STABILIZATION OF NONAUTONOMOUS PARABOLIC EQUATIONS BY A SINGLE MOVING ACTUATOR

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ABSTRACT. It is shown that an internal control based on a moving indicator function is able to stabilize the state of parabolic equations evolving in rectangular domains. For proving the stabilizability result, we start with a control obtained from an oblique projection feedback based on a finite number of static actuators, then we used the continuity of the state when the control varies in relaxation metric to construct a switching control where at each given instant of time only one of the static actuators is active, finally we construct the moving control by traveling between the static actuators.

Numerical computations are performed by a concatenation procedure following a receding horizon control approach. They confirm the stabilizing performance of the moving control.

1. INTRODUCTION

Stabilizability of controlled parabolic-like equations of the form

$$\dot{y} + Ay + A_{\mathrm{rc}}(t)y = u(t)\Phi(t), \qquad y(0) = y_0, \qquad t > 0, \tag{1.1}$$

where the state evolves in a Hilbert space H, that is, $y(t) \in H$ for all $t \geq 0$ is investigated. The pair (u, Φ), with $u(t) \in \mathbb{R}$ and $\Phi(t) \in H$, with $|\Phi(t)|_H = 1$, is at our disposal. We shall look for a continuous function $\Phi\colon [0, +\infty) \to H$, where $\Phi(t)$ represents the actuator moving on a compact subset of the unit sphere in H.

The goal consists in constructing a (signed) magnitude control function $u(t)$ and a moving actuator $\Phi(t)$ so that the solution of (1.1) satisfies

$$|y(t)|_H \leq C\mathrm{e}^{-\mu t} |y_0|_H \,, \qquad \text{for all} \qquad t \geq 0, \tag{1.2a}$$

with

$$u \in L^2(\mathbb{R}_0, \mathbb{R}), \quad \dot{\Phi} \in L^\infty((0, +\infty), H), \quad \text{and} \quad \ddot{\Phi} \in L^\infty((0, +\infty), H). \tag{1.2b}$$

In particular $\Phi \in \mathcal{C}^1([0, +\infty), H)$, which implies that the actuator moves in a regular way.

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1.1. **Example.** As an illustration we consider a parabolic equation whose state evolves in $H = L^2(\Omega)$, with $\Omega \in \mathbb{R}^d$, $d \in \{1,2,3\}$, a regular bounded domain.

$$\dot{y} - \nu\Delta y + ay + b\cdot\nabla y = u\widehat{1}_{\omega(c)}, \quad \mathcal{G}y\big|_\Gamma = 0, \quad y(0,\cdot) = y_0, \tag{1.3}$$

where $y = y(t,x) \in \mathbb{R}$, $y(t,\cdot) \in L^2(\Omega)$, $a = a(t,x) \in \mathbb{R}$, $b = b(t,x) \in \mathbb{R}^d$, and $\mathcal{G}$ denotes either Dirichlet or Neumann conditions on the boundary Γ of Ω, i.e. $\mathcal{G}y\big|_\Gamma = y(t,\bar{x})$ or $\mathcal{G}y\big|_\Gamma = \mathbf{n}(\bar{x})\cdot\nabla y(t,\bar{x})$, where $\mathbf{n}(\bar{x})$ stands for the unit outward vector normal at $\overline{x} \in \Gamma$.

Here the actuator is chosen as $\Phi(t) = \widehat{1}_{\omega(c(t))}$, where $\widehat{1}_{\omega(c(t))}$ denotes the normalized indicator function whose support is the rectangle $\overline{\omega(c(t))}$. This rectangle $\omega(c(t)) \coloneqq c(t) + \omega_0 \subset \Omega$ is the translation of a rectangular reference domain $\omega_0 \subset \mathbb{R}^d$, with $0 \in \omega_0$, and $\|\omega_0\| \coloneqq \int_{\omega_0} 1 \,\mathrm{d}\mathbb{R}^d$. Then

$$\widehat{1}_{\omega(c(t))}(x) \coloneqq \begin{cases} \|\omega_0\|^{-\frac{1}{2}}, & \text{if} \quad x \in \omega(c(t)), \\ 0, & \text{if} \quad x \notin \overline{\omega(c(t))}, \end{cases} \qquad \text{and} \qquad \left|\widehat{1}_{\omega(c(t))}\right|_{L^2(\Omega)} = 1.$$

To simplify the exposition let us also assume that 0 is the center mass of ω_0, so that we can simply say that 0 is the center of ω_0. Since $c(t) \in \omega(c(t))$, this justifies to call $c(t) \in \mathbb{R}^d$ the *center* of the actuator. Hence the motion of the actuator $\Phi(t) = \widehat{1}_{\omega(c(t))}$ is described by the center of $\omega(c(t))$. See Figure 1, where we have taken $\omega_0 \subset \mathbb{R}^d$ as a small rectangular domain.

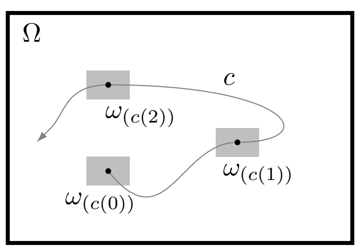


FIGURE 1. An internal moving actuator with support $\overline{\omega(c(t))} \subset \overline{\Omega}$.

The main result of this paper, when applied to (1.3), implies the following Theorem 1.1, concerning parabolic equations evolving in the bounded rectangular domain

$$\Omega \coloneqq \bigtimes_{n=1}^{d} (0, L_n) \subset \mathbb{R}^d, \qquad L \coloneqq (L_1, L_2, \dots, L_d) \in (0,+\infty)^d \subset \mathbb{R}^d. \tag{1.4a}$$

For any given $r \in [0,1]$ we further define the subsets

$$r\Omega \coloneqq \bigtimes_{n=1}^{d} (0, rL_n) \subset \Omega, \qquad (1-r)\Omega + \tfrac{r}{2}L \coloneqq \bigtimes_{n=1}^{d} (\tfrac{r}{2}L_n, L_n - \tfrac{r}{2}L_n) \subset \Omega, \tag{1.4b}$$

$$\omega_0 \coloneqq r\Omega - \tfrac{r}{2}L = \bigtimes_{n=1}^{d} (-\tfrac{r}{2}L_n, \tfrac{r}{2}L_n). \tag{1.4c}$$

Observe that $c + \omega_0 \subset \Omega$ if, and only if, $c \in (1-r)\Omega + \frac{r}{2}L$.

Theorem 1.1. *Let Ω be a bounded rectangular domain as in* (1.4a)*, and let $a \in L^\infty((0,+\infty)\times\Omega,\mathbb{R})$ and $b \in L^\infty((0,+\infty)\times\Omega,\mathbb{R}^d)$. Then for each sufficiently small $r\in(0,1)$, and for each initial state $y_0 \in L^2(\Omega)$, each initial actuator position $c(0)=c_0 \in (1-r)\Omega + \frac{r}{2}L$ with initial actuator velocity $\dot c(0) = 0 \in \mathbb{R}^d$, there exists an actuator motion function c and a magnitude control function u, with*

$$c(t) \in (1-r)\Omega + \tfrac{r}{2}L, \qquad \text{and} \qquad u(t)\in\mathbb{R}$$

such that the solution of (1.3)*, with*

$$\omega(c(t)) \coloneqq r\Omega + c(t) - \tfrac{r}{2}L = \bigtimes_{n=1}^{d} (c_n(t) - \tfrac{r}{2}L_n, c_n(t)+\tfrac{r}{2}L_n)$$

satisfies

$$|y(t,\cdot)|_{L^2(\Omega)} \le C\mathrm{e}^{-\mu t}\,|y_0|_{L^2(\Omega)}, \qquad \text{for all} \qquad t\ge 0, \tag{1.5a}$$

with

$$u\in L^2((0,+\infty),\mathbb{R}), \quad \dot c \in L^\infty((0,+\infty),\mathbb{R}^d), \quad \text{and} \quad \ddot c \in L^\infty((0,+\infty),\mathbb{R}^d). \tag{1.5b}$$

Furthermore, the mapping $y_0 \mapsto u(y_0)$ is continuous from $L^2(\Omega)$ into $L^2((0,+\infty),\mathbb{R})$ with $|u(y_0)|_{L^2((0,+\infty),\mathbb{R})} \le C_u\,|y_0|_H$ and $|\dot c|_{L^\infty(\mathbb{R}_0,\mathbb{R}^M)} + |\ddot c|_{L^\infty(\mathbb{R}_0,\mathbb{R}^M)} \le C_c$. Above the constants C, C_u, C_c, and $\mu>0$ are independent of y_0 and c_0.

Besides the theoretical result we also discuss the numerical computation and implementation of a stabilizing control input based on a moving indicator function. Note that the control input $u(t)\widehat{1}_{\omega(c(t))}$ depends nonlinearly on the control functions (u,c). In order to realize the geometrical constraint $\omega(c(t))\subset\Omega$, which can be obtained through constraints on the velocity $\dot c$ and acceleration $\ddot c$, it will be convenient to introduce a new auxiliary function

$$\eta = \ddot c + \varsigma \dot c + \epsilon c, \qquad \text{for given} \quad \epsilon \ge 0, \quad \varsigma \ge 0.$$

We shall consequently consider system (1.3) in the extended form

$$\dot y - \nu\Delta y + ay + b\cdot\nabla y = u\widehat{1}_{\omega(c)}, \quad \mathcal{G}y|_\Gamma = 0, \qquad y(0,\cdot) = y_0, \tag{1.6a}$$

$$\ddot c + \varsigma\dot c + \epsilon c = \eta, \qquad c(0)=c_0, \quad \dot c(0) = 0, \tag{1.6b}$$

with proper constraints on the newly introduced additional control η, in order to force the actuator to move in an appropriate way. Note that looking for c is equivalent to looking for η, as soon as the initial actuator position c_0 is given. An analogous extension argument is used in [5,21,24], with a first order ODE, $\dot c+\epsilon c = \eta$ in order to deal with boundary controls problems.

Observe that system (1.3) is linear in the state variable y and nonlinear in the control variable (u,c). Instead system (1.6) is linear in the control variable (u,η) and nonlinear in the state variable (y,c), because $c(t)\mapsto 1_{\omega(c(t))}$ is nonlinear from $\mathbb{R}^d$ into $L^2(\Omega)$.

In order to compute the pairs (u,η), the stabilization problem will be formulated as an infinite horizon optimal control problem (see (5.1)–(5.2)) whose solution will be a stabilizing pair of (u,η). To deal with the resulting infinite-horizon problem a receding horizon control framework will be employed. In this framework, a stabilizing moving control is constructed through the concatenation of solutions of open-loop problems defined on overlapping temporal domains covering $[0,\infty)$.

1.2. **Related literature.** Moving controls have been considered, for example, in [9,17] where suitable moving Dirac delta functions are taken as actuators. In [17], both approximate controllability and exact null controllability results are proven for a semilinear 1D parabolic equation by means of two moving Dirac functions. Both Dirac delta functions and indicator functions are typical actuators in applications, see for instance [17] (cf. [17, Eqs. (1.2) and (1.3)]). Such actuators lead to *lumped* controls, which are essentially characterized by the temporal behavior only. Concerning again the terminology, in [17] the Dirac delta functions based controls are called *point* controls, and the indicator functions based controls are called *average* controls or *zone* controls. In [9] approximate controllability results for higher dimensional linear autonomous parabolic equations, by means of moving point controls and, more generally, with controls moving in a lower-dimensional submanifold, are presented. For semilinear 1D parabolic equations evolving in the spatial interval $(0,1) \in \mathbb{R}$, approximate controllability results have been derived in [16] by means of a single static average control $u(t)1_{\widehat{\omega}}$, $\widehat{\omega} = (l_1, l_2) \subset (0,1)$. The results are obtained under the condition that $l_1 \pm l_2$ are irrational numbers.

Concerning partial differential equations which are not of parabolic type we refer to [20], where controllability properties for 1D damped wave equations, under periodic boundary conditions, $\Omega = \mathbb{T}$, are derived by means of a control based on a single moving point actuator $u(t)\Phi(t)$. The actuator is either a Dirac delta $\Phi(t) = \delta_{c(t)}$, see [20, Thm. 1.4], or a single moving function $\Phi(t) = \phi_{c(t)} \in L^2(\mathbb{T})$, see [20, Thm. 1.1] where we can also see that the function $\phi_{c(t)}$ is required to have zero mean. We refer also to [8,10,19] where a moving average control is considered, but where the magnitude control function $u = u(t,x)$ depends on both time and space variables. By means of such a moving control, in [10] the approximate controllability of higher dimensional damped wave equations is derived and, in [8] the inner null controllability of the one-dimensional wave equation is investigated theoretically and numerically. In [19] the null controllability is derived for a 1D coupled PDE-ODE system of FitzHugh–Nagumo type, again with a magnitude control function $u = u(t,x)$ depending on both time and space variables. We recall that such systems are not null controllable by means of static average controls.

It is well known that observability properties and null controllability properties are related. In this respect we refer to the observability results in [15] for the autonomous higher dimensional case with point observations. We recall that often the tools used to derive controllability/observability results for autonomous systems are not appropriate or are not valid to deal with the nonautonomous case. See for example the solution representation in [15, Eq. (2.1)], and the discussion in [9, Sect. 6, §1].

Our result in Theorem 1.1 is of different nature, when compared to the ones mentioned above. Approximate and null controllability are properties concerning the state $y(T)$ at a given time T. Instead, our goal in (1.5) is concerned with the asymptotic behavior of the state (as time goes to $+\infty$). Of course, if we have a control driving the state to $y(T) = 0$ at time $t = T$, then by switching the control off, for $t > T$, results in a stabilizing control. Thus exact controllability is a stronger property than stabilizability.

On the other hand for practical considerations controls driving the system to 0 at time T may not be enough for applications, since, due to noise or computational error, the control may not drive the state exactly to the origin. If the latter is unstable and the control is nonetheless switched off then the state may diverge as time tends to infinity. Therefore, a control is still needed which stabilizes the state once it is close to the origin, or which keeps it in a small neighborhood of 0 which is proportional to the magnitude of noise and disturbances.

1.3. **On (lack of) stabilizability with a single static actuator.** In this section we provide examples where a single static actuator is not sufficient to stabilize the system, no matter what its shape or placement in the spatial domain is. This negative result can be seen as a motivation for our work in this manuscript, where we show that we can still stabilize the system if we are allowed to dynamically move a given indicator function as actuator.

Here we consider only the particular case of controlled autonomous diffusion-reaction systems of the form

$$\tfrac{\partial}{\partial t}y(t,x) - \nu\Delta y(t,x) + (a_0 + a(x))y(t,x) = u(t)\Psi, \qquad t > 0, \tag{1.7a}$$

$$y(0,\cdot) = y_0, \qquad \mathcal{G}y|_\Gamma = 0, \tag{1.7b}$$

evolving in a regular enough bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}_0$, and with

$$\nu > 0, \qquad a_0 \in \mathbb{R}, \qquad \Psi \in L^2(\Omega), \tag{1.8}$$

$$u \in L^2_{\mathrm{loc}}((0,+\infty),\mathbb{R}), \qquad y_0 \in L^2(\Omega), \qquad \text{and} \qquad a \in L^\infty(\Omega). \tag{1.9}$$

In (1.7) above $\mathcal{G}$ stands for either the Dirichlet or the Neumann trace operator.

Let $\{\widetilde{e}_i \mid i = 1, 2, \dots\}$ be a countable complete linearly independent system of eigenfunctions of the operator

$$\mathcal{A} \coloneqq -\nu\Delta + a(x)\mathbf{1}\colon \mathrm{D}(\mathcal{A}) \to L^2(\Omega),$$

with domain $\mathrm{D}(\mathcal{A}) = \{z \in H^2(\Omega) \mid \mathcal{G}z|_\Gamma = 0\}$. Let $\widetilde{\alpha}_i$ be the corresponding eigenvalues

$$\mathcal{A}\widetilde{e}_i = \widetilde{\alpha}_i\widetilde{e}_i, \qquad \widetilde{\alpha}_1 \le \widetilde{\alpha}_2 \le \widetilde{\alpha}_3 \le \dots., \qquad \lim_{i\to+\infty}\widetilde{\alpha}_i = +\infty.$$

The following result implies that system (1.7) is not stabilizable, for any given static actuator $\Psi \in L^2(\Omega)$.

Proposition 1.2. *If there exists a nonsimple eigenvalue $\widetilde{\alpha}_j$ and if $-a_0 > 0$ is large enough, then for each $\Psi \in L^2(\Omega)$ we can find $y_0 \in L^2(\Omega)$ such that for all $u \in L^2_{\mathrm{loc}}((0,+\infty),\mathbb{R})$ the weak solution y of* (1.7) *satisfies $\lim\limits_{t\to+\infty}|y(t,\cdot)|_H = +\infty$.*

Next, for the sake of completeness, we present/recall also a positive result for stabilization with a single actuator Ψ.

Proposition 1.3. *If all the nonpositive eigenvalues of $\mathcal{A} + a_0\mathbf{1}$ are simple, and if none of the corresponding eigenfunctions is orthogonal to Ψ, then system* (1.7) *is stabilizable.*

The proofs of Propositions 1.2 and 1.3 are given in the Appendix, Section A.1.

1.4. **Contents and notation.** In Section 2 we present the assumptions we require for the operators A and A_{rc} in (1.1). Our main exponential stabilization result is proved in Section 3. In Section 4 this is applied to the concrete parabolic equations as (1.3) and Theorem 1.1 is proved. Section 5 is devoted to the numerical computation of a moving control based on the receding horizon framework which shows the exponentially stabilizing performance.

Concerning notation, we write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we set $\mathbb{R}_r := (r, +\infty)$ with $r \in \mathbb{R}$, whose closure is denoted by $\overline{\mathbb{R}}_r := [r, +\infty)$. Finally, we set $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

Given two Banach spaces X and Y, if the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \overset{\mathrm{d}}{\hookrightarrow} Y$, respectively $X \overset{\mathrm{c}}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact.

