On the value function for optimal control of semilinear parabolic equations

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Abstract

The value function for an infinite horizon tracking type optimal control problem with semilinear parabolic equation is investigated. In view of a possible nonconvexity of the optimal control problem, a local version of the value function is considered. Its differentiability is proved for initial data in a neighborhood around the nominal initial value, provided a second order sufficient optimality condition is fulfilled for the nominal locally optimal control. Based on the differentiability of the value function, a Hamilton-Jacobi-Bellman equation is derived.

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1. Introduction

In this paper, we study the value function associated with the optimal control problem

$$\text{(P)} \min_{u \in L^2(Q_\omega)} J(u) := \frac{1}{2}\int_Q (y_u - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{Q_\omega} u^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where $Q = \Omega \times (0,\infty)$ and $Q_\omega = \omega \times (0,\infty)$ with Ω being a bounded domain of $\mathbb{R}^n$, $1 \le n \le 3$, and ω a measurable subset of Ω of positive Lebesgue measure. We assume that $\kappa > 0$ and $y_d \in L^2(Q)$. We denote by y_u the state associated with u, solution of the following Neumann initial-boundary value problem

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) = g + \chi_\omega u & \text{in } Q, \\ \partial_n y = 0 \quad \text{on } \Sigma,\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Above, the notation $\Sigma = \Gamma \times (0,\infty)$ is used, where Γ is the boundary of Ω that we assume to be Lipschitz. Assumptions on the other data will be given later.

The main goal of this paper is to investigate the sensitivity of the optimal solution with respect to the initial datum and subsequently regularity properties of the value function associated with the optimal control problem (P). The value function plays a fundamental role in optimal feedback control of evolution equations. Indeed, an optimal feedback control can be obtained by means of the verification theorem which involves the gradient of the value function. The value function itself satisfies a Hamilton-Jacobi-Bellman (HJB) equation, possibly only in the viscosity sense. Its numerical realisation is a significant challenge, since inevitably one is confronted with a curse of dimensionality.

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Nonetheless, investigating the HJB equation is an essential first step towards numerical techniques for obtaining approximating solutions.

Value functions were extensively studied in the literature. For the control of ordinary differential equations, we mention exemplarily [12]. Infinite-dimensional control systems were discussed in the general expositions [1, 2, 6], and in [3, 4, 5]. For semilinear parabolic equations, we mention [13] that inspired the investigations in our paper and [3, 11] that are related to parabolic equations, as well.

In the references cited above, the problems are posed with initial data in $L^2(\Omega)$. This is a convenient setting since, in the case that the value function is differentiable, its operator representation of the derivatives is again in $L^2(\Omega)$. As a consequence of this choice of initial data in $L^2(\Omega)$, dimension-dependent restrictions on the degree of the nonlinearity f arise. Therefore in the present work, the parameter space for the initial conditions is chosen to be $L^\infty(\Omega)$, which significantly enlarges the range of the admissible degree of nonlinearities for the nonlinearity f. Moreover, as it turns out, once the differentiability of the value function is guaranteed, the derivative enjoys extra regularity and allows a convenient representation in terms of the adjoint state. One of the difficulties which has to be overcome along this approach results from the fact that the control-to-state mapping is not in general differentiable for L^2-controls, while the optimization in (P) is subject to controls of $L^2(Q_\omega)$. Therefore, in several steps we show for an auxiliary problem that any of its locally optimal controls is automatically essentially bounded, i.e. it belongs to $L^\infty(Q_\omega)$ although the controls are not restricted. The reader is referred to [10] where this issue has been deeply studied for finite horizon optimal control problems and arbitrary dimension $n \geq 2$.

Concerning the techniques involved to study the regularity of the value function, let us point out that in [3, 5], for instance, the approach rests primarily on a direct study of the value function, verifying, at first Lipschitz and semi-concavity properties. This is followed by results on the superdifferential of the value function along optimal trajectories. Under additional assumptions the superdifferentials are in fact Frechet derivatives. In [13] and the present paper, the differentiability of the value function is essentially obtained by the chain rule applied to the mapping from the initial conditions to the optimal values of the cost-functional, with optimal controls and optimal states as intermediate quantities.

The plan of the paper is as follows: Section 2 deals with the well-posedness of the state equation in suitable spaces and with the differentiability of the control-to-state mapping for controls of $L^p(0,\infty;L^2(\omega))$ with $p > \frac{4}{4-n}$ if $n \in \{2,3\}$ or $p \geq 2$ if $n = 1$. In Section 3, the existence of optimal controls in $L^2(Q_\omega)$ is proved. Moreover, it will be confirmed that they belong to $L^p(0,\infty;L^2(\omega)) \cap L^2(Q_\omega) \cap L^\infty(Q_\omega)$. First-order necessary and second-order sufficient optimality conditions are established in Section 4. Second-order sufficient conditions are indispensable for the stability of locally optimal controls. Section 5 contains the main result of our paper – the differentiability of the local value function and the representation of its derivative in terms of the adjoint state function. In Section 6 we discuss the Hamilton-Jacobi-Bellman equation associated to (P) for the special case $y_d = g = 0$.

2. Preliminary results

In this section, we analyze the well-posedness of the state equation and the differentiability of the associated control-to-state mapping. For this purpose, we assume the following hypotheses:

(A1) We suppose that $a \in L^\infty(Q)$, $a \geq 0$, $a \not\equiv 0$, $g, y_d \in L^2(Q) \cap L^p(0,\infty;L^2(\Omega))$ with $p > \frac{4}{4-n}$ if $n = 2$ or 3 and $p \geq 2$ if $n = 1$, and $y_0 \in L^\infty(\Omega)$. Along this paper this condition on p will be assumed.

(A2) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2, $f'(s) \geq 0$ for all $s \in \mathbb{R}$, and $f(0) = 0$.

The number $p \in (\frac{4}{4-n}, \infty]$ introduced in Assumption (A1) remains fixed throughout this paper. The boundedness of y_0 is required to make the nonlinearity of the parabolic equation well defined.

Due to the assumptions on the function a, we know that there exists a constant $C_a > 0$ such that

$$C_a \|y\|_{H^1(\Omega)} \leq \Big(\int_\Omega [|\nabla y|^2 + a y^2] \,\mathrm{d}x \Big)^{\frac{1}{2}} \quad \forall y \in H^1(\Omega). \tag{2.1}$$

In equation (1.1), χ_ω denotes the characteristic function of the set ω. Hence, we have $(\chi_\omega u)(x,t) = u(x,t)$ if $(x,t) \in Q_\omega$ and $\chi_\omega u$ is zero in $Q \setminus Q_\omega$.

