Boundary Control of Infinite Horizon Semilinear Parabolic Equations*

Eduardo Casas[1] and Karl Kunisch[2]

***Abstract*— This work concentrates on a class of Neumann optimal control problems for semilinear parabolic equations on an infinite horizon domain $Q = \Omega \times (0,\infty)$ subject to a control constraint of the form $\alpha \le u(x,t) \le \beta$ for $(x,t) \in \Gamma \times (0,\infty)$, where Γ is the boundary of Ω. Existence of a solution, first- and second-order optimality conditions are established. Finally, the approximation of the solution by finite horizon control problems is addressed and some error estimates are provided.**

I. INTRODUCTION

We investigate the boundary optimal control problem

$$\text{(P)} \quad \inf_{u \in U_{ad}} J(u)$$

$$J(u) := \frac{1}{2}\int_Q (y_u - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_\Sigma u^2 \,\mathrm{d}x\,\mathrm{d}t,\ \kappa > 0,$$

$$U_{ad} = \{u \in L^2(\Sigma) : \alpha \le u(x,t) \le \beta \text{ a.e.}\},\ \alpha < 0 < \beta,$$

and y_u is the solution of the semilinear parabolic equation

$$\left\{\begin{array}{l} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(x,t,y) = g \text{ in } Q = \Omega \times (0,\infty), \\ \partial_\nu y = u \text{ on } \Sigma = \Gamma \times (0,\infty),\ y(0) = y_0 \text{ in } \Omega. \end{array}\right. \tag{1.1}$$

Hereafter $\nu(x) = (\nu_j(x))_{j=1}^n$ denotes the outward normal unit vector to Γ at the point x. It is assumed that Ω is a bounded, connected, and open subset of $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz boundary Γ.

The fact of considering infinite horizon problems is motivated by the fact that the choice of T in finite horizon problems is frequently ad hoc. Nowadays the infinite horizon problems are widely studied because of their importance in biology, economy, and engineering. The reader is referred to [3] and [4] for references. Despite the large number of papers for infinite horizon control of ordinary differential equations, there are only a few papers dealing with partial differential equations. In particular, as far as we know, the case of boundary control has not been studied.

II. ANALYSIS OF THE STATE EQUATION

To carry out the analysis of the state equation we make the following assumptions. We assume that $y_0 \in L^\infty(\Omega)$, $a \in L^\infty(\Omega)$ with $0 \le a \not\equiv 0$, and $g \in L^{p,2}(Q) = L^2(Q) \cap L^p(0,\infty; L^2(\Omega))$, where $p \in (\frac{4}{4-n}, \infty]$. We suppose that the nonlinear function $f : Q \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and of class C^2 with respect to the last variable satisfying the following properties:

$$f(x,t,0) = 0 \text{ and } \exists M_f \ge 0 \text{ such that} \tag{2.1}$$

$$\frac{\partial f}{\partial y}(x,t,y) \ge 0 \text{ and } f(x,t,y)y \ge 0\ \forall |y| \ge M_f, \tag{2.2}$$

$$\forall M > 0\, \exists C_M : \left|\frac{\partial^j f}{\partial y^j}(x,t,y)\right| \le C_M\ \forall |y| \le M, \tag{2.3}$$

$$\left\{\begin{array}{l} \exists m_f > 0,\ \exists \delta_f \in [0,1), \text{ and } \exists C_f > 0 \text{ such that} \\ \dfrac{\partial f}{\partial y}(x,t,s) \ge -C_f|s| - \delta_f a(x,t)\ \forall |s| \le m_f, \end{array}\right. \tag{2.4}$$

for almost all $(x,t) \in Q$ and $j = 1, 2$. Let us observe that (2.2) and (2.3) yield

$$\frac{\partial f}{\partial y}(x,t,y) \ge -C_{M_f}\ \forall y \in \mathbb{R} \text{ a.e. in } Q. \tag{2.5}$$

Moreover, (2.1) and (2.3) along with the mean value theorem imply that $\forall |y| \le M$ and for a.a. $(x,t) \in Q$

$$|f(x,t,y)| = \left|\frac{\partial f}{\partial y}(x,t,\theta(x,t)y)y\right| \le C_M M. \tag{2.6}$$

We note the generalized Poincaré inequality

$$C_a\|y\|_{H^1(\Omega)} \le \left(\int_\Omega [|\nabla y|^2 + ay^2]\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.7}$$

As usual, for $0 < T \le \infty$ we note the Hilbert space

$$W(0,T) = \{y \in L^2(0,T; H^1(\Omega)) : \partial_t y \in L^2(0,T; H^1(\Omega)^*)\}.$$

We recall that $W(0,T)$ is continuously embedded in $C([0,T]; L^2(\Omega))$ and for $T < \infty$ is compactly embedded in $L^2(0,T; L^2(\Omega))$; see [7, §5.2].

We mention two examples of functions f satisfying the above assumptions: $f(x,t,y) = b(x,t)(\mathrm{e}^y - 1)$ with $b \ge 0$ and $b \in L^\infty(Q)$; $f(x,t,y) = \sum_{k=0}^{2m+1} a_k(x,t)y^k$ with $m \ge 1$ integer, $a_k \in L^\infty(Q)$ for every k, and $a_{2m+1} \ge \mu > 0$. This in particular includes the Allen-Cahn equation, which arises in phase separation in material science and biology.

DEFINITION 2.1: Given $u \in L^q(\Sigma)$ with $q > n + 1$, we say that y a solution to (1.1) if $y \in W(0,T) \cap L^\infty(Q_T)$ for every $T \in (0,\infty)$, where $Q_T = \Omega \times (0,T)$, and satisfies the following equation in the variational sense

$$\left\{\begin{array}{l} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(x,t,y) = g \text{ in } Q_T, \\ \partial_\nu y = u \text{ on } \Sigma_T = \Gamma \times (0,T),\ y(0) = y_0 \text{ in } \Omega. \end{array}\right. \tag{2.8}$$

The choice $y_0 \in L^\infty(\Omega)$ rather than $L^2(\Omega)$ is necessary to guaranty that the state y is in $L^\infty(Q)$, which in term is

*The first author was supported by MICIU/AEI/10.13039/501100011033/ under research project PID2023-147610NB-I00.

[1]E. Casas is with Departamento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39005 Santander, Spain eduardo.casas@unican.es

[2]K. Kunisch is with Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, A-8010 Graz, Austria, and Johann Radon Institute, Austrian Academy of Sciences, Linz, Austria karl.kunisch@uni-graz.at

crucial in the proof of the differentiability of the control-to-state mapping.