Let $X \subseteq Z$ and $Y \subseteq Z$ be continuous inclusions, where Z is a Hausdorff topological space. Then we define the Banach spaces $X \times Y$, $X \cap Y$, and $X + Y$, endowed with the norms $|(h, g)|_{X \times Y} := \left(|h|_X^2 + |g|_Y^2\right)^{\frac{1}{2}}$, $|\hat{h}|_{X \cap Y} := |(\hat{h}, \hat{h})|_{X \times Y}$, and $|\tilde{h}|_{X+Y} := \inf_{(h,g) \in X \times Y} \left\{|(h, g)|_{X \times Y} \mid \tilde{h} = h + g\right\}$, respectively. In case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead.

For a given interval $I \subset \mathbb{R}$, we denote $W(I, X, Y) := \{f \in L^2(I, X) \mid \dot{f} \in L^2(I, Y)\}$, endowed withe the norm $|f|_{W(I,X,Y)} := \left|(f, \dot{f})\right|_{L^2(I,X) \times L^2(I,Y)}$.

The space of continuous linear mappings from X into Y is denoted by $\mathcal{L}(X, Y)$. In case $X = Y$ we write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

The continuous dual of X is denoted $X' := \mathcal{L}(X, \mathbb{R})$. The adjoint of an operator $L \in \mathcal{L}(X, Y)$ will be denoted $L^* \in \mathcal{L}(Y', X')$.

The space of continuous functions from X into Y is denoted by $\mathcal{C}(X, Y)$.

The orthogonal complement to a given subset $B \subset H$ of a Hilbert space H, with scalar product $(\cdot, \cdot)_H$, is denoted by $B^\perp := \{h \in H \mid (h, s)_H = 0 \text{ for all } s \in B\}$.

Given two closed subspaces $F \subseteq H$ and $G \subseteq H$ of the Hilbert space given by $H = F \oplus G$, we denote by $P_F^G \in \mathcal{L}(H, F)$ the oblique projection in H onto F along G. That is, writing $h \in H$ as $h = h_F + h_G$ with $(h_F, h_G) \in F \times G$, we have $P_F^G h := h_F$. The orthogonal projection in H onto F is denoted by $P_F \in \mathcal{L}(H, F)$. Notice that $P_F = P_F^{F^\perp}$.

By $\overline{C}_{[a_1,\dots,a_n]}$ we denote a nonnegative function that increases in each of its nonnegative arguments $a_i \geq 0$, $1 \leq i \leq n$.

Finally, C, C_i, $i = 0, 1, \dots$, stand for unessential positive constants.

2. Assumptions

The results will follow under general assumptions on the plant dynamics operators A and A_{rc}, and on a particular stabilizability assumption of (1.1) by means of controls based on a large enough finite number M of suitable static actuators.

The Hilbert space H, in which system (1.1) is evolving in, will be set as a pivot space, that is, we identify, $H' = H$. Let V be another Hilbert space with $V \subset H$.

Assumption 2.1. *$A \in \mathcal{L}(V, V')$ is symmetric and $(y, z) \mapsto \langle Ay, z\rangle_{V',V}$ is a complete scalar product in V.*

From now on, we suppose that V is endowed with the scalar product $(y, z)_V := \langle Ay, z\rangle_{V',V}$, which still makes V a Hilbert space. Necessarily, $A\colon V \to V'$ is an isometry.

Assumption 2.2. *The inclusion $V \subseteq H$ is dense, continuous, and compact.*

Necessarily, we have that

$$\langle y, z\rangle_{V',V} = (y, z)_H, \quad \text{for all } (y, z) \in H \times V,$$

and also that the operator A is densely defined in H, with domain $\mathrm{D}(A)$ satisfying

$$\mathrm{D}(A) \xhookrightarrow{\mathrm{d,\,c}} V \xhookrightarrow{\mathrm{d,\,c}} H \xhookrightarrow{\mathrm{d,\,c}} V' \xhookrightarrow{\mathrm{d,\,c}} \mathrm{D}(A)'.$$

Further, A has a compact inverse $A^{-1}\colon H \to \mathrm{D}(A)$, and we can find a nondecreasing system of (repeated accordingly to their multiplicity) eigenvalues $(\alpha_n)_{n\in\mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_n)_{n\in\mathbb{N}_0}$:

$$0 < \alpha_1 \le \alpha_2 \le \dots \le \alpha_n \le \alpha_{n+1} \to +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n. \tag{2.1}$$

We can define, for every $\zeta \in \mathbb{R}$, the fractional powers A^ζ, of A, by

$$y = \sum_{n=1}^{+\infty} y_n e_n, \quad A^\zeta y = A^\zeta \sum_{n=1}^{+\infty} y_n e_n \coloneqq \sum_{n=1}^{+\infty} \alpha_n^\zeta y_n e_n,$$

and the corresponding domains $\mathrm{D}(A^{|\zeta|}) \coloneqq \{y \in H \mid A^{|\zeta|} y \in H\}$, and $\mathrm{D}(A^{-|\zeta|}) \coloneqq \mathrm{D}(A^{|\zeta|})'$. We have that $\mathrm{D}(A^\zeta) \xhookrightarrow{\mathrm{d,\,c}} \mathrm{D}(A^{\zeta_1})$, for all $\zeta > \zeta_1$, and we can see that $\mathrm{D}(A^0) = H$, $\mathrm{D}(A^1) = \mathrm{D}(A)$, $\mathrm{D}(A^{\frac12}) = V$.

For the time-dependent operator we assume the following:

Assumption 2.3. *For almost every $t > 0$ we have $A_{\mathrm{rc}}(t) \in \mathcal{L}(V, H)$, and we have a uniform bound, that is, $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(V,H))} \eqqcolon C_{\mathrm{rc}} < +\infty$.*

Finally, we will need the following norm squeezing property, by means of controls based on static actuators.

Assumption 2.4. *There exist:*

- *a positive integer M, and positive real numbers $T > 0$ and $\theta \in (0, 1)$,*
- *a linearly independent family $\{\widehat{\Phi}_j \mid j \in \{1, 2, \dots, M\}\} \subset H$ with $\left|\widehat{\Phi}\right|_H = 1$,*
- *a family of functions $\{v_k \in \mathcal{L}(V, L^\infty((kT, kT + T), \mathbb{R}^M)) \mid k \in \mathbb{N}\}$, with $\sup\limits_{k\in\mathbb{N}} |v_k|_{\mathcal{L}(V, L^\infty((kT,kT+T),\mathbb{R}^M))} \le \mathfrak{K}$,*

such that: for all $k \in \mathbb{N}$, the solution of

$$\dot y + Ay + A_{\mathrm{rc}}(t)y = \sum_{j=1}^{M} v_{k,j}(\mathfrak{v})(t)\widehat{\Phi}_j, \qquad y(kT) = \mathfrak{v}, \qquad t \in (kT, kT + T), \tag{2.2}$$

satisfies

$$|y(kT + T)|_V \le \theta\, |\mathfrak{v}|_V\,, \qquad \text{for all} \quad \mathfrak{v} \in V. \tag{2.3}$$

Remark 2.5. Assumptions 2.1–2.4 are satisfiable for parabolic equations as (1.3) evolving in bounded rectangular domains $\Omega \subset \mathbb{R}^d$. The satisfiability of such assumptions shall be revisited/proven later on, in Section 4, where we give the proof of Theorem 1.1, concerning standard parabolic equations.

Remark 2.6. Alternatively, in Assumption (2.3) we can take a reaction-convection term $A_{\mathrm{rc}}(t) \in L^\infty(\mathbb{R}_0, \mathcal{L}(H, V'))$. The proof will however involve slightly different steps. Motivations and further details are given later in Section 4.4.

3. Existence of a moving stabilizing control

Hereafter $\mathfrak{S}_H$ denote the unit sphere in H,

$$\mathfrak{S}_H \coloneqq \{h \in H \mid |h|_H = 1\}.$$

We prove our main result, which is the following.

Theorem 3.1. *Under Assumptions 2.1–2.4, there exist a magnitude control function u and a continuous moving actuator Φ satisfying*

$$\begin{aligned}&u \in L^2(\mathbb{R}_0,\mathbb{R}), \quad \dot\Phi \in L^\infty(\mathbb{R}_0,H), \quad \ddot\Phi \in L^\infty(\mathbb{R}_0,H),\\ &\Phi(0) = \widehat\Phi_1, \quad \dot\Phi(0) = 0, \quad \Phi(t) \in \mathfrak{S}_H \quad \textit{for} \quad t \ge 0,\end{aligned}$$

and constants $C \ge 1$ and $\mu > 0$, such that the solution of the system (1.1),

$$\dot y + Ay + A_{\rm rc}(t)y = u(t)\Phi(t), \qquad y(0) = y_0 \in V, \qquad t > 0, \tag{3.1}$$

satisfies (1.2),

$$|y(t)|_V \le C\mathrm{e}^{-\mu t}\,|y_0|_V\,, \quad \textit{for all} \quad t \ge 0, \tag{3.2a}$$

and the mapping $y_0 \mapsto u(y_0)$ is continuous,

$$|u|_{\mathcal{C}(V,L^2(\mathbb{R}_0,\mathbb{R}))} \eqqcolon \mathfrak{N}_0 < +\infty. \tag{3.2b}$$

Furthermore, $\left|\dot\Phi\right|_{L^\infty(\mathbb{R}_0,H)} + \left|\ddot\Phi\right|_{L^\infty(\mathbb{R}_0,H)} \le C_\Phi$ with C_Φ independent of y_0.

Note that Theorem 3.1 gives us stabilizability in the V-norm. The stabilizability in H-norm as stated in (1.2a) follows as a consequence.

Corollary 3.2. *Let $\Phi^* \in \mathcal{C}^1([0,1],H)$ satisfy*

$$\ddot\Phi^* \in L^\infty((0,1),H), \quad \Phi^*(1) = \widehat\Phi_1, \quad \dot\Phi^*(1) = 0, \quad \Phi^*(t) \in \mathfrak{S}_H, \quad \textit{for} \quad t \in [0,1].$$

Under Assumptions 2.1–2.4, there exist a magnitude control function $u^{\rm e}$ and a continuous moving actuator $\Phi^{\rm e}$ satisfying

$$\begin{aligned}&u^{\rm e} \in L^2(\mathbb{R}_0,\mathbb{R}), \quad \dot\Phi^{\rm e} \in L^\infty(\mathbb{R}_0,H), \quad \ddot\Phi^{\rm e} \in L^\infty(\mathbb{R}_0,H),\\ &\Phi^{\rm e}|_{[0,1]} = \Phi^*, \quad \Phi^{\rm e}(t) \in \mathfrak{S}_H \quad \textit{for} \quad t \ge 0,\end{aligned}$$

and constants $C^{\rm e} \ge 1$ and $\mu > 0$, such that the solution of the system (1.1),

$$\dot y + Ay + A_{\rm rc}(t)y = u^{\rm e}(t)\Phi^{\rm e}(t), \qquad y(0) = y_0 \in H, \qquad t > 0, \tag{3.3}$$

satisfies (1.2),

$$|y(t)|_H \le C^{\rm e}\mathrm{e}^{-\mu t}\,|y_0|_H\,, \quad \textit{for all} \quad t \ge 0, \tag{3.4a}$$

and the mapping $y_0 \mapsto u(y_0)$ is continuous,

$$|u|_{\mathcal{C}(H,L^2(\mathbb{R}_0,\mathbb{R}))} \eqqcolon \mathfrak{N}_0^{\rm e} < +\infty. \tag{3.4b}$$

Moreover, $\left|\dot\Phi^e\right|_{L^\infty(\mathbb{R}_0,H)} + \left|\ddot\Phi^e\right|_{L^\infty(\mathbb{R}_0,H)} \le \max\left\{C_\Phi, \left|\dot\Phi^*\right|_{L^\infty((0,1),H)} + \left|\ddot\Phi^*\right|_{L^\infty((0,1),H)}\right\}$ with C_Φ independent of $(y_0, \Phi^*(0))$.

Proof. For $t \in [0,1]$ we choose the control $u(t)\Phi^*$ with $u = 0$. Using the smoothing property of parabolic-like equations (cf. [7, Lem. 2.4]), we arrive at a state $y(1) =: y_1 \in V$, with

$$|y(1)|_V \le \overline{C}_{[C_{\mathrm{rc}}]} |y(0)|_H . \tag{3.5}$$

In the time interval $\mathbb{R}_1$, we can find a control $u^1 \in L^2(\mathbb{R}_1, \mathbb{R})$ and a moving actuator Φ^1 as in Theorem 3.1, with $\Phi^1(1) = \widehat{\Phi}_1$ and $\dot{\Phi}^1(1) = 0$, giving us

$$|y(t)|_V \le \overline{C}_{[C_{\mathrm{rc}}]} \mathrm{e}^{-\mu(t-1)} |y(1)|_V , \qquad t \ge 1. \tag{3.6}$$

Indeed it is enough to consider a shift in time variable and use Theorem 3.1 to the function $w(\tau) = y(1+\tau)$, which solves the system

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau} w + Aw + \widetilde{A}_{\mathrm{rc}} w = u(\tau)\Phi(\tau), \quad w(0) = y_1, \quad \tau > 0,$$

with $\widetilde{A}_{\mathrm{rc}}(\tau) = A_{\mathrm{rc}}(s+\tau)$. Hence obtaining

$$|w(\tau)|_V \le \overline{C}_{[C_{\mathrm{rc},1}]} \mathrm{e}^{-\mu\tau} |w(0)|_V , \qquad \tau \ge 0,$$

with $C_{\mathrm{rc},1} = \left|\widetilde{A}_{\mathrm{rc}}\right|_{L^\infty(\mathbb{R}_0, \mathcal{L}(V,H))} = |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_1, \mathcal{L}(V,H))} \le C_{\mathrm{rc}}$ which implies (3.6), by taking for $t \ge 1$, $u^1(t) = u(t-1)$ and $\Phi^1(t) = \Phi(t-1)$.

Next, defining

$$\begin{aligned} &u^e(t) = 0 &&\text{and} &&\Phi^e(t) = \Phi^*(t), &&\text{for} \quad t \in [0,1),\\ &u^e(t) = u^1(t) &&\text{and} &&\Phi^e(t) = \Phi^1(t), &&\text{for} \quad t \ge 1, \end{aligned}$$

we obtain, using (3.6) and (3.5),

$$\begin{aligned} |y(t)|_H &\le |\mathbf{1}|_{\mathcal{L}(V,H)} |y(t)|_V \le \overline{C}_{\left[C_{\mathrm{rc}}, |\mathbf{1}|_{\mathcal{L}(V,H)}\right]} \mathrm{e}^{-\mu(t-1)} |y(1)|_V \\ &\le \overline{C}_{\left[C_{\mathrm{rc}}, |\mathbf{1}|_{\mathcal{L}(V,H)}, \mu\right]} \mathrm{e}^{-\mu t} |y(0)|_H , \qquad t \ge 1. \end{aligned}$$

and (cf. [7, Lem. 2.2])

$$\begin{aligned} |y(t)|_H &\le \overline{C}_{[C_{\mathrm{rc}}]} |y(0)|_H \le \overline{C}_{[C_{\mathrm{rc}}]} \mathrm{e}^{\mu} \mathrm{e}^{-\mu t} |y(0)|_V \\ &\le \overline{C}_{[C_{\mathrm{rc}}, \mu]} \mathrm{e}^{-\mu t} |y(0)|_H , \qquad t \in [0,1). \end{aligned}$$

We can see that we can take C^{e} of the form $\overline{C}_{\left[C_{\mathrm{rc}}, |\mathbf{1}|_{\mathcal{L}(V,H)}, \mu\right]}$ in (3.4a).

Using $\Phi^*(1) = \Phi^1(1) = \widehat{\Phi}_1$ and $\dot{\Phi}^*(1) = \dot{\Phi}^1(1) = 0$, we can conclude that $\Phi^e \in \mathcal{C}^1([0,+\infty), H)$. Finally, by Theorem 3.1 we have that $\left|\dot{\Phi}^1\right|_{L^\infty(\mathbb{R}_1, H)} + \left|\ddot{\Phi}^1\right|_{L^\infty(\mathbb{R}_1, H)} \le C_\Phi$ with C_Φ independent of y_1 (and of Φ^*). □

We are going to use Assumption 2.4 together with a concatenation argument, and will prove that Theorem 3.1 is a corollary of the following result concerning the restriction of our system to the intervals

$$I_k \coloneqq (kT, kT+T), \quad \overline{I_k} \coloneqq [kT, kT+T] \qquad k \in \mathbb{N}. \tag{3.7}$$

Theorem 3.3. *Under Assumptions 2.1–2.4, there exist a magnitude control function u_k and a continuous moving actuator Φ_k satisfying*

$$\begin{aligned} &u_k \in L^2(I_k, \mathbb{R}), \quad \dot{\Phi}_k \in L^\infty(I_k, H), \quad \ddot{\Phi}_k \in L^\infty(I_k, H),\\ &\Phi_k(kT) = \Phi_k(kT+T), \quad \dot{\Phi}_k(kT) = \dot{\Phi}_k(kT+T), \quad \Phi_k(t) \in \mathfrak{S}_H \quad \textit{for} \quad t \in \overline{I_k}, \end{aligned}$$

such that the solution of the system

$$\dot y + Ay + A_{\rm rc}(t)y = u_k(t)\Phi_k(t), \qquad\qquad y(kT) = \mathfrak{v} \in V, \tag{3.8}$$

satisfies

$$|y(kT+T)|_V \le \tfrac{\theta+1}{2}\,|\mathfrak{v}|_V\,, \tag{3.9a}$$

and, the mapping $\mathfrak{v} \mapsto u_k(\mathfrak{v})$ *is continuous,*

$$|u_k|_{\mathcal{C}(V,L^2(I_k,\mathbb{R}))} =: \mathfrak{N}_1 < +\infty. \tag{3.9b}$$

with $\mathfrak{N}_1$ *independent of* $k \in \mathbb{N}$.

Proof of Theorem 3.1. We consider the concatenation of controls u_k given by Theorem 3.3 as follows

$$u(y_0) = u_k(y(kT)), \quad \text{if} \quad t \in I_k,$$

where the construction of u is to be understood in a sequential manner: first we take $u(y_0)|_{I_0} = u_0(y(0T)) = u_0(y_0)$, then we consider the corresponding state $y(T)$ at final time $t = T$, which we then use to define $u(y_0)|_{I_1} = u_1(y(1T))$, in this way, by concatenation, we have constructed a control on the interval $I_0 \bigcup I_1 = (0, 2T)$. Once we have constructed the control $u(y_0)|_{(0,kT)}$ on $(0, kT)$, we take $u(y_0)|_{I_k} = u_k(y(kT))$ and have a control defined for time $t \in (0,(k+1)T)$. Eventually we will have $u(y_0)$ defined in the entire time interval $\mathbb{R}_0$.