For every $T \in (0,\infty]$, we define $W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) : \frac{\partial y}{\partial t} \in L^2(0,T;H^1(\Omega)^*)\}$. We know that $W(0,T)$ is a Hilbert space endowed with the Hilbertian norm

$$\|y\|_{W(0,T)} = \Big(\|y\|^2_{L^2(0,T;H^1(\Omega))} + \Big\|\frac{\partial y}{\partial t}\Big\|^2_{L^2(0,T;H^1(\Omega)^*)}\Big)^{\frac{1}{2}}.$$

Definition 2.1. *Given $u \in L^2(Q_\omega)$, we say that a function $y : Q \longrightarrow \mathbb{R}$ is a solution to (1.1) if for every $T \in (0,\infty)$ the restriction of y to $Q_T = \Omega \times (0,T)$ belongs to $W(0,T)$, $f(y_{|_{Q_T}}) \in L^2(Q_T)$, and y satisfies the following equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) = g + \chi_\omega u \text{ in } Q_T, \\ \partial_n y = 0 \text{ on } \Sigma_T,\ y(0) = y_0 \text{ in } \Omega, \end{cases} \tag{2.2}$$

where $\Sigma_T = \Gamma \times (0,T)$.

We have the following result on well-posedness of the state equation (1.1).

Theorem 2.2. *Under the assumptions (A1) and (A2), for every control $u \in L^2(Q_\omega)$ there exists a unique solution $y_u \in W(0,\infty)$ of (1.1). Moreover, $f(y_u)$ belongs to $L^2(Q)$ and there is a constant C independent of u and y_0 such that*

$$\|y_u\|_{C([0,\infty);L^2(\Omega))} + \|y_u\|_{L^2(0,\infty;H^1(\Omega))} \le C\Big(\|y_0\|_{L^2(\Omega)} + \|g + \chi_\omega u\|_{L^2(Q)}\Big), \tag{2.3}$$

$$\|y_u\|_{W(0,\infty)} + \|f(y_u)\|_{L^2(Q)} \le C\Big(\|f(y_0)\|_{L^\infty(\Omega)} + \|y_0\|_{L^\infty(\Omega)} + \|g + \chi_\omega u\|_{L^2(Q)}\Big). \tag{2.4}$$

In addition, if $u \in L^p(0,\infty;L^2(\omega))$, then $y_u \in L^\infty(Q)$ and the following estimate holds with a constant C_∞ independent of u and y_0

$$\|y_u\|_{L^\infty(Q)} \le C_\infty\Big(\|y_0\|_{L^\infty(\Omega)} + \|g + \chi_\omega u\|_{L^2(Q)} + \|g + \chi_\omega u\|_{L^p(0,\infty;L^2(\Omega))}\Big). \tag{2.5}$$

Proof. The uniqueness is an immediate consequence of the monotonicity of f. Let us prove the existence and the associated estimates. For every integer $k \ge 1$, we define $f_k(s) = f(\text{Proj}_{[-k,+k]}(s))$, where $\text{Proj}_{[-k,+k]}(s)$ denotes the projection of s on the interval $[-k,+k]$. We also take a sequence $\{\tilde y_{0k}\}_{k=1}^\infty \subset H^1(\Omega)$ such that $\tilde y_{0k} \to y_0$ in $L^2(\Omega)$. For $M = \|y_0\|_{L^\infty(\Omega)}$ we define $y_{0k}(x,t) = \text{Proj}_{[-M,+M]}(\tilde y_{0k}(x,t))$. Then, we still have that $\{y_{0k}\}_{k=1}^\infty \subset H^1(\Omega)$ and $y_{0k} \to y_0$ in $L^2(\Omega)$. In addition, the inequality $\|y_{0k}\|_{L^\infty(\Omega)} \le \|y_0\|_{L^\infty(\Omega)}$ holds. For every $T \in (0,\infty)$, we consider the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f_k(y) = g + \chi_\omega u \text{ in } Q_T, \\ \partial_n y = 0 \text{ on } \Sigma_T,\ y(0) = y_{0k} \text{ in } \Omega. \end{cases} \tag{2.6}$$

By an easy application of Schauder's fixed point theorem, we infer the existence of a solution $y_k \in W(0,T)$ of (2.6). Moreover, since $y_{0k} \in H^1(\Omega)$ and $g+\chi_\omega u - f_k(y_k) \in L^2(Q_T)$, we deduce from [16, Proposition III 2.5] that $y_k \in H^1(Q_T)$. Testing the equation (2.6) with y_k, using (2.1), and that $f_k(s)s \ge 0$ for all $s \in \mathbb{R}$ (notice that $f(0) = 0$), we get

$$\begin{aligned} &\frac{1}{2}\|y_k(T)\|^2_{L^2(\Omega)} + C_a^2\|y_k\|^2_{L^2(0,T;H^1(\Omega))} \\ &\le \frac{1}{2}\|y_k(T)\|^2_{L^2(\Omega)} + \int_{Q_T}[|\nabla y_k|^2 + ay_k^2]\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f_k(y_k)y_k\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T}(g+\chi_\omega u)y_k\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|y_{0k}\|^2_{L^2(\Omega)} \\ &\le \frac{1}{2C_a^2}\|g+\chi_\omega u\|^2_{L^2(Q)} + \frac{C_a^2}{2}\|y_k\|^2_{L^2(Q)} + \frac{1}{2}\|y_{0k}\|^2_{L^2(\Omega)}. \end{aligned}$$

Since $y_{0k} \to y_0$ in $L^2(\Omega)$ and $T \in (0,\infty)$ is arbitrary this inequality implies

$$\|y_k\|_{L^\infty(0,\infty;L^2(\Omega))} + \|y_k\|_{L^2(0,\infty;H^1(\Omega))} \le 2\max\Big\{1, \frac{1}{C_a^4}\Big\}\Big(\|y_0\|_{L^2(\Omega)} + \|g+\chi_\omega u\|_{L^2(Q)}\Big). \tag{2.7}$$

Now, we define the function $F_k : \mathbb{R} \longrightarrow \mathbb{R}$ by $F_k(s) = \int_0^s f_k(\theta)\,d\theta$. The monotonicity of f_k and the fact that $f_k(0) = 0$ imply that $0 \le F_k(s) \le s f_k(s) \le s f(s)$. Using these properties, we get

$$\int_{Q_T} \frac{\partial y_k}{\partial t} f_k(y_k)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \frac{d}{dt}\int_\Omega F_k(y_k)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega F_k(y_k(x,T))\,\mathrm{d}x - \int_\Omega F_k(y_k(x,0))\,\mathrm{d}x$$
$$\ge -\int_\Omega y_{0k}(x) f(y_{0k}(x))\,\mathrm{d}x \ge -|\Omega|\,\|f(y_0)\|_{L^\infty(\Omega)}\|y_0\|_{L^\infty(\Omega)}.$$

