Hölder continuity for non-autonomous parabolic equations

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Abstract

Hölder continuity of the solution to linear non-autonomous parabolic equations is proven with special emphasis on the spatio-temporal regularity assumptions on the coefficients in the leading term of the differential equation. These results can be used to establish new regularity properties for the solutions of optimal control problems where the controls act on lower dimensional manifolds of the physical system.

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1. Introduction and main results

The aim of this paper is to prove Hölder regularity for the solutions in space and time of non-autonomous parabolic equations

$$u'(t) + A_t u(t) = f(t), \quad u(0) = 0, \tag{1}$$

the right hand side f being a function from a space $L^q(J; X)$ and the operators $A(t)$ complemented by mixed boundary conditions.

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In case of pure – even inhomogeneous – Dirichlet conditions this is classical, see the fundamental work [24, Ch. III.10]. In [28] we succeeded in proving this also for the case of mixed boundary conditions – but under comparably severe conditions on the geometry.
Here we go an entirely different way: we restrict the right hand sides – spatially – to belong to either $X = L^2$ or $X = W^{-1,q}$ with a q larger than the space dimension and demand some continuity properties for the coefficients in time. In space no additional conditions aside boundedness and uniform ellipticity are needed. This continuity property in time can be relaxed in the $W^{-1,q}$ case to *piecewise* continuity, see the last point in the Concluding Remarks.
The two approaches are rather different in both philosophy and in the required technical means. Let us explain this in some detail.
First, it is clear that $L^2(\Omega)$ continuously embeds in $W^{-1,q}(\Omega)$ (if q is not too large). In fact, $W^{-1,q}(\Omega)$ contains many objects, not contained in $L^2(\Omega)$, but which are of great interest in applications. In particular, this applies to measures which are concentrated on singular sets, i.e. on sets of Lebesgue measure zero. This is carried out in great detail in [23] and [31]. The disadvantage is here – besides the presupposed continuity of the coefficient function in time – that one must assume that the domain of the second order divergence operator, when considered on $W^{-1,q}_{\mathfrak{D}}(\Omega)$, is the best possible, namely $W^{1,q}_{\mathfrak{D}}(\Omega)$ – for some q larger than the space dimension d. It turns out that this is, under very general assumptions on the domain, no real restriction in the case of two spatial dimensions, see Prop. 2.1 below, see also [19]. In three space dimensions it is generically false that the domain of the second order operator, when considered on $W^{-1,q}_{\mathfrak{D}}(\Omega)$, equals $W^{1,q}_{\mathfrak{D}}(\Omega)$ for a $q \geq 3$. This is the reason for considering in three space dimensions right hand sides which are spatially L^2. Here *nothing* is supposed on the domains of the elliptic operators; in particular they can be quite different for different time points, see the discussion in the Concluding Remarks. Investing here Hölder continuity – in time – for the coefficient function, one is enabled to use a theorem of Haak/Ouhabaz [17] to obtain maximal parabolic regularity over $L^2(\Omega)$. Then one is in the position to employ the continuous embeddings

$$dom(A_t) \hookrightarrow dom_{W^{-1,q}_{\mathfrak{D}}}(-\nabla \cdot \mu(t,\cdot)\nabla) \hookrightarrow C^\alpha(\Omega) \tag{2}$$

which hold uniformly in t and afterwards to use a well-known embedding of the space of maximal parabolic regularity in a Hölder space in time with values in a suitable interpolation space between $L^2(\Omega)$ and $C^\alpha(\Omega)$, see Prop. 2.3. In this spirit, everything comes down to assure the second embedding in (2). This is an elliptic Hölder result in case of mixed boundary conditions, which is taken from [11], see also [19] for a simplified version in dimensions $d = 2, 3, 4$. This approach demands the restriction to *real* coefficients. Indeed, in the case of complex coefficients one generally cannot expect Hölder

continuity. To the best of our knowledge, the geometric conditions in both these papers, assuring elliptic Hölder continuity, are the most general ones known in case of mixed boundary conditions, with the predecessors [6], [22], [29], [13], [14], [15], [25], [18]. Moreover, the paper of Stampacchia [32] should not be ignored, where under very general conditions also elliptic Hölder continuity is proved. Unfortunately, the conditions of his main theorem are very implicit and extremely difficult to control in examples if the geometry of the underlying domain becomes complicated.
The original motivation of this work comes from optimal control. In [23] we investigated maximal parabolic regularity in the H^s scale with right hand sides including measures measures which live on lower dimensional subsets. Resting on these results, it was shown in [23] that related problems from optimal control admit solutions which possess as much regularity as one can expect in this context. However, in three spatial dimensions, we had to impose an elliptic regularity result which is a severe restriction in practice, see [23, Assumption 5.11 (b)]. The Hölder result for non-autonomous parabolic equations, proved here, allows to avoid this regularity assumption for the corresponding elliptic operators and pose only a – very weak – geometric condition on the interface between the Dirichlet boundary part $\mathfrak{D}$ and its complement instead.

Throughout this paper we denote by $d \in \{2, 3\}$ the dimension of the *bounded* domain Ω and by $\mathcal{H}$ the $(d-1)$-dimensional Hausdorff measure.

Finally, for two Banach spaces X, Y, with Y continuously embedded into X, we denote by $(X, Y)_{\theta,r}$ the usual real interpolation space and by $[X, Y]_\theta$ the corresponding complex interpolation space (see [33, Ch. I]).

Below $B(x, r)$ denotes the ball in $\mathbb{R}^d$ with center x and radius r. Let us introduce the following

Assumption 1.1. (a) Ω is a Lipschitz domain (see [16, Def. 1.2.1.2]or [26, Ch. 1.1.9 Def. 3]), in detail: for $x \in \partial\Omega$, there is an open neighbourhood U_x of x and a bi-Lipschitz map Φ_x from U_x onto the cube $K :=]-1, 1[^d$, such that the following three conditions are satisfied:

$$\Phi_x(x) = 0,$$
$$\Phi_x(U_x \cap \Omega) = \{x \in K : x_d < 0\},$$
$$\Phi_x(U_x \cap \partial\Omega) = \{x \in K : x_d = 0\}.$$

(b) $\mathfrak{D}$ is a closed subset of $\partial\Omega$ which satisfies the *Ahlfors–David condition*, i.e. there are two constants $c_\bullet, c^\bullet$ such that

$$c_\bullet \, r^{d-1} \le \mathcal{H}(\mathfrak{D} \cap B(x; r)) \le c^\bullet \, r^{d-1}, \quad x \in \mathfrak{D},\ r \in]0, 1].$$