By (3.9a), we find that the solution associated to $u(y_0)$ satisfies

$$|y(kT+T)|_V \le \tfrac{\theta+1}{2}\,|y(kT)|_V \le (\tfrac{\theta+1}{2})^{k+1}\,|y(0)|_V\,, \tag{3.10}$$

that is, since $0 < \theta < 1$,

$$|y(kT+T)|_H \le \mathrm{e}^{-\overline{\mu}(k+1)T}\,|y(0)|_H\,, \quad \text{with} \quad \overline{\mu} := -\tfrac{1}{T}\log(\tfrac{\theta+1}{2}) > 0. \tag{3.11}$$

By a standard continuity argument (e.g., see [7, Lem. 2.3], recalling that $\mathcal{C}(\overline{I_k}, V) \hookrightarrow W(I_k, \mathrm{D}(A), H)$), we find that for $t \ge 0$,

$$|y(t)|_V^2 \le \overline{C}_{[T,C_{\rm rc}]}\left(|y(kT)|_V^2 + |u_k\Phi|_{L^2(I_k,H)}^2\right), \quad \text{with} \quad k = \lfloor \tfrac{t}{T} \rfloor, \tag{3.12}$$

where $\lfloor r \rfloor$ denotes the integer satisfying $k \le r < k+1$, $r \in \mathbb{R}$. Since $|\Phi(t)|_H = 1$, it follows that, by using (3.10),

$$\begin{aligned}
|y(t)|_V^2 &\le \overline{C}_{[T,C_{\rm rc},M]}\left(|y(kT)|_V^2 + |u_k|_{L^2(I_k,\mathbb{R}^M)}^2\right) \le \overline{C}_{[T,C_{\rm rc},M,\mathfrak{N}_1]}\,|y(kT)|_V^2\\
&\le \overline{C}_{[T,C_{\rm rc},M,\mathfrak{N}_1]}\mathrm{e}^{-\overline{\mu}kT}\,|y(0)|_V^2 = \overline{C}_{[T,C_{\rm rc},M,\mathfrak{N}_1]}\mathrm{e}^{\overline{\mu}(t-kT)}\mathrm{e}^{-\overline{\mu}t}\,|y(0)|_V^2\\
&\le \overline{C}_{[T,C_{\rm rc},M,\mathfrak{N}_1,\overline{\mu}]}\mathrm{e}^{-\overline{\mu}t}\,|y(0)|_V^2\,,
\end{aligned}$$

because $\mathrm{e}^{\overline{\mu}(t-kT)} \le \mathrm{e}^{\overline{\mu}T} = \overline{C}_{[T,\overline{\mu}]}$. Therefore, (3.2a) holds true.

It remains to show (3.2b). Due to (3.9b) and (3.10) we find

$$\begin{aligned}
|u(y_0)|_{L^2(\mathbb{R}_0,\mathbb{R})} &= \sum_{k=0}^{\infty} |u_k(y(kT))|_{L^2(I_k,\mathbb{R})}\\
&\le \mathfrak{N}_1 \sum_{k=0}^{\infty} |y(kT)|_V \le \mathfrak{N}_1\,|y(0)|_V \sum_{k=0}^{\infty} \mathrm{e}^{-\frac{\overline{\mu}}{2}kT} = \mathfrak{N}_1 \tfrac{1}{1-\mathrm{e}^{-\frac{\overline{\mu}}{2}T}}\,|y_0|_V\,,
\end{aligned}$$

which gives us (3.2b), with $\mathfrak{N}_0 \le \mathfrak{N}_1 \frac{1}{1-\mathrm{e}^{-\frac{\overline{\mu}}{2}T}}$. $\square$

Proof of Theorem 3.3. Let us fix an arbitrary $k \in \mathbb{N}$. By Assumption 2.4, we have that $\mathcal{V}_k(\mathfrak{v})(t) = \sum_{j=1}^{M} v_{k,j}(\mathfrak{v})(t)\widehat{\Phi}_j$, is a control function driving system (2.2) from $\mathfrak{v} \in V$ at time $t = kT$ to a state $y(kT+T)$ at time $t = kT+T$, with a norm squeezed by a factor $\theta \in (0,1)$. The proof will follow by successive approximations of such control, hence we start by denoting $\mathcal{V}_k^0 = \mathcal{V}_k$, where the superscript underlines that $\mathcal{V}_k^0$ is our starting control. Since k has been fixed, for simplicity we will omit the subscript k in the control, $\mathcal{V}^0 = \mathcal{V}_k^0 = \mathcal{V}_k$.

$$\mathcal{V}^0(\mathfrak{v})(t) = \sum_{j=1}^{M} v_j^0(\mathfrak{v})(t)\widehat{\Phi}_j, \qquad v_j^0(\mathfrak{v})(t) \coloneqq v_{k,j}(\mathfrak{v})(t), \qquad (\mathfrak{v},t) \in V \times I_k. \tag{3.13}$$

Let us consider our dynamical system (2.2) with a general external forcing f as follows,

$$\dot{y} + Ay + A_{\mathrm{rc}}y = f, \qquad y(kT) = \mathfrak{v} \in V, \qquad t \in I_k, \tag{3.14}$$

Denoting by $y = \mathfrak{Y}_k(\mathfrak{v}, f)$, $y(t) = \mathfrak{Y}_k(\mathfrak{v}, f)(t)$ the solution of (3.14). We can write

$$\left|\mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^0(\mathfrak{v}))(kT+T)\right|_V \le \theta \left|\mathfrak{v}\right|_V \tag{3.15a}$$

where $0 \le \theta < 1$. We see that $\mathcal{V}^0(\mathfrak{v})$ is a control based on the static actuators $\widehat{\Phi}_j$, and recall that

$$\mathcal{V}^0(\mathfrak{v})(t) \in \mathcal{S}_{\widehat{\Phi}} \coloneqq \operatorname{span}\{\widehat{\Phi}_j \mid 1 \le j \le M\} \subset H, \tag{3.15b}$$

$$v^0 = (v_1^0, v_2^0, \dots, v_M^0) \in \mathcal{L}(V, L^\infty(I_k, \mathbb{R}^M)), \tag{3.15c}$$

The proof is completed into 5 main steps in Sections 3.1–3.5, where we construct suitable approximations of $\mathcal{V}^0$: $\mathcal{V}^0 \approx \mathcal{V}^1 \approx \mathcal{V}^2 \approx \mathcal{V}^3 \approx \mathcal{V}^4 \approx \mathcal{V}^5$, arriving at a moving control $\mathcal{V}^5 = \mathcal{V}^5(\mathfrak{v})$, taking values $\mathcal{V}^5(\mathfrak{v})(t)$ in H, with

$$\left|\mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^5(\mathfrak{v}))(kT+T)\right|_V \le \tfrac{1+\theta}{2} \left|\mathfrak{v}\right|_V.$$

That is, $\mathcal{V}^5(\mathfrak{v})$ drives the system from $\mathfrak{v} \in H$ at initial time $t = kT$ to a state $y(kT+T)$ at final time $t = kT+T$ with a norm squeezed by a factor $\frac{1+\theta}{2} \in (\theta, 1)$. $\square$

In each of remaining steps of the proof of Theorem 3.3 we will use a continuity argument for system (3.14). The main contents, in each step, are as follows.

ⓢ Step 1: *Taking auxiliary static actuators in* $\mathrm{D}(A)$. In Section 3.1, we replace (i.e., approximate) our static actuators $\widehat{\Phi}_j \in H$ by suitable static actuators $\widetilde{\Phi}_j \in \mathrm{D}(A) \subset H$. In this way we obtain a control $\mathcal{V}^1(\mathfrak{v})(t) = \sum_{j=1}^{M} v_j^0(\mathfrak{v})(t)\widetilde{\Phi}_j$, taking values in $\mathcal{S}_{\widetilde{\Phi}} \coloneqq \operatorname{span}\{\widetilde{\Phi}_j \mid 1 \le j \le M\} \subset \mathrm{D}(A)$. Taking actuators in $\mathrm{D}(A)$ is needed for technical reasons, which play a role in Step 2 of the proof.

ⓢ Step 2: *Piecewise constant static control in* $\mathrm{D}(A)$. In Section 3.2, we approximate the control $\mathcal{V}^1(\mathfrak{v})$, by a right-continuous piecewise constant control $\mathcal{V}^2(\mathfrak{v})$ taking values $\mathcal{V}^2(\mathfrak{v})(t)$ in the set $\{s\widetilde{\Phi}_j \mid 1 \le j \le M,\ -K \le s \le K\} \subset \mathrm{D}(A)$ for a suitable constant $K > 0$, for all $t \in I_k$.

ⓢ Step 3: *Piecewise constant static control in H. Back to original actuators.* In Section 3.3, we replace back the $\widetilde{\Phi}_j$s by the $\widehat{\Phi}_j$s. In this way we arrive at a piecewise constant control $\mathcal{V}^3(\mathfrak{v})$ defined as $\mathcal{V}^3(\mathfrak{v})(t) \coloneqq s\widehat{\Phi}_j$ if $\mathcal{V}^2(\mathfrak{v})(t) = s\widetilde{\Phi}_j$ taking values in the set $\{s\widehat{\Phi}_j \mid 1 \le j \le M,\ -K \le s \le K\} \subset H$.

ⓢ Step 4: *A piecewise constant static control with nondegenerate intervals of constancy.* In Section 3.4 we construct a piecewise constant control $\mathcal{V}^4(\mathfrak{v})$ taking values $\mathcal{V}^4(\mathfrak{v})(t)$ in H, where the lengths of the intervals of constancy are all larger than a suitable positive constant.

ⓢ Step 5: *A moving control in H.* In Section 3.5, we construct a moving control $\mathcal{V}^5 = \mathcal{V}^5(\mathfrak{v}) = u(t)\Phi(t)$ which visits (several times) the positions of the static actuators $\widehat{\Phi}_j$, spending a suitable amount of time at those positions, and travels, in H, fast enough between those positions. In this way we obtain a moving control $\mathcal{V}^5 = \mathcal{V}^5(\mathfrak{v})$, taking values $\mathcal{V}^5(\mathfrak{v})(t)$ in H.

Steps 1 and 3 are needed only if some of our static actuators are in $H \setminus \mathrm{D}(A)$. This will, in general, be the case for indicator functions 1_ω, $\omega \subset \Omega$, for scalar parabolic equations evolving in bounded domains $\Omega \subset \mathbb{R}^d$.

The continuity arguments in Steps 1, 3, 4, and 5 are standard, namely the continuity of the solution of system (3.14) on the right-hand side as $y = \mathfrak{Y}_k(\mathfrak{v}, \cdot) \in \mathcal{C}(L^2(I_k, H), \mathcal{C}(\overline{I_k}, V))$. The continuity argument in Step 2 is less standard, involving the continuity of the solution when the right-hand side varies in the so called relaxation metric, details will be given in Section 3.2.

3.1. **A static control taking values in** $\mathrm{D}(A)$**.** Observe that $\frac{1+\theta}{2} - \theta = \frac{1-\theta}{2} > 0$.

Recall that (cf. [7, Lem. 2.3], recalling that $\mathcal{C}(\overline{I_k}, V) \hookrightarrow W(I_k, \mathrm{D}(A), H)$) the solution of system (3.14) satisfies

$$|\mathfrak{Y}_k(\mathfrak{v}, f)(t)|_V^2 \le D_Y \left(|\mathfrak{v}|_V^2 + |f|_{L^2(I_k,H)}^2 \right), \quad t \in I_k = (kT, kT+T), \tag{3.16}$$

with $D_Y = \overline{C}_{[T, C_{\mathrm{rc}}]}$ independent of k, where C_{rc} is defined in Assumption 2.3.

By Assumption 2.4, the actuators $\widehat{\Phi}_j$ are in the unit sphere $\mathfrak{S}_H$ of H. Then, from $\mathrm{D}(A) \overset{\mathrm{d}}{\hookrightarrow} H$ we can choose a family $\{\widetilde{\Phi}_j \mid 1 \le j \le M\}$ such that

$$\widetilde{\Phi}_j \in \mathrm{D}(A) \bigcap \mathfrak{S}_H, \tag{3.17a}$$

$$\left| \widetilde{\Phi}_j - \widehat{\Phi}_j \right|_H \le \tfrac{1-\theta}{10} D_Y^{-\frac{1}{2}} T^{-\frac{1}{2}} \mathfrak{K}^{-1} M^{-1}, \qquad 1 \le j \le M. \tag{3.17b}$$

The fact that the $\widetilde{\Phi}_j$ can be taken in the unit sphere is a corollary of the following result, whose proof is given in Section A.2.

Proposition 3.4. *Let $X \subset H$ be a vector space. Then, the density of $X \subset H$ implies the density of $X \bigcap \mathfrak{S}_H \subset \mathfrak{S}_H$.*

Now we recall the control $\mathcal{V}^0$ in (3.13), and define a new control as

$$\mathcal{V}^1(\mathfrak{v})(t) = \sum_{j=1}^{M} v_j^1(\mathfrak{v})(t) \widetilde{\Phi}_j, \qquad v_j^1(\mathfrak{v})(t) \coloneqq v_j^0(\mathfrak{v})(t), \qquad (\mathfrak{v}, t) \in V \times I_k. \tag{3.18}$$

where we replace each actuator $\widehat{\Phi}_j \in H$ by the auxiliary actuator $\widetilde{\Phi}_j \in \mathrm{D}(A) \subset H$.

Note that, by Assumption 2.4, for $v^1 \coloneqq (v_1^1, v_2^1, \dots, v_M^1)$, we find

$$\sup_{t \in I_k} \left| v^1(\mathfrak{v})(t) \right|_{\mathbb{R}^M} = \sup_{t \in I_k} \left| v^0(\mathfrak{v})(t) \right|_{\mathbb{R}^M} \le \mathfrak{K} \left| \mathfrak{v} \right|_V. \tag{3.19}$$

Let us denote $d^1 \coloneqq \mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^1(\mathfrak{v})) - \mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^0(\mathfrak{v}))$, which satisfies

$$\dot{d}^1 + A d^1 + A_{\mathrm{rc}} d^1 = \mathcal{V}^1(\mathfrak{v}) - \mathcal{V}^0(\mathfrak{v}), \quad \text{for} \quad t \in I_k, \qquad d^1(kT) = 0. \tag{3.20}$$

By (3.16) and (3.17), we find that, since $M \ge 1$,

$$\begin{aligned}\left|d^1(t)\right|_V &\le D_Y^{\frac12}\left|\mathcal{V}^1(\mathfrak{v})-\mathcal{V}^0(\mathfrak{v})\right|_{L^2(I_k,H)} \le \tfrac{1-\theta}{10}T^{-\frac12}\mathfrak{K}^{-1}M^{-1}\left|v^0(\mathfrak{v})\right|_{L^2(I_k,\mathbb{R}^M)}\\ &\le \tfrac{1-\theta}{10}T^{-\frac12}\mathfrak{K}^{-1}\left|v^0(\mathfrak{v})\right|_{L^2(I_k,\mathbb{R}^M)} \le \tfrac{1-\theta}{10}\mathfrak{K}^{-1}\left|v^0(\mathfrak{v})\right|_{L^\infty(I_k,\mathbb{R}^M)}.\end{aligned}$$

Thus, using Assumption 2.4,

$$\left|\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^1(\mathfrak{v}))(t)-\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^0(\mathfrak{v}))(t)\right|_V \le \tfrac{1-\theta}{10}\left|\mathfrak{v}\right|_V, \quad \text{for all} \quad t\in\overline{I_k}. \tag{3.21}$$

3.2. **A piecewise constant static control taking values in** $\mathrm{D}(A)$**.** Let us denote the closed unit ball in $\mathrm{D}(A)$ by $\overline{\mathfrak{B}}_{\mathrm{D}(A)}$. Recall the control $\mathcal{V}^1(\mathfrak{v})(t)$, defined in (3.18), taking values in $\mathcal{S}_{\widetilde\Phi}=\operatorname{span}\{\widetilde\Phi_j \mid 1\le j\le M\}$. We will prove that the solution of (3.14) varies continuously in $\mathcal{C}(\overline{I_k},V)$ when the external forcing varies continuously in the so called (weak) relaxation metric (cf. [12, Ch. 3])

$$\mathfrak{D}^{\mathrm{wrx}}_{I_k}(f,g) := \sup_{t\in\overline{I_k}}\left|\int_{kT}^t f(s)-g(s)\,\mathrm{d}s\right|_{\mathrm{D}(A)}, \qquad \{f,g\}\subset L^\infty(I_k,\mathcal{S}_{\widetilde\Phi}\textstyle\bigcap K\overline{\mathfrak{B}}_{\mathrm{D}(A)}), \tag{3.22}$$

for a given $K>0$. Hence we will approximate $\mathcal{V}^1(\mathfrak{v})$ by a piecewise constant control $\mathcal{V}^2(\mathfrak{v})$ in such a metric. We underline here that f and g above are functions taking their values in the *bounded* subset $\mathcal{S}_{\widetilde\Phi}\bigcap K\overline{\mathfrak{B}}_{\mathrm{D}(A)}$ of the *finite dimensional* subspace $\mathcal{S}_{\widetilde\Phi}\subset \mathrm{D}(A)$.

As the reference [12] shows, such continuity is known in control theory of ordinary differential equations. It has also been used to derive (approximate) controllability results for partial differential equations, see for example [1, Sect. 12.3], [2, Sect. 6.3], [22, Sect. 9], [23, Sect. 3.2.2].

We follow a variation of the procedure in [12, Ch. 3], which allows us to construct a piecewise constant control taking values in $\{s\widetilde\Phi_j \mid 1\le j\le M,\ -K\le s\le K\}$, for a suitable fixed $K>0$, see (3.39). The fact that the control takes its values in a subset of the cone $\{r\widetilde\Phi_j \mid 1\le j\le M,\ r\in\mathbb{R}\}$ will be important in Section 3.5. With respect to this, we would like to refer also to [27, Lemma 3.5], for a different approximation involving piecewise constant controls, but where the control is allowed to take values which are not necessarily in the cone above.