In the sequel, we will denote $U_q = L^2(\Sigma) \cap L^q(\Sigma)$ with $q > n+1$.

PROPOSITION 2.1: For every $u \in U_q$ (1.1) has a unique solution y. In addition, if $y \in L^2(Q)$, then $y \in W(0,\infty) \cap L^\infty(Q)$ and $f(\cdot,\cdot,y) \in L^2(Q) \cap L^\infty(Q)$ holds. Moreover, the following estimates are satisfied

$$\|y\|_Q \le K_1\Big(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} + \|u\|_{L^2(\Sigma)} + \|y\|_{L^2(Q)}\Big), \tag{2.9}$$

$$\|y\|_{L^\infty(Q)} \le K_2\Big(\|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^{p,2}(Q)} + \|u\|_{U_q} + \|y\|_{L^2(Q)} + M_f\Big), \tag{2.10}$$

$$\|f(\cdot,\cdot,y)\|_{L^\infty(Q)} \le C_{K_\infty}\|y\|_{L^\infty(Q)}, \tag{2.11}$$

$$\|f(\cdot,\cdot,y)\|_{L^2(Q)} \le C_{K_\infty}\|y\|_{L^2(Q)}, \tag{2.12}$$

$$\lim_{t\to\infty}\|y(t)\|_{L^2(\Omega)} = 0, \tag{2.13}$$

where M_f is given by (2.2), $K_\infty = \|y\|_{L^\infty(Q)}$, C_{K_∞} as in (2.3) with $M = K_\infty$, and

$$\|y\|_Q = \Big(\|y\|^2_{L^\infty(0,\infty;L^2(\Omega))} + \|y\|^2_{L^2(0,\infty;H^1(\Omega))}\Big)^{\frac{1}{2}}.$$

Proof: The existence and uniqueness of a solution y of (1.1) is a consequence of [3, Theorem 2.1]. For the proof of the estimate (2.9) the reader is referred to [4, Theorem A.2]. In that proof the control was located in Q. To deal with this difference it is enough to take into account that

$$\Big|\int_\Sigma yu\,\mathrm{d}x\,\mathrm{d}t\Big| \le \frac{C_a^2}{2}\|y\|^2_{L^2(0,\infty;H^1(\Omega))} + C\|u\|^2_{L^2(\Sigma)}$$

for some constant C. The $L^\infty(Q)$ estimate (2.10) follows from [5]. The inequalities of (2.11) and (2.12) are immediate consequences of the mean value theorem and assumptions (2.1) and (2.3). Finally, inequality (2.13) is a straightforward consequence of the fact that $y \in W(0,\infty)$; see [2, Theorem 2.4] for details. ■

We define

$$\mathcal{U}_q = \{u \in U_q \text{ such that } y_u \in L^2(Q)\},$$
$$Y_{p,q} = \{y \in W(0,\infty)\cap L^\infty(Q) \text{ such that } \frac{\partial y}{\partial t} - \Delta y + ay \in L^{p,2}(Q) \text{ and } \partial_\nu y \in U_q\}.$$

$Y_{p,q}$ is a Banach space when endowed with the corresponding graph norm. We also define the mapping $G_{p,q} : \mathcal{U}_q \longrightarrow Y_{p,q}$ by $G_{p,q}(u) = y_u$, where y_u is the solution of (1.1).

THEOREM 2.1: Let us assume that $\mathcal{U}_q$ is not empty. Then, $\mathcal{U}_q$ is an open subset of U_q and the mapping $G_{p,q}$ is of class C^2. Moreover, given $u \in \mathcal{U}_q$ and $v, v_1, v_2 \in U_q$, $z_{u,v} = DG_{p,q}(u)v$ and $z_{u,(v_1,v_2)} = D^2G_{p,q}(u)(v_1,v_2)$ are the unique solutions of

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + \dfrac{\partial f}{\partial y}(x,t,y_u)z = 0 \text{ in } Q,\\ \partial_\nu z = v \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega, \end{cases} \tag{2.14}$$

and

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + \dfrac{\partial f}{\partial y}(x,t,y_u)z \\ \quad = -\dfrac{\partial^2 f}{\partial y^2}(x,t,y_u)z_{u,v_1}z_{u,v_2} \text{ in } Q,\\ \partial_\nu z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega. \end{cases} \tag{2.15}$$

This proof can be obtained applying the implicit function theorem; see [4] for a similar proof in the case of distributed controls.

LEMMA 2.1: For every $u \in \mathcal{U}_q$ and $(h,v,z_0) \in L^2(0,\infty;H^1(\Omega)^*) \times L^2(\Sigma) \times L^2(\Omega)$ the following equation

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + \dfrac{\partial f}{\partial y}(x,t,y_u)z = h \text{ in } Q,\\ \partial_\nu z = v \text{ on } \Sigma,\ z(0) = z_0 \text{ in } \Omega \end{cases} \tag{2.16}$$

has a unique solution $z \in W(0,\infty)$ satisfying for some constant $C > 0$ depending in a monotone way of $\|y_u\|_{L^\infty(Q)}$, but independent of (h,v,z_0),

$$\|z\|_{W(0,\infty)} \le C(\|h\|_{L^2(0,\infty;H^1(\Omega)^*)} + \|v\|_{L^2(\Sigma)} + \|z_0\|_{L^2(\Omega)}) \tag{2.17}$$

Proof: Due to the fact that $y_u \in L^\infty(Q)$ and (2.3) we have that $\frac{\partial f}{\partial y}(x,t,y_u) \in L^\infty(Q)$, and hence the existence and uniqueness of $z \in W(0,T)\cap L^\infty(Q_T)$ holds for every $T < \infty$. Let us prove that $z \in W(0,\infty)$. We put $K = \|y_u\|_{L^\infty(Q)}$ and $C_K = \left\|\frac{\partial f}{\partial y}(x,t,y_u)\right\|_{L^\infty(Q)}$. For $\delta_f \in [0,1)$ as in (2.4), there exists a constant C_{a,δ_f} such that $\forall w \in H^1(\Omega)$

$$C_{a,\delta_f}\|w\|_{H^1(\Omega)} \le \left(\int_\Omega [|\nabla w|^2 + (1-\delta_f)aw^2]\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.18}$$

