^k\| \leqslant \lambda_{p+1} \|u_k - u^*\|^2 \text{ for all } j \in \{0, \ldots, p+1\}. \tag{A.5}$$

First, suppose that $u_{k+j} \in \mathscr{B}_\tau(u^*)$ for $j = 1, 2, \ldots, p$. By (2.7) and (2.13), we have

$$\begin{aligned} \|u_{k+p+1} - u^*\| &\leqslant \|u_{k+p} - u^*\| + \|\mathscr{S}_{k+p}\| \leqslant \|u_{k+p} - u^*\| + \frac{1}{|\alpha_{k+p}|}\|\mathscr{G}_{k+p}\| \\ &\leqslant (1 + \frac{\alpha_{\sup}}{\alpha_{\inf}})\|u_{k+p} - u^*\| \leqslant \cdots \leqslant (1 + \frac{\alpha_{\sup}}{\alpha_{\inf}})^{p+1}\|u_k - u^*\|. \end{aligned}$$

Therefore, due to the definition of η_{p+1}, it follows that $u_\ell \in \mathscr{B}_\tau(u^*)$ for every $\ell \in \{0, 1, \ldots, p+1\}$ and any $u_k \in \mathscr{B}_{\eta_{p+1}}(u^*)$ and $u_{k-1} \in \mathscr{B}_\tau(u^*)$. It remains to verify (A.5). In fact, due to (A.4), the induction hypothesis, and the fact that $\eta_{q+1} \leqslant \eta_q$, (A.5) holds for any arbitrary $\lambda_{p+1} \geqslant \lambda_p$ and $j \leqslant p$. Hence, it suffices to show that

$$\|u_{k+p+1} - \hat{u}^k_{p+1}\| \leqslant \lambda_{p+1}\|u_k - u^*\|^2 \tag{A.6}$$

for some $\lambda_{p+1} \geqslant \lambda_p$.

By using (1.2) and the triangle inequality, we obtain

$$\begin{aligned} \|u_{k+p+1} - \hat{u}^k_{p+1}\| &\leqslant \|u_{k+p} - \frac{1}{\alpha_{k+p}}\mathscr{G}(u_{k+p}) - (\hat{u}^k_p - \frac{1}{\hat{\alpha}^k_p}\hat{\mathscr{G}}(\hat{u}^k_p))\| \\ &\leqslant \|u_{k+p} - \hat{u}^k_p\| + \frac{1}{|\hat{\alpha}^k_p|}\|\mathscr{G}(u_{k+p}) - \hat{\mathscr{G}}(\hat{u}^k_p)\| + \left|\frac{1}{\alpha_{k+p}} - \frac{1}{\hat{\alpha}^k_p}\right| \|\mathscr{G}(u_{k+p})\|. \end{aligned} \tag{A.7}$$

From now on, we define c as a positive generic constant which depends only on τ, $\alpha_{\inf}$, $\alpha_{\sup}$ and m , but not on k, and the choice of $u_{k-1}, u_k \in \mathscr{B}_\tau(u^*)$. We shall show that

$$\frac{1}{|\hat{\alpha}^k_p|}\|\mathscr{G}(u_{k+p}) - \hat{\mathscr{G}}(\hat{u}^k_p)\| \leqslant c\|u_k - u^*\|^2, \tag{A.8}$$

$$\left|\frac{1}{\alpha_{k+p}} - \frac{1}{\hat{\alpha}^k_p}\right| \|\mathscr{G}(u_{k+p})\| \leqslant c\|u_k - u^*\|^2. \tag{A.9}$$