In order to construct a piecewise constant control, we start with a partition of the time interval I_k into N subintervals of constant size $\frac{T}{N}$,

$$I_{k,n} := (kT+(n-1)\tfrac{T}{N}, kT+n\tfrac{T}{N}), \qquad 1\le n\le N.$$

and we denote $\overline{I_{k,n}} := [kT+(n-1)\frac{T}{N}, kT+n\frac{T}{N}]$. We are going to construct a piecewise constant control on each of the subintervals $I_{k,n}$ with exactly $2M$ subintervals of constancy (possibly with vanishing length) where each of the M actuators $\widetilde\Phi_j$, $1\le j\le M$ will be active in exactly two of such intervals.

We start by defining the nonnegative constant

$$\Sigma_n(\mathfrak{v}) := \frac{N}{T}\sum_{m=1}^M\left|\int_{I_{k,n}} v^1_m(\mathfrak{v})(t)\,\mathrm{d}t\right|_{\mathbb{R}}. \tag{3.23}$$

Observe that

$$\Sigma_n(\mathfrak{v}) \le \frac{N}{T}\int_{I_{k,n}}\sum_{m=1}^M\left|v^1_m(\mathfrak{v})(t)\right|_{\mathbb{R}}\,\mathrm{d}t \le \frac{N}{T}M\int_{I_{k,n}}\left|v^1(\mathfrak{v})(t)\right|_{\mathbb{R}^M}\,\mathrm{d}t$$

and, by (3.19), it follows that

$$\Sigma_n(\mathfrak{v}) \le M\mathfrak{K}\left|\mathfrak{v}\right|_V. \tag{3.24}$$

Let us denote

$$l_{k,n,j} = l_{k,n,2M+1-j} \coloneqq \tfrac{1}{2\Sigma_n(\mathfrak{v})}\int_{I_{k,n}} v_j^1(\mathfrak{v})(t)\,\mathrm{d}t, \qquad 1 \le j \le M. \tag{3.25}$$

Next, we consider the cases $\Sigma_n(\mathfrak{v}) \ne 0$ and $\Sigma_n(\mathfrak{v}) = 0$ separately.

• The case $\Sigma_n(\mathfrak{v}) \ne 0$. We rewrite our control $\mathcal{V}^1(\mathfrak{v})$, as

$$\mathcal{V}^1(\mathfrak{v})(t) = \sum_{j=1}^{M} v_j^1(\mathfrak{v})(t)\widetilde{\Phi}_j = \sum_{j=1}^{M} \tfrac{v_j^1(\mathfrak{v})(t)}{\Sigma_n(\mathfrak{v})}\Sigma_n(\mathfrak{v})\widetilde{\Phi}_j.$$

We define a piecewise constant control in each interval $I_{k,n} \subset I_k$, where the lengths of the intervals of constancy are given by

$$|l_{k,n,j}| \coloneqq |l_{k,n,j}|_{\mathbb{R}}. \tag{3.26a}$$

Observe that

$$\sum_{j=1}^{2M} |l_{k,n,j}| = 2\tfrac{1}{2\Sigma_n(\mathfrak{v})}\sum_{j=1}^{M}\left|\int_{I_{k,n}} v_j^1(\mathfrak{v})(t)\,\mathrm{d}t\right|_{\mathbb{R}} = \tfrac{T}{N}. \tag{3.26b}$$

Note also that some of the lengths may vanish.

Next, we denote the switching time instants $t_{k,n,j} = t_{k,n,j}^{[N]}$ as follows

$$t_{k,n,0} \coloneqq kT + (n-1)\tfrac{T}{N}, \tag{3.27a}$$

$$t_{k,n,j} \coloneqq kT + (n-1)\tfrac{T}{N} + \sum_{m=1}^{j} |l_{k,n,m}|, \qquad 1 \le j \le 2M. \tag{3.27b}$$

In particular, we have $t_{k,n,2M} = kT + n\frac{T}{N}$.

To simplify the exposition we denote

$$\widetilde{\Phi}_{2M+1-j} \coloneqq \widetilde{\Phi}_j, \qquad 1 \le j \le M. \tag{3.28}$$

We define

$$I_{k,n,j} \coloneqq [t_{k,n,j-1}, t_{k,n,j}), \qquad 1 \le j \le 2M. \tag{3.29a}$$

$$\mathcal{V}_{[N]}(\mathfrak{v})(t) = \operatorname{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widetilde{\Phi}_j, \quad \text{if} \quad t \in I_{k,n,j}, \qquad 1 \le j \le 2M. \tag{3.29b}$$

• The case $\Sigma_n(\mathfrak{v}) = 0$. We define $\mathcal{V}_{[N]}(\mathfrak{v})(t) = 0$, for all $t \in I_{k,n}$. Which we can still rewrite as a piecewise constant control as follows.

Firstly we define

$$t_{k,n,0} \coloneqq kT + (n-1)\tfrac{T}{N}, \quad t_{k,n,j} \coloneqq kT + (n-1)\tfrac{T}{N} + j\tfrac{T}{2MN}, \quad 1 \le j \le 2M. \tag{3.30}$$

and, then we set analogously to (3.29),

$$I_{k,n,j} \coloneqq [t_{k,n,j-1}, t_{k,n,j}) = [kT + (n-1)\tfrac{T}{N} + j\tfrac{T}{2MN}), \qquad 1 \le j \le 2M. \tag{3.31a}$$

$$\mathcal{V}_{[N]}(\mathfrak{v})(t) = \operatorname{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widetilde{\Phi}_j = 0\widetilde{\Phi}_j, \quad \text{if} \quad t \in I_{k,n,j}, \qquad 1 \le j \le 2M. \tag{3.31b}$$

In either case we obtain a piecewise constant control in the entire interval I_k. Observe that $\mathcal{V}_{[N]}(\mathfrak{v})(t)$ tells us that we activate the actuators $\widetilde{\Phi}_j$ in each interval $I_{k,n}$ in the order

$$\widetilde{\Phi}_1 \to \widetilde{\Phi}_2 \to \cdots \to \widetilde{\Phi}_{M-1} \to \widetilde{\Phi}_M \to \widetilde{\Phi}_{M+1} \to \widetilde{\Phi}_{M+2} \to \cdots \to \widetilde{\Phi}_{2M-1} \to \widetilde{\Phi}_{2M},$$

which is the same, by (3.28), as the cycle

$$\widetilde{\Phi}_1 \to \widetilde{\Phi}_2 \to \dots \to \widetilde{\Phi}_{M-1} \to \widetilde{\Phi}_M \to \widetilde{\Phi}_M \to \widetilde{\Phi}_{M-1} \to \dots \to \widetilde{\Phi}_2 \to \widetilde{\Phi}_1. \tag{3.32}$$

Some actuators may be active in degenerate intervals of length zero. The actuators are activated with the same input of constant magnitude $\operatorname{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})$.

Next, we show that $\mathcal{V}_{[N]}(\mathfrak{v})(t)$ approaches $\mathcal{V}^1(\mathfrak{v})(t)$ in the relaxation metric (3.22). We set

$$\mathcal{I}_{[N]}(t) \coloneqq \int_{kT}^{t} \left(\mathcal{V}_{[N]}(\mathfrak{v})(s) - \mathcal{V}^1(\mathfrak{v})(s)\right) \mathrm{d}s.$$

Then

$$\mathfrak{D}^{\mathrm{wrx}}_{I_k}(\mathcal{V}_{[N]}(\mathfrak{v}), \mathcal{V}^1(\mathfrak{v})) = \sup_{t\in\overline{I_k}} \left|\mathcal{I}_{[N]}(t)\right|_{\mathrm{D}(A)}.$$

We show now that $\mathcal{I}_{[N]}$ vanishes at the extrema of the intervals $I_{k,n}$. Clearly

$$\mathcal{I}_{[N]}(kT) = 0. \tag{3.33}$$

Further, if we assume that $\mathcal{I}_{[N]}(kT + n\frac{T}{N}) = 0$ for a given $0 \le n \le N-1$, then:
- if $\Sigma_n(\mathfrak{v}) \ne 0$ we obtain

$$\begin{aligned}
\mathcal{I}_{[N]}(kT + (n+1)\tfrac{T}{N}) &= \int_{I_{k,n}} \left(\mathcal{V}_{[N]}(\mathfrak{v})(s) - \mathcal{V}^1(\mathfrak{v})(s)\right)\, \mathrm{d}s \\
&= \sum_{j=1}^{2M} |l_{k,n,j}| \operatorname{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widetilde{\Phi}_j - \int_{I_{k,n}} \mathcal{V}^1(\mathfrak{v})(s)\,\mathrm{d}s \\
&= \sum_{j=1}^{2M} \frac{1}{2}\int_{I_{k,n}} v_j^1(\mathfrak{v})(s)\,\mathrm{d}s\widetilde{\Phi}_j - \int_{I_{k,n}} \mathcal{V}^1(\mathfrak{v})(s)\,\mathrm{d}s = 0.
\end{aligned}$$

- if $\Sigma_n(\mathfrak{v}) = 0$ we obtain

$$\mathcal{I}_{[N]}(kT + (n+1)\tfrac{T}{N}) = \int_{I_{k,n}} \left(0 - \mathcal{V}^1(\mathfrak{v})(s)\right)\, \mathrm{d}s = 0.$$

Therefore, in either case we have that

$$\mathcal{I}_{[N]}(kT + n\tfrac{T}{N}) = 0 \quad\implies\quad \mathcal{I}_{[N]}(kT + (n+1)\tfrac{T}{N}) = 0, \qquad 0 \le n \le N-1. \tag{3.34}$$

From (3.33) and (3.34), by induction we can conclude that

$$\mathcal{I}_{[N]}(kT + n\tfrac{T}{N}) = 0, \quad \text{for all} \quad n \in \{0,1,2,\dots,N\}. \tag{3.35}$$

Now for an arbitrary $t \in I_{k,n}$ we find

$$\begin{aligned}
\left|\mathcal{I}_{[N]}(t)\right|_{\mathrm{D}(A)} &\le \tfrac{T}{N}\left(\sup_{s\in I_{k,n}} \left|\mathcal{V}_{[N]}(\mathfrak{v})(s)\right|_{\mathrm{D}(A)} + \sup_{s\in I_{k,n}} \left|\mathcal{V}^1(\mathfrak{v})(s)\right|_{\mathrm{D}(A)}\right) \\
&\le \tfrac{T}{N}\left(\Sigma_n(\mathfrak{v}) \sup_{1\le j\le M} \left|\widetilde{\Phi}_j\right|_{\mathrm{D}(A)} + \sup_{s\in I_k} \left|v^1(\mathfrak{v})(s)\right|_{\mathbb{R}^M} \sup_{1\le j\le M} \left|\widetilde{\Phi}_j\right|_{\mathrm{D}(A)}\right)
\end{aligned}$$

and by (3.24) and Assumption 2.4,

$$\begin{aligned}
\left|\mathcal{I}_{[N]}(t)\right|_{\mathrm{D}(A)} &\le \tfrac{T}{N}\left(M\mathfrak{K}\left|\mathfrak{v}\right|_V + \mathfrak{K}\left|\mathfrak{v}\right|_V\right) \sup_{1\le j\le M} \left|\widetilde{\Phi}_j\right|_{\mathrm{D}(A)} \\
&\le \tfrac{T}{N}(M+1)\left\|\widetilde{\Phi}\right\| \mathfrak{K}\left|\mathfrak{v}\right|_V, \qquad t \in \overline{I_k}, \qquad \left\|\widetilde{\Phi}\right\| \coloneqq \sup_{1\le j\le M} \left|\widetilde{\Phi}_j\right|_{\mathrm{D}(A)}.
\end{aligned} \tag{3.36}$$

Next we show the continuity of the solution when the right-hand side control varies in the relaxation metric. Let us denote $d_N \coloneqq \mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}_{[N]}(\mathfrak{v})) - \mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^1(\mathfrak{v}))$, and observe that d_N satisfies (3.14), as

$$\dot{d}_N + Ad_N + A_{\mathrm{rc}}d_N = \mathcal{V}_{[N]}(\mathfrak{v}) - \mathcal{V}^1(\mathfrak{v}), \qquad d_N(kT) = 0.$$

With $z_N \coloneqq d_N - \mathcal{I}_{[N]}$, we see that $\dot{z}_N = \dot{d}_N - \dot{\mathcal{I}}_{[N]} = -Ad_N - A_{\mathrm{rc}}d_N$, which implies

$$\dot{z}_N + Az_N + A_{\mathrm{rc}}z_N = -A\mathcal{I}_{[N]} - A_{\mathrm{rc}}\mathcal{I}_{[N]}, \qquad z_N(kT) = 0,$$

and also, by (3.35),

$$z_N(kT + n\tfrac{T}{N}) = d_N(kT + n\tfrac{T}{N}), \qquad 0 \le n \le N. \tag{3.37}$$

Therefore, for $z_N = \mathfrak{Y}_k(0, -A\mathcal{I}_{[N]} - A_{\mathrm{rc}}\mathcal{I}_{[N]})$ we obtain, see (3.16),

$$|z_N(t)|_V^2 \le D_Y \left|-A\mathcal{I}_{[N]} - A_{\mathrm{rc}}\mathcal{I}_{[N]}\right|_{L^2(I_k,H)}^2, \qquad t \in \overline{I_k}.$$

By standard computations we find, for all $t \in \overline{I_k}$,

$$\begin{aligned}
|z_N(t)|_V^2 &\le D_Y \left(\left|A\mathcal{I}_{[N]}\right|_{L^2(I_k,H)} + \left|A_{\mathrm{rc}}\mathcal{I}_{[N]}\right|_{L^2(I_k,H)}\right)^2 \\
&\le D_Y \left(\left|\mathcal{I}_{[N]}\right|_{L^2(I_k,\mathrm{D}(A))} + C_{\mathrm{rc}} \left|\mathcal{I}_{[N]}\right|_{L^2(I_k,V)}\right)^2 \\
&= D_Y(1 + C_{\mathrm{rc}} \left|\mathbf{1}\right|_{\mathcal{L}(\mathrm{D}(A),V)})^2 T \left|\mathcal{I}_{[N]}\right|_{L^\infty(I_k,\mathrm{D}(A))}^2,
\end{aligned}$$

and using (3.36),

$$|z_N(t)|_V \le \frac{1}{N} D_Y^{\frac{1}{2}}(1 + C_{\mathrm{rc}} \left|\mathbf{1}\right|_{\mathcal{L}(\mathrm{D}(A),V)}) T^{\frac{3}{2}}(M+1) \left\|\widetilde{\Phi}\right\| \mathfrak{K} \left|\mathfrak{v}\right|_V.$$

Now we can take N large enough, namely

$$N = \widehat{N} \ge D_Y^{\frac{1}{2}}(1 + C_{\mathrm{rc}} \left|\mathbf{1}\right|_{\mathcal{L}(\mathrm{D}(A),V)}) T^{\frac{3}{2}}(M+1) \left\|\widetilde{\Phi}\right\| \mathfrak{K} \tfrac{10}{1-\theta}, \tag{3.38}$$

in order to obtain $\left|z_{\widehat{N}}(t)\right|_V \le \frac{1-\theta}{10} \left|\mathfrak{v}\right|_V$. Then, we set

$$\mathcal{V}^2(\mathfrak{v})(t) \coloneqq \mathcal{V}_{[\widehat{N}]}(\mathfrak{v})(t) = \mathrm{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widetilde{\Phi}_j, \quad \text{if } \ t \in I_{k,n,j}, \ \ 1 \le j \le 2M, \tag{3.39}$$

where the intervals $I_{k,n,j}$ are defined as in (3.29) and (3.31),

$$I_{k,n,j} = I_{k,n,j}^{[\widehat{N}]} = [t_{k,n,j-1}^{[\widehat{N}]}, t_{k,n,j}^{[\widehat{N}]}) = [t_{k,n,j-1}, t_{k,n,j}). \tag{3.40}$$

We find, using (3.36),

$$\begin{aligned}
&\left|\mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^2(\mathfrak{v}))(t) - \mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^1(\mathfrak{v}))(t)\right|_V = \left|d_{\widehat{N}}(t)\right|_V \le \left|z_{\widehat{N}}(t)\right|_V + \left|\mathcal{I}_{[\widehat{N}]}(t)\right|_V \\
&\qquad \le \tfrac{1-\theta}{10} \left|\mathfrak{v}\right|_V + \tfrac{T}{\widehat{N}}(M+1) \left\|\widetilde{\Phi}\right\| \mathfrak{K} \left|\mathfrak{v}\right|_V, \quad \text{for all} \quad t \in \overline{I_k},
\end{aligned} \tag{3.41a}$$

and, using (3.37),

$$\begin{aligned}
&\left|\mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^2(\mathfrak{v}))(kT+T) - \mathfrak{Y}_k(\mathfrak{v}, \mathcal{V}^1(\mathfrak{v}))(kT+T)\right|_V = \left|d_{\widehat{N}}(kT+T)\right|_V \\
&\qquad = \left|z_{\widehat{N}}(kT+T)\right|_V \le \tfrac{1-\theta}{10} \left|\mathfrak{v}\right|_V.
\end{aligned} \tag{3.41b}$$

3.3. **A piecewise constant static control taking values in H.** To simplify the exposition we denote

$$\widehat{\Phi}_{2M+1-j} \coloneqq \widehat{\Phi}_j, \qquad 1 \le j \le M. \tag{3.42}$$

Recall that $\mathcal{V}^2(\mathfrak{v})$ takes its values $\mathcal{V}^2(\mathfrak{v})(t)$ in the set $\{\pm\Sigma_n(\mathfrak{v})\widetilde{\Phi}_j\} \subset \mathrm{D}(A)$, for $t \in \overline{I_k}$. We define a new piecewise constant control $\mathcal{V}^3(\mathfrak{v})(t)$, taking its values in the set $\{\pm\Sigma_n(\mathfrak{v})\widehat{\Phi}_j\} \subset H$, by

$$\mathcal{V}^3(\mathfrak{v})(t) = \mathrm{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widehat{\Phi}_j, \qquad \text{for} \quad \mathcal{V}^2(\mathfrak{v})(t) = \mathrm{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widetilde{\Phi}_j. \tag{3.43}$$