We select $\varepsilon > 0$ such that $\max\{C_f, \frac{C_K}{m_f}\}C_1^2\varepsilon \le \frac{C^2_{a,\delta_f}}{4}$, where m_f and C_f are given in (2.4) and C_1 is the embedding constant for $H^1(\Omega) \subset L^4(\Omega)$. We also denote by C_2 the norm of the trace operator $H^1(\Omega) \longrightarrow L^2(\Gamma)$. Using (2.13) we deduce the existence of $T_\varepsilon > 0$ such that

$$\|y(t)\|_{L^2(\Omega)} \le \varepsilon \quad \forall t \ge T_\varepsilon. \tag{2.19}$$

For $t > 0$ we set $\Omega_{m_f}(t) = \{x \in \Omega : |y(x,t)| \le m_f\}$. Now, we test (2.16) with z and integrate over $\Omega \times (T_\varepsilon, t)$ for every $t > T_\varepsilon$. Using assumption (2.3) and (2.19) we get

$$\begin{aligned} &\frac{1}{2}\|z(t)\|^2_{L^2(\Omega)} + C^2_{a,\delta_f}\int_{T_\varepsilon}^t \|z(t)\|^2_{H^1(\Omega)}\,\mathrm{d}s \\ &\le \frac{1}{2}\|z(t)\|^2_{L^2(\Omega)} + \int_{T_\varepsilon}^t\int_\Omega [|\nabla z|^2 + (1-\delta_f)az^2]\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)} + \int_{T_\varepsilon}^t \langle h(s), z(s)\rangle\,\mathrm{d}s + \int_{T_\varepsilon}^t\int_\Sigma vz\,\mathrm{d}x\,\mathrm{d}s \\ &+ C_f\int_{T_\varepsilon}^t\int_{\Omega_{m_f}(t)} |y|z^2\,\mathrm{d}x\,\mathrm{d}s + C_K\int_{T_\varepsilon}^t\int_{\Omega\setminus\Omega_{m_f}(t)} z^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)} + \|h\|_{L^2(0,\infty;H^1(\Omega)^*)}\|z\|_{L^2(T_\varepsilon,t;H^1(\Omega))} \\ &+ \|v\|_{L^2(\Sigma)}\|z\|_{L^2(T_\varepsilon,t;L^2(\Gamma))} \end{aligned}$$

$$+\max\{C_f,\frac{C_K}{m_f}\}\int_{T_\varepsilon}^t\int_\Omega|y|z^2\,\mathrm{d}x\,\mathrm{d}s\le\frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)}$$
$$+\frac{2}{C^2_{a,\delta_f}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)}+\frac{C^2_{a,\delta_f}}{8}\|z\|^2_{L^2(T_\varepsilon,t;H^1(\Omega))}$$
$$+\frac{2C_2^2}{C^2_{a,\delta_f}}\|v\|^2_{L^2(\Sigma)}+\frac{C^2_{a,\delta_f}}{8}\|z\|^2_{L^2(T_\varepsilon,t;H^1(\Omega))}$$
$$+\max\{C_f,\frac{C_K}{m_f}\}\int_{T_\varepsilon}^t\|y(s)\|_{L^2(\Omega)}\|z(s)\|^2_{L^4(\Omega)}\,\mathrm{d}s$$
$$\le\frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)}+\frac{2}{C^2_{a,\delta_f}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)}$$
$$+\frac{2C_2^2}{C^2_{a,\delta_f}}\|v\|^2_{L^2(\Sigma)}+\frac{C^2_{a,\delta_f}}{4}\int_{T_\varepsilon}^t\|z\|^2_{H^1(\Omega)}\,\mathrm{d}s$$
$$+C_1^2\max\{C_f,\frac{C_K}{m_f}\}\varepsilon\int_{T_\varepsilon}^t\|z(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s$$
$$\le\frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)}+\frac{2}{C^2_{a,\delta_f}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)}$$
$$+\frac{2C_2^2}{C^2_{a,\delta_f}}\|v\|^2_{L^2(\Sigma)}+\frac{C^2_{a,\delta_f}}{2}\int_{T_\varepsilon}^t\|z\|^2_{H^1(\Omega)}\,\mathrm{d}s.$$

This implies

$$\|z(t)\|^2_{L^2(\Omega)}+C^2_{a,\delta_f}\int_{T_\varepsilon}^t\|z(t)\|^2_{H^1(\Omega)}\,\mathrm{d}s\le\|z(T_\varepsilon)\|^2_{L^2(\Omega)}$$
$$+\frac{4}{C^2_{a,\delta_f}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)}+\frac{4C_2^2}{C^2_{a,\delta_f}}\|v\|^2_{L^2(\Sigma)}.$$

Since z solves (2.16) in $(0,T_\varepsilon)$, we have $z\in W(0,T_\varepsilon)$ and $\|z\|_{W(0,T_\varepsilon)}$ can be estimated by $\|h\|_{L^2(0,\infty;H^1(\Omega)^*)}+\|v\|_{L^2(\Sigma)}+\|z_0\|_{L^2(\Omega)}$. This along with the above estimate implies the desired estimate of z in $L^2(0,\infty;H^1(\Omega))\cap L^\infty(0,\infty;L^2(\Omega))$. From the equation (2.16) we infer that $\frac{\partial z}{\partial t}\in L^2(0,\infty;H^1(\Omega)^*)$ and estimate (2.17) follows. ■

REMARK 2.1: Lemma 2.1 implies that for every $u\in\mathcal{U}_q$ the mapping $DG_{p,q}(u):U_q\longrightarrow Y_{p,q}$ can be extended to a continuous linear mapping $D_{p,q}(u):L^2(\Sigma)\longrightarrow W(0,\infty)$ by the same equation (2.14).

III. ANALYSIS OF THE CONTROL PROBLEM

In this section, we prove the existence of an optimal solution of (P) and derive the first and second order optimality conditions. Let us observe that Theorem 2.1 implies the existence of a unique state y_u for every control $u\in U_{ad}$. However, it could happen that $y_u\notin L^2(Q)$ and, consequently, $J(u)=\infty$. Therefore, the assumption about the existence of a control $u_0\in U_{ad}$ such that $J(u_0)<\infty$ is needed. This issue will not be addressed in this paper, the reader is referred, for instance, to [1] and [6] for this question. We will say that u is a feasible control if $u\in U_{ad}$ and $J(u)<\infty$.

THEOREM 3.1: Let us assume that there exists a feasible control u_0. Then, (P) has at least one solution.