In case of $d = 3$ we have to introduce one more geometric condition; this, rather intriguing, concerns the interface between the Dirichlet boundary part $\mathfrak{D}$ and the Neumann boundary part $N = \partial\Omega \setminus \mathfrak{D}$ in the boundary of Ω:

Assumption 1.2. In case $d = 3$, supposing that Assu. 1.1 (a) holds, we demand, in that terminology, the existence of two constants $c_0 \in]0,1[$ and $c_1 > 0$ such that for every point $x \in E := \mathfrak{D} \cap \overline{N}$, every $y \in \mathbb{R}^{d-1}$ such that $(y,0) \in \Phi_x(E \cap U_x)$ and every $s \in (0,1]$ it holds

$$\lambda_2\Big(\big\{z \in \tilde{B}_s(y) \colon \operatorname{dist}\big(z, \Phi_x(N \cap U_x)\big) > c_0 s\big\}\Big) \geq c_1 s^2. \tag{3}$$

Here, $\tilde{B}_r(y) \subset \{y \in \mathbb{R}^3 : y_3 = 0\}$, denotes the open ball of radius r in $\mathbb{R}^2$ with its center at $y \in \mathbb{R}^2$, and in the distance function we tacitly consider $\phi_x(N \cap V_x) \subset [z_3 = 0]$ as a subset of $\mathbb{R}^2$ in the obvious manner. λ_2 is the two-dimensional Lebesgue measure on $\{y \in \mathbb{R}^3 : y_3 = 0\}$.

Condition (3) seems a bit strange at first. However, it turned out recently that this – or quite similar conditions in character – are adequate for the treatment of mixed boundary value problems, see the detailed discussion in [5].

Definition 1.3. Assume $q \in]1,\infty[$. For $\mathfrak{D}$ a closed subset of $\partial\Omega$ let us define $C^\infty_{\mathfrak{D}}(\Omega)$ by

$$C^\infty_{\mathfrak{D}}(\Omega) := \{u|_\Omega : u \in C^\infty_0(\mathbb{R}^d),\ \operatorname{supp} u \cap \mathfrak{D} = \emptyset\}$$

and $W^{1,q}_{\mathfrak{D}}(\Omega)$ as the closure of $C^\infty_{\mathfrak{D}}(\Omega)$ in the Sobolev space $W^{1,q}(\Omega)$. Moreover, $W^{-1,q}_{\mathfrak{D}}(\Omega)$ stands for the space of continuous antilinear forms on $W^{1,q'}_{\mathfrak{D}}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Definition 1.4. Assume $q \in]1,\infty[$. Let $\mu \in L^\infty(\Omega;\mathbb{C}^{d\times d})$. Then we define the operator

$$-\nabla\cdot\mu\nabla + 1 : W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{-1,q}_{\mathfrak{D}}(\Omega) \tag{4}$$

by

$$\langle -\nabla\cdot\mu\nabla\psi + \psi, \varphi\rangle = \int_\Omega \mu\nabla\psi\cdot\nabla\overline{\varphi} + \psi\overline{\varphi}, \quad \psi \in W^{1,q}_{\mathfrak{D}}(\Omega),\ \varphi \in W^{1,q'}_{\mathfrak{D}}(\Omega). \tag{5}$$

Remark 1.5.
1. It is clear that the operators are consistent for different q's. Therefore we use for all of them the symbol introduced above, and the corresponding q will be made clear from the context.
2. If μ satisfies the usual (strong) ellipticity condition

$$\Re(\mu(x)\xi,\xi)_{\mathbb{C}^d} \geq \mu_\bullet |\xi|^2, \quad \xi \in \mathbb{C}^d \tag{6}$$

for a constant $\mu_\bullet > 0$ and uniformly for almost all $x \in \Omega$, then (4) is a topological isomorphism by Lax/Milgram for $q = 2$.
3. The mapping

$$L^\infty(\Omega) \ni \mu \mapsto -\nabla\cdot\mu\nabla + 1 \in \mathcal{L}(W^{1,q}_{\mathfrak{D}}(\Omega); W^{-1,q}_{\mathfrak{D}}(\Omega)) \tag{7}$$

is, by Hölder's inequality, of norm not larger than 1 and, hence, continuous.

4. Due to the interpolation properties of the spaces $W^{1,q}_{\mathfrak{D}}(\Omega)/W^{-1,q}_{\mathfrak{D}}(\Omega)$ (see [19, Ch. 3]) the set of numbers for which (4) is a topological isomorphism, forms an interval, in the sequel denoted by $\mathcal{I}_\mu$. A priori it is not clear that this interval contains more points than only the number 2. But we see later (cf. Prop. 2.1) that, under very general assumptions, it is an *open* one.

Throughout the rest of this paper let $T > 0$ and set $J =]0, T[$. Let us start by recalling the following (standard) definition.

Definition 1.6. If X is a Banach space and $r \in]1, \infty[$, then we denote by $L^r(J; X)$ the space of X-valued functions f on J which are Bochner-measurable and for which $\int_J \|f(t)\|^r \, dt$ is finite. We define $W^{1,r}(J; X) := \{u \in L^r(J; X) : \frac{\partial u}{\partial t} \in L^r(J; X)\}$, where $\frac{\partial u}{\partial t}$ is to be understood as the time derivative of u in the sense of X-valued distributions (cf. [1, Section III.1]). Moreover, we introduce the subspace

$$W^{1,r}_0(J; X) := \{u \in W^{1,r}(J; X) : u(0) = 0\}.$$

We equip this subspace with the norm $u \mapsto \|\frac{\partial u}{\partial t}\|_{L^r(J;X)}$.

From now on we fix a continuous function $\overline{J} \ni t \to L^\infty(\Omega; \mathbb{C}^{d\times d})$ and denote it $\hat{\mu}$. Moreover, we assume that the functions $\hat{\mu}(t, \cdot)$ admit a uniform – in t – ellipticity constant $\mu_\bullet$. Let us now formulate the main results of this work.

Theorem 1.7. *Suppose $d = 2$ and adopt Assu. 1.1.*
Then, there is a $q_0 > 2$ such that, for $q \in [2, q_0]$, $r \in [2, \infty[$ and $f \in L^r(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$, the solution u of

$$u'(t) - \nabla \cdot \hat{\mu}(t, \cdot)\nabla u(t) = f(t), \text{ a.e. on } J, \; u(0) = 0 \tag{8}$$

exists and is unique. It belongs to the class $W^{1,r}(J; W^{-1,q}_{\mathfrak{D}}(\Omega)) \cap L^r(J; W^{1,q}_{\mathfrak{D}}(\Omega))$, and the assignment

$$f \in L^r(J; W^{-1,q}_{\mathfrak{D}}(\Omega)) \mapsto u \in W^{1,r}_0(J; W^{-1,q}_{\mathfrak{D}}(\Omega)) \cap L^r(J; W^{1,q}_{\mathfrak{D}}(\Omega)) \tag{9}$$

is continuous.