Using this inequality and testing (2.6) with $f_k(y_k)$, we obtain

$$\|f_k(y_k)\|^2_{L^2(Q_T)} \le \int_{Q_T} \frac{\partial y_k}{\partial t} f_k(y_k)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} [f_k'(y_k)|\nabla y_k|^2 + a y_k f_k(y_k)]\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f_k(y_k)^2\,\mathrm{d}x\,\mathrm{d}t + |\Omega|\,\|f(y_0)\|_{L^\infty(\Omega)}\|y_0\|_{L^\infty(\Omega)}$$
$$= \int_{Q_T} (g + \chi_\omega u) f_k(y_k)\,\mathrm{d}x\,\mathrm{d}t + |\Omega|\,\|f(y_0)\|_{L^\infty(\Omega)}\|y_0\|_{L^\infty(\Omega)}$$
$$\le \frac{1}{2}\|g + \chi_\omega u\|^2_{L^2(Q)} + \frac{1}{2}\|f_k(y_k)\|^2_{L^2(Q_T)} + \frac{|\Omega|}{2}\Big(\|f(y_0)\|^2_{L^\infty(\Omega)} + \|y_0\|^2_{L^\infty(\Omega)}\Big).$$

Once again, since T is arbitrary we infer from the above estimate

$$\|f_k(y_k)\|_{L^2(Q)} \le \max\{1, \sqrt{|\Omega|}\}\Big(\|f(y_0)\|_{L^\infty(\Omega)} + \|y_0\|_{L^\infty(\Omega)} + \|g + \chi_\omega u\|_{L^2(Q)}\Big). \tag{2.8}$$

Combining (2.7) and (2.8), we get from (2.6)

$$\|y_k\|_{W(0,\infty)} \le C\Big(\|f(y_0)\|_{L^\infty(\Omega)} + \|y_0\|_{L^\infty(\Omega)} + \|g + \chi_\omega u\|_{L^2(Q)}\Big). \tag{2.9}$$

From (2.8) and (2.9), we infer the existence of a subsequence of $\{y_k\}_{k=1}^\infty$, denoted in the same way, such that $y_k \rightharpoonup y$ in $W(0,\infty)$ and $f_k(y_k) \rightharpoonup \phi$ in $L^2(Q)$. Since the embedding $W(0,T) \subset L^2(Q_T)$ is compact, we deduce that $y_k \to y$ in $L^2(Q_T)$ for every $T < \infty$. Hence, taking a new subsequence, we have that $y_k(x,t) \to y(x,t)$ for almost all $(x,t) \in Q_T$. This implies that $f_k(y_k(x,t)) \to f(y(x,t))$ for almost every $(x,t) \in Q_T$. Therefore, the identity $\phi = f(y)$ holds. Using these convergences, it is straightforward to pass to the limit in (2.6) and to get that y satisfies (2.2) for every $T > 0$. Hence, y is the solution of (1.1). Moreover, the estimates (2.3) and (2.4) are immediately obtained from (2.7)–(2.9). Finally, the $L^\infty(Q)$ regularity of y and the estimate (2.5) follow from [7, Theorem A.2 and Remark 5.2]. □

While Theorem 2.3 ensures well-posedness of the state equation in $L^2(Q_\omega)$, the sensitivity analysis requires the differentiability of the mapping $u \mapsto y_u$ that cannot be proved for $u \in L^2(Q_\omega)$. This is why we introduce below the dense subspace $\mathcal{U}_p$ of $L^2(Q_\omega)$, where differentiability can be shown. We will prove later that locally optimal controls belong to $L^\infty(Q_\omega)$ so that first- and second-order optimality conditions can be derived. This is the roadmap for the sensitivity analysis.

We set $\mathcal{U}_p = L^p(0,\infty; L^2(\omega)) \cap L^2(Q_\omega)$ and endow $\mathcal{U}_p$ with the norm

$$\|u\|_{\mathcal{U}_p} = \|u\|_{L^2(Q_\omega)} + \|u\|_{L^p(0,\infty;L^2(\omega))}.$$

Then, $\mathcal{U}_p$ is a Banach space that is reflexive and separable if $p < \infty$. We notice that $L^2(Q_\omega) \cap L^\infty(Q_\omega) \subset \mathcal{U}_p$. Indeed, this follows from the inequality

$$\|u\|_{L^p(0,\infty;L^2(\omega))} \le \|u\|^{\frac{p-2}{p}}_{L^\infty(Q_\omega)}\|u\|^{\frac{2}{p}}_{L^2(Q_\omega)} \le \frac{p-2}{p}\|u\|_{L^\infty(Q_\omega)} + \frac{2}{p}\|u\|_{L^2(Q_\omega)}. \tag{2.10}$$

According to Theorem 2.2, the mapping $G : \mathcal{U}_p \longrightarrow W(0,\infty) \cap L^\infty(Q)$ associating to every control u the corresponding state, $G(u) = y_u$, is well defined. Concerning the differentiability of G, the next result follows from [7, Theorems 2.2 and 3.1, and Remark 5.2].

Theorem 2.3. *The mapping G is of class C^2. Moreover, given $u, v, v_1, v_2 \in \mathcal{U}_p$, the derivatives $z_v = G'(u)v$ and $z_{v_1,v_2} = G''(u)(v_1, v_2)$ are the unique solutions of the equations*

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z = \chi_\omega v \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega, \end{cases} \tag{2.11}$$

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z = -f''(y_u)z_{v_1}z_{v_2} \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega, \end{cases} \tag{2.12}$$

where $z_{v_i} = G'(u)v_i$ for $i = 1, 2$.

By this theorem and the chain rule, we obtain the differentiability properties of the cost functional J.

Corollary 2.4. *The functional $J : \mathcal{U}_p \longrightarrow \mathbb{R}$ is of class C^2. For every $u, v, v_1, v_2 \in \mathcal{U}_p$, its derivatives are given by*

$$J'(u)v = \int_{Q_\omega} (\varphi_u + \kappa u)v \,\mathrm{d}x\,\mathrm{d}t, \tag{2.13}$$

$$J''(u)(v_1, v_2) = \int_Q \big[1 - f''(y_u)\varphi_u\big] z_{u,v_1} z_{u,v_2} \,\mathrm{d}x\,\mathrm{d}t + \kappa \int_{Q_\omega} v_1 v_2 \,\mathrm{d}x\,\mathrm{d}t, \tag{2.14}$$

where $z_{u,v_i} = G'(u)v_i$, $i = 1, 2$, and $\varphi_u \in W(0,\infty) \cap BC(\bar\Omega \times [0,\infty))$ satisfies the adjoint equation

$$\begin{cases} -\dfrac{\partial \varphi_u}{\partial t} - \Delta\varphi_u + a\varphi_u + f'(y_u)\varphi_u = y_u - y_d \text{ in } Q, \\ \partial_n \varphi_u = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \|\varphi_u(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{2.15}$$

where $BC(\bar\Omega \times [0,\infty))$ denotes the space of bounded and continuous functions in $\bar\Omega \times [0,\infty)$.