Proof: Let $\{u_k\}_{k=1}^\infty\subset U_{ad}$ be a minimizing sequence of feasible controls with associated states $\{y_{u_k}\}_{k=1}^\infty$. Since $J(u_k)\to\inf\text{(P)}\le J(u_0)<\infty$, then the boundedness of $\{u_k\}_{k=1}^\infty$ and $\{y_{u_k}\}_{k=1}^\infty$ in $L^2(\Sigma)$ and $L^2(Q)$, respectively, follows. Then, taking subsequences we can assume that $(u_k,y_{u_k})\rightharpoonup(\bar u,\bar y)$ in $L^2(\Sigma)\times L^2(Q)$. Since U_{ad} is a closed and convex subset of $L^2(\Sigma)$, we infer that $\bar u\in U_{ad}$. Due to the weak lower semicontinuity of J with respect to (y,u) in $L^2(Q)\times L^2(\Sigma)$, it is enough to establish that $\bar y$ is the state associated to $\bar u$ to conclude the proof. For this purpose we have to show that $\bar y$ satisfies (2.8) with $\bar u$ on Σ for every $T<\infty$. The only delicate point in this respect is to prove the convergence of $f(x,t,y_{u_k})\to f(x,t,\bar y)$ in $L^2(Q_T)$ for every $T>0$. Using the boundedness of $\{(u_k,y_{u_k})\}_{k=1}^\infty$ in $[L^2(\Sigma)\cap L^\infty(\Sigma)]\times L^2(Q)$ we deduce from (2.9)–(2.12) the boundedness of $\{y_{u_k}\}_{k=1}^\infty$ in $W(0,\infty)\cap L^\infty(Q)$ and $\{f(\cdot,\cdot,y_{u_k})\}_{k=1}^\infty$ in $L^\infty(Q)\cap L^2(Q)$. Hence, using the compactness of the embedding $W(0,T)\subset L^2(Q_T)$ the desired convergence follows. ■

Since (P) is not a convex problem, we will also deal with local minimizers. In this paper, a local minimizer of (P) is understood in the $L^2(\Sigma)$ sense. Before writing the first-order optimality conditions satisfied by a local minimizer of (P), we address the differentiability of the function $J:\mathcal{U}_q\longrightarrow\mathbb{R}$.

THEOREM 3.2: If $\mathcal{U}_q$ is not empty, then J is of class C^2 and for every $u\in\mathcal{U}_q$ and $v,v_1,v_2\in L^2(\Sigma)\cap L^\infty(\Sigma)$ its derivatives are given by

$$J'(u)v=\int_\Sigma(\varphi_u+\kappa u)v\,\mathrm{d}x\,\mathrm{d}t,\tag{3.1}$$

$$J''(u)(v_1,v_2)=\int_Q\Big[1-\frac{\partial^2 f}{\partial y^2}(x,t,y_u)\varphi_u\Big]z_{v_1}z_{v_2}\,\mathrm{d}x\,\mathrm{d}t+\kappa\int_\Sigma v_1v_2\,\mathrm{d}x\,\mathrm{d}t,\tag{3.2}$$

where $\varphi_u\in W(0,\infty)\cap L^\infty(Q)$ satisfies

$$\begin{cases}-\dfrac{\partial\varphi_u}{\partial t}-\Delta\varphi_u+a\varphi_u+\dfrac{\partial f}{\partial y}(x,t,y_u)\varphi_u=y_u-y_d\text{ in }Q,\\ \partial_\nu\varphi_u=0\text{ on }\Sigma,\ \lim_{t\to\infty}\|\varphi_u(t)\|_{L^2(\Omega)}=0.\end{cases}\tag{3.3}$$

This theorem follows from Theorem 2.1 and the chain rule.

REMARK 3.1: From Remark 2.1 we infer that for every $u\in\mathcal{U}_q$, $J'(u)$ and $J''(u)$ can be extended to linear and bilinear forms, respectively, on $L^2(\Sigma)$.

Now, using the convexity of U_{ad} and (3.1) we deduce the first-order optimality conditions:

THEOREM 3.3: Let $\bar u$ be a local minimizer of (P). Then, there exist $\bar y,\bar\varphi\in W(0,\infty)\cap L^\infty(Q)$ such that

$$\begin{cases}\dfrac{\partial\bar y}{\partial t}-\Delta\bar y+a\bar y+f(x,t,\bar y)=g\text{ in }Q,\\ \partial_\nu\bar y=\bar u\text{ on }\Sigma,\ \bar y(0)=y_0\text{ in }\Omega,\end{cases}\tag{3.4}$$

$$\begin{cases}-\dfrac{\partial\bar\varphi}{\partial t}-\Delta\bar\varphi+a\bar\varphi+\dfrac{\partial f}{\partial y}(x,t,\bar y)\bar\varphi=\bar y-y_d\text{ in }Q,\\ \partial_\nu\bar\varphi=0\text{ on }\Sigma,\ \lim_{t\to\infty}\|\bar\varphi(t)\|_{L^2(\Omega)}=0,\end{cases}\tag{3.5}$$

$$\int_\Sigma(\bar\varphi+\kappa\bar u)(u-\bar u)\,\mathrm{d}x\,\mathrm{d}t\ge0\quad\forall u\in U_{ad}.\tag{3.6}$$

Next we focus on the second order analysis for (P). Given a control $\bar{u} \in U_{ad}$ satisfying the first-order optimality conditions (3.4)–(3.6) we introduce the cone of critical directions:

$$C_{\bar{u}} = \{v \in L^2(\Sigma) : J'(\bar{u})v = 0 \text{ and } v(x,t) \left\{ \begin{array}{ll} \geq 0 & \text{if } \bar{u}(x,t) = \alpha \\ \leq 0 & \text{if } \bar{u}(x,t) = \beta \end{array} \right\}.$$

We prove the second-order necessary optimality condition.

THEOREM 3.4: If $\bar{u}$ is a local minimizer of (P), then $J''(\bar{u})v^2 \geq 0$ for all $v \in C_{\bar{u}}$.