Corollary 1.8. *Adopt the assumptions of Theorem 1.7 and suppose $q > 2$. Then for r sufficiently large, and $f \in L^r(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$, the solution u of (8) belongs to $C^\beta(J; C^\gamma(\Omega))$ for some $\beta, \gamma > 0$.*
In this case, the mapping $f \in L^r(J; W^{-1,q}_{\mathfrak{D}}(\Omega)) \mapsto u \in C^\beta(J; C^\gamma(\Omega))$ is continuous.

We turn to dimension three next.

Theorem 1.9. *Suppose $d = 3$ and adopt Assumption 1.1 (a). Moreover, assume that the – time dependent – coefficient function $\hat{\mu}$ is* real *and satisfies*

$$\|\hat{\mu}(t_1,\cdot) - \hat{\mu}(t_2,\cdot)\|_{L^\infty(\Omega;\mathcal{L}(C^d))} \le c|s-t|^\alpha, \quad t_1, t_2 \in J \tag{10}$$

for some $\alpha > \frac{1}{2}$. Let A_t denote the closed, sectorial operator which is induced on $L^2(\Omega)$ by the quadratic form

$$W^{1,2}_{\mathfrak{D}}(\Omega) \ni \psi \mapsto \int_\Omega \hat{\mu}(t,\cdot)\,\nabla\psi \cdot \nabla\overline{\psi} + \psi\,\overline{\psi}. \tag{11}$$

i) *Then, for $r \in [2,\infty[$ and $f \in L^r(J; L^2(\Omega))$ the solution u of*

$$u' + A_t u = f, \quad u(0) = 0 \tag{12}$$

exists and is unique. It belongs to the class $W^{1,r}_0(J; L^2(\Omega))$, and the assignment $f \in L^r(J; L^2(\Omega)) \mapsto u \in W^{1,r}_0(J; L^2(\Omega))$ is continuous.

ii) *Let now also Assumption 1.2 be in power. If r is sufficiently large, then u even belongs to $C^\beta(J; C^\gamma(\Omega))$ for some $\beta, \gamma > 0$. In this case, the assignment $f \in L^r(J; L^2(\Omega)) \mapsto u \in C^\beta(J; C^\gamma(\Omega))$ is continuous.*

Remark 1.10. We are aware that the condition (10) restricts the class of admissible coefficients in comparison to only 'measurable in t' considerably. Prototypically, in the latter a (scalar) coefficient function $\hat{\mu}$ is allowed which is identically 1 up to a time point $t_0 \in J$ and from t_0 identical 1 on a subdomain $\Omega_\bullet$ and 2 on $\Omega \setminus \Omega_\bullet$. Obviously, such $\hat{\mu}$ does not satisfy (10). But the following is allowed: Consider a ball $\Omega \subset \mathbb{R}^2$ around 0. Let the coefficient function $\hat{\mu}(t,\cdot)$ be defined as follows: for $t \in]0,1]$, it is the identity matrix, and for $t > 1$ and $x \in \Omega$ it is given by,

$$\hat{\mu}(t,x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} & \text{if} \quad x_1, x_2 \ge 0 \\ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} & \textit{else}. \end{cases} \tag{13}$$

2. The proofs

Central for the proof of Thm. 1.7 is the following result.

Proposition 2.1. *Let μ be a bounded, measurable, strongly elliptic coefficient function on Ω, satisfying (6) for some positive constant $\mu_\bullet$.*
Then, under Assumption 1.1, the set of q's for which

$$-\nabla \cdot \mu\nabla + I : W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{-1,q}_{\mathfrak{D}}(\Omega) \tag{14}$$

is a surjection and, hence, a topological isomorphism, is an open *interval $\mathcal{I}_\mu =]2-\delta, 2+\varepsilon[$ with $\delta > 0, \varepsilon > 0$.*

For every set $\mathcal{C}$ of coefficient functions μ which admit a uniform L^∞ bound and also a uniform ellipticity constant $\mu_\bullet$, the numbers δ and ε may be taken uniformly with respect to all coefficient functions from $\mathcal{C}$.
In addition,

$$\sup_{\mu \in \mathcal{C}} \|(-\nabla \cdot \mu \nabla + I)^{-1}\|_{\mathcal{L}(W^{-1,q}_{\mathfrak{D}};W^{1,q}_{\mathfrak{D}})} < \infty$$

for all q from the corresponding uniform interval $\mathcal{I}_\mu$.

Proof. see [19, Ch. 5] . □

We start with the proof of Thm. 1.7, which will rest on the famous Prüss/Schnaubelt result [30]. In order to make this applicable we need some preliniaries.

For $q > 2$, let B_q be the part of $-\nabla \cdot \mu \nabla : W^{1,2}_{\mathfrak{D}}(\Omega) \to W^{-1,2}_{\mathfrak{D}}(\Omega)$ in $W^{-1,q}_{\mathfrak{D}}(\Omega)$. If μ is elliptic, then $B_q + 1$ is injective. It is also a surjection at least for $q \in [2,6]$, thanks to the embedding $W^{-1,q}_{\mathfrak{D}}(\Omega) \hookrightarrow W^{-1,2}_{\mathfrak{D}}(\Omega)$ and, consequently, $\mathrm{dom}(B_q) \hookrightarrow W^{1,2}_{\mathfrak{D}}(\Omega) \hookrightarrow W^{-1,q}_{\mathfrak{D}}(\Omega)$.

Proposition 2.2. *(see [12, Thm. 7.7]) Let μ be an elliptic $d \times d$ matrix and assume that $q \in [2,4]$. Then B_q satisfies maximal parabolic regularity on $W^{-1,q}_{\mathfrak{D}}(\Omega)$, i.e. for every $r \in]1,\infty[$ and $f \in L^r(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$, there is a unique $u \in W^{1,r}_0(J; W^{-1,q}_{\mathfrak{D}}(\Omega) \cap L^r(J, dom(B_q))$ satisfying*

$$u' + B_q(t)u = f, \quad u(0) = 0. \tag{15}$$

Consequently the claim of Thm. 1.7 holds for the function $\hat{\mu}$ which is constant for $t \in J$ with value μ.