Verification of inequality (A.8)**:** First, by adding and subtracting $\hat{\mathscr{G}}(u_{k+p})$, using the triangle inequality, $L1$ and $L3$, we obtain

$$\begin{aligned} \|\mathscr{G}(u_{k+p}) - \hat{\mathscr{G}}(\hat{u}^k_p)\| &\leqslant \|\mathscr{G}(u_{k+p}) - \hat{\mathscr{G}}(u_{k+p})\| + \|\hat{\mathscr{G}}(u_{k+p}) - \hat{\mathscr{G}}(\hat{u}^k_p)\| \\ &\leqslant \|\mathscr{G}(u_{k+p}) - \mathscr{A}^{\mathscr{F}}_{u^*}(u_{k+p} - u^*)\| + \|\mathscr{A}^{\mathscr{F}}_{u^*}(u_{k+p} - \hat{u}^k_p)\| \\ &\leqslant L\|u_{k+p} - u^*\|^2 + \alpha_{\sup}\|u_{k+p} - \hat{u}^k_p\| \leqslant c\|u_k - u^*\|^2, \end{aligned} \tag{A.10}$$

where $\hat{\mathscr{G}}(u) = \mathscr{A}^{\mathscr{F}}_{u^*}(u - u^*)$. In the last estimate, we have used the induction hypothesis and the fact that

$$\|u_{k+p} - u^*\| \leqslant (1 + \frac{\alpha_{\sup}}{\alpha_{\inf}})^p\|u_k - u^*\| \leqslant (1 + \frac{\alpha_{\sup}}{\alpha_{\inf}})^m\|u_k - u^*\|. \tag{A.11}$$

Now by using (A.10) and (2.12), we can infer that

$$\frac{1}{|\hat{\alpha}^k_p|}\|\mathscr{G}(u_{k+p}) - \hat{\mathscr{G}}(\hat{u}^k_p)\| \leqslant \frac{1}{\alpha_{\inf}}\|\mathscr{G}(u_{k+p}) - \hat{\mathscr{G}}(\hat{u}^k_p)\| \leqslant c\|u_k - u^*\|^2.$$

Verification of inequality (A.9)**:** Here we need only to show that

$$\left|\frac{1}{\alpha_{k+p}} - \frac{1}{\hat{\alpha}^k_p}\right| \leqslant c\|u_k - u^*\|. \tag{A.12}$$

Then, thanks to (2.13), (A.11), and (A.12), we obtain

$$\left|\frac{1}{\alpha_{k+p}} - \frac{1}{\hat{\alpha}^k_p}\right| \|\mathscr{G}(u_{k+p})\| \leqslant c\alpha_{\sup}\|u_k - u^*\|\|u_{k+p} - u^*\| \leqslant c\|u_k - u^*\|^2.$$

which implies (A.9). Due to (A.1) we have only these two cases :

1. $\hat{\alpha}_p^k = \hat{\alpha}_p^{BB1,k}$ and $\alpha_{k+p} = \alpha_{k+p}^{BB1}$,

2. $\hat{\alpha}_p^k = \hat{\alpha}_p^{BB2,k}$ and $\alpha_{k+p} = \alpha_{k+p}^{BB2}$.

We investigate the first case. Due to (1.3), we have

$$\frac{1}{\alpha_{k+p}} = \frac{(\mathscr{S}_{k+p-1}, \mathscr{S}_{k+p-1})}{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}, \text{ and } \frac{1}{\hat{\alpha}_p^k} = \frac{(\hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{S}}_{p-1}^k)}{(\hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}. \tag{A.13}$$

Due to (A.4) and the induction hypothesis i.e., property (P_q) for $q = p$, we have

$$\|\mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k\| \leqslant \|u_{k+p} - \hat{u}_p^k\| + \|u_{k+p-1} - \hat{u}_{p-1}^k\| \leqslant 2\lambda_p \|u_k - u^*\|^2, \tag{A.14}$$

and, as a consequence, we obtain

$$\begin{aligned} \left| \|\mathscr{S}_{k+p-1}\|^2 - \|\hat{\mathscr{S}}_{p-1}^k\|^2 \right| &\leqslant \left| 2(\mathscr{S}_{k+p-1}, \mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k) - \|\hat{\mathscr{S}}_{p-1}^k - \mathscr{S}_{k+p-1}\|^2 \right| \\ &\leqslant c\|u_k - u^*\|^3. \end{aligned} \tag{A.15}$$