Proof: Since $\bar{u}$ is a local minimizer of (P), there exists $\varepsilon > 0$ such that $J(\bar{u}) \leq J(u)$ for all $u \in U_{ad} \cap B_\varepsilon(\bar{u})$, where $B_\varepsilon(\bar{u}) = \{u \in L^2(\Sigma) : \|u - \bar{u}\|_{L^2(\Sigma)} < \varepsilon\}$. Given $q \in (n+1, \infty)$ we have for every $u \in U_{ad} \cap B_\varepsilon(\bar{u})$

$$\|u - \bar{u}\|_{L^q(\Sigma} \leq (\beta - \alpha)^{1-\frac{2}{q}} \|u - \bar{u}\|_{L^2(\Sigma)}^{\frac{2}{q}} < (\beta - \alpha)^{1-\frac{2}{p}} \varepsilon^{\frac{2}{q}}.$$

Using that $\mathcal{U}_q$ is an open set, we select $\varepsilon > 0$ small enough such that $U_{ad} \cap B_\varepsilon(\bar{u}) \subset \mathcal{U}_q$ holds. Now, given $v \in C_{\bar{u}}$ we define for every integer $k \geq 1$ the function v_k by

$$v_k(x,t) = \left\{ \begin{array}{l} 0 \text{ if } \bar{u}(x,t) \in (\alpha, \alpha + \frac{1}{k}) \cup (\beta - \frac{1}{k}, \beta) \\ \text{Proj}_{[-k,+k]}(v(x,t)) \text{ otherwise.} \end{array} \right.$$

We get $\{v_k\}_{k=1}^\infty \subset L^\infty(\Sigma) \cap L^2(\Sigma)$ and $v_k \to v$ in $L^2(\Sigma)$ as $k \to \infty$. Further, if we set $\rho_k = \min\{\frac{1}{k^2}, \frac{\beta-\alpha}{k}, \frac{\varepsilon}{\|v\|_{L^2(\Sigma)}}\}$, then $\bar{u} + \rho v_k \in U_{ad} \cap B_\varepsilon(\bar{u})$ for every $\rho \in (0, \rho_k)$. In view of (3.6), it is straightforward to check that the condition $J'(\bar{u})v = 0$ in the definition of $C_{\bar{u}}$ is equivalent to $(\bar{\varphi} + \nu\bar{u})(x,t)v(x,t) = 0$ for almost all $(x,t) \in \Sigma$. Using this fact, it is immediate that $J'(\bar{u})v_k = 0$ for every k. Then, performing a Taylor expansion we get for every $\rho \in (0, \rho_k)$

$$0 \leq J(\bar{u} + \rho v_k) - J(\bar{u}) = \rho J'(\bar{u})v_k + \frac{\rho^2}{2} J''(\bar{u} + \theta_{\rho,k}\rho v_k)v_k^2$$
$$= \frac{\rho^2}{2} J''(\bar{u} + \theta_{\rho,k}\rho v_k)v_k^2 \text{ with } \theta_{\rho,k} \in [0,1].$$

Dividing by $\frac{\rho^2}{2}$ we deduce $J''(\bar{u} + \theta_{\rho,k}\rho v_k)v_k^2 \geq 0$. Since $\bar{u} + \theta_{\rho,k}\rho v_k \to \bar{u}$ in U_q as $\rho \to 0$, we deduce $J''(\bar{u})v_k^2 \geq 0$. Moreover, since $v_k \to v$ in $L^2(\Sigma)$ we infer that $z_{v_k} \to z_v$ in $L^2(\Sigma)$; see Lemma 2.1. Hence, we can pass to the limit in the previous inequality and obtain $J''(\bar{u})v^2 \geq 0$. ∎

We conclude this section by proving the second-order sufficient condition for local optimality.

THEOREM 3.5: Let $\bar{u} \in U_{ad} \cap \mathcal{U}_\infty$ satisfy the first order optimality conditions (3.4)–(3.6) and the second order condition $J''(\bar{u})v^2 > 0$ for every $v \in C_{\bar{u}} \setminus \{0\}$. Then, there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|_{L^2(\Sigma)}^2 \leq J(u) \quad \forall u \in U_{ad} \cap B_\varepsilon(\bar{u}), \quad (3.7)$$

where $B_\varepsilon(\bar{u}) = \{u \in L^2(\Sigma) : \|u - \bar{u}\|_{L^2(\Sigma)} \leq \varepsilon\}$.

Proof: We argue by contradiction and assume that (3.7) does not hold. Then, for every integer $k \geq 1$ there exists a control $u_k \in U_{ad}$ such that

$$\left\{ \begin{array}{l} \rho_k = \|u_k - \bar{u}\|_{L^2(\Sigma)} < \frac{1}{k} \text{ and} \\ J(u_k) < J(\bar{u}) + \frac{1}{2k}\|u_k - \bar{u}\|_{L^2(\Sigma)}^2. \end{array} \right. \quad (3.8)$$

We define $v_k = \frac{1}{\rho_k}(u_k - \bar{u})$. Since $\|v_k\|_{L^2(\Sigma)} = 1$ for every k, taking a subsequence, we can assume that $v_k \rightharpoonup v$ in $L^2(\Sigma)$. From (3.8) we deduce that $\{y_{u_k}\}_{k=1}^\infty$ is a bounded sequence in $L^2(Q)$, hence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_\infty$. Moreover, given $q \in (n+1, \infty)$ we have

$$\|u_k - \bar{u}\|_{L^q(\Sigma)} \leq \|u_k - \bar{u}\|_{L^\infty(\Sigma)}^{\frac{q-2}{q}} \|u_k - \bar{u}\|_{L^2(\Sigma)}^{\frac{2}{q}} \stackrel{k\to\infty}{\longrightarrow} 0.$$

Then, $y_{u_k} = G_{p,q}(u_k) \to G_{p,q}(\bar{u}) = \bar{y}$ in $Y_{p,q}$. Consequently, there exists a ball $B_r(\bar{u}) \subset \mathcal{U}_q$ and $k_0 \geq 1$ such that $\{u_k\}_{k \geq k_0} \subset B_r(\bar{u})$. The rest of the proof is split into three steps.

Step I - $v \in C_{\bar{u}}$. From (3.1) and (3.6) we infer that

$$0 \leq J'(\bar{u})v_k \to J'(\bar{u})v. \quad (3.9)$$

Using the mean value theorem and (3.8) we get

$$\int_\Sigma (\varphi_{\theta_k} + \kappa u_{\theta_k})v_k \,\mathrm{d}x\,\mathrm{d}t = J'(u_{\theta_k})v_k$$
$$= \frac{J(u_k) - J(\bar{u})}{\rho_k} < \frac{\rho_k}{2k} \to 0,$$

where $\theta_k \in [0,1]$, $u_{\theta_k} = \bar{u} + \theta_k(u_k - \bar{u})$, and φ_{θ_k} is the adjoint state corresponding to u_{θ_k}. Since $y_{\theta_k} = G_{p,q}(u_{\theta_k}) \to G_{p,q}(\bar{u}) = \bar{y}$ in $Y_{p,q}$, we deduce from [4, Theorem A4] that $\varphi_{\theta_k} \to \bar{\varphi}$ in $W(0,\infty) \cap L^\infty(Q)$ as $k \to \infty$. Then, it is straightforward to pass to the limit in the above expression and to get $J'(\bar{u})v \leq 0$. This inequality and (3.9) imply that $J'(\bar{u})v = 0$. On the other hand, since the set of elements of $L^2(\Sigma)$ satisfying the sign conditions given in the definition of $C_{\bar{u}}$ is convex and closed in $L^2(\Sigma)$, it is weakly closed. Due to the fact that every $v_k = (u_k - \bar{u})/\rho_k$ satisfies the sign conditions, they are also satisfied by v. Hence we have that $v \in C_{\bar{u}}$.