Proof of Theorem 1.7. First, due to Prop. 2.1 there is a $q_0 > 2$ such that, for all $t \in \overline{J}$ and all $q \in [2, q_0]$ the domain of $B_q(t)$ is $W^{1,q}_{\mathfrak{D}}(\Omega)$. Thus by Rem. 1.5, the operator function

$$J \ni t \mapsto B_q(t) = -\nabla \cdot \hat{\mu}(t, \cdot)\nabla \in \mathcal{L}(W^{1,q}_{\mathfrak{D}}(\Omega); W^{-1,q}_{\mathfrak{D}}(\Omega))$$

is continuous due to the continuity of $\hat{\mu}$ in time. So, taking into account Prop. 2.2 the proof is implied by the Prüss/Schnaubelt theorem, see [2, Thm. 7.1] or [3, Thm. 2.7] for modern versions of this. □

We next turn to the verification of Cor. 1.8 and start by recalling some abstract embedding properties for spaces of maximal parabolic regularity.

Proposition 2.3. *Let X, Y be Banach spaces and assume that Y is continuously embedded into X.*

i) *If $r \in]1,\infty[$, then*

$$W^{1,r}(J; X) \cap L^r(J; Y) \hookrightarrow C(\overline{J}; (X,Y)_{1-\frac{1}{r},r}), \tag{16}$$

(see [1, Ch. III Thm. 4.10].

ii) *If* $r \in]1,\infty[$ *and* $\theta \in]0, 1 - \frac{1}{r}[$, *then*

$$W^{1,r}(J;X) \cap L^r(J;Y) \hookrightarrow C^\beta(J;(X,Y)_{\theta,1}), \tag{17}$$

where $\beta = 1 - \frac{1}{r} - \theta$, *(see [8, Lemma 2.11]).*

Proposition 2.4. *$W^{1,q}_{\mathfrak{D}}(\Omega)$ admits a linear extension operator $\mathcal{E}$ to $W^{1,q}(\Omega)(\mathbb{R}^d)$. By continuity this defines a continuous extension operator $\mathcal{E} : L^p(\Omega) \to L^p(\mathbb{R}^d)$, which is universal in $p, q \in]1,\infty[$.*

Proof. see [10, Thm. 3.4] or [4, Lemma 3.2]. □

Lemma 2.5. *Adopt Assumption 1.1. Let $d \in \{2, 3\}$, $q \in]d, 4[$ and $\theta > \frac{1}{2}(1 + \frac{d}{q})$. Then*

$$(W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{\theta,1} \hookrightarrow C^\alpha(\Omega), \tag{18}$$

with $\alpha = 2\theta - 1 - \frac{d}{q}$.

Proof. Consider the (negative, shifted) Laplacian $-\Delta + 1 : W^{1,2}_{\mathfrak{D}}(\Omega) \to W^{-1,2}_{\mathfrak{D}}(\Omega)$ and its part $-\Delta_q + 1$ in $W^{-1,q}_{\mathfrak{D}}(\Omega)$. Evidently, the injectivity is maintained. Moreover, $-\Delta_q + 1$ is also surjective, since

$$\mathrm{dom}(\Delta_q) \subset \mathrm{dom}(\Delta) = W^{1,2}_{\mathfrak{D}}(\Omega) \hookrightarrow W^{-1,q}_{\mathfrak{D}}(\Omega).$$

Here we use that the space dimension does not exceed 3 and $q \in [2, 4[$. We write, according to the re-iteration theorem

$$(W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{\theta,1} = \big((W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{\frac{1}{2},1} W^{1,q}_{\mathfrak{D}}(\Omega)\big)_{2\theta-1,1}. \tag{19}$$

According to (7), one knows the embedding $W^{1,q}_{\mathfrak{D}}(\Omega) \hookrightarrow \mathrm{dom}(\Delta_q)$ – the latter equipped with the graph norm. So the expression in (19) continuously embeds into

$$\big((W^{-1,q}_{\mathfrak{D}}(\Omega), \mathrm{dom}(\Delta_q))_{\frac{1}{2},1}, W^{1,q}_{\mathfrak{D}}(\Omega)\big))_{\theta,1} \tag{20}$$

Thanks to [4, Thm. 11.5], $-\Delta_q + 1$ is a *positive* operator in the sense of Triebel, see [33, Ch. 1.14.1]. Hence, we may employ [33, Thm. 1.15.2], which tells us that $(W^{-1,q}_{\mathfrak{D}}(\Omega), \mathrm{dom}(\Delta_q))_{\frac{1}{2},1} \hookrightarrow \mathrm{dom}\big((-\Delta_q + 1)^{1/2}\big)$ But [4, Thm. 5.1] gives $\mathrm{dom}\big((-\Delta_q + 1)^{1/2}\big) = L^q(\Omega)$ and, consequently, (20) embeds into

$$(L^q(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{2\theta-1,1}. \tag{21}$$

It remains to show that this space continuously embeds into $C^\alpha(\Omega)$. First, the condition on q allows, according to [33, Thm. 2.8.1], the estimate,

$$\|u\|_{C^\alpha(\Omega)} \le \|\mathcal{E}u\|_{C^\alpha(\mathbb{R}^d)} \le c\,\|\mathcal{E}u\|_{H^{2\theta-1}_q(\mathbb{R}^d)}, \tag{22}$$

$H^{2\theta-1}_q(\mathbb{R}^d)$ being the corresponding Bessel potential space, see also [33, Ch. 2.3.1/Ch. 2.3.3], and $\mathcal{E}$ as in Proposition 2.4. But $H^{2\theta-1}_q(\mathbb{R}^d)$ is identical with the interpolation space $[L^q(\mathbb{R}^d), W^{1,q}(\mathbb{R}^d)]_{2\theta-1} = [L^q(\mathbb{R}^d), H^1_q(\mathbb{R}^d)]_{2\theta-1}$ (see [33, Ch. 2.4.2]. So the right hand side of (22) is not larger than

$$c\,\|\mathcal{E}u\|^{2(1-\theta)}_{L^q(\mathbb{R}^d)}\,\|\mathcal{E}u\|^{2\theta-1}_{W^{1,q}(\mathbb{R}^d)} \le c\|u\|^{2(1-\theta)}_{L^q(\Omega)}\,\|u\|^{2\theta-1}_{W^{1,q}_{\mathfrak{D}}(\Omega)}.$$

This, combined with (22), shows that indeed the space in (21) continously embeds into $C^\alpha(\Omega)$ by a universal interpolation principle, see [33, 1.10.1]. □

Cor. 1.8 is now a straightforward consequence: one diminishes q so that it belongs to the set $\cap_{t\in J}\mathcal{I}_{\hat{\mu}(t,\cdot)}$, see Prop. 2.1. Afterwards one applies Theorem 1.7, Proposition 2.3, and Lemma 2.5.