Proof. The reader is referred to [7, Theorem A.4 and Remark 5.2] for the existence proof of a unique solution $\varphi_u \in W(0,\infty) \cap L^\infty(Q)$ of (2.15). The expressions (2.13) and (2.14) follow from (2.11), (2.12), and (2.15) in the standard way. It remains to prove the continuity of φ_u. To do this we define $z_T(x,t) = t\varphi_u(x, T - t)$ for every $T \in (0,\infty)$. From (2.15) we deduce that z_T satisfies the following equation

$$\begin{cases} \dfrac{\partial z_T}{\partial t} - \Delta z_T + az_T + f'(\hat y_u)z_T = t(\hat y_u - \hat y_d) + \hat\varphi_u \text{ in } Q_T, \\ \partial_n z_T = 0 \text{ on } \Sigma_T,\ z_T(0) = 0 \text{ in } \Omega, \end{cases}$$

where $(\hat y_u, \hat y_d, \hat\varphi_u)(t) = (y_u, y_d, \varphi_u)(T - t)$. Since $t(\hat y_u - \hat y_d) + \hat\varphi_u \in L^p(0,T; L^2(\Omega))$ we infer that $z_T \in C(\bar Q_T)$; see, for instance, [9] or [14, Chapter 3]. Finally, the identity $\varphi_u(x,t) = \frac{1}{T-t} z_T(x, T-t)$ implies the continuity of φ_u in $\bar\Omega \times [0, T)$. Since T was taken arbitrarily in $(0,\infty)$, the continuity of φ_u in $\bar\Omega \times [0,\infty)$ follows. □

Remark 2.5. *From [7, Theorem A.3 and Remark 5.2], we deduce that the mapping $G'(u) : \mathcal{U}_p \longrightarrow W(0,\infty)$ can be extended to a linear continuous mapping $G'(u) : L^2(Q_\omega) \longrightarrow W(0,\infty)$. Using this, the expressions (2.13) and (2.14), and the fact that y_u, φ_u belong to $L^\infty(Q) \cap L^2(Q)$ for every $u \in \mathcal{U}_p$, we infer that the linear and bilinear forms $J'(u) : \mathcal{U}_p \longrightarrow \mathbb{R}$ and $J''(u) : \mathcal{U}_p \times \mathcal{U}_p \longrightarrow \mathbb{R}$ can be extended to continuous forms $J'(u) : L^2(Q_\omega) \longrightarrow \mathbb{R}$ and $J''(u) : L^2(Q_\omega) \times L^2(Q_\omega) \longrightarrow \mathbb{R}$, respectively. We mention that $\mathcal{U}_p$ is continuously and densely embedded in $L^2(Q_\omega)$. The continuity is obvious. The density is proved as follows: given $v \in L^2(Q_\omega)$, we set $v_k(x,t) = \mathrm{Proj}_{[-k,+k]}(v(x,t))$ for every integer $k \geq 1$. Then, it is obvious that $\{v_k\}_{k=1}^\infty \subset L^2(Q_\omega) \cap L^\infty(Q_\omega) \subset \mathcal{U}_p$ and $v_k \to v$ in $L^2(Q_\omega)$.*

3. Solvability of (P)

In this section, first we prove the existence of solutions to (P) in $L^2(Q_\omega)$. In a second step, we will prove that every local minimizer belongs to $L^\infty(Q_\omega)$. This information is needed to later prove that the notion of local optimality in the sense of $L^2(Q_\omega)$ is equivalent to that in the sense of $\mathcal{U}_p$, the space where J is of class C^2. This proof will occupy the major part of this section.

Theorem 3.1. *The optimal control problem* (P) *has at least one solution* $\bar{u} \in L^2(Q_\omega)$.

Proof. Let $\{u_k\}_{k=1}^\infty$ be a sequence such that $J(u_k) \searrow \inf(\mathrm{P})$. Then, the inequality $\frac{\kappa}{2}\|u_k\|^2_{L^2(Q_\omega)} \le J(u_k) \le J(u_1)$ for every k implies the boundedness of $\{u_k\}_{k=1}^\infty$ in $L^2(Q_\omega)$. Taking a subsequence that we denote in the same way, we obtain that $u_k \rightharpoonup \bar{u}$ in $L^2(Q_\omega)$. Let us write $y_k = y_{u_k}$. Thanks to (2.4), the sequences $\{y_k\}_{k=1}^\infty$ and $\{f(y_k)\}_{k=1}^\infty$ are bounded in $W(0,\infty)$ and $L^2(Q)$, respectively. Then, selecting again a subsequence, if needed, we have the convergences $y_k \rightharpoonup \bar{y}$ in $W(0,\infty)$ and $f(y_k) \rightharpoonup \phi$ in $L^2(Q)$. Arguing as in the proof of Theorem 2.2, we get $\phi = f(\bar{y})$. Passing to the limit in the state equation satisfied by y_k we conclude that $\bar{y} = y_{\bar{u}}$. Finally, we deduce by standard semicontinuity arguments that $\bar{u}$ is a solution of (P). □

Let us mention the obvious fact that any solution $\bar{u}$ of (P) is also a local solution in the $L^2(Q_\omega)$-sense. The latter means that there exists $\varepsilon > 0$ such that $J(\bar{u}) \le J(u)$ if $\|u - \bar{u}\|_{L^2(Q_\omega)} \le \varepsilon$.

Theorem 3.2. *Let $\bar{u}$ be a local solution of* (P) *in the $L^2(Q_\omega)$-sense. Then $\bar{u}$ belongs to $L^\infty(Q_\omega) \cap \mathcal{U}_p$.*

Proof. Let $\bar{u}$ be a local solution of (P) and fix $\varepsilon > 0$ such that $J(\bar{u}) \le J(u)$ for all $u \in \bar{B}_\varepsilon(\bar{u})$, where $\bar{B}_\varepsilon(\bar{u})$ is the closed ball of $L^2(Q_\omega)$ centered at $\bar{u}$ with radius ε. Define $K_M = \{u \in \bar{B}_\varepsilon(\bar{u}) : \|u\|_{L^\infty(Q_\omega)} \le M\}$. For every $M > 0$ we consider the auxiliary control problems

$$(\mathrm{P}_M) \quad \min_{u \in K_M} I(u) := J(u) + \frac{1}{2}\int_{Q_\omega} (u - \bar{u})^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Since K_M is closed and convex in $L^2(Q_\omega)$, arguing as in the proof of Theorem 3.1 we deduce the existence of a solution u_M of (P_M) for every $M > 0$. The remaining proof of the theorem is split into two steps.

Step I - $\lim_{M\to\infty} \|u_M - \bar{u}\|_{L^2(Q_\omega)} = 0$. Since $K_M \subset \bar{B}_\varepsilon(\bar{u})$ we deduce that $\{u_M\}_{M>0}$ is bounded in $L^2(Q_\omega)$. Hence, there exists a sequence $\{M_k\}_{k=1}^\infty$ tending to ∞ such that $u_{M_k} \rightharpoonup \tilde{u}$ in $L^2(Q_\omega)$ with some $\tilde{u} \in \bar{B}_\varepsilon(\bar{u})$. As in the proof of Theorem 3.1, we have that $y_{M_k} = y_{u_{M_k}} \rightharpoonup \tilde{y} = y_{\tilde{u}}$ in $W(0,\infty)$.