Step II - $J''(\bar{u})v^2 \leq 0$. Performing a Taylor expansion and using (3.8) and (3.6) we infer for some $\vartheta_k \in [0,1]$

$$\frac{1}{2k}\|u_k - \bar{u}\|_{L^2(\Sigma)}^2 > J(u_k) - J(\bar{u}) = J'(\bar{u})(u_k - \bar{u})$$
$$+ \frac{1}{2}J''(\bar{u} + \vartheta_k(u_k - \bar{u}))(u_k - \bar{u})^2$$
$$\geq \frac{1}{2}J''(\bar{u} + \vartheta_k(u_k - \bar{u}))(u_k - \bar{u})^2.$$

Dividing the above inequality by $\frac{\rho_k^2}{2}$ we get

$$J''(\bar{u} + \vartheta_k(u_k - \bar{u}))v_k^2 \leq \frac{1}{k}. \quad (3.10)$$

Denoting by $u_{\vartheta_k} = \bar{u} + \vartheta_k(u_k - \bar{u})$, y_{ϑ_k} its associated state, and φ_{ϑ_k} the corresponding adjoint state, we get from (3.7)

$$J''(\bar{u} + \vartheta_k(u_k - \bar{u}))v_k^2 = \kappa\|v_k\|_{L^2(\Sigma)}^2$$
$$+ \int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k}\Big] z_{\vartheta_k,v_k}^2 \,\mathrm{d}x\,\mathrm{d}t, \quad (3.11)$$

where z_{ϑ_k,v_k} satisfies the equation

$$\left\{ \begin{array}{l} \dfrac{\partial z_{\vartheta_k,v_k}}{\partial t} - \Delta z_{\vartheta_k,v_k} + a z_{\vartheta_k,v_k} + \dfrac{\partial f}{\partial y}(x,t,y_{\vartheta_k}) z_{\vartheta_k,v_k} = 0 \\ \partial_\nu z_{\vartheta_k,v_k} = v_k \text{ on } \Sigma, \ z_{\vartheta_k,v_k}(0) = 0 \text{ in } \Omega. \end{array} \right. \quad (3.12)$$

Now, we study the lower limit of (3.10). From Lemma 2.1 and the boundedness of $\{v_k\}_{k=1}^\infty$ and $\{y_{\vartheta_k}\}_{k=1}^\infty$ in $L^2(\Sigma)$ and $L^\infty(Q)$, respectively, we infer the boundedness of $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ in $W(0,\infty)$. Therefore, we can extract a subsequence, that we denote in the same way, such that $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ converges weakly in $W(0,\infty)$. Moreover, the convergence $u_{\vartheta_k} \to \bar u$ in U_q implies $y_{\vartheta_k} = G_{p,q}(u_{\vartheta_k}) \to G_{p,q}(\bar u) = \bar y$ in $Y_{p,q}$. Using this and the convergence $v_k \rightharpoonup v$ in $L^2(\Sigma)$, it is straightforward to pass to the limit in (3.12) and to deduce that $z_{\vartheta_k,v_k} \rightharpoonup z_v$ in $W(0,\infty)$, where z_v is the solution of (2.14). Further, the convergence of $y_{\vartheta_k} \to \bar y$ in $Y_{p,q}$ implies the convergence in $L^p(0,\infty;L^2(\Omega)) \cap L^\infty(Q)$. Then, from [4, Theorem A.4] we infer that $\varphi_{\vartheta_k} \to \bar\varphi$ in $W(0,\infty) \cap L^\infty(Q)$. Indeed, subtracting the equations satisfied by φ_{ϑ_k} and $\bar\varphi$ we get for $\psi_k = \varphi_{\vartheta_k} - \bar\varphi$

$$\left\{\begin{array}{l} -\dfrac{\partial \psi_k}{\partial t} - \Delta\psi_k + a\psi_k + \dfrac{\partial f}{\partial y}(x,t,\bar y)\psi_k \\ \qquad = y_{\vartheta_k} - \bar y + \Big[\frac{\partial f}{\partial y}(x,t,\bar y) - \frac{\partial f}{\partial y}(x,t,y_{\vartheta_k})\Big]\varphi_{\vartheta_k} \text{ in } Q, \\ \partial_\nu \psi_k = 0 \text{ on } \Sigma,\ \lim_{t\to\infty}\|\psi_k(t)\|_{L^2(\Omega)} = 0. \end{array}\right.$$

Then, using (3.3), the established convergence $y_{\vartheta_k} \to \bar y$ and [4, Theorem A.4] we get the claimed convergence of $\{\varphi_{\vartheta_k}\}_{k=1}^\infty$ to $\bar\varphi$.

Now, we take the lower limit in (3.10). For this purpose we take into account that $z_{\vartheta_k,v_k} \rightharpoonup z_v$ in $L^2(Q)$ and $v_k \rightharpoonup v$ in $L^2(\Sigma)$. Hence, we get by (3.10)

$$\begin{aligned} &0 \ge \liminf_{k\to\infty} J''(\bar u + \vartheta_k(u_k - \bar u))v_k^2 \ge \liminf_{k\to\infty} \|z_{\vartheta_k,v_k}\|^2_{L^2(Q)} \\ &+ \liminf_{k\to\infty} \int_Q -\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \kappa \liminf_{k\to\infty} \|v_k\|^2_{L^2(\Sigma)} \ge \|z_{\bar u,v}\|^2_{L^2(Q)} + \kappa\|v\|^2_{L^2(\Sigma)} \\ &+ \liminf_{k\to\infty} \int_Q -\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.13}$$

Below we prove that

$$\begin{aligned} &\lim_{k\to\infty} \int_Q \frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q \frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi z_v^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.14}$$

Thus, (3.2) and (3.13)-(3.14) yield $J(\bar u)v^2 \le 0$.