Using (3.24), we can see that the corresponding solutions satisfy

$$\begin{aligned}
&\left|\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^3(\mathfrak{v}))(t) - \mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^2(\mathfrak{v}))(t)\right|_V^2 \le D_Y \left|\mathcal{V}^3(\mathfrak{v}) - \mathcal{V}^2(\mathfrak{v})\right|_{L^2(I_k,H)}^2\\
&\qquad \le D_Y T \sup_{1\le n\le N} \Sigma_n(\mathfrak{v})^2 \max_{1\le j\le M} \left|\widetilde{\Phi}_j - \widehat{\Phi}_j\right|_H^2\\
&\qquad \le D_Y T M^2 \mathfrak{K}^2 \left|\mathfrak{v}\right|_V^2 \max_{1\le j\le M} \left|\widetilde{\Phi}_j - \widehat{\Phi}_j\right|_H^2
\end{aligned}$$

and by (3.17),

$$\left|\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^3(\mathfrak{v}))(t) - \mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^2(\mathfrak{v}))(t)\right|_V \le \tfrac{1-\theta}{10} \left|\mathfrak{v}\right|_V, \qquad t \in \overline{I_k}. \tag{3.44a}$$

Note also that

$$\left|\mathcal{V}^3(\mathfrak{v})(t)\right|_V \le \sup_{1\le n\le N} \Sigma_n(\mathfrak{v}) \le M\mathfrak{K}\left|\mathfrak{v}\right|_V, \qquad t \in \overline{I_k}, \tag{3.44b}$$

$$\mathcal{V}^3(\mathfrak{v})(kT + (n-1)\tfrac{T}{N}) \in \{\pm\Sigma_n(\mathfrak{v})\widehat{\Phi}_1\}, \qquad 1 \le n \le \widehat{N}. \tag{3.44c}$$

Observe that $\mathcal{V}^2$ switches between the actuators $\widetilde{\Phi}_j$ as described in (3.32), and hence $\mathcal{V}^3$ switches between the actuators $\widehat{\Phi}_j$ as in the analogous cycle

$$\widehat{\Phi}_1 \to \widehat{\Phi}_2 \to \cdots \to \widehat{\Phi}_{M-1} \to \widehat{\Phi}_M \to \widehat{\Phi}_M \to \widehat{\Phi}_{M-1} \to \cdots \to \widehat{\Phi}_2 \to \widehat{\Phi}_1, \tag{3.45}$$

which, due to (3.28), results in

$$\widehat{\Phi}_1 \to \widehat{\Phi}_2 \to \cdots \to \widehat{\Phi}_{M-1} \to \widehat{\Phi}_M \to \widehat{\Phi}_{M+1} \to \widehat{\Phi}_{M+2} \to \cdots \to \widehat{\Phi}_{2M-1} \to \widehat{\Phi}_{2M}.$$

3.4. **A control with no degenerate intervals of constancy.** By construction, the length $|l_{k,n,j}|$ vanishes if $v_j^1(\mathfrak{v})$ has zero average on $I_{k,n}$, see (3.25). It will be convenient, also to simplify the exposition in Section 3.5 below, that all the intervals of constancy have a length larger than a suitable positive constant.

Recall the switching time instants $t_{k,n,j}$ on each interval $\overline{I_{k,n}} \subset \overline{I_k}$, see (3.40),

$$t_{k,n,0} \le t_{k,n,1} \le t_{k,n,2} \le \cdots \le t_{k,n,2M-1} \le t_{k,n,2M}, \qquad 1 \le n \le \widehat{N}, \tag{3.46a}$$

and recall that

$$\mathbf{t}_{k,n-1} \coloneqq t_{k,n,0} = kT + (n-1)\tfrac{T}{\widehat{N}} \quad \text{and} \quad \mathbf{t}_{k,n} \coloneqq t_{k,n,2M} = kT + n\tfrac{T}{\widehat{N}}. \tag{3.46b}$$

To guarantee nondegenerate intervals of constancy we will define new switching time instants satisfying

$$t^\varepsilon_{k,n,0} < t^\varepsilon_{k,n,1} < t^\varepsilon_{k,n,2} < \cdots \le t^\varepsilon_{k,n,2M-1} < t^\varepsilon_{k,n,2M} \tag{3.47a}$$

with

$$t^\varepsilon_{k,n,0} \coloneqq \mathbf{t}_{k,n-1} \quad \text{and} \quad t^\varepsilon_{k,n,2M} \coloneqq \mathbf{t}_{k,n}. \tag{3.47b}$$

To do so we fix a positive number $\varepsilon > 0$, and define

$$t^{\varepsilon}_{k,n,j} = \mathbf{t}_{k,n-1} + \vartheta_\varepsilon(t_{k,n,j} - \mathbf{t}_{k,n-1} + \tfrac{j+1}{2} j\varepsilon), \qquad 0 \le j \le 2M, \tag{3.48a}$$

with

$$\vartheta_\varepsilon \coloneqq \tfrac{T}{T+\widehat{N}(2M+1)M\varepsilon}. \tag{3.48b}$$

Now we show that, indeed, the sequence (3.48) satisfies (3.47). We find

$$\begin{aligned} t^{\varepsilon}_{k,n,0} &= \mathbf{t}_{k,n-1} + \vartheta_\varepsilon 0 = \mathbf{t}_{k,n-1}, && (3.49a)\\ t^{\varepsilon}_{k,n,2M} &= \mathbf{t}_{k,n-1} + \vartheta_\varepsilon(\tfrac{T}{\widehat{N}} + (2M+1)M\varepsilon) = \mathbf{t}_{k,n-1} + \tfrac{T}{\widehat{N}} = \mathbf{t}_{k,n}, && (3.49b)\\ t^{\varepsilon}_{k,n,j} - t^{\varepsilon}_{k,n,j-1} &= \vartheta_\varepsilon(t_{k,n,j} - t_{k,n,j-1} + (\tfrac{j+1}{2} j - \tfrac{j}{2}(j-1))\varepsilon), \\ &\ge \vartheta_\varepsilon(\tfrac{j+1}{2} j - \tfrac{j}{2}(j-1))\varepsilon = \varepsilon\vartheta_\varepsilon j > 0, \qquad 1 \le j \le 2M. && (3.49c) \end{aligned}$$

From (3.49) we see that (3.47) is satisfied.

Next we define the piecewise constant control, for time $t \in I_k$, as follows

$$\mathcal{V}_\varepsilon(\mathfrak{v})(t) \coloneqq \operatorname{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widehat{\Phi}_j, \quad \text{if} \quad t \in [t^{\varepsilon}_{k,n,j-1}, t^{\varepsilon}_{k,n,j}), \qquad \text{for} \quad 1 \le n \le \widehat{N}, \quad 1 \le j \le 2M. \tag{3.50}$$

where the intervals of constancy have a positive minimum length, see (3.49),

$$\min_{1\le j\le 2M} \{t^{\varepsilon}_{k,n,j-1} - t^{\varepsilon}_{k,n,j-1}\} \ge \varepsilon\vartheta_\varepsilon > 0. \tag{3.51}$$

Observe that, from (3.39) and (3.43), we have that

$$\mathcal{V}^3(\mathfrak{v})(t) = \operatorname{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widehat{\Phi}_j, \quad \text{if} \quad t \in [t_{k,n,j-1}, t_{k,n,j}).$$

Note that as $\varepsilon \to 0$ we have $t^{\varepsilon}_{k,n,j} \to t_{k,n,j}$. Now we show that we also have $\mathcal{V}_\varepsilon(\mathfrak{v})(t) \to \mathcal{V}^3(\mathfrak{v})(t)$ in $L^2(I_k, H)$, as $\varepsilon \to 0$.

Proposition 3.5. *Let $[a, b] \in \mathbb{R}$ be a nonempty interval, $a < b$, let X be a Banach space, and let K be positive integer. Let us be given a finite sequence in X*

$$\phi_j \in X, \qquad 1 \le j \le K,$$

and two finite sequences in $[a, b]$,

$$a = \tau_0 \le \tau_1 \le \cdots \le \tau_{K-1} \le \tau_K = b \quad \textit{and} \quad a = \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{K-1} \le \sigma_K = b.$$

Then, for the following two functions defined for $t \in (a, b)$ by

$$f_\tau(t) \coloneqq \phi_j \quad \textit{if} \quad t \in [\tau_{j-1}, \tau_j) \qquad \textit{and} \qquad f_\sigma(t) \coloneqq \phi_j \quad \textit{if} \quad t \in [\sigma_{j-1}, \sigma_j),$$

we have the estimate

$$|f_\tau - f_\sigma|_{L^2((a,b),X)} \le K^{\frac12}\mathcal{R}^{\frac12}\mathcal{X},$$

with $\mathcal{R} \coloneqq \max_{0\le j\le K} |\tau_j - \sigma_j|_{\mathbb{R}}$ and $\mathcal{X} \coloneqq \max_{1\le i,j\le K} |\phi_j - \phi_i|_X$.

The proof of Proposition 3.5 is given in Section A.3.

From Proposition 3.5 it follows that

$$\begin{aligned}
&\left|\mathcal{V}_\varepsilon(\mathfrak{v})-\mathcal{V}^3(\mathfrak{v})\right|_{L^2(I_k,H)}\\
&\qquad\le (2M\widehat{N})^{\frac12}\max_{\substack{0\le j\le 2M\\1\le n\le\widehat{N}}}\left|t^\varepsilon_{k,n,j}-t_{k,n,j}\right|_{\mathbb{R}}^{\frac12}\max_{\substack{1\le i,j\le M\\1\le n\le\widehat{N}}}\left|\Sigma_n(\mathfrak{v})\widehat{\Phi}_j-\Sigma_n(\mathfrak{v})\widehat{\Phi}_i\right|_H\\
&\qquad\le 2(2M\widehat{N})^{\frac12}\max_{1\le n\le\widehat{N}}\Sigma_n(\mathfrak{v})\max_{\substack{0\le j\le 2M\\1\le n\le\widehat{N}}}\left|t^\varepsilon_{k,n,j}-t_{k,n,j}\right|_{\mathbb{R}}^{\frac12}.
\end{aligned}$$

Recalling (3.24), we arrive at

$$\left|\mathcal{V}_\varepsilon(\mathfrak{v})-\mathcal{V}^3(\mathfrak{v})\right|_{L^2(I_k,H)}\le (2M)^{\frac32}\widehat{N}^{\frac12}\mathfrak{K}\left|\mathfrak{v}\right|_V\max_{\substack{0\le j\le 2M\\1\le n\le\widehat{N}}}\left|t^\varepsilon_{k,n,j}-t_{k,n,j}\right|_{\mathbb{R}}^{\frac12}. \tag{3.52}$$

Next, from (3.48) we find that

$$\begin{aligned}
\left|t^\varepsilon_{k,n,j}-t_{k,n,j}\right|_{\mathbb{R}}&=\left|\mathbf{t}_{k,n-1}-t_{k,n,j}+\vartheta_\varepsilon(t_{k,n,j}-\mathbf{t}_{k,n-1}+\tfrac{j+1}{2}j\varepsilon)\right|_{\mathbb{R}}\\
&=\left|(\vartheta_\varepsilon-1)(t_{k,n,j}-\mathbf{t}_{k,n-1})+\tfrac{j+1}{2}j\varepsilon\vartheta_\varepsilon\right|_{\mathbb{R}}\\
&\le(1-\vartheta_\varepsilon)\tfrac{T}{\widehat{N}}+(2M+1)M\varepsilon\vartheta_\varepsilon=:\varTheta(\varepsilon).
\end{aligned}\tag{3.53}$$

By combining (3.52) and (3.53), we obtain that

$$\left|\mathcal{V}_\varepsilon(\mathfrak{v})-\mathcal{V}^3(\mathfrak{v})\right|_{L^2(I_k,H)}\le (2M)^{\frac32}\widehat{N}^{\frac12}\mathfrak{K}\left|\mathfrak{v}\right|_V\varTheta(\varepsilon)^{\frac12}, \tag{3.54}$$

and by using (3.16) it follows that

$$\left|\mathfrak{Y}_k(\mathfrak{v},V_e(\mathfrak{v}))(t)-\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^3(\mathfrak{v}))(t)\right|_V\le(8M^3D_Y\widehat{N})^{\frac12}\mathfrak{K}\varTheta(\varepsilon)^{\frac12}\left|\mathfrak{v}\right|_V,\quad t\in\overline{I_k}. \tag{3.55}$$

From (3.48) we also find

$$1-\vartheta_\varepsilon=\tfrac{\widehat{N}(2M+1)M\varepsilon}{T+\widehat{N}(2M+1)M\varepsilon},\qquad \varepsilon\vartheta_\varepsilon=\tfrac{T\varepsilon}{T+\widehat{N}(2M+1)M\varepsilon},$$

and

$$1-\vartheta_\varepsilon\to0,\quad \varepsilon\vartheta_\varepsilon\to0,\quad\text{and}\quad\varTheta(\varepsilon)\to0,\qquad\text{as}\quad\varepsilon\to0.$$

Therefore, there exists $\widehat{\varepsilon}$ small enough, so that

$$\varTheta(\widehat{\varepsilon})\le(8M^3D_Y\widehat{N})^{-1}\mathfrak{K}^{-2}(\tfrac{1-\theta}{10})^2. \tag{3.56}$$

Now we set the control

$$\mathcal{V}^4(\mathfrak{v})=\mathcal{V}_{\widehat{\varepsilon}}(\mathfrak{v}),\qquad t\in I_k, \tag{3.57a}$$

with nondegenerate intervals of constancy (cf. (3.51)),

$$\min_{1\le j\le 2M}\{t^{\widehat{\varepsilon}}_{k,n,j}-t^{\widehat{\varepsilon}}_{k,n,j-1}\}\ge\widehat{\varepsilon}\vartheta_{\widehat{\varepsilon}}>0. \tag{3.57b}$$

From (3.55), and (3.56), it follows that

$$\left|\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^4(\mathfrak{v}))(t)-\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^3(\mathfrak{v}))(t)\right|_V\le\tfrac{1-\theta}{10}\left|\mathfrak{v}\right|_V,\qquad t\in\overline{I_k}. \tag{3.58}$$

3.5. **A continuously moving control taking values in H.** We will travel in H between the static actuators $\widehat{\Phi}_i$, following the cycle (3.45).

For traveling we fix a set of roads, in the unit sphere $\mathfrak{S}_H$, connecting the static actuators, as follows:

$$\mathfrak{R}_j\colon \mathcal{C}^p([0,1],\mathfrak{S}_H), \qquad 1\le j\le M-1,\quad p\in\mathbb{N}, \tag{3.59a}$$

$$\text{with}\quad \mathfrak{R}_j(0)=\widehat{\Phi}_j\quad\text{and}\quad \mathfrak{R}_j(1)=\widehat{\Phi}_{j+1}, \tag{3.59b}$$

$$\mathfrak{R}_M(s)=\widehat{\Phi}_M, \qquad s\in[0,1], \tag{3.59c}$$

$$\mathfrak{R}_{M+j}(s)=\mathfrak{R}_{M-j}(1-s), \qquad 1\le j\le M-1. \tag{3.59d}$$

Note that by (3.42), we also have

$$\mathfrak{R}_{M+j}(0)=\mathfrak{R}_{M-j}(1)=\widehat{\Phi}_{M-j+1}=\widehat{\Phi}_{M+j}, \tag{3.59e}$$

$$\mathfrak{R}_{M+j}(1)=\mathfrak{R}_{M-j}(0)=\widehat{\Phi}_{M-j}=\widehat{\Phi}_{M+j+1}. \tag{3.59f}$$

We also introduce the scalar function

$$r_{k,n,j}^{\widehat{\varepsilon},\xi}(t) \coloneqq \left(\tfrac{\xi+t_{k,n,j}^{\widehat{\varepsilon}}-t}{2\xi}\right)\operatorname{sign}(l_{k,n,j})+\left(\tfrac{\xi-t_{k,n,j}^{\widehat{\varepsilon}}+t}{2\xi}\right)\operatorname{sign}(l_{k,n,j+1}), \tag{3.60a}$$

with, recall (3.57),

$$\xi\in(0,\tfrac{\widehat{\varepsilon}\vartheta_{\widehat{\varepsilon}}}{2}). \tag{3.60b}$$

Then we define a moving control $\mathcal{V}_\xi(\mathfrak{v})$, for $t\in\overline{I_k}$ as follows:

$$\mathcal{V}_\xi(\mathfrak{v})(t)\coloneqq \operatorname{sign}(l_{k,n,1})\Sigma_n(\mathfrak{v})\widehat{\Phi}_1, \tag{3.61a}$$
$$\text{if}\quad t\in[kT+(n-1)\tfrac{T}{\widehat{N}},t_{k,n,1}^{\widehat{\varepsilon}}-\xi],\qquad 1\le n\le\widehat{N}.$$

$$\mathcal{V}_\xi(\mathfrak{v})(t)\coloneqq \operatorname{sign}(l_{k,n,2M})\Sigma_n(\mathfrak{v})\widehat{\Phi}_1, \tag{3.61b}$$
$$\text{if}\quad t\in[t_{k,n,2M-1}^{\widehat{\varepsilon}}+\xi,kT+n\tfrac{T}{\widehat{N}}],\qquad 1\le n\le\widehat{N}.$$

$$\mathcal{V}_\xi(\mathfrak{v})(t)\coloneqq \operatorname{sign}(l_{k,n,j})\Sigma_n(\mathfrak{v})\widehat{\Phi}_j, \tag{3.61c}$$
$$\text{if}\quad t\in[t_{k,n,j-1}^{\widehat{\varepsilon}}+\xi,t_{k,n,j}^{\widehat{\varepsilon}}-\xi],\qquad 1\le n\le\widehat{N},\quad 2\le j\le 2M-1.$$

$$\mathcal{V}_\xi(\mathfrak{v})(t)\coloneqq r_{k,n,j}^{\widehat{\varepsilon},\xi}(t)\Sigma_n(\mathfrak{v})\mathfrak{R}_j(\tfrac{\xi-t_{k,n,j}^{\widehat{\varepsilon}}+t}{2\xi}), \tag{3.61d}$$
$$\text{if}\quad t\in[t_{k,n,j}^{\widehat{\varepsilon}}-\xi,t_{k,n,j}^{\widehat{\varepsilon}}+\xi],\qquad 1\le n\le\widehat{N},\quad 1\le j\le 2M-1.$$