We next turn to the proof of Theorem 1.9. It is obtained by a combination of elliptic Hölder regularity theory and the above mentioned result on non-autonomous parabolic regularity, proved in [17]. We establish some preliminaries, beginning with an elliptic Hölder result.

Proposition 2.6. *(see [11], see also [20] for a simplified version for dimensions up to* 4*) Let $d = 3$ and suppose that Ω and $\mathfrak{D}$ satisfy Assumption 1.1 (a) and Assumption 1.2. Moreover, assume that $\mu \in L^\infty(\Omega;\mathbb{R}^{3\times 3})$ and that it is elliptic.*
Then, for $q > 3$, there is $\alpha = \alpha(q) > 0$ such that for every $f \in W^{-1,q}_{\mathfrak{D}}(\Omega)$ the equation

$$(B_q + 1)v = f \tag{23}$$

has a unique solution $v \in W^{1,2}_{\mathfrak{D}}(\Omega)$ that belongs to the Hölder space $C^\alpha(\Omega)$. Moreover, the mapping $W^{-1,q}_{\mathfrak{D}}(\Omega) \ni f \mapsto v \in C^\alpha(\Omega)$ is continuous and its norm is uniform with respect to the L^∞-bound and ellipticity constant $\mu_\bullet$ of μ.

The reader should carefully notice that here – and in the sequel – *no explicit* knowledge on the domain of B_q is invested.

Lemma 2.7. *$C^\alpha(\Omega)$ admits a linear extension operator $\mathcal{F}$ to $C^\alpha(\mathbb{R}^d)$, this providing – by continuity – a continuous extension operator $\mathcal{F} : L^p(\Omega) \to L^p(\mathbb{R}^d)$. The extended functions have their supports in a sufficiently large ball. $\mathcal{F}$ is universal in $\alpha \in]0,1[$ and $p \in]1,\infty[$.*

Proof. Although $\mathcal{F}$ is essentially $\mathcal{E}$ from Proposition 2.4, we give a separate proof for the convenience of the reader. Based on the Lipschitz property of Ω, let U_x be an open neighbourhood of $x \subset \partial\Omega$ and let Φ_x be bi-Lipschitz map from U_x onto the cube $K := \,]-1,1[^d$, as supposed to exist in Assumption 1.1. Choose a finite subcovering $U_{x_1},\dots,U_{x_n}$ of $\partial\Omega$. Let $\Pi \subset \Omega$ be open, with positive distance to $\partial\Omega$, such that $\Pi \cup \left(\cup_j U_{x_j}\right)$ forms a covering of $\overline{\Omega}$. Let $\zeta,\eta_1,\dots,\eta_n$ be a partition of unity over $\overline{\Omega}$ such that $\operatorname{supp}\zeta \subset \Pi$ and $\operatorname{supp}\eta_j \subset U_{x_j}$. Let now $\psi \in C^\alpha(\Omega)$ be arbitrary. Then, by construction, the function $\zeta\psi|_\Pi$ is identically zero on a neighbourhood of $\partial\Omega$. So we may extend it by 0 outside this set to a function $\psi_\bullet$ on the whole $\mathbb{R}^d$– while maintaining its C^α-norm. Let us now construct extensions for the functions $\eta_j\psi$. We consider the restriction φ_j of $\eta_j\psi$ to $U_{x_j} \cap \Omega$. Then we define, via the transformation Φ_{x_j}, a function $\check{\varphi}_j$ on the cube K as follows: we put

$$\check{\varphi}_j(x) = \varphi(\Phi_{x_j}^{-1}(x)), \quad \text{if} \quad x_d \le 0$$

and $\check{\varphi}_j(x_1, \ldots, x_{d-1}, -x_d)$ if $x_d > 0$. It is clear that then $\check{\varphi}_j \in C^\alpha(K)$ and its norm is controlled by that of ψ, the C^1 norm of η_j and the Lipschitz constant of $\Phi_{x_j}^{-1}$. Moreover, from the support property of η_j it follows that $\check{\varphi}_j$ has its support in K. We transform $\check{\varphi}_j$ back via Φ_{x_j} to a function $\widehat{\varphi_j}$ on U_{x_j}. Evidently, $\widehat{\varphi_j}$ has its support within U_{x_j} and it belongs to $C^\alpha(U_{x_j})$. So we may extend $\widehat{\varphi_j}$ by zero outside U_{x_j} to a function $\widetilde{\varphi_j}$ on the whole $\mathbb{R}^d$. It is clear by construction that $\widetilde{\varphi_j}$ coincides on Ω with $\eta_j\psi$. So $\Psi := \psi_\bullet + \sum_j \widetilde{\varphi_j}$ is an extension of ψ to $\mathbb{R}^d$ which also belongs to $C^\alpha(\mathbb{R}^d)$ and has compact support. Finally, it is not difficult to see that the C^α-norm of Ψ may be estimated in terms of the C^α-norm of ψ.
The $L^p(\Omega)$ statement follows analogously. □

Next we quote, for the convenience of the reader, the pioneering result on *non-autonomous* parabolic regularity result by Haak/Ouhabaz (see [17]).

Proposition 2.8. *Let $V \hookrightarrow Y \hookrightarrow V^*$ be a Gelfand triple of Hilbert spaces with dense embeddings. Assume that we are given, for each $t \in J$, a continuous, coercive sesquilinear form $\mathfrak{s}_t$ on V, with a common coercivity constant for $t \in J$. Moreover, suppose that*

$$\sup_{\|\varphi\|_V = \|\psi\|_V = 1} \big|\mathfrak{s}_t(\varphi,\psi) - \mathfrak{s}_s(\varphi,\psi)\big| \le c|s-t|^\alpha, \quad s,t \in J \tag{24}$$

for some $\alpha > \frac{1}{2}$. Let $\mathcal{A}_t$ be the sectorial operator which is induced by $\mathfrak{s}_t$ on Y and $r \in]1,\infty[$.
Then, for every $f \in L^r(J,Y) \hookrightarrow L^r(J;V^*)$ the solution of the equation

$$u'(t) + \mathcal{A}_t u(t) = f(t), \quad u(0) = 0 \tag{25}$$

exists, it is unique and satisfies $u \in W^{1,r}(J,Y)$, and, consequently,

$$J \ni t \mapsto \mathcal{A}_t\big(u(t)\big) \in L^r(J,Y). \tag{26}$$

Moreover, the mapping $L^r(J;Y) \ni f \mapsto u \in W^{1,r}_0(J,Y)$ is continuous.