Let us prove (3.14). Given $\varepsilon > 0$, (3.3) implies the existence of $T_\varepsilon > 0$ such that $\|\bar\varphi(t)\|_{L^2(\Omega)} < \varepsilon$ for every $t \ge T_\varepsilon$. Further, the convergence $z_{\vartheta_k,v_k} \rightharpoonup z_{\bar u,v}$ in $W(0,\infty)$ implies the convergence $z_{\vartheta_k,v_k} \to z_{\bar u,v}$ in $L^2(Q_{T_\varepsilon})$. Using these properties and (2.3) with $M = \|\bar y\|_{L^\infty(Q)}$ we get

$$\begin{aligned} &\int_Q \Big|\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k} - \frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi z^2_{\bar u,v}\Big|\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_Q \Big|\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} - \frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi\Big| z^2_{\vartheta_k,v_k}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{Q_{T_\varepsilon}} \Big|\frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi\Big| |z^2_{\vartheta_k,v_k} - z^2_{\bar u,v}|\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{T_\varepsilon}^\infty \int_\Omega \Big|\frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi\Big| |z^2_{\vartheta_k,v_k} - z^2_{\bar u,v}|\,\mathrm{d}x\,\mathrm{d}t \\ &\le \Big\|\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} - \frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi\Big\|_{L^\infty(Q)} \|z_{\vartheta_k,v_k}\|^2_{L^2(Q)} \\ &+ C_M\|\bar\varphi\|_{L^\infty(Q)}\|z_{\vartheta_k,v_k} - z_{\bar u,v}\|_{L^2(Q_{T_\varepsilon})}\|z_{\vartheta_k,v_k} + z_{\bar u,v}\|_{L^2(Q_{T_\varepsilon})} \\ &+ C_M\varepsilon\|z_{\vartheta_k,v_k} - z_{\bar u,v}\|_{L^2(Q)}\|z_{\vartheta_k,v_k} + z_{\bar u,v}\|_{L^2(Q)} \\ &= I_1 + I_2 + I_3 \end{aligned}$$

The convergence $(y_{\vartheta_k},\varphi_{\vartheta_k}) \to (\bar y,\bar\varphi)$ in $L^\infty(Q)^2$ and the boundedness of $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ in $W(0,\infty)$ yield $I_1 \to 0$ as $k \to \infty$. The convergence $z_{\vartheta_k,v_k} \to z_{\bar u,v}$ in $L^2(Q_{T_\varepsilon})$ implies that $I_2 \to 0$ as well. Finally, we have $|I_3| \le C\varepsilon$, where we have used again the boundedness of $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ in $W(0,\infty)$. Since $\varepsilon > 0$ is arbitrarily small, (3.14) follows.

Step III - Final contradiction. The facts proved in Steps I and II along with the assumption $J''(\bar u)v^2 > 0$ for every $v \in C_{\bar u}\setminus\{0\}$ lead to $v = 0$ and $z_v = 0$. Therefore, looking at the relations (3.13) we obtain with (3.14) and $\|v_k\|_{L^2(\Sigma)} = 1$

$$0 \ge \liminf_{k\to\infty} J''(\bar u + \vartheta_k(u_k - \bar u))v_k^2 \ge \liminf_{k\to\infty} \kappa\|v_k\|^2_{L^2(\Sigma)} = \kappa,$$

which contradicts the hypothesis $\kappa > 0$. ■

IV. APPROXIMATION BY FINITE HORIZON PROBLEMS

For every $0 < T < \infty$ we consider the control problem

$$(\mathrm{P}_T) \qquad \min_{u\in U_{T,ad}} J_T(u),$$

where $U_{T,ad} = \{u \in L^2(\Sigma_T) : \alpha \le u(x,t) \le \beta \text{ a.e.}\}$,

$$J_T(u) = \frac{1}{2}\int_{Q_T} (y_{T,u} - y_d)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{\Sigma_T} u^2\,\mathrm{d}x\,\mathrm{d}t$$

and $y_{T,u}$ denotes the solution of the equation

$$\left\{\begin{array}{l} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(x,t,y) = g \text{ in } Q_T, \\ \partial_\nu y = u \text{ on } \Sigma_T,\ y(0) = y_0 \text{ in } \Omega. \end{array}\right. \tag{4.1}$$

The next two theorems establish the convergence of the approximating problems (P_T) to (P) as $T \to \infty$.

THEOREM 4.1: For every $T > 0$ the control problem (P_T) has at least one solution u_T. If (P) has a feasible control u_0, then the extensions by zero $\{\hat u_T\}_{T>0}$ of any family of solutions are bounded in $L^2(\Sigma)$. Every weak limit $\bar u$ in $L^2(\Sigma)$ of a sequence $\{\hat u_{T_k}\}_{k=1}^\infty$ with $T_k \to \infty$ as $k \to \infty$ is a solution of (P). Moreover, strong convergence $\hat u_{T_k} \to \bar u$ in U_q holds for every $q \in [2,\infty)$.

Proof: The proof of existence of a solution for (P_T) is standard. If u_0 is a feasible control for (P), then using that $J_T(u_T) \le J_T(u_0)$ we infer that $\{\hat u_T\}_{T>0}$ is bounded in $L^2(\Sigma)$. Therefore, there exist sequences $\{\hat u_{T_k}\}_{k=1}^\infty$ with $T_k \to \infty$ weakly converging to some $\bar u \in L^2(\Sigma)$. Arguing similarly as in the proof of [4, Theorem 4.1] we deduce that $\bar u$ is a solution of (P) and $J(\hat u_{T_k}) \to J(\bar u)$. By the structure of the cost, that includes the Tikhonov term, it is immediate that $\hat u_T \to \bar u$ strongly in $L^2(\Sigma)$. Since $\{\hat u_{T_k}\}$ is bounded in $L^\infty(\Sigma)$, the convergence in U_q follows. ■

THEOREM 4.2: Let $\bar u$ be a strict local minimizer of (P). Then, there exist $T_0 \in (0,\infty)$ and a family $\{u_T\}_{T>T_0}$ of local minimizers to (P$_T$) such that the convergence $\hat u_T \to \bar u$ in U_q holds as $T \to \infty$ for every $q \in [2,\infty)$.