Observe that $\mathcal{V}_\xi$ differs from $\mathcal{V}^4$ only in the intervals $(t_{k,n,j}^{\widehat{\varepsilon}}-\xi,t_{k,n,j}^{\widehat{\varepsilon}}+\xi)$, $1\le n\le\widehat{N}$, $\quad 1\le j\le 2M-1$, when we travel from the static actuator $\widehat{\Phi}_j$ to the static actuator $\widehat{\Phi}_{j+1}$. These are exactly $\widehat{N}(2M-1)$ intervals, where each has length 2ξ. Thus

$$\left|\mathcal{V}_\xi(\mathfrak{v})-\mathcal{V}^4(\mathfrak{v})\right|_{L^2(I_k,H)}^2\le 2\xi\widehat{N}(2M-1)\max_{\substack{1\le n\le\widehat{N}\\1\le j\le 2M-1}}\left\{\left|r_{k,n,j}^{\widehat{\varepsilon},\xi}(t)\right|_{\mathbb{R}}^2\Sigma_n(\mathfrak{v})^2\left|\widehat{\Phi}_j\right|_H^2\right\}.$$

Since $\left|r_{k,n,j}^{\widehat{\varepsilon},\xi}(t)\right|_{\mathbb{R}}\le 1$ and $\left|\widehat{\Phi}_j\right|_H=1$, using (3.24), we arrive at

$$\left|\mathcal{V}_\xi(\mathfrak{v})-\mathcal{V}^4(\mathfrak{v})\right|_{L^2(I_k,H)}^2\le 2\xi\widehat{N}(2M-1)M^2\mathfrak{K}^2\left|\mathfrak{v}\right|_H^2.$$

Recalling (3.16), we obtain

$$\begin{aligned}\left|\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}_\xi(\mathfrak{v}))(t)-\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^4(\mathfrak{v}))(t)\right|_V^2&\le D_Y\left|\mathcal{V}_\xi(\mathfrak{v})-\mathcal{V}^4(\mathfrak{v})\right|_{L^2(I_k,H)}^2\\&\le D_Y 2\xi\widehat{N}(2M-1)M^2\mathfrak{K}^2\left|\mathfrak{v}\right|_V^2,\qquad t\in I_k.\end{aligned}\tag{3.62}$$

Now choosing small enough ξ, namely

$$\xi=\widehat{\xi}\coloneqq\min\left\{\frac{\widehat{\varepsilon}\vartheta_{\widehat{\varepsilon}}}{2},\left(D_Y 2\widehat{N}(2M-1)M^2\mathfrak{K}^2\right)^{-1}(\tfrac{1-\theta}{10})^2\right\},\tag{3.63}$$

and setting

$$\mathcal{V}^5(\mathfrak{v}))(t)\coloneqq\mathcal{V}_{\widehat{\xi}}(\mathfrak{v}))(t),\qquad t\in I_k,\tag{3.64}$$

we find

$$\left|\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^5(\mathfrak{v}))(t)-\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^4(\mathfrak{v}))(t)\right|_H\le\tfrac{1-\theta}{10}\left|\mathfrak{v}\right|_H.\tag{3.65}$$

Finally, note that $\mathcal{V}^5(\mathfrak{v})$ is a moving control of the form

$$\mathcal{V}^5(\mathfrak{v})(t)=:u(t)\Phi(t),\quad t\in\overline{I_k},\quad\text{with}\quad\left|u(t)\right|_H\le M\mathfrak{K}\left|\mathfrak{v}\right|_V,\quad\Phi(t)\in\mathfrak{S}_H,\tag{3.66a}$$

for suitable $u\in L^\infty(\overline{I_k},\mathbb{R})$ and $\Phi\in\mathcal{C}(\overline{I_k},\mathfrak{S}_H)$. Furthermore, by choosing $p\ge0$ in (3.59) we can obtain a regular motion of the actuator. Namely, if we have

$$\tfrac{\mathrm{d}^q}{\mathrm{d}s^q}\big|_{s=0}\mathfrak{R}_j=0=\tfrac{\mathrm{d}^q}{\mathrm{d}s^q}\big|_{s=1}\mathfrak{R}_j,\qquad 1\le q\le p,\tag{3.66b}$$

then

$$\Phi\in\mathcal{C}^p(\overline{I_k},\mathfrak{S}_H)\tag{3.66c}$$

and

$$\max_{\tau\in\overline{I_k}}\left|\tfrac{\mathrm{d}^p}{\mathrm{d}t^p}\big|_{t=\tau}\Phi\right|_H\le(\tfrac{1}{2\widehat{\xi}})^p\max_{1\le j\le M}\max_{s_0\in[0,1]}\left|\tfrac{\mathrm{d}^p}{\mathrm{d}s^p}\big|_{s=s_0}\mathfrak{R}_j\right|_H.$$

In particular, we have that

$$\left|\Phi\right|_{\mathcal{C}^p(\overline{I_k},\mathfrak{S}_H)}\le(\tfrac{1}{2\widehat{\xi}})^p\max_{1\le j\le M}\left|\mathfrak{R}_j\right|_{\mathcal{C}^p([0,1],H)}.\tag{3.66d}$$

Conclusion of the proof of Theorem 3.3. By using (3.21), (3.41), (3.44), (3.58), and (3.65), together with the triangle inequality, we arrive at

$$\left|\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^5(\mathfrak{v}))(kT+T)-\mathfrak{Y}_k(\mathfrak{v},\mathcal{V}^0(\mathfrak{v}))(kT+T)\right|_H\le5\tfrac{1-\theta}{10}\left|\mathfrak{v}\right|_H=\tfrac{1-\theta}{2}\left|\mathfrak{v}\right|_H.$$

Finally, note that the choice of $\widehat{\varepsilon}$ in (3.56) and that of ξ in (3.63) are independent of y_0. This finishes the proof of Theorem 3.3. □

Remark 3.6. Observe that the actuators $\widetilde{\Phi}_j$ in (3.17), the integer $\widehat{N}$ in (3.38), the parameter $\widehat{\varepsilon}$ in (3.56), and the parameter $\widehat{\xi}$ in (3.63), were all chosen independently of $k\in\mathbb{N}$. Furthermore, from (3.66) we can also see that $\left|\Phi\right|_{\mathcal{C}^p(\overline{I_k},\mathfrak{S}_H)}$ is bounded by a constant independent of $k\in\mathbb{N}$. Again from (3.66), by recalling Assumption 2.4 we also have $\left|u\right|_{L^\infty(I_k,H)}\le M\mathfrak{K}\left|\mathfrak{v}\right|_V=M\mathfrak{K}\left|y(kT)\right|_V$ with the product $M\mathfrak{K}$ independent of $k\in\mathbb{N}$.

4. Proof of Theorem 1.1

We start by writing (1.3) as

$$\dot y + Ay + A_{\rm rc}y = u\widehat{1}_{\omega(c)}, \quad y(0) = y_0, \quad t > 0, \tag{4.1}$$

with $A \coloneqq -\nu\Delta + \mathbf{1}$ and $A_{\rm rc}z = A_{\rm rc}(t)z \coloneqq (a(t,\cdot) - 1)z + b(t,\cdot)\cdot\nabla z$.

It is not hard to check that Assumptions 2.1, 2.2, and 2.3, are satisfied by $A \in \mathcal{L}(V, V')$ and $A_{\rm rc} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V, H))$, namely with $V = H^1_0(\Omega)$ in the case of Dirichlet boundary conditions and with $V = H^1(\Omega)$ (see, e.g., [25, Sect.5.1]).

4.1. **Satisfiability of Assumption 2.4.** Assumption 2.4 follows from the results in [25, Thm. 4.5] (when applied to linear equations), from which we know that

$$\dot y + Ay + A_{\rm rc}y = P_{U_M}^{E_M^\perp}\left(A_{\rm rc}y - \lambda y\right), \quad y(0) = y_0, \quad t > 0, \tag{4.2}$$

is a stable system for a suitable oblique projection $P_{U_M}^{E_M^\perp}$, where $\lambda > 0$. Namely, its solution satisfies,

$$|y(t)|_V \le C\mathrm{e}^{-\mu(t-s)}\,|y(s)|_V\,, \qquad t \ge s \ge 0, \tag{4.3}$$

with $C \ge 1$ and $\mu \ge 0$ independent of (t,s). Actually in [25, Thm. 4.5] only the case $s = 0$ is mentioned, however by a time shift argument $t =: s+\tau$, $w(\tau) = y(s+\tau)$, $\widetilde A_{\rm rc}(\tau) = A_{\rm rc}(s+\tau)$, we can rewrite (4.2) as

$$\tfrac{\mathrm d}{\mathrm d\tau}w + Aw + \widetilde A_{\rm rc}w = P_{U_M}^{E_M^\perp}\left(\widetilde A_{\rm rc}w - \lambda w\right), \quad w(0) = y(s), \quad \tau > 0,$$

and the results in [25, Thm. 4.5] give us

$$|w(\tau)|_V \le C_s\mathrm{e}^{-\mu\tau}\,|y(s)|_V\,, \qquad \tau \ge 0,$$

which is equivalent to (4.3). The constant C_s is of the form $\overline{C}_{\left[\left|\widetilde A_{\rm rc}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(V,H))}\right]}$, and so $C_s \le C_0$, that is we can take C independent of s in (4.3).

This stability result in [25, Thm. 4.5] holds for large enough M, where $U_M = \operatorname{span}\{1_{\omega_j} \mid 1 \le j \le M\}$ is the span of suitable indicator functions supported in small rectangles $\omega_j \subset \Omega$. The operator $P_{U_M}^{E_M}$ is the oblique projection in $L^2(\Omega)$ onto U_M along an auxiliary space $E_M^\perp$, where E_M is the span of a suitable set of eigenfunctions of the diffusion A defined in $L^2(\Omega)$, where Ω is a bounded rectangular domain. For precise definitions of suitable U_M and E_M we refer to [18, Sect. 4.8.1] and [25, Sect. 2.2]. Furthermore, for such choice we have

$$\sup_{M\ge1}\left|P_{U_M}^{E_M^\perp}\right|_{\mathcal{L}(H)} =: \|P\| < +\infty. \tag{4.4}$$

Observe that U_M is the range of $P_{U_M}^{E_M}$, hence our control is of the form

$$P_{U_M}^{E_M^\perp}\left(A_{\rm rc}y - \lambda y\right) =: \sum_{j=1}^{M}\widetilde v_j(t)1_{\omega_j} = \sum_{j=1}^{M}v_j(t)\widehat\Phi_j =: \mathbf{v}(t)$$

In particular, from (4.3), for any given $\theta \in (0,1)$ we have that, for all $k \in \mathbb{N}$,

$$|y(kT+T)|_V \le C\mathrm{e}^{-\mu T}\,|y(kT)|_V \le \theta\,|y(kT)|_V\,, \quad \text{if} \quad T \ge \mu^{-1}\log(\tfrac{C}{\theta}).$$

To prove that Assumption 2.4 is satisfied, it remains to show that the $v_j(t)$ are appropriately essentially bounded. From (4.3) and $P_{U_M}^{E_M^\perp} = P_{U_M}^{E_M^\perp} P_{E_M}$, we find

$$\begin{aligned}|\mathbf{v}(t)|_H &\le \|P\| \left|P_{E_M}(A_{\mathrm{rc}}y - \lambda y)\right|_H\\ &\le \Big(|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(V,H))} + \lambda\,|\mathbf{1}|_{\mathcal{L}(V,H)}\Big)\|P\|\,|y(t)|_V\\ &\le \Big(C_{\mathrm{rc}} + \lambda\,|\mathbf{1}|_{\mathcal{L}(V,H)}\Big)\|P\|\,C\mathrm{e}^{-\mu(t-kT)}\,|y(kT)|_V\,, \qquad t \ge kT. \end{aligned} \tag{4.5}$$

which implies that

$$|\mathbf{v}(t)|_H \le \mathfrak{K}_0\,|y(kT)|_V\,, \quad t \in \overline{I_k} = [kT, kT+T]. \tag{4.6}$$

with $\mathfrak{K}_0 = \Big(C_{\mathrm{rc}} + \lambda\,|\mathbf{1}|_{\mathcal{L}(V,H)}\Big)\|P\|\,C$ independent of k.

Therefore, Assumption 2.4 holds for M large enough, and with $T = \mu^{-1}\log(\frac{C}{\theta})$ and $\mathfrak{K} = \mathfrak{K}_0$ as above.

4.2. **Illustration of a path for the moving actuator.** We consider the static actuators $1_{\omega_i} = 1_{\omega(c^i)}$ with center c^i as in [18, Sect. 4.8.1], illustrated in figure 2. Then we order the actuators, for example as illustrated in the figure, and consider the corresponding cycle, starting at the first actuator in the bottom-left corner, going up until the Mth actuator in the top-right corner and returning back to the bottom-left corner.

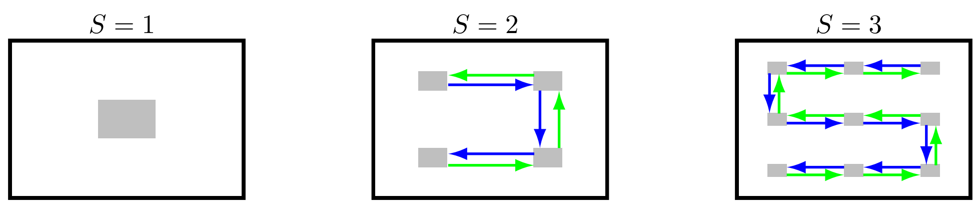


FIGURE 2. Supports of the static actuators. Case $\Omega \subset \mathbb{R}^2$.

In this way we are considering roads, see (3.66),

$$\mathfrak{R}_j(s) = \widehat{1}_{\omega(c_j(s))}, \qquad \omega(c_j(s)) \subset \Omega, \qquad c_j \in \mathcal{C}^2([0,1],\mathbb{R}^d),$$

and

$$c_j(0) = c^j, \qquad c_j(1) = c^{j+1}, \qquad \dot{c}_j(0) = \dot{c}_j(1) = \ddot{c}_j(0) = \ddot{c}_j(1) = 0.$$

Note that for such roads, we may take

$$c_j(s) = c_j + \phi(s)(c_{j+1} - c_j), \qquad 1 \le j \le M,$$

where $\phi \in \mathcal{C}^2([0,1],[0,1])$ is increasing and satisfies the relations $\phi(0) = 0$, $\phi(1) = 1$, and $\dot{\phi}(0) = \dot{\phi}(1) = \ddot{\phi}(0) = \ddot{\phi}(1) = 0$. Furthermore, we have

$$|c|_{\mathcal{C}^m(\mathbb{R}_0,\mathbb{R}^d} \le (\tfrac{1}{2\widehat{\xi}})^m\,|\phi|_{\mathcal{C}^m([0,1],\mathbb{R})}\,, \qquad m \in \{0,1,2\}. \tag{4.8}$$

Recall also that $\widehat{\xi}$ can be chosen independent of y_0.

4.3. **Conclusion of proof of Theorem 1.1.** Let us fix $y_0 \in H$ and $c_0 \in \mathbb{R}^M$ with $\omega(c_0) \subset \Omega$, and let c^1 be the center of the static actuator 1_{ω_1}. For $\widehat{c}(t) \coloneqq c_0 + t^2(2-t)^2(c^1 - c_0)$ we have that

$$\widehat{c}(0) = c_0, \quad \widehat{c}(1) = c^1, \qquad \dot{\widehat{c}}(0) = \dot{\widehat{c}}(1) = 0, \qquad \ddot{\widehat{c}} \in L^\infty((0,1), \mathbb{R}^M).$$

We proceed as in Corollary 3.2 by taking the actuator path $\Phi^*(t) = \widehat{1}_{\widehat{c}(t)}$, for time $t \in [0,1]$, and the path illustrated in section 4.2 for time $t \geq 1$ where we use Theorem 3.1. Note that $\Phi^*(1) = \widehat{1}_{\omega(\widehat{c}(1))} = \widehat{1}_{\omega_1}$ and $\dot{\Phi}^*(1) = 0$.

Observe also that $\left|\dot{\widehat{c}}\right|_{W^{1,\infty}((0,1),\mathbb{R}^M)} \leq |\dot{\varphi}|_{W^{1,\infty}(0,1)} \left|c^1 - c_0\right|_{\mathbb{R}^M} \leq C_3$ with $\varphi(t) \coloneqq t^2(2-t)^2$, where C_3 can be taken independent of c_0 because Ω is bounded. Therefore, we have that $|\dot{c}|_{W^{1,\infty}(\mathbb{R}_0,\mathbb{R}^M)} \leq \max\{C_3, |\cdot c|_{W^{1,\infty}(\mathbb{R}_1,\mathbb{R}^M)}\} \leq C_4$, with C_4 independent of (y_0, c_0), because $|\dot{c}|_{W^{1,\infty}(\mathbb{R}_1,\mathbb{R}^M)}$ is independent of $y(1)$ (cf. (4.8)), hence independent of $y(0)$. □

4.4. **A remark on Assumption 2.3 and weak solutions.** Instead of the reaction-convection operator $A_{\mathrm{rc}} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V,H))$, we can also take $A_{\mathrm{rc}} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))$ which is the case for a convection term as $\nabla \cdot (by)$ under homogeneous Dirichlet boundary conditions, with $b \in L^\infty(\mathbb{R}_0, \mathbb{R}^d)$. For the latter case we can repeat the procedure and prove the stabilizability result in the H-norm. That is, we must work with weak solutions $y \in \mathcal{C}(\overline{\mathbb{R}}_0, H)$ instead of strong solutions $y \in \mathcal{C}(\overline{\mathbb{R}}_0, V)$. In particular we would just need to replace V by H in Assumption (2.4) and in (3.16). Recall that, with $\widetilde{C}_{\mathrm{rc}} = |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$ and $\widetilde{D}_Y = \overline{C}_{\left[T,\widetilde{C}_{\mathrm{rc}}\right]}$, we will have (cf. [21, Lem. 2.2], recalling that $\mathcal{C}(\overline{I_k}, H) \hookrightarrow W(I_k, V, V')$)

$$|\mathfrak{Y}_k(\mathfrak{v}, f)(t)|_H^2 \leq \widetilde{D}_Y \left(|\mathfrak{v}|_H^2 + |f|_{L^2(I_k,V')}^2 \right), \quad t \in I_k = (kT, kT+T).$$

Concerning parabolic equations we can see that instead of (4.5) we would obtain

$$\begin{aligned}
|\mathbf{v}(t)|_{V'} &\leq \|P\| \left(|P_{E_M}|_{\mathcal{L}(V',H)} |A_{\mathrm{rc}} y|_{V'} + \lambda |y|_H \right) \\
&\leq \left(|P_{E_M}|_{\mathcal{L}(V',H)} |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} + \lambda \right) \|P\| \, |y(t)|_H \\
&\leq \left(|P_{E_M}|_{\mathcal{L}(V',H)} C_{\mathrm{rc}} + \lambda \right) \|P\| \, C\mathrm{e}^{-\mu(t-kT)} |y(kT)|_H, \qquad t \geq kT. \quad (4.9)
\end{aligned}$$

which implies that

$$|\mathbf{v}(t)|_{V'} \leq \mathfrak{K}_0 |y(kT)|_H, \quad t \in \overline{I_k} = [kT, kT+T]. \quad (4.10)$$

Such inequality implies that the control is essentially bounded as required in Assumption (2.4).