Define $\bar{u}_{M_k}(x,t) = \mathrm{Proj}_{[-M_k,+M_k]}(\bar{u}(x,t))$. It is clear that $\{\bar{u}_{M_k}\}$ converges pointwise to $\bar{u}$. From the Lebesgue dominated convergence theorem, we obtain that $\bar{u}_{M_k} \to \bar{u}$ in $L^2(Q_\omega)$. Therefore, an integer k_ε exists such that $\|\bar{u}_{M_k} - \bar{u}\|_{L^2(Q_\omega)} \le \varepsilon$ for every $k \ge k_\varepsilon$. Consequently, $\bar{u}_{M_k}$ belongs to K_{M_k} for every $k \ge k_\varepsilon$.

Thanks to the local and global optimality of $\bar{u}$ and u_{M_k}, respectively, we infer

$$J(\bar{u}) \le J(\tilde{u}) \le I(\tilde{u}) \le \liminf_{k\to\infty} I(u_{M_k}) \le \limsup_{k\to\infty} I(u_{M_k}) \le \limsup_{k\to\infty} I(\bar{u}_{M_k}) = I(\bar{u}) = J(\bar{u}).$$

These inequalities imply that $\tilde{u} = \bar{u}$. With [8, Lemma 5.2], the convergence of the three summands of $I(u_{M_k})$ follows, in particular $\lim_{k\to\infty} \|u_{M_k} - \bar{u}\|_{L^2(Q_\omega)} = 0$. Since this holds for all weakly convergent sequences $\{u_{M_k}\}_{k=1}^\infty$, the convergence $\lim_{M\to\infty} \|u_M - \bar{u}\|_{L^2(Q_\omega)} = 0$ is obtained.

Step II - Optimality conditions for u_M. We know from Step I that there exists $M_\varepsilon > 0$ such that $\|u_M - \bar{u}\|_{L^2(Q_\omega)} < \varepsilon$ holds for all $M > M_\varepsilon$. Since $u_M \in L^2(Q_\omega) \cap L^\infty(Q_\omega) \subset \mathcal{U}_p$, Corollary 2.4 implies the differentiability of I at u_M. Hence, we have $I'(u_M)(u - u_M) \ge 0$ for every $u \in L^2(Q_\omega)$ satisfying $\|u\|_{L^\infty(Q_\omega)} \le M$. This leads to the optimality conditions

$$\begin{cases} \dfrac{\partial y_M}{\partial t} - \Delta y_M + a y_M + f(y_M) = g + \chi_\omega u_M & \text{in } Q, \\ \partial_n y_M = 0 \ \text{ on } \Sigma, \ y_M(0) = y_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

$$\begin{cases} -\dfrac{\partial \varphi_M}{\partial t} - \Delta \varphi_M + a\varphi_M + f'(y_M)\varphi_M = y_M - y_d \text{ in } Q, \\ \partial_n \varphi_M = 0 \ \text{ on } \Sigma, \ \lim_{t\to\infty} \|\varphi_M(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{3.2}$$

$$\int_{Q_\omega} (\varphi_M + \kappa u_M + u_M - \bar{u})(u - u_M)\,\mathrm{d}x\,\mathrm{d}t \ge 0 \ \ \forall u \in L^2(Q_\omega) \text{ such that } |u(x,t)| \le M \ \text{ for a.a. } (x,t) \in Q_\omega. \tag{3.3}$$

Applying Theorem 2.2 to equation (3.1), we deduce from (2.3) and the boundedness of $\{u_M\}_{M>0}$ in $L^2(Q_\omega)$ that $\|y_M\|_{L^\infty(0,\infty;L^2(\Omega))} + \|y_M\|_{L^2(Q)} \le C_1$ for some real number C_1 independent of M. Using again [7, Theorem A.4 and

Remark 5.2] for (3.2) we infer that $\|\varphi_M\|_{L^\infty(Q)} \le C_2 < \infty$ for all $M \ge M_\varepsilon$. Now, (3.3) implies that

$$u_M(x,t) = \mathrm{Proj}_{[-M,+M]}\Big(-\frac{1}{\kappa}(\varphi_M + u_M - \bar u)(x,t)\Big).$$

Taking $M > \max\{C_2/\kappa, M_\varepsilon\}$ we obtain from the above identity

$$|u_M(x,t)| \le \Big|\mathrm{Proj}_{[-M,+M]}\big(-\frac{1}{\kappa}\varphi_M(x,t)\big)\Big| \\ + \Big|\mathrm{Proj}_{[-M,+M]}\big(-\frac{1}{\kappa}(\varphi_M + u_M - \bar u)(x,t)\big) - \mathrm{Proj}_{[-M,+M]}\big(-\frac{1}{\kappa}\varphi_M(x,t)\big)\Big| \le \frac{C_2}{\kappa} + \frac{1}{\kappa}|u_M(x,t) - \bar u(x,t)|.$$

Now we select an increasing sequence $\{M_k\}_{k=1}^\infty$ of real numbers tending to infinity such that $u_{M_k}(x,t) \to \bar u(x,t)$ for almost all $(x,t) \in Q_\omega$. Passing to the limit in the above inequality with M replaced by M_k, we infer that $|\bar u(x,t)| \le C_2/\kappa$ and, hence, $\bar u \in L^\infty(Q_\omega)$. Finally, the inclusion $\bar u \in \mathcal{U}_p$ follows from (2.10). □

So far, we have considered local solutions of (P) in the $L^2(Q_\omega)$-sense. However, since the functional J is differentiable on $\mathcal{U}_p$, it is natural to ask for local optimality in the $\mathcal{U}_p$-sense, which is defined analogously to the $L^2(Q_\omega)$-sense just substituting the $\mathcal{U}_p$-norm for the $L^2(Q_\omega)$-norm. Now we prove that both concepts are equivalent.

Theorem 3.3. *A control $\bar u \in L^2(Q_\omega)$ is a local solution of* (P) *in the $L^2(Q_\omega)$-sense if and only if $\bar u \in \mathcal{U}_p$ and it is a $\mathcal{U}_p$-local solution of* (P).

Proof. If $\bar u$ is a local solution of (P) in the $L^2(Q_\omega)$-sense, Theorem 3.2 implies that $\bar u \in \mathcal{U}_p$. Moreover, the fact that $\bar u$ is also a local solution of (P) in the $\mathcal{U}_p$-sense is a straightforward consequence of the inequality of the norms $\|u\|_{L^2(Q_\omega)} \le \|u\|_{\mathcal{U}_p}$.