Further, by (2.12), (A.4), we have

$$\begin{aligned} \|\hat{\mathscr{S}}_{p-1}^k\| = \frac{1}{|\hat{\alpha}_{p-1}^k|} \|\hat{\mathscr{G}}_{p-1}^k\| &\geqslant \frac{1}{\alpha_{\text{sup}}} \|\mathscr{A}_{u^*}^{\mathscr{F}}(\hat{u}_{p-1}^k - u^*)\| \geqslant \frac{\alpha_{\text{inf}}}{\alpha_{\text{sup}}} \|(\hat{u}_{p-1}^k - u^*)\| \\ &\geqslant \frac{\alpha_{\text{inf}}}{2\alpha_{\text{sup}}} \|(\hat{u}_0^k - u^*)\| = \frac{\alpha_{\text{inf}}}{2\alpha_{\text{sup}}} \|(u_k - u^*)\|. \end{aligned} \tag{A.16}$$

From (A.15) and (A.16), it follows that

$$\left| 1 - \frac{\|\mathscr{S}_{k+p-1}\|^2}{\|\hat{\mathscr{S}}_{p-1}^k\|^2} \right| \leqslant c\|u_k - u^*\|. \tag{A.17}$$

Now observe that

$$\begin{aligned} (\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1}) - (\hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k) &= (\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k) + (\mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k) \\ &= (\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k) + (\mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k, \mathscr{A}_{u^*}^{\mathscr{F}} \hat{\mathscr{S}}_{p-1}^k). \end{aligned} \tag{A.18}$$

Using (A.14) and $L3$, we obtain

$$\begin{aligned} &\left| (\mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k, \mathscr{A}_{u^*}^{\mathscr{F}} \hat{\mathscr{S}}_{p-1}^k) \right| \\ &= \left| (\mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k, \mathscr{A}_{u^*}^{\mathscr{F}} \mathscr{S}_{k+p-1}) - (\mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k, \mathscr{A}_{u^*}^{\mathscr{F}}(\mathscr{S}_{k+p-1} - \hat{\mathscr{S}}_{p-1}^k)) \right| \\ &\leqslant c\|u_k - u^*\|^3, \end{aligned} \tag{A.19}$$

and, by (A.10) and the induction hypothesis, we have

$$\left| (\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k) \right| \leqslant \|\mathscr{S}_{k+p-1}\| (\|\mathscr{G}_{k+p} - \hat{\mathscr{G}}_p^k\| + \|\mathscr{G}_{k+p-1} - \hat{\mathscr{G}}_{p-1}^k\|) \leqslant c\|u_k - u^*\|^3. \tag{A.20}$$

Hence, using (A.18), (A.19), and (A.20), we have

$$\left|(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1}) - (\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{Y}}^k_{p-1})\right| \leqslant c\|u_k - u^*\|^3. \tag{A.21}$$

Moreover, by using $L3$, (2.7), (2.8), (2.13), and the facts that $u_{k+p}, u_{k+p-1} \in \mathscr{B}_\tau(u^*)$ and $\alpha_k \leqslant \alpha_{\sup}$ for all $k \geqslant 1$, we can write that

$$\begin{aligned}
&(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1}) = (\mathscr{S}_{k+p-1}, \mathscr{G}_{k+p} - \mathscr{G}_{k+p-1}) \\
&\geqslant \alpha_{\inf}\|\mathscr{S}_{k+p-1}\|^2 = \alpha_{\inf}\left|\frac{1}{\alpha_{k+p-1}}\right|^2 \|\mathscr{G}_{k+p-1}\|^2 \\
&\geqslant \frac{\alpha_{\inf}}{\alpha^2_{\sup}}\|\mathscr{G}_{k+p-1}\|^2 = \frac{\alpha_{\inf}}{\alpha^2_{\sup}}\|\mathscr{G}(u_{k+p-1}) - \mathscr{G}(u^*)\|^2 \geqslant \frac{\alpha^3_{\inf}}{\alpha^2_{\sup}}\|u_{k+p-1} - u^*\|^2.
\end{aligned} \tag{A.22}$$