Such weak solutions are also defined for $A_{\mathrm{rc}} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V,H))$, but we cannot show that the control remains essentially bounded in the case we only know that $|y(t)|_H$ remains bounded. For that we would need to bound (4.5) by $|y(kT)|_H$ instead of $|y(kT)|_V$, but this seems to be not possible in general.

5. NUMERICAL SIMULATIONS

According to the construction in Sections 3 and 4, once we have fixed a set of roads $\mathfrak{R}_j$, see (4.7), we could compute the moving control $\mathcal{V}^5$ in (3.64) from the control $\mathcal{V}^0$ given by (4.2) simply by setting $N = \widehat{N}$ large enough, $\varepsilon = \widehat{\varepsilon}$ small enough and $\xi = \widehat{\xi}$, and by computing the scalars $l_{k,n,j}$ in (3.61), from which we could also compute $\Sigma_n(y(kT))$, the switching times in (3.48). However, we would obtain an actuator $\mathcal{V}^5$ which would be moving very fast by visiting all initial static actuators twice in a each interval of time $i\frac{T}{N}$, $i \in \mathbb{N}$. In applications, this is likely not the "best" motion for the actuator, sometimes it would be better to stay longer in a particular region or it would be better to leave the roads $\mathfrak{R}_j$ in order to cover other regions of Ω. Therefore, we are going to compute the center $c = c(t)$ of the moving actuator and the control magnitude $u = u(t)$ using tools from optimal control.

5.1. **Computation of a stabilizing single actuator based receding horizon control.** We deal with system (1.6), where now we will consider (y, c) as the state of the system and (u, η) as the control. Note that η can be seen as a control on the acceleration of c, which also makes sense from the applications point of view, where we cannot change instantaneously the velocity of a device, but instead we can apply a force/acceleration to it. Then, to compute the the force η and magnitude u, we formulate the following infinite-horizon optimal control problem defined by minimizing the performance index function defined by

$$J_\infty(u, \eta : (y_0, c_0, 0)) \coloneqq \frac{1}{2}\int_0^\infty |\nabla y(t,\cdot)|^2_{L^2(\Omega,\mathbb{R}^d)} + \beta |u(t)|^2 \,\mathrm{d}t. \tag{5.1}$$

That is, we define the infinite-horizon optimization problem

$$\inf_{(u,\eta)\in L^2(\mathbb{R}_0,\mathbb{R})\times L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbb{R}^d)} J_\infty(u, \eta : (y_0, c_0, 0)) \tag{5.2a}$$

subject to

$$\begin{cases} \dot{y} - \nu\Delta y + ay + b\cdot\nabla y = u1_{\omega(c)}, \quad y|_\Gamma = 0, \\ \ddot{c} + \varsigma\dot{c} + \epsilon c = \eta, \\ y(0,\cdot) = y_0, \quad c(0) = c_0, \quad \dot{c}(0) = 0, \end{cases} \tag{5.2b}$$

as well as to the constraints

$$\begin{cases} c \in \mathcal{C} \coloneqq \{g \in C(\overline{\mathbb{R}}_0, \mathbb{R}^d) \mid \omega(g(s)) \subset \Omega, \text{ for } s \in \overline{\mathbb{R}}_0\}, \\ \eta \in \mathcal{X} \coloneqq \{\kappa \in L^2_{\mathrm{loc}}(\mathbb{R}_0, \mathbb{R}^d) \mid \|\kappa(s)\| \le K \text{ for a.e. } s \in \mathbb{R}_0\}, \end{cases} \tag{5.2c}$$

where $K = (K_1, K_2, \dots, K_d) \in \mathbb{R}^d$ is a vector with coordinates $K_i > 0$, for all $1 \le i \le d$, and where by $\|\kappa(s)\| \le K$ we mean that $|\kappa_i(s)| \le K_i$ for all $1 \le i \le d$. For tackling this infinite-horizon problem we employ a receding horizon framework. This framework relies on successively solving finite-horizon open-loop problems on bounded time-intervals as follows. Let us fix $T > 0$ and let an initial vector of the form $\mathcal{I}_0 \coloneqq (t_0, y_0, c_0, c_0^1) \in \overline{\mathbb{R}}_0 \times L^2(\Omega) \times \mathbb{R}^d \times \mathbb{R}^d$ be given. We define the time interval $I_{t_0} \coloneqq (t_0, t_0 + T)$, and the finite-horizon cost functional

$$J_T(u, \eta : \mathcal{I}_0) \coloneqq \frac{1}{2}\int_{t_0}^{t_0+T} |\nabla y(t,\cdot)|^2_{L^2(\Omega,\mathbb{R}^d)} + \beta |u(t)|^2 \,\mathrm{d}t,$$

and introduce the finite-horizon optimization problem

$$\min_{(u,\eta)\in L^2(I_{t_0},\mathbb{R}^{1+d})} J_T(u, \eta : \mathcal{I}_0) \tag{5.3a}$$

subjected to the dynamical constraints

$$\begin{cases} \dot{y} - \nu\Delta y + ay + b\cdot\nabla y = u1_{\omega(c)}, \quad y|_\Gamma = 0,\\ \ddot{c} + \varsigma\dot{c} + \epsilon c = \eta,\\ y(t_0,\cdot) = y_0, \quad c(t_0) = c_0, \quad \dot{c}(t_0) = c_0^1, \end{cases} \tag{5.3b}$$

in the time interval I_{t_0}, as well as to the constraints

$$\begin{cases} c \in \mathcal{C}_{t_0,T} := \{g \in C(\overline{I}_{t_0}, \mathbb{R}^d) \mid \omega(g(s)) \subset \Omega \text{ for } s \in \overline{I}_{t_0}\}\\ \eta \in \mathcal{X}_{t_0,T} := \{\kappa \in L^2(I_{t_0}, \mathbb{R}^d) \mid \|\kappa(s)\| \le K \text{ for a.e. } s \in I_{t_0}\}. \end{cases} \tag{5.3c}$$

The steps of the RHC are described in Algorithm 1, where we use the subset

$$\mathbb{R}^d_{[\omega]} := \{c \in \mathbb{R}^d \mid \omega(c) \in \Omega\}.$$

Algorithm 1 Receding Horizon Algorithm

Require: The prediction horizon $T > 0$, the sampling time $\delta < T$, and an initial vector $\mathcal{I}_\infty = (y_0, c_0) \in H \times \mathbb{R}^d_{[\omega]}$.

Ensure: The suboptimal RHC pair (u_{rh}, η_{rh}).

1: Set $t_0 = 0$ and $\mathcal{I}_0 = (t_0, y_0, c_0, 0)$;

2: Find the solution $(y_T^*(\cdot;\mathcal{I}_0), u_T^*(\cdot;\mathcal{I}_0), c_T^*(\cdot;\mathcal{I}_0), \eta_T^*(\cdot;\mathcal{I}_0))$ over the time horizon I_{t_0} by solving the open-loop problem (5.3);

3: For all $\tau \in [t_0, t_0 + \delta)$, set $u_{rh}(\tau) = u_T^*(\tau;\mathcal{I}_0)$ and $\eta_{rh}(\tau) = \eta_T^*(\tau;\mathcal{I}_0)$;

4: Update: $t_0 \leftarrow t_0 + \delta$;

5: Update: $\mathcal{I}_0 \leftarrow (t_0, y_T^*(t_0;\mathcal{I}_0), c_T^*(t_0;\mathcal{I}_0), \dot{c}_T^*(t_0;\mathcal{I}_0))$;

5.2. **Numerical discretization and implementation.** Here we report on numerical experiments related to Algorithm 1. These experiments confirm the capability of the moving control computed by Algorithm 1. In all examples, we deal with one-dimensional controlled systems of the form (1.3) defined on $\Omega := (0, 1)$ which are exponentially unstable without control. Moreover, we compare the performance of one single moving control with finitely many static actuators. Throughout, the spatial discretization was done by the standard Galerkin method using piecewise linear and continuous basis functions with mesh-size $h = 0.0025$. Moreover, for temporal discretization we used the Crank–Nicolson/Adams–Bashforth scheme [13] with step-size $t_{\text{step}} = 0.001$. In this scheme, the implicit Crank–Nicolson scheme is used except for the nonlinear term $u1_{\omega(c)}$ and convection term $b \cdot \nabla y$ which are treated with the explicit Adams–Bashforth scheme. To deal with open-loop problems 5.3a, we considered the reduced formulation of the problem with respect to the independent variables (η, u). The state constraints $\mathcal{C}_{t_0,T}$ were treated using the Moreau–Yosida [14] regularization with parameter $\mu = 10^{-5}$. Moreover, the box constraints $|u(t)| \le K$ were handled using projection. We used the projected Barzilai–Borwein gradient method [4, 6, 11] equipped with a nonmonotone line search strategy. Further, we terminated the algorithm as the L^2-norm of the projected gradient for the reduced problem was smaller than 10^{-4} times of the norm of the projected gradient for initial iterate.

For the case with static actuators, we choose the indicator functions 1_{ω_i} with the placements

$$\omega_i := \left(\tfrac{1}{2M}(2i-1) - \tfrac{r}{2}, \tfrac{1}{2M}(2i-1) + \tfrac{r}{2}\right) \qquad \text{for } i = 1, \ldots, M, \tag{5.4}$$

where $r > 0$ and integer $M \in \mathbb{N}$. This is motivated by the stabilizability results given in [26, Thm. 4.4] and [18, Sect. 4.8.1]. Further, for every $t \geq 0$, the moving actuator $1_{\omega(c(t))}$ is described by

$$\omega(c(t)) := \left(c(t) - \tfrac{r}{2}, c(t) + \tfrac{r}{2}\right).$$

For all actuators, namely moving and fixed ones, we chose $r = 0.04$. Thus the support of every actuator covers only four percent of the whole of domain. In the case of the static actuators, we employed the receding horizon framework given in [3, Alg. 1] for the choice of $|\cdot|_* = |\cdot|_{\ell_2}$ with control cost parameter β. In all numerical experiments, we chose $T = 1.25$ and $\delta = 0.5$.

Example 5.1. In this example, we set (cf. (5.2b)–(5.2c))

$$\nu = 0.1, \qquad \varsigma = 1, \qquad \epsilon = 0,$$
$$a(t,x) = -3 - 2|\sin(t+x)|, \qquad b(t,x) = |\cos(t+x)|, \qquad K = 500.$$

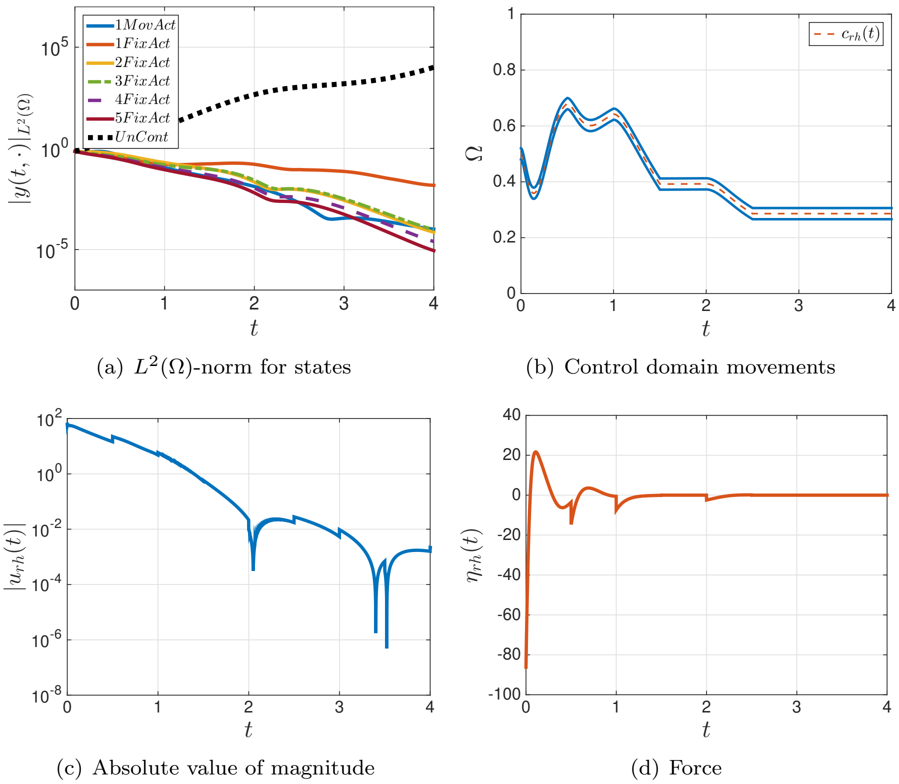
Further, we chose the initial conditions

$$y_0(x) := \sin(\pi x), \quad (c_0, c_0^1) := (0.5, 0).$$

Figures 3 and 4 correspond to the choices $\beta = 0.1$ and $\beta = 0.5$, respectively. Figures 3(a) and 4(a) illustrate the evolution of the $L^2(\Omega)$-norm for the states corresponding to uncontrolled system, one single moving actuator, and fixed actuators ($M = 1, \dots, 5$). The black dotted line in both figures corresponds to the uncontrolled state. It shows that the uncontrolled state is exponentially unstable. For both cases $\beta = 0.1$ and $\beta = 0.5$, we can see that the moving control obtained by Algorithm 1 is stabilizing and its stabilization rate is smaller than the one corresponding to one single static actuator ($M = 1$), and comparable to the cases $M = 2, 3, 4$. Further, by comparing Figures 3(a) and 4(a), we can infer that $\beta = 0.1$ leads to a faster stabilization compared to the case $\beta = 0.5$. As can be seen from Figure 4(a), it is not clear for the case $\beta = 0.5$ that one fixed actuator is asymptotically stabilizing. Moreover, for $M = 4, 5$ we have better stabilization results compared to the single moving control. Figures 3(b) and 4(b) illustrate the time evolution of the control domain $\omega(c_{rh}(t))$. From Figures 3(b), we can observe that, at some point ($t = 2.5$), the actuator stops moving. This corresponds to Figure 3(d), which demonstrates the evolution of the force. In this case, the receding horizon framework moved the actuator until some degree of stabilization ($|y_{rh}(t,\cdot)|_{L^2(\Omega)} \leq 10^{-3}$) was reached and, then, decided to steer the system with only a fixed actuator. In this case, the $|y_{rh}(t,\cdot)|_{L^2(\Omega)}$-norm corresponding to $M = 4$ and $M = 5$ is smaller than the one corresponding to the single actuator which is free to move, once $t \geq 3$, see Figure 3(a). For the case $\beta = 0.5$, we have a different scenario. In this case, the control remains moving throughout the whole simulation (see Figure 4(b)). This fact can also be seen from Figure 4(d) which shows that, here a stronger force was needed compared to the case $\beta = 0.1$. Figures 3(c) and 4(c) show the evolution of the absolute value of the magnitude control u_{rh}.

Example 5.2. In this example, motivated by Proposition 1.2, we present a situation in which one single fixed actuator is not stabilizing, or more precisely, it has no influence. A moving control steers, however, the system to zero. Here we used the same setting as in the previous example, except that we put

$$a(t,x) = -5, \qquad b(t,x) = 0, \qquad y_0(x) = \sin(2\pi x).$$



(a) $L^2(\Omega)$-norm for states

(b) Control domain movements

(c) Absolute value of magnitude

(d) Force

FIGURE 3. Example 5.1: Numerical results for $\beta = 0.1$.

From Figure 5(a), we can see that the curves corresponding to the uncontrolled state and one single fixed actuator are overlapping each other completely. This means that their corresponding states are exponentially unstable and the fixed actuator centered at 0.5 has no influence. Interestingly, we observed that all open-loop problems within the receding horizon framework for this single fixed actuator were solved easily with $u^* = 0$ also for long time horizons. This suggests that $u^* = 0$ is the unique minimum for all finite horizon open-loop problems which corresponds to the result given in Proposition 1.2. On the contrary, we can see from Figures 5(b) and 5(a) that a single moving control is able to steer the system exponentially to zero by moving the actuator. Figures 5(c) and 5(d) depict the evolution of the absolute value of the magnitude u_{rh} and the evolution of the force η_{rh}, respectively.

Summarizing, we can assert that the single moving actuator obtained by Algorithm 1 is able to stabilize the system to zero, confirming our theoretical findings.