Let us prove the converse implication. By definition of a local solution in the sense of $\mathcal{U}_p$, there exists $\varepsilon > 0$ such that

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U}_p \text{ with } \|u - \bar u\|_{\mathcal{U}_p} \le \varepsilon. \tag{3.4}$$

For $r > 0$ specified below, we consider the following control problem:

$$(\mathrm{P}_r) \qquad \min_{u \in \bar B_r(\bar u)} J(u),$$

where $\bar B_r(\bar u) = \{u \in L^2(Q_\omega) : \|u - \bar u\|_{L^2(Q_\omega)} \le r\}$. As in Theorem 3.1, we obtain the existence of a solution $\tilde u$ of (P_r). Below, we will prove that $\tilde u \in L^\infty(Q_\omega)$ and that there exists a constant M independent of r such that $\|\tilde u\|_{L^\infty(Q_\omega)} \le M$. From this and (2.10) we get that $\tilde u \in \mathcal{U}_p$ and

$$\|\tilde u - \bar u\|_{L^p(0,\infty;L^2(\omega))} \le \|\tilde u - \bar u\|_{L^\infty(Q_\omega)}^{\frac{p-2}{p}} \|\tilde u - \bar u\|_{L^2(Q_\omega)}^{\frac{p}{2}} \le \big(M + \|\bar u\|_{L^\infty(Q_\omega)}\big)^{\frac{p-2}{p}} r^{\frac{p}{2}}.$$

Selecting r satisfying

$$\big(M + \|\bar u\|_{L^\infty(Q_\omega)}\big)^{\frac{p-2}{p}} r^{\frac{p}{2}} \le \varepsilon,$$

we infer that $\|\tilde u - \bar u\|_{\mathcal{U}_p} \le \varepsilon$. Hence, (3.4) and the optimality of $\tilde u$ for (P_r) imply $J(\bar u) \le J(\tilde u) = \inf(\mathrm{P}_r)$. Therefore, $\bar u$ is also a solution of (P_r) and, consequently, $\bar u$ is an $L^2(Q_\omega)$-local solution of (P). It remains to show the estimate $\|\tilde u\|_{L^\infty(Q_\omega)} \le M$. The remainder of this section is devoted to prove this. □

For every integer $k \ge 1$ we define the truncation function $h_k : \mathbb{R} \longrightarrow \mathbb{R}$ of class C^2 by

$$h_k(s) = \begin{cases} k+1 & \text{if } s \ge k+1, \\ 3(s-k)^5 - 7(s-k)^4 + 4(s-k)^3 + s & \text{if } s \in (k, k+1), \\ s & \text{if } s \in [-k, +k], \\ 3(s+k)^5 + 7(s+k)^4 + 4(s+k)^3 + s & \text{if } s \in (-k-1, -k), \\ -k-1 & \text{if } s \le -k-1. \end{cases}$$

The verification of the C^2-property and of the following facts is left to the reader:

$$h_k(0) = 0, \quad |h_k(s)| \leq 1.512|s|, \quad 0 \leq h_k'(s) \leq 1.512, \quad |h_k''(s)| \leq 3.95. \tag{3.5}$$

Associated with h_k we set $f_k(s) = f(h_k(s))$. Then, $f_k : \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2 and

$$f_k(0) = 0,\ |f_k(s)| \leq C'_{f,k}|s|,\ 0 \leq f_k'(s) \leq C'_{f,k} = 1.512 \max_{|\theta|\leq k+1} f'(\theta),\ |f_k''(s)| \leq C''_{f,k} = 2.3 \max_{|\theta|\leq k+1} |f''(\theta)| + 3.95C'_{f,k}. \tag{3.6}$$

Given $u \in L^2(Q_\omega)$, we denote by $y_{k,u}$ the solution of the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f_k(y) = g + \chi_\omega u & \text{in } Q, \\ \partial_n y = 0 \quad \text{on } \Sigma,\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{3.7}$$

Existence and uniqueness of $y_{k,u} \in W(0,\infty)$ with $f_k(y_{k,u}) \in L^2(Q)$ follow from Theorem 2.2. Next, we show the differentiability of the mapping $G_k : L^2(Q_\omega) \longrightarrow W(0,\infty)$ defined by $G_k(u) = y_{k,u}$. Notice that this does not follow from Theorem 2.3, since G is defined on $\mathcal{U}_p$, while we need the differentiability of G_k in $L^2(Q_\omega)$.

Lemma 3.4. *The mapping G_k is of class C^1. Given $u, v \in L^2(Q_\omega)$, the derivative $z_{k,v} = G_k'(u)v$ is the solution of the equation*

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f_k'(y_{k,u})z = \chi_\omega v \ \text{in } Q, \\ \partial_n z = 0 \ \text{on } \Sigma,\ z(0) = 0 \ \text{in } \Omega. \end{cases} \tag{3.8}$$

Proof. We apply the implicit function theorem to the following mapping:

$$\mathcal{F} : W(0,\infty) \times L^2(Q_\omega) \longrightarrow L^2(0,\infty; H^1(\Omega)^*) \times L^2(\Omega),$$
$$\mathcal{F}(y,u) = \Big(\frac{\partial y}{\partial t} - \Delta y + ay + f_k(y) - g - \chi_\omega u, y(0) - y_0\Big).$$

We prove that $\mathcal{F}$ is of class C^1. To this end, it is enough to check that the mapping $F_k : W(0,\infty) \longrightarrow L^2(Q)$ with $F_k(y)(x,t) = f_k(y(x,t))$ is of class C^1, because f_k is the only nonlinear term that appears in $\mathcal{F}$. We prove this for $n = 3$, the proof for $n < 3$ is similar and even simpler. In this case, we use the Gagliardo inequality

$$\|y\|_{L^4(\Omega)} \leq C\|y\|^{\frac{1}{4}}_{L^2(\Omega)}\|y\|^{\frac{3}{4}}_{H^1(\Omega)}. \tag{3.9}$$

Let us confirm that $[F_k'(y)z](x,t) = f_k'(y(x,t))z(x,t)$. We get with the mean value theorem, (3.9), Hölder's, and Young's inequalities

$$\|F_k(y+z) - F_k(y) - F_k'(y)z\|^2_{L^2(Q)} = \int_Q |f_k(y+z) - f_k(y) - f_k'(y)z|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_Q |f_k'(y+\theta_k z) - f_k'(y)|^2 z^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_0^\infty \|f_k'(y+\theta_k z) - f_k'(y)\|^2_{L^4(\Omega)}\|z\|^2_{L^4(\Omega)}\,\mathrm{d}t \leq C^2 \int_0^\infty \|f_k'(y+\theta_k z) - f_k'(y)\|^2_{L^4(\Omega)}\|z\|^{\frac{1}{2}}_{L^2(\Omega)}\|z\|^{\frac{3}{2}}_{H^1(\Omega)}\,\mathrm{d}t$$
$$\leq C^2\|z\|^{\frac{1}{2}}_{L^\infty(0,\infty;L^2(\Omega))}\Big(\int_0^\infty \|f_k'(y+\theta_k z) - f_k'(y)\|^8_{L^4(\Omega)}\,\mathrm{d}t\Big)^{\frac{1}{4}}\|z\|^{\frac{3}{2}}_{L^2(0,\infty;H^1(\Omega))}$$
$$\leq \frac{3C^2}{4}\|f_k'(y+\theta_k z) - f_k'(y)\|^2_{L^8(0,\infty;L^4(\Omega))}\Big(\|z\|^2_{L^\infty(0,\infty;L^2(\Omega))} + \|z\|^2_{L^2(0,\infty;H^1(\Omega))}\Big) \leq \frac{3C^2}{4}\|f_k'(y+\theta_k z) - f_k'(y)\|^2_{L^8(0,\infty;L^4(\Omega))}\|z\|^2_{W(0,\infty)}.$$