Proof: Since $\bar u$ is a strict local minimizer of (P), there exists $\rho > 0$ such that $J(\bar u) < J(u)$ for every $u \in U_{ad} \cap B_\rho(\bar u)$ with $u \neq \bar u$, where $B_\rho(\bar u)$ is the closed ball in $L^2(Q_\omega)$ centered at $\bar u$ and radius $\rho > 0$. We consider the control problems

$$(\mathrm{P}_\rho) \min_{u\in B_\rho(\bar u)\cap U_{ad}} J(u) \quad \text{and} \quad (\mathrm{P}_{T,\rho}) \min_{u\in B_{T,\rho}(\bar u)\cap U_{T,ad}} J_T(u),$$

where $B_{T,\rho}(\bar u) = \{u \in L^2(\Sigma_T) : \|u-\bar u\|_{L^2(\Sigma_T)} \le \rho\}$. Obviously $\bar u$ is the unique solution of (P$_\rho$). Existence of a solution u_T of (P$_{T,\rho}$) is straightforward. Then, arguing as in the proof of Theorem 4.1 and using the uniqueness of the solution of (P$_\rho$), we deduce the convergence $\hat u_T \to \bar u$ in $L^2(\Sigma)$ as $T \to \infty$. This implies the existence of $T_0 > 0$ such that $\|u_T - \bar u\|_{L^2(\Sigma_T)} \le \|\hat u_T - \bar u\|_{L^2(\Sigma)} < \rho$ for all $T > T_0$. Hence, u_T is also a local minimizer of (P$_T$) for $T > T_0$. The strong convergence $\hat u_T \to \bar u$ in U_q follows from the convergence in $L^2(\Sigma)$ and the boundedness of U_{ad} in $L^\infty(\Sigma)$. ■

THEOREM 4.3: Suppose that $\bar u$ is a local minimizer of (P) satisfying the second order sufficient optimality condition. We assume that $\frac{\partial f}{\partial y}(x,t,y) \ge 0$ holds for all $y \in \mathbb{R}$ and almost all $(x,t) \in Q$. Let $\{u_T\}_{T>T_0}$ be a sequence of local minimizers of problems (P$_T$) such that $\hat u_T \to \bar u$ in $L^2(\Sigma)$. Then, there exist $T^* \in [T_0,\infty)$ and a constant C such that for every $T \ge T^*$

$$\|\hat u_T - \bar u\|_{L^2(\Sigma)} + \|\hat y_T - \bar y\|_{W(0,\infty)} \le C\Big(\|y_T(T)\|_{L^2(\Omega)} + \|y_d\|_{L^2(T,\infty;L^2(\Omega))} + \|g\|_{L^2(T,\infty;L^2(\Omega))}\Big), \tag{4.2}$$

where $\hat y_T \in Y_{p,q}$ is the solution of the equation

$$\begin{cases} \dfrac{\partial \hat y_T}{\partial t} - \Delta \hat y_T + a\hat y_T + f(x,t,\hat y_T) = g \text{ in } Q_T, \\ \partial_\nu \hat y_T = \hat u_T \text{ on } \Sigma,\ \hat y_T(0) = y_0 \text{ in } \Omega. \end{cases} \tag{4.3}$$

Proof: As proved in [4, Theorem 4.3], there exists $T_0^* < \infty$ such that $J_T(u_T) \le J_T(\bar u)$ for all $T \ge T_0^*$. We are going to use inequality (3.7). For this purpose, we take $T^* \in [T_0^*,\infty)$ such that $\|\hat u_T - \bar u\|_{L^2(\Sigma)} < \varepsilon$ for all $T \ge T^*$. Then, given $T \ge T^*$, (3.7) yields

$$\begin{aligned}
&\frac{\delta}{2}\|\hat u_T - \bar u\|^2_{L^2(\Sigma)} \le J(\hat u_T) - J(\bar u) = J_T(u_T) - J_T(\bar u) \\
&+ \frac12 \int_T^\infty \|\hat y_T(t) - y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&- \frac12 \int_T^\infty \|\bar y(t) - y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t - \frac{\kappa}{2}\int_T^\infty \|\bar u(t)\|^2_{L^2(\omega)}\,\mathrm{d}t \\
&\le \frac12 \int_T^\infty \|\hat y_T(t) - y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t,
\end{aligned}$$

which leads to

$$\|\hat u_T - \bar u\|_{L^2(\Sigma)} \le \frac{1}{\sqrt\delta}\|\hat y_T - y_d\|_{L^2(T,\infty;L^2(\Omega))}. \tag{4.4}$$

To prove the first estimate of (4.2) we observe that $\hat y_T$ satisfies the equation

$$\begin{cases} \dfrac{\partial \hat y_T}{\partial t} - \Delta \hat y_T + a\hat y_T + f(x,t,\hat y_T) = g \text{ in } \Omega\times(T,\infty), \\ \partial_n \hat y_T = 0 \text{ on } \Gamma\times(T,\infty),\ \hat y_T(T) = y_T(T) \text{ in } \Omega. \end{cases}$$

Testing this equation with $\hat y_T$, and using that $f(x,t,\hat y_T)\hat y_T \ge 0$ due to the monotonicity of f with respect to y and (2.1), it follows that

$$\begin{aligned}
&\frac12\|\hat y_T(t)\|^2_{L^2(\Omega)} + \int_T^\infty\int_\Omega [|\nabla \hat y_T|^2 + a\hat y_T^2]\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac12 \|y_T(T)\|^2_{L^2(\Omega)} + \int_T^\infty\int_\Omega g\hat y_T\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using this we infer with (2.7) that $\|\hat y_T\|_{L^2(T,\infty;L^2(\Omega))} \le C'\Big(\|y_T(T)\|_{L^2(\Omega)} + \|g\|_{L^2(T,\infty;L^2(\Omega))}\Big)$. This inequality and (4.4) imply the estimate of the controls in (4.2). To get the estimate for the states we observe that $\phi_T = \hat y_T - \bar y$ satisfies the equation

$$\begin{cases} \dfrac{\partial \phi_T}{\partial t} - \Delta\phi_T + a\phi_T + \dfrac{\partial f}{\partial y}(x,t,y_{T,\theta})\phi_T = 0 \ \text{ in } Q, \\ \partial_\nu \phi_T = \hat u_T - \bar u \text{ on } \Sigma,\ \phi_T(0) = 0 \text{ in } \Omega, \end{cases}$$

where $y_{T,\theta} = \bar y + \theta_T(\hat y_T - \bar y)$ with $\theta_T : Q \longrightarrow [0,1]$ measurable. Then, applying Theorem A3 and Remark 5.2 of [4] we infer $\|\phi_T\|_{W(0,\infty)} \le K_3\|\hat u_T - \bar u\|_{L^2(\Sigma)}$. Combining this estimate with the one established for the controls we deduce (4.2). ■

REMARK 4.1: We note that all the results of this paper remain valid if the system is controlled from a subset $\Sigma_\omega = \omega\times(0,\infty)$ with ω a measurable subset of Γ.

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