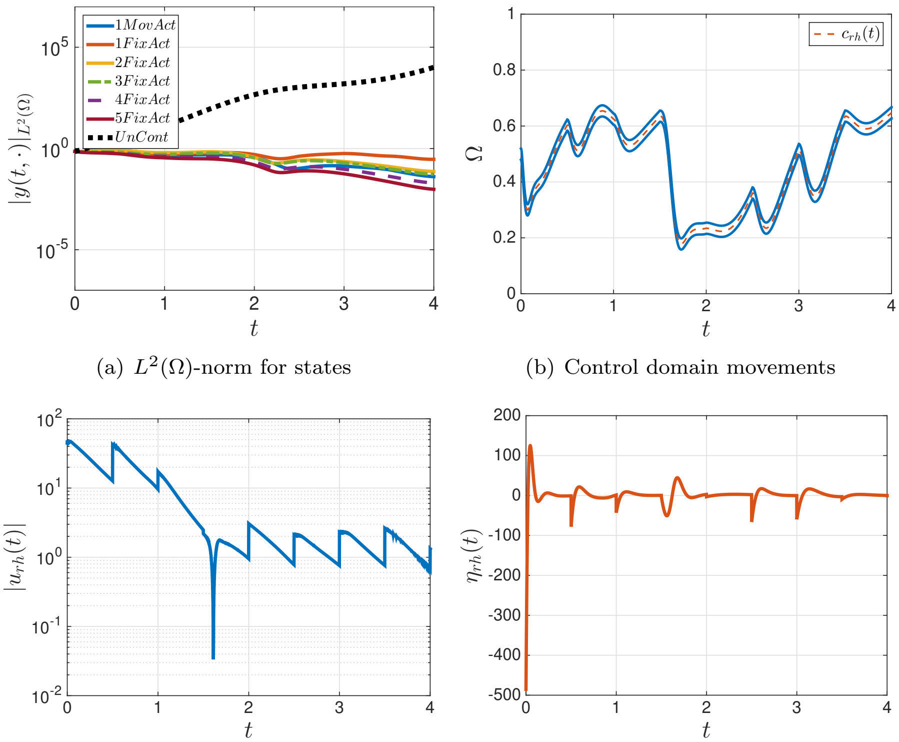
APPENDIX

A.1. **Proofs of Propositions 1.2 and 1.3.**

Recall system (1.7),

$$\tfrac{\partial}{\partial t}y(t,x) - \nu\Delta y(t,x) + (a_0 + a(x))y(t,x) = u(t)\Psi, \tag{A.1a}$$

$$y(0,\cdot) = y_0, \qquad \mathcal{G}y|_\Gamma = 0, \tag{A.1b}$$



(a) $L^2(\Omega)$-norm for states

(b) Control domain movements

(c) Absolute value of magnitude

(d) Force

FIGURE 4. Example 5.1: Numerical results for $\beta = 0.5$

and the system of eigenfunctions $\widetilde{e}_i$ and increasing sequence of eigenvalues $\widetilde{\alpha}_i$ of the operator $\mathcal{A} = -\nu\Delta + a(x)\mathbf{1}$, $\mathcal{A}\widetilde{e}_i = \widetilde{\alpha}_i\widetilde{e}_i$.

We start with the following auxiliary result.

Lemma A.1. *If there exists a nonsimple eigenvalue $\widetilde{\alpha}_j$, then there exists one associated eigenfunction $\overline{e}_j$ such that $(\overline{e}_j, \Psi)_{L^2(\Omega} = 0$.*

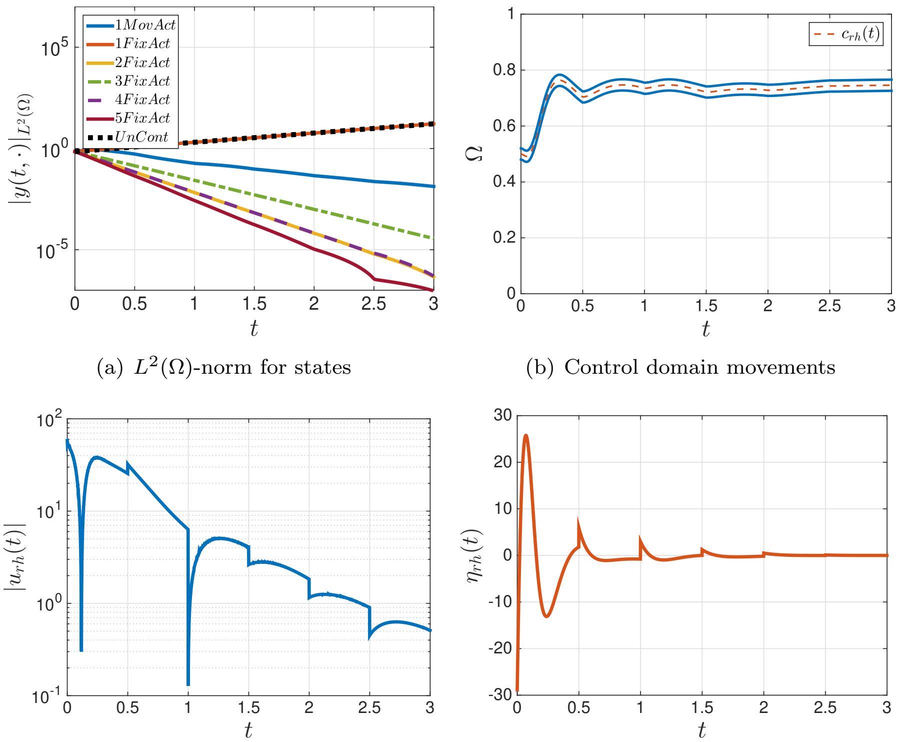
Proof. If $\widetilde{\alpha}_j$ is a nonsimple eigenvalue, we can assume that $\widetilde{\alpha}_j = \widetilde{\alpha}_{j+1}$. Then, in case $(\widetilde{e}_j, \Psi)_{L^2(\Omega)} = 0$ or $(\widetilde{e}_{j+1}, \Psi)_{L^2(\Omega)} = 0$ the proof is finished. It remains to consider the case $(\widetilde{e}_j, \Psi)_{L^2(\Omega} =: \beta_j \neq 0 \neq \beta_{j+1} := (\widetilde{e}_{j+1}, \Psi)_{L^2(\Omega}$. In this case we simply take the eigenfunction $\overline{e}_j := \beta_{j+1}\widetilde{e}_j - \beta_j\widetilde{e}_{j+1}$ which satisfies $(\overline{e}_j, \Psi)_{L^2(\Omega)} = \beta_{j+1}\beta_j - \beta_j\beta_{j+1} = 0$. □

Proof of Proposition 1.2. We take the eigenfunction given by Lemma A.1 as initial condition, $y_0 := \overline{e}_j$, $(\overline{e}_j, \Psi)_{L^2(\Omega} = 0$. Note that the eigenfunctions of $\mathcal{A}$ coincide with those of $\mathcal{A}+a_0\mathbf{1}$. So we can decompose the solution into orthogonal components $y = q + Q$, with $q \in \operatorname{span}\{\overline{e}_j\}$ and $Q \in \{\overline{e}_j\}^\perp$, leading us to

$$\tfrac{\partial}{\partial t}q(t,x) + (\mathcal{A} + a_0\mathbf{1})q(t,x) = 0, \qquad q(0) = \overline{e}_j, \tag{A.2a}$$

$$\tfrac{\partial}{\partial t}Q(t,x) + (\mathcal{A} + a_0\mathbf{1})Q(t,x) = u(t)\Psi, \qquad Q(0) = 0, \tag{A.2b}$$

$$\mathcal{G}q\big|_\Gamma = 0 = \mathcal{G}Q\big|_\Gamma\,. \tag{A.2c}$$



(a) $L^2(\Omega)$-norm for states (b) Control domain movements

(c) Absolute value of magnitude (d) Force

FIGURE 5. Example 5.1: Numerical results for $\beta = 0.01$

Observe that the dynamics of the component q is independent of u, and such component is then given by $q(t,\cdot) = \mathrm{e}^{-(a_0+\widetilde{\alpha}_j)t}\overline{e}_j$, $t > 0$. Now, for the norm of the entire state, and for any magnitude control u we obtain

$$|y(t,\cdot)|^2_{L^2(\Omega)} = |q(t,\cdot)|^2_{L^2(\Omega)} + |Q(t,\cdot)|^2_{L^2(\Omega)} \geq \mathrm{e}^{-2(a_0+\widetilde{\alpha}_j)t} \left|\overline{e}_j\right|^2_{L^2(\Omega)}, \qquad t > 0.$$

Finally, if $-a_0$ is large enough we find that $-a_0 - \widetilde{\alpha}_j > 0$, which implies the divergence $|y(t,\cdot)|^2_{L^2(\Omega)} \to +\infty$, regardless of the control u. □

Proof of Proposition 1.3. Let $j_0 := \min\{j \in \mathbb{N} \mid a_0 + \widetilde{\alpha}_j > 0\}$. If $j_0 = 1$ then the free dynamics is exponentially stable. If $j_0 > 1$, then we consider the dynamics onto the linear span of the first $j_0 - 1$ eigenfunctions

$$\mathcal{E}_{j_0-1} := \operatorname{span}\{\widetilde{e}_j \mid 1 \leq j \leq j_0 - 1\}, \qquad q(t) := P_{\mathcal{E}_{j_0-1}} y(t) \in \mathcal{E}_{j_0-1}$$

where $P_{\mathcal{E}_{j_0-1}} \in \mathcal{L}(L^2(\Omega), \mathcal{E}_{j_0-1})$ denotes the orthogonal projection in $L^2(\Omega)$ onto the subspace $\mathcal{E}_{j_0-1}$. We decompose the system as

$$\tfrac{\partial}{\partial t} q(t,x) + (\mathcal{A} + a_0\mathbf{1})q(t,x) = u(t)P_{\mathcal{E}_{j_0-1}}\Psi, \qquad q(0) = P_{\mathcal{E}_{j_0-1}} y_0, \tag{A.3a}$$

$$\tfrac{\partial}{\partial t} Q(t,x) + (\mathcal{A} + a_0\mathbf{1})Q(t,x) = u(t)P_{\mathcal{E}^\perp_{j_0-1}}\Psi, \qquad Q(0) = P_{\mathcal{E}^\perp_{j_0-1}} y_0, \tag{A.3b}$$

$$\mathcal{G}q\big|_\Gamma = 0 = \mathcal{G}Q\big|_\Gamma\,, \tag{A.3c}$$

with $Q = y - q = P_{\mathcal{E}_{j_0-1}^\perp} y$. Next we prove that the finite dimensional system (A.3a) is null controllable. Writing

$$q = \sum_{k=1}^{j_0-1} q_k \widetilde{e}_k, \qquad \overline{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{j_0-1} \end{bmatrix}$$

we obtain the system

$$\dot{\overline{q}} = A\overline{q} + Bu, \tag{A.4}$$

with $A \in \mathbb{R}^{(j_0-1)\times(j_0-1)}$ and $B \in \mathbb{R}^{(j_0-1)\times 1}$ as follows

$$A = \operatorname{diag}\left(a_0 + \widetilde{\alpha}_1, a_0 + \widetilde{\alpha}_2, \dots, a_0 + \widetilde{\alpha}_{j_0-1}\right) \quad \text{and} \quad B = \begin{bmatrix} (\widetilde{e}_1, \Psi) \\ (\widetilde{e}_2, \Psi) \\ \vdots \\ (\widetilde{e}_{j_0-1}, \Psi) \end{bmatrix}.$$

The matrix A is diagonal with entries $A_{(i,i)} = a_0 + \widetilde{\alpha}_i$.

For any given $T > 0$, we have that system (A.4) is controllable at time T. Indeed, this follows from Kalman rank condition (see, e.g., [28, Sect 1.3, Thm. 1.2]), because we have that

$$\det(\begin{bmatrix} A \mid B \end{bmatrix}) = \left(\bigtimes_{k=1}^{j_0-1} (\widetilde{e}_k, \Psi) \right) \det \mathcal{V}$$

where

$$\begin{bmatrix} A \mid B \end{bmatrix} \coloneqq \begin{bmatrix} B & AB & \dots & A^{j_0-2}B \end{bmatrix}$$

and $\mathcal{V}$ is the Vandermonde matrix whose entries are

$$\mathcal{V}_{(i,j)} = (a_0 + \widetilde{\alpha}_i)^{j-1}.$$

Hence

$$\det(\begin{bmatrix} A \mid B \end{bmatrix}) = \left(\bigtimes_{k=1}^{j_0-1} (\widetilde{e}_k, \Psi) \right) \left(\bigtimes_{1 \le i < j \le j_0-1} (\widetilde{\alpha}_j - \widetilde{\alpha}_i) \right) \neq 0.$$

Therefore, we can choose a control u such that $\overline{q}(T, \cdot) = 0$, which implies $q(T, \cdot) = 0$. Then, we take the concatenated control defined as: $u^c(t)$ if $t \in [0, T]$, and $u^c(t) = 0$ for $t > T$. For time $t \ge T$ we have that $q(t, \cdot) = 0$ and

$$|y(t, \cdot)|_{L^2(\Omega)}^2 = |Q(t, \cdot)|_{L^2(\Omega)}^2 = \mathrm{e}^{-(a_0 + \widetilde{\alpha}_{j_0})(t-T)} \left|Q(T, \cdot)\right|_{L^2(\Omega)}, \qquad t \ge T.$$

Since, by definition of j_0, we have that $a_0 + \widetilde{\alpha}_{j_0} > 0$, it follows that $|y(t, \cdot)|_{L^2(\Omega)}^2$ converges exponentially to zero, as $t \to +\infty$. That is, u^c is a stabilizing (open-loop) control. $\square$

A.2. **Proof of Proposition 3.4.** Let us fix an arbitrary $\delta > 0$ and let $h \in \mathfrak{S}_H$ be in the unit ball of H. Since X is dense in H, we can choose $\overline{h} \in X \setminus \{0\}$ such that $\left|\overline{h} - h\right|_H \le \frac{\delta}{2}$. Now, for $\widetilde{h} \coloneqq \left|\overline{h}\right|_H^{-1} \overline{h} \in \mathfrak{S}_H$ we find

$$\begin{aligned} \left|\widetilde{h} - h\right|_H &\le \left|\left|\overline{h}\right|_H^{-1} \overline{h} - \overline{h}\right|_H + \left|\overline{h} - h\right|_H \le \left|\left|\overline{h}\right|_H^{-1} - 1\right|_{\mathbb{R}} \left|\overline{h}\right|_H + \tfrac{\delta}{2} \\ &= \left|\overline{h}\right|_H^{-1} \left|1 - \left|\overline{h}\right|_H\right|_{\mathbb{R}} \left|\overline{h}\right|_H + \tfrac{\delta}{2} = \left|\left|h\right|_H - \left|\overline{h}\right|_H\right|_{\mathbb{R}} + \tfrac{\delta}{2} \le \left|h - \overline{h}\right|_H + \tfrac{\delta}{2} = \delta. \end{aligned}$$

Hence we can conclude that $X \bigcap \mathfrak{S}_H$ is dense in $\mathfrak{S}_H$. $\square$

A.3. **Proof of Proposition 3.5.** We start by defining

$$\bar{t}_j \coloneqq \max\{\tau_j, \sigma_j\}, \qquad \underline{t}_j \coloneqq \min\{\tau_j, \sigma_j\}, \qquad 0 \le j \le K,$$

and by writing, with $g \coloneqq f_\tau - f_\sigma$,

$$|f_\tau - f_\sigma|^2_{L^2((a,b),X)} = \sum_{j=1}^{K} \int_{\bar{t}_{j-1}}^{\bar{t}_j} |f_\tau(t) - f_\sigma(t)|^2_X \,\mathrm{d}t = \sum_{j=1}^{K} \int_{\bar{t}_{j-1}}^{\bar{t}_j} |g(t)|^2_X \,\mathrm{d}t. \tag{A.5}$$

We proceed by Induction. Firstly, we find that

$$\int_a^{\bar{t}_1} |g(t)|^2_X \,\mathrm{d}t = \int_{\bar{t}_0}^{\bar{t}_1} |g(t)|^2_X \,\mathrm{d}t = \int_{\underline{t}_1}^{\bar{t}_1} |g(t)|^2_X \,\mathrm{d}t \le \mathcal{R}\mathcal{X}^2, \tag{A.6}$$

where in the last inequality we used the fact that $g(t) = f_\tau(t) - f_\sigma(t) = \phi_1 - \phi_1 = 0$ for $t \in [\bar{t}_0, \underline{t}_1)$.

Next, we assume that for a given $i \in \{1, 2, \dots, K-1\}$ we have

$$\int_a^{\bar{t}_i} |g(t)|^2_X \,\mathrm{d}t \le i\mathcal{R}\mathcal{X}^2. \tag{$\mathfrak{H}$}$$

Then we obtain

$$\int_a^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t \le i\mathcal{R}\mathcal{X}^2 + \int_{\bar{t}_i}^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t,$$

which implies that

$$\int_a^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t \le i\mathcal{R}\mathcal{X}^2 + \int_{\bar{t}_i}^{\underline{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t + \int_{\underline{t}_{i+1}}^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t, \quad \text{if} \quad \bar{t}_i < \underline{t}_{i+1} \le \bar{t}_{i+1},$$

and

$$\int_a^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t \le i\mathcal{R}\mathcal{X}^2 + \int_{\underline{t}_{i+1}}^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t, \quad \text{if} \quad \underline{t}_{i+1} \le \bar{t}_i \le \bar{t}_{i+1},$$

Observe that, if $\bar{t}_i < \underline{t}_{i+1}$, then $g(t) = f_\tau(t) - f_\sigma(t) = \phi_{i+1} - \phi_{i+1} = 0$, for $t \in (\bar{t}_i, \underline{t}_{i+1}) \subseteq (\tau_i, \tau_{i+1}) \bigcap (\sigma_i, \sigma_{i+1})$. Therefore, in either case we have

$$\int_a^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t \le i\mathcal{R}\mathcal{X}^2 + \int_{\underline{t}_{i+1}}^{\bar{t}_{i+1}} |g(t)|^2_X \,\mathrm{d}t \le i\mathcal{R}\mathcal{X}^2 + \mathcal{R}\mathcal{X}^2 = (i+1)\mathcal{R}\mathcal{X}^2. \tag{$\mathfrak{T}$}$$

Hence, assumption ($\mathfrak{H}$) implies ($\mathfrak{T}$), which together with (A.6) imply, by Induction,

$$\int_a^{\bar{t}_j} |g(t)|^2_X \,\mathrm{d}t \le j\mathcal{R}\mathcal{X}^2, \quad \text{for all} \quad j \in \{1, 2, \dots, K\}. \tag{A.7}$$

In particular, since $\bar{t}_K = b$, for $j = K$ we obtain $|f_\tau - f_\sigma|_{L^2((a,b),X)} \le K^{\frac{1}{2}}\mathcal{R}^{\frac{1}{2}}\mathcal{X}$. $\square$

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