Then, we have

$$\frac{\|F_k(y+z) - F_k(y) - F_k'(y)z\|_{L^2(Q)}}{\|z\|_{W(0,\infty)}} \leq \frac{\sqrt{3}C}{2}\|f_k'(y+\theta_k z) - f_k'(y)\|_{L^8(0,\infty;L^4(\Omega))}.$$

To prove that the right hand side of the above inequality tends to zero as $\|z\|_{W(0,\infty)} \to 0$, we use (3.6) to get

$$\|f_k'(y+\theta_k z) - f_k'(y)\|_{L^4(\Omega)} \leq 2C'_{f,k}|\Omega|^{\frac{1}{4}}$$

and

$$\int_0^\infty \|f_k'(y+\theta_k z) - f_k'(y)\|^8_{L^4(\Omega)}\,\mathrm{d}x\,\mathrm{d}t \le [2C'_{f,k}]^4|\Omega| \int_0^\infty \|f_k'(y+\theta_k z) - f_k'(y)\|^4_{L^4(\Omega)}\,\mathrm{d}x\,\mathrm{d}t$$
$$\le [2C'_{f,k}]^6|\Omega| \int_0^\infty \|f_k'(y+\theta_k z) - f_k'(y)\|^2_{L^2(\Omega)}\,\mathrm{d}x\,\mathrm{d}t \le [2C'_{f,k}]^6|\Omega| C''^2_{f,k} \int_0^\infty \|z\|^2_{L^2(\Omega)}\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } \|z\|_{W(0,\infty)} \to 0.$$

Due to the continuity of the derivative $F_k' : W(0,\infty) \longrightarrow L^2(Q)$, the C^1 property of F_k follows. Thus, $\mathcal{F}$ is of class C^1 and

$$\frac{\partial \mathcal{F}}{\partial y}(y,u) : W(0,\infty) \longrightarrow L^2(0,\infty; H^1(\Omega)^*) \times L^2(\Omega)$$
$$\frac{\partial \mathcal{F}}{\partial y}(y,u)z = \Big(\frac{\partial z}{\partial t} - \Delta z + az + f_k'(y)z, z(0)\Big).$$

The linear mapping $\frac{\partial \mathcal{F}}{\partial y}(y,u)$ is an isomorphism, if the equation

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f_k'(y)z = h \text{ in } Q,\\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = z_0 \text{ in } \Omega \end{cases}$$

has a unique solution $z \in W(0,\infty)$ continuously depending on $(h, z_0) \in L^2(0,\infty; H^1(\Omega)^*) \times L^2(\Omega)$. This was proved in [7, Theorem A.3 and Remark 5.2]. The statement of the lemma follows from the implicit function theorem. □

Lemma 3.5. *If $\{u_k\}_{k=1}^\infty \subset L^2(Q_\omega)$ converges weakly in $L^2(Q_\omega)$ to u, then $\{y_{k,u_k}\}_{k=1}^\infty$ converges weakly in $W(0,\infty)$ to y_u, the solution of (1.1).*

Proof. Proceeding as in the proof of Theorem 2.2, with some constants C_1, C_2 we get the formulas

$$\|y_{k,u_k}\|_{L^\infty(0,\infty;L^2(\Omega))} + \|y_{k,u_k}\|_{L^2(0,\infty;H^1(\Omega))} \le \frac{\sqrt{2}}{\min\{1, C_a^2\}}\Big(\|y_0\|_{L^2(\Omega)} + \|g + \chi_\omega u_k\|_{L^2(Q)}\Big) \le C_1$$
$$\|f_k(y_{k,u_k})\|_{L^2(Q)} \le \max\{1, \sqrt{|\Omega|}\}\Big(\|f(y_0)\|_{L^\infty(\Omega)} + \|y_0\|_{L^\infty(\Omega)} + \|g + \chi_\omega u_k\|_{L^2(Q)}\Big) \le C_2$$

that are analogous to (2.7) and (2.8). This implies that $\{y_{k,u_k}\}_{k=1}^\infty$ is bounded in $W(0,\infty)$. Once again, arguing as in the proof of Theorem 2.2, we conclude that $y_{k,u_k} \rightharpoonup y_u$ in $W(0,\infty)$. □

Lemma 3.6. *For arbitrary $u \in L^2(Q_\omega)$, the sequence $\{y_{k,u}\}_{k=1}^\infty$ converges strongly in $L^2(0,\infty; H^1(\Omega))$ to y_u.*

Proof. The function $\phi_k = y_{k,u} - y_u$ satisfies

$$\begin{cases} \dfrac{\partial \phi_k}{\partial t} - \Delta\phi_k + a\phi_k + f_k'(y_u + \theta_k(y_{k,u} - y_u))\phi_k = f(y_u) - f_k(y_u) \text{ in } Q,\\ \partial_n\phi_k = 0 \text{ on } \Sigma,\ \phi_k(0) = 0 \text{ in } \Omega. \end{cases}$$

In view of $|f_k(y_u(x,t))| \le |f(y_u(x,t))|$ and $f_k(y_u(x,t)) \to f(y_u(x,t))$ almost everywhere in Q, and $f(y_u) \in L^2(Q)$, we deduce with the Lebesgue dominated convergence theorem that $f_k(y_u) \to f(y_u)$ in $L^2(Q)$ as $k \to \infty$. Testing the above equation with ϕ_k, we get with the constant C_a introduced in (2.1)

$$\frac{1}{2}\|\phi_k(T)\|^2_{L^2(\Omega)} + C_a^2\|\phi_k\|^2_{L^2(0,T;H^1(\Omega))} \le \frac{1}{2C_a^2}\|f(y_u) - f_k(y_u)\|^2_{L^2(Q)} + \frac{C_a^2}{2}\|\phi_k\|^2_{L^2(Q)} \quad \forall T \in (0,\infty),$$

that obviously implies our claim. □

Now, we introduce the following family of control problems

$$(\mathrm{P}_{r,k}) \qquad \min_{u\in \bar B_r(\bar u)} J_k(u) := \frac{1}{2}\int_Q (y_{k,u}-y_d)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{Q_\omega} u^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{Q_\omega}(u-\tilde u)^2\,\mathrm{d}x\,\mathrm{d}t,$$

where $\tilde u$ is the solution of (P_r) introduced in the proof of Theorem 3.3.