Remark 2.9. A key ingredient of this theorem lies in the fact that the operators $\mathcal{A}_t$ need not – and will generally not – have the same domain as required in the main result of [30]. A striking example where the condition of the theorem is fulfilled but, $\mathrm{dom}(\mathcal{A}_s)$ is not equal to $\mathrm{dom}(\mathcal{A}_t)$ for $s \ne t$ is given in Remark 1.10. This depends on the fact that, for increasing t, the singularities in $0 \in \mathbb{R}^2$ of certain elements of $\mathrm{dom}(\mathcal{A}_t)$ become worse and worse, see [9, Ch. 4].

Having collected all essential preparations, we now present the proof of Theorem 1.9: i) follows straight forward from Prop. 2.8, there taking $V = W^{1,2}_{\mathfrak{D}}(\Omega)$, $Y = L^2(\Omega)$ and $\mathcal{A}_t = A_t$. In order to achieve Hölder regulartiy in ii) in space and time it is first necessary to 'convert' the measurability of the functions $u \in C(J; L^2(\Omega))$ and $J \ni t \mapsto A_t\big(u(t)\big)$ into the measurability of the function $J \ni t \mapsto u(t) \in X$, the Banach space X being suitably chosen. For doing so, we need the subsequent

Lemma 2.10. *Let $v : J \to L^2(\mathbb{R}^d)$ be a measurable mapping. Assume that the reflexive, separable space Banach space X of functions continuously injects into $L^2(\mathbb{R}^d)$, and that the $X - X^*$ duality extends the L^2 scalar product. Finally, suppose that $v(t) \in X$ for almost every $t \in J$. Then $v : J \to X$ is also measurable.*

Proof. By separability, it suffices to show the weak measurability. In this spirit, we first prove that $L^2(\mathbb{R}^d)$ is dense in X^* by the following argument: If the claim was wrong, then there exists a nontrivial element $\Psi \in X^{**}$ which annihilates $L^2(\mathbb{R}^d)$, i.e. one has $\langle \psi, \Psi \rangle_{X^* \times X^{**}} = 0$ for all $\psi \in L^2(\mathbb{R}^d)$. By the reflexivity of X, $\Psi \in X^{**}$ may be represented by an element $\eta \in X$. So

$$\langle \psi, \Psi \rangle_{X^* \times X^{**}} = \langle \psi, \eta \rangle_{X^* \times X} = \int_{\mathbb{R}^d} \psi \, \overline{\eta} = 0 \quad \text{for all } \psi \in L^2(\mathbb{R}^d).$$

Thus, η and, consequently, Ψ must be the zero element, and we have a contradiction. By assumption, the mapping v is measurable when considered with values in $L^2(\Omega)$. So, in particular, the functions

$$J \ni t \mapsto \int_{\mathbb{R}^d} v(t) \, \overline{\psi} = \langle v(t), \psi \rangle_{X \times X^*}, \quad \psi \in L^2(\mathbb{R}^d) \tag{27}$$

are measurable. Thus, the measurability of the functions $J \ni \mapsto \langle v(t), \psi \rangle_{X \times X^*}$ holds true for all $\psi \in X^*$ due to the density of $L^2(\Omega)$ in X^*. □

Turning to the proof of Theorem 1.9 recall that Proposition 2.8 already gives the existence and uniqueness for the solution u of (25) together with the estimate

$$\|u\|_{W^{1,r}(J;L^2(\Omega))} + \|A(\cdot)u\|_{L^r(J;L^2(\Omega))} \le c_r \|f\|_{L^r(J;L^2(\Omega))}, \tag{28}$$

where $r \in [2, \infty[$ can be arbitrarily chosen.

Proof of Theorem 1.9. Let $\mathcal{F}$ be the extension operator from Lemma 2.7. Again we have $u \in W^{1,r}(J; L^2(\Omega))$, and, hence, $\mathcal{F}u \in W^{1,r}(J; L^2(\mathbb{R}^3))$. On the other hand, we have

$$\|\psi\|_{C^\alpha(\Omega)} \le c\|A_t\psi\|_{W^{-1,p}_{\mathfrak{D}}(\Omega)} \le c\|A_t\psi\|_{L^2}, \quad \psi \in dom(A_t) \tag{29}$$

uniformly for almost all $t \in J$, see Prop. 2.6.
Unfortunately, (29) together with the property $J \ni t \mapsto A_t(u(t)) \in L^r(J; L^2)$ does not directly imply that the function $J \ni t \mapsto u(t) \in C^\alpha(\Omega)$ also belongs to $L^r(J; C^\alpha(\Omega))$ since we presently do not know its measurability. So we must go the following detour.

We consider $\mathcal{F}$ as a mapping from $W^{1,r}(J; L^2(\mathbb{R}^3)) \hookrightarrow C(\overline{J}; L^2(\mathbb{R}^3))$, taking its value in $C^\alpha(\mathbb{R}^d)$, for almost every $t \in J$. It satisfies the estimate

$$\|\mathcal{F}u(t)\|_{C^\alpha(\mathbb{R}^3)} \le c\|u(t)\|_{C^\alpha(\Omega)}, \tag{30}$$

for a constant c independent of t. Recall that $\mathcal{F}$ takes its values in the set of functions whose support is contained in a sufficiently large ball, say B. If $\kappa \in]0, \alpha[$, then the straight forward estimate

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi(x)-\psi(y)|^o}{|x-y|^{3+\kappa o}} dx\, dy = \int_B\int_B \frac{|\psi(x)-\psi(y)|^o}{|x-y|^{3+\kappa o}} dx\, dy \le$$

$$\le \|\psi\|^o_{C^\alpha(\mathbb{R}^3)} \int_B\int_B \frac{1}{|x-y|^{3+(\kappa-\alpha)o}} dx\, dy, \quad \psi \in C^\alpha(\mathbb{R}^3),\ supp\,\psi \subset B \tag{31}$$

shows that $C^\alpha(\mathbb{R}^3) \cap \{\psi : supp\,\psi \subset B\}$ continuously embeds into the Sobolev-Slobodetskii space $W^{\kappa,o}(\mathbb{R}^3)$, for *every finite* $o \ge 1$.

By the separability (see [33, Ch. 2.3.2]) and reflexivity (see [33, Ch. 2.6.1]) of $W^{\kappa,o}(\mathbb{R}^3)$ one may exploit Lemma 2.10, taking X as $W^{\kappa,o}(\mathbb{R}^3)$. This shows that

$$J \ni t \mapsto \mathcal{F}u(t) \in W^{\kappa,o}(\mathbb{R}^3) \tag{32}$$

is measurable. Thus, (29) in combination with (30) yields that (32) belongs to $L^r(J; W^{\kappa,o}(\mathbb{R}^3))$. So $\mathcal{F}u$ belongs to $L^r(J; W^{\kappa,o}(\mathbb{R}^3)) \cap W^{1,r}_0(J; L^2(\mathbb{R}^3))$.