Further, by $L3$, the definition of η_{p+1} in (A.2), (A.4) and (P_q) with $q = p$, we have

$$\begin{aligned}
&\|u_{k+p-1} - u^*\|^2 \geqslant \frac{1}{2}\|\hat{u}^k_{p-1} - u^*\|^2 - \|u_{k+p-1} - \hat{u}^k_{p-1}\|^2 \\
&\geqslant \frac{1}{8}\|\hat{u}^k_0 - u^*\|^2 - \lambda_p^2\|u_k - u^*\|^4 \geqslant (\frac{1}{8} - \lambda_p^2\eta^2_{p+1})\|u_k - u^*\|^2 = \frac{1}{16}\|u_k - u^*\|^2,
\end{aligned} \tag{A.23}$$

Combining (A.22) and (A.23) we have

$$(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1}) \geqslant \frac{\alpha^3_{\inf}}{\alpha^2_{\sup}}\|u_{k+p-1} - u^*\|^2 \geqslant \frac{\alpha^3_{\inf}}{16\alpha^2_{\sup}}\|u_k - u^*\|^2. \tag{A.24}$$

From (A.21) and (A.24) we can write

$$\left|1 - \frac{(\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{Y}}^k_{p-1})}{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}\right| \leqslant c\|u_k - u^*\|. \tag{A.25}$$

Now, observe that by (A.13)

$$\begin{aligned}
\left|\frac{1}{\alpha_{k+p}} - \frac{1}{\hat{\alpha}^k_p}\right| &= \left|\frac{(\mathscr{S}_{k+p-1}, \mathscr{S}_{k+p-1})}{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})} - \frac{(\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{S}}^k_{p-1})}{(\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{Y}}^k_{p-1})}\right| \\
&= \frac{1}{|\hat{\alpha}^k_p|}\left|1 - \left(\frac{(\mathscr{S}_{k+p-1}, \mathscr{S}_{k+p-1})}{(\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{S}}^k_{p-1})}\right)\left(\frac{(\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{Y}}^k_{p-1})}{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}\right)\right| \\
&= \frac{1}{\alpha_{\inf}}\left|\phi_1(1-\phi_2) + \phi_2\right| \leqslant \frac{1}{\alpha_{\inf}}(|\phi_1| + |\phi_2| + |\phi_1\phi_2|),
\end{aligned} \tag{A.26}$$

where

$$\phi_1 := 1 - \frac{(\mathscr{S}_{k+p-1}, \mathscr{S}_{k+p-1})}{(\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{S}}^k_{p-1})} \quad \text{and} \quad \phi_2 := 1 - \frac{(\hat{\mathscr{S}}^k_{p-1}, \hat{\mathscr{Y}}^k_{p-1})}{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}. \tag{A.27}$$

By (A.17), (A.25), (A.26), and (A.27), we can infer that estimate (A.12) holds for the case that $\alpha_{k+p} = \alpha^{BB1}_{k+p}$ and $\hat{\alpha}^k_p = \hat{\alpha}^{BB1,k}_p$ are chosen.

Now we deal with the second case, i.e., $\alpha_{k+p} = \alpha_{k+p}^{BB2}$ and $\hat{\alpha}_p^k = \hat{\alpha}_p^{BB2,k}$. First due to (1.3), we have

$$\frac{1}{\alpha_{k+p}} = \frac{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}{(\mathscr{Y}_{k+p-1}, \mathscr{Y}_{k+p-1})}, \text{ and } \frac{1}{\hat{\alpha}_p^k} = \frac{(\hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}{(\hat{\mathscr{Y}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}.$$

By using the fact that $u_{k+p}, u_{k+p-1}, \hat{u}_{p-1}^k, \hat{u}_p^k \in \mathscr{B}_\tau(u^*)$, and the hypothesis of induction which is applicable due to (A.4), we can write

$$\|\mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k\| \leqslant \|\mathscr{G}_{k+p} - \hat{\mathscr{G}}_p^k\| + \|\mathscr{G}_{k+p-1} - \hat{\mathscr{G}}_{p-1}^k\| \leqslant c\|u_k - u^*\|^2. \tag{A.28}$$