Now Proposition 2.3 tells us that

$$\mathcal{F}u \in C^\beta(J; (L^2(\mathbb{R}^3), W^{\kappa,o}(\mathbb{R}^3))_{\theta,1}) \hookrightarrow C^\beta(J; [L^2(\mathbb{R}^3), W^{\kappa,o}(\mathbb{R}^3)]_\theta)$$

with $\theta \in]0, 1-\frac{1}{r}[$ and $\beta = 1 - \frac{1}{r} - \theta$. One obtains – in the terminology of [33] –

$$[L^2(\mathbb{R}^3), W^{\kappa,o}(\mathbb{R}^3))]_\theta = [H^0_2(\mathbb{R}^3), B^\kappa_{o,o}(\mathbb{R}^3)]_\theta = F^{\theta\kappa}_{\iota,\iota}(\mathbb{R}^3), \quad \frac{1}{\iota} = \frac{1-\theta}{2} + \frac{\theta}{o}, \tag{33}$$

$F^{\theta\kappa}_{\iota,\iota}$ being the corresponding Lizorkin-Triebel space, see [33, Ch. 2.4.2]).

We employ now the following well known embedding result

$$F^{\frac{3}{\iota}+\gamma}_{\iota,\iota}(\mathbb{R}^3) \hookrightarrow C^\gamma(\mathbb{R}^3), \quad \iota \in]1,\infty[,\ \gamma \in]0,1[, \tag{34}$$

(see [33, Ch. 2.8.1]). On may chose $\kappa \in]0,\alpha[$ arbitrarily close to α. Taking r sufficiently large, one can achieve that θ is arbitrarily close to 1 and, consequently, $\theta\kappa$ becomes arbitrarily close to α and $\frac{1-\theta}{2}$ becomes arbitrarily small. Now choosing o sufficiently large, one can arrange that $\kappa\theta > \frac{3}{\iota}$. Hence, in this case (34) provides a continuous mapping $F^{\theta\kappa}_{\iota,\iota}(\mathbb{R}^3) \hookrightarrow C^\gamma(\mathbb{R}^3)$ for some $\gamma > 0$. Overall, we get $\mathcal{F}u \in C^\beta(J; C^\gamma(\mathbb{R}^3))$ for some positive β, γ, if r is large enough. Finally, restricting the functions $\mathcal{F}u(t)$ again to Ω, one obtains $u \in C^\beta(J; C^\gamma(\Omega))$. □

3. Optimal control

In this section we briefly illustrate how the results of the previous section can be utilized to achieve regularity results for the adjoint state in optimal control problems. This is important since the adjoint state typically arises in the sensitivity analysis of optimal control problems and further the regularity of the adjoint state can provide additional information about the regularity of the optimal control. The result presented here does not improve the quality of the adjoint state, but avoids the very restrictive assumption posed for the three-dimensional case in [23, Assumption 5.11 (b)].

Let us now specify the setting of the optimal control problem. For an arbitrary $u_d \in L^2(J; L^2)$ we consider

$$\min_{\xi \in L^2(J;Y)} \mathcal{J}(\xi) = \frac{1}{2}\int_0^T \|u(\xi)(t) - u_d(t)\|^2_{L^2}\, dt + \frac{\rho}{2}\int_0^T \|\xi\|^2_Y\, dt, \tag{$\mathcal{P}$}$$

where $u(\xi)$ is the solution to

$$\frac{\partial u(t)}{\partial t} - \mathcal{A}(t)\, u(t) = B(t)\xi(t), \quad u(0) = u_0. \tag{35}$$

Here $\rho > 0$, $J = (0,T)$, Y is a Hilbert space and the time dependent control operator B is an element of $L^\infty(J; \mathcal{L}(Y, H^{-1+\tau}_{\mathfrak{D}}))$, where $\tau \in (0,1)$ is sufficiently small. Here and below we write $H^s_{\mathfrak{D}}$ for the Sobolev–Slobodetskii space $W^{s,2}_{\mathfrak{D}}(\Omega)$ and we set, $\mathcal{A}(t) = -\nabla \cdot \hat{\mu}(t)\nabla \in \mathcal{L}(H^{1+\tau}_{\mathfrak{D}}, H^{-1+\tau}_{\mathfrak{D}})$. For the moment, Ω is a two or three dimensional domain and it satisfies Assumption 1.1. Further the initial state u_0 is chosen in the interpolation space $(H^{-1+\tau}_{\mathfrak{D}}, H^{-1+\tau}_{\mathfrak{D}})_{\frac{1}{2},2}$. Concerning the coefficient function $\hat{\mu}$, in addition to the assumptions specified above Theorem 1.7, it is required that there exists $\delta \in]0, \frac{1}{2}[$ such that for every $t \in J$ the components $\hat{\mu}_{i,j}(t,\cdot)$ are multipliers on H^s, $s \in], 0, \delta[$ and the corresponding multiplier norms are uniformly bounded in $t \in J$ and $s \in], 0, \delta[$.

Then from [23, Theorem 3.18] we recall that there exists a unique solution $u(\xi) \in W^{1,2}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1+\tau})$ to the differential equation in (35) and we have

$$\|u\|_{W^{1,2}(J;H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J;H_{\mathfrak{D}}^{1+\tau})} \le C(\|u_0\|_{(H_{\mathfrak{D}}^{-1+\tau}, H_{\mathfrak{D}}^{1+\tau})_{\frac{1}{2},2}} + \|f\|_{L^2(J;H_{\mathfrak{D}}^{-1+\tau})}), \tag{36}$$

for a constant C independent of u_0 and f.

By the standard technique of the calculus of variations the existence of a unique optimal solution $\xi^* \in L^2(J; Y)$ to problem $(\mathcal{P})$ can be argued. This solution is characterized by the optimality condition expressed in the following lemma.

Lemma 3.1. *Associated to the optimal solution ξ^* there exists an adjoint state $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1+\tau})$ such that the triple $(\xi^*, u(\xi^*), \varphi)$ satisfies the system*

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t) - \mathcal{A}(t)u(t) = B(t)\xi^*(t), && u(0) = u_0,\\
&-\frac{\partial \varphi}{\partial t}(t) + \widehat{\mathcal{A}}\varphi(t) = -(u(\xi^*)(t) - u_d(t)), && \varphi(T) = 0,\\
&\rho\,\xi^*(t) = B^*(t)\varphi(t),
\end{aligned}$$

where $\widehat{\mathcal{A}}$ arises from $\mathcal{A}$ by transposition of the coefficient function.