In addition, by using (2.13), (A.11), and the triangle inequality we obtain

$$\begin{aligned} \|\mathscr{Y}_{k+p-1}\| = \|\mathscr{G}_{k+p} - \mathscr{G}_{k+p-1}\| &\leqslant \|\mathscr{G}_{k+p} - \mathscr{G}(u^*)\| + \|\mathscr{G}_{k+p-1} - \mathscr{G}(u^*)\| \\ &\leqslant \alpha_{\sup}\left(\|u_{k+p} - u^*\| + \|u_{k+p-1} - u^*\|\right) \leqslant c\|u_k - u^*\|. \end{aligned} \tag{A.29}$$

From (A.28), (A.29), we deduce

$$\begin{aligned} &\left|\|\mathscr{Y}_{k+p-1}\|^2 - \|\hat{\mathscr{Y}}_{p-1}^k\|^2\right| \leqslant \left|(\mathscr{Y}_{k+p-1}, \mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k) + (\mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)\right| \\ &\leqslant \|\mathscr{Y}_{k+p-1}\|\|\mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k\| + \|\hat{\mathscr{Y}}_{p-1}^k\|\|\mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k\| \leqslant c\|u_k - u^*\|^3, \end{aligned} \tag{A.30}$$

where in the last line we have used the fact that

$$\|\hat{\mathscr{Y}}_{p-1}^k\| \leqslant \|\mathscr{Y}_{k+p-1} - \hat{\mathscr{Y}}_{p-1}^k\| + \|\mathscr{Y}_{k+p-1}\|.$$

Furthermore, by using (2.5), (2.12), and (A.16), we obtain

$$\|\hat{\mathscr{Y}}_{p-1}^k\| = \|\mathscr{A}_{u^*}^{\mathscr{F}} \hat{\mathscr{S}}_{p-1}^k\| \geqslant \alpha_{\inf}\|\hat{\mathscr{S}}_{p-1}^k\| \geqslant \frac{\alpha_{\inf}^2}{2\alpha_{\sup}}\|(u_k - u^*)\|, \tag{A.31}$$

and, as a consequence, it follows from (A.30) and (A.31) that

$$\left|1 - \frac{\|\mathscr{Y}_{k+p-1}\|^2}{\|\hat{\mathscr{Y}}_{p-1}^k\|^2}\right| \leqslant c\|u_k - u^*\|. \tag{A.32}$$

Now similarly to the case for $BB1$, by (2.7) we can write

$$\begin{aligned} \left|\frac{1}{\alpha_{k+p}} - \frac{1}{\hat{\alpha}_p^k}\right| &= \left|\frac{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}{(\mathscr{Y}_{k+p-1}, \mathscr{Y}_{k+p-1})} - \frac{(\hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}{(\hat{\mathscr{Y}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}\right| \\ &= \frac{1}{|\alpha_{k+p}|}\left|1 - \left(\frac{(\mathscr{Y}_{k+p-1}, \mathscr{Y}_{k+p-1})}{(\hat{\mathscr{Y}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}\right)\left(\frac{(\hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}\right)\right| \\ &= \frac{1}{\alpha_{\inf}}\left|\phi_1(1-\phi_2) + \phi_2\right| \leqslant \frac{1}{\alpha_{\inf}}(|\phi_1| + |\phi_2| + |\phi_1\phi_2|), \end{aligned} \tag{A.33}$$

where

$$\phi_1 := 1 - \frac{(\mathscr{Y}_{k+p-1}, \mathscr{Y}_{k+p-1})}{(\hat{\mathscr{Y}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)} \quad \text{and} \quad \phi_2 := 1 - \frac{(\hat{\mathscr{S}}_{p-1}^k, \hat{\mathscr{Y}}_{p-1}^k)}{(\mathscr{S}_{k+p-1}, \mathscr{Y}_{k+p-1})}. \tag{A.34}$$

By (A.25), (A.32), (A.33), and (A.34), we infer that (A.12) holds for the case $BB2$.

Hence, we are finished with the verification of (A.9). Now from (A.7), (A.8), and (A.9), estimate (A.6) follows and, thus, property (P_q) holds for $q = p+1$. Since m is fixed and finite, we can choose λ and η independent of k and ℓ, and, thus the proof is complete. □