Proof. The verification is standard with optimal control techniques. One can follow, for example [23, Lemma 5.9, Theorem 5.10] where a very similar cases are treated. Note that in our case $B^* \in L^2(J; \mathcal{L}(H_{\mathfrak{D}}^{1-\tau}, Y^*))$. Moreover since $\mathcal{P}$ is a linear-quadratic problem, the above system is a necessary and sufficient optimality system. □

The regularity results of the previous section now provide extra regularity of the adoint state.

Corollary 3.2. *Let $u_d \in L^\infty(J; L^2(\Omega))$, and consider the case $d = 3$ with* (10) *and Assumption 1.2 holding. Then the adjoint state satisfies $\varphi \in C^\beta(J; C^\gamma(\Omega))$ for some $\beta > 0, \gamma > 0$.*

Proof. We shall apply Theorem 1.9 to the time-reversed adjoint equation. First we show that $u(\xi^*) \in C(\overline{J}; L^2(\Omega))$, which implies that $u(\xi^*) - u_d \in L^r(J; L^2(\Omega))$ for all finite r. Indeed, since $B \in L^\infty(J; \mathcal{L}(Y, H_{\mathfrak{D}}^{-1+\tau})$ the right hand side $B\xi^*$ of the first equation of the optimality system belongs to $L^2(J; H_{\mathfrak{D}}^{-1+\tau})$. So (36) tells us that $u = u(\xi^*) \in W^{1,2}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1+\tau})$. By [23, Theorem 3.19] we know that this space embeds continuously (in fact: even compactly) into $C(\overline{J}; L^2(\Omega))$. Having this at hand, we employ Theorem 1.9, and the claim is verified. □

Remark 3.3. By introducing as specific choice for the control operator B and the space Y we next demonstrate, for a prototypical example, how the regularity result of Corollary 3.2 can be utilized to obtain regularity of a control which acts on a subset of codimension 1 in dimension $d = 3$. For this purpose we set $Y = L^2(M; \mathcal{H}_{d-1})$ and define the time dependent control operators for a.e. t by

$$B(t) : L^2(M; \mathcal{H}_{d-1}) \to H^{-1+\tau}_{\mathfrak{D}} \text{ with } B(t) = \mathcal{I}_t V_t,$$

with $V_t : L^2(M; \mathcal{H}_{d-1}) \to L^2(M_t; \mathcal{H}_{d-1})$, and $\mathcal{I}_t : L^2(M_t; \mathcal{H}_{d-1}) \to H^{-1+\tau}_{\mathfrak{D}}$. Here we abbreviate $L^2(M_t; \mathcal{H}_{d-1}) := L^2(M_t; \mathcal{H}_{d-1}|M)$, where $\mathcal{H}_{d-1}|M$ denotes the restriction of the $d-1$-dimensional Hausdorff measure to the closed upper $d-1$ set $M \subset \Omega \subset \mathbb{R}^d$, see [21]. Further, $M_t \subset \Omega$, with $t \in J$, arise from M through bi-Lipschitz diffeomorphisms ϕ_t from M onto M_t. The Lipschitz constants of the ϕ_t's and their inverses ϕ_t^{-1}, are uniformly bounded in $t \in J$. It is also assumed that the mappings $t \to \phi_t(x) \in \Omega$ are measurable for every $x \in M$. Finally V_t is defined as

$$\big(V_t(\varphi)\big)(x) = \varsigma_t(x)\varphi(\phi_t^{-1}(x)), \quad \varphi \in L^2(M; \mathcal{H}_{d-1}),\, x \in M_t, \tag{37}$$

for some $\mathcal{H}_{d-1}$-measurable ς which realises the forward image of the Hausdorff measure on M under ϕ_t onto M_t. Finally, $\mathcal{I}_t$ is the canonical continuous embedding. As shown in [23], in this case the third equation in the optimality system results in

$$\rho\xi^* = \varphi(t, \phi_t(\cdot)) \text{ for a.e. } t \in J.$$

Since $\phi_t \in L^\infty(J; W^{1,\infty}(\Omega))$, Corollary 3.2 implies the regularity $\xi^* \in L^\infty(J; C^\gamma)$ for the optimal control. If we assume that $\phi_t \in W^{1,\infty}(J \times \Omega)$, then $\xi^* \in C^\beta(J; C^\gamma)$. In dimension $d = 2$ the same regularity of ξ^* can be obtained utilizing Propostion 2.1 and Theorem 1.8

4. Concluding remarks

1. The geometric setting of assuming Ω to be a Lipschitz domain is not the most general – and admissible – one in this context. But, if relaxing this to the demand that only the elements of $\partial\Omega \setminus \mathfrak{D}$ shall possess bi-Lipschitzian boundary charts one is forced to pose quite a list of additional (technical) assumptions. For problems arising in practise, we do not see an advantage in doing so.
2. The result in Theorem 1.7 carries over one-to-one to systems, since the elliptic regularity result Proposition 2.1 continues to hold for them, see [19]. Concerning Proposition 2.6, there is no analogue for systems and it is even false for complex coefficients, compare [27].
3. We did not aim at determining the minimal value of r in Theorem 1.8. One could have achieved this in terms of q, but also the requirements on q are highly implicit. In very special cases bounds in the $W^{1,q}$–norm are obtained in [7].

4. Also in Theorem 1.9: one could aim for the minimal r in terms of α, but α itself is widely unknown. Consequently the philosophy in both cases is as follows: one knows – mostly for external reasons – that the integrability exponent r is either ∞ or it is understood as finite, but 'sufficiently large'.
5. The results of Theorem 1.7, Theorem 1.8, and Theorem 1.9 also hold for non-zero initial values and for coefficient functions which are only piecewise Hölder continuous. Concerning Theorem 1.7 this is easy: in [30] everything is already formulated as a general initial-boundary value problem with the initial value taken – as usual – from the interpolation space $\big(W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega)\big)_{1-\frac{1}{r},r}$. Here the equality of the domains allows to concatenate different intervals of continuity for the coefficient function, thus obtaining a *global* result.
   Unfortunately, for Theorem 1.9 the situation is quite different and much more complicated. Here also the quality of the initial value is related to the integrability index r. In order to get large indices r one must assume that $u_0 = u(0) \in (L^2(\Omega), dom\big(A_0\big))_{1-\frac{1}{r}}$. This is problematic in case of non-smooth domains/coefficients and for mixed boundary conditions since in general $\mathrm{dom}(A_0)$ cannot be determined explicitly. Also the determination of $(L^2(\Omega), dom\big(A_0\big))_{1-\frac{1}{r}}$ is a difficult problem.

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