Lemma 3.7. *Problem* $(\mathrm{P}_{r,k})$ *has at least one solution* u_k *and it holds* $\lim_{k\to\infty}\|u_k-\tilde u\|_{L^2(Q_\omega)} = 0$.

Proof. Once again, arguing as in Theorem 3.1 we infer the existence of a solution u_k for $(\mathrm{P}_{r,k})$. We invoke the optimality of u_k for $(\mathrm{P}_{r,k})$, Lemma 3.6, and the optimality of $\tilde u$ for (P_r) to obtain

$$\frac{\kappa}{2}\|u_k\|^2_{L^2(Q_\omega)} \le J_k(u_k) \le J_k(\tilde u) \to J(\tilde u) \le J(\bar u). \tag{3.10}$$

This implies the boundedness of $\{u_k\}_{k=1}^\infty$ in $L^2(Q_\omega)$. Hence, we can select a subsequence denoted in the same way such that $u_k \rightharpoonup u$ in $L^2(Q_\omega)$. We know from Lemma 3.5 that $y_{k,u_k} \rightharpoonup y_u$ in $L^2(Q)$. Using this fact, Lemma 3.6, that $u \in \bar B_r(\bar u)$, and that $\tilde u$ is a solution of (P_r), we get

$$J(u) + \frac{1}{2}\|u-\tilde u\|^2_{L^2(Q_\omega)} \le \liminf_{k\to\infty} J_k(u_k) \le \limsup_{k\to\infty} J_k(u_k) \le \limsup_{k\to\infty} J_k(\tilde u) = J(\tilde u) \le J(u).$$

This implies that $u = \tilde u$ and $\lim_{k\to\infty} J_k(u_k) = J(\tilde u)$. Consequently, the strong convergence $u_k \to \tilde u$ follows. This holds for every weakly converging subsequence, hence the result is valid for the whole sequence. □

Now we are ready to confirm the estimate that is needed for completing the proof of Theorem 3.3.

Lemma 3.8. *The control* $\tilde u$ *belongs to* $L^\infty(Q_\omega)$ *and there exists a constant* M *independent of* r *such that* $\|\tilde u\|_{L^\infty(Q_\omega)} \le M$.

Proof. From Lemma 3.4 and the chain rule, we deduce that $J_k : L^2(Q_\omega) \longrightarrow \mathbb{R}$ is of class C^1. The optimality of u_k yields $J_k'(u_k)(u-u_k) \ge 0$ for every $u \in \bar B_r(\bar u)$. This leads to the optimality system

$$\begin{cases} \dfrac{\partial y_{k,u_k}}{\partial t} - \Delta y_{k,u_k} + a y_{k,u_k} + f_k(y_{k,u_k}) = g + \chi_\omega u_k & \text{in } Q,\\ \partial_n y_{k,u_k} = 0 \ \text{ on } \Sigma,\ y_{k,u_k}(0) = y_0 & \text{in } \Omega, \end{cases} \tag{3.11}$$

$$\begin{cases} -\dfrac{\partial \varphi_{k,u_k}}{\partial t} - \Delta\varphi_{k,u_k} + a\varphi_{k,u_k} + f'(y_{k,u_k})\varphi_{k,u_k} = y_{k,u_k} - y_d \text{ in } Q,\\ \partial_n\varphi_{k,u_k} = 0 \ \text{ on } \Sigma,\ \lim_{t\to\infty}\|\varphi_{k,u_k}(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{3.12}$$

$$\int_{Q_\omega} (\varphi_{k,u_k} + \kappa u_k + u_k - \tilde u)(u-u_k)\,\mathrm{d}x\,\mathrm{d}t \ge 0 \quad \forall u \in \bar B_\delta(\bar u). \tag{3.13}$$

It is well known that the variational inequality (3.13) is equivalent to

$$u_k = \mathrm{Proj}_{\bar B_r(\bar u)}\Big(-\frac{1}{\kappa}[\varphi_{k,u_k} + u_k - \tilde u]\Big),$$

where $\mathrm{Proj}_{\bar B_r(\bar u)}$ stands for the projection onto $\bar B_r(\bar u)$ in the $L^2(Q_\omega)$ sense. This means

$$u_k = \begin{cases} -\dfrac{1}{\kappa}[\varphi_{k,u_k} + u_k - \tilde u] & \text{if } \Big\|\dfrac{1}{\kappa}[\varphi_{k,u_k} + u_k - \tilde u] + \bar u\Big\|_{L^2(Q_\omega)} \le r,\\ \bar u - r\dfrac{\frac{1}{\kappa}[\varphi_{k,u_k} + u_k - \tilde u] + \bar u}{\Big\|\frac{1}{\kappa}[\varphi_{k,u_k} + u_k - \tilde u] + \bar u\Big\|_{L^2(Q_\omega)}} & \text{else.} \end{cases} \tag{3.14}$$

Thanks to (3.10) we get $\|u_k\|_{L^2(Q_\omega)} \le C_1$ for a constant C_1 independent of r. Then, (3.11) and (2.3) imply that $\|y_{k,u_k}\|_{L^\infty(0,T;L^2(\Omega))} \le C_2$, where C_2 is also independent of r. This bound can be applied to the adjoint equation (3.12) to obtain that $\|\varphi_{k,u_k}\|_{L^\infty(Q)} \leq C_3$ with C_3 independent of r. This estimate follows from [7, Theorem A.4 and Remark 5.2]. Inserting this in (3.14) we infer

$$|u_k(x,t)| \leq \frac{C_3}{\kappa} + 2\|\bar{u}\|_{L^\infty(Q_\omega)} + |u_k(x,t) - \tilde{u}(x,t)| \quad \text{for a.a. } (x,t) \in Q_\omega.$$

Notice that the boundedness of $\bar{u}$ follows from Theorem 3.2. From Lemma 3.7, we know that $u_k \to \tilde{u}$ in $L^2(Q_\omega)$. Hence, taking a subsequence such that $u_k(x,t) \to \tilde{u}(x,t)$ for almost all $(x,t) \in Q_\omega$, the above estimate yields

$$|\tilde{u}(x,t)| = \lim_{k\to\infty} |u_k(x,t)| \leq \frac{C_3}{\kappa} + 2\|\bar{u}\|_{L^\infty(Q_\omega)} + \lim_{k\to\infty} |u_k(x,t) - \tilde{u}(x,t)| = \frac{C_3}{\kappa} + 2\|\bar{u}\|_{L^\infty(Q_\omega)} = M.$$

This concludes the proof of lemma. □

4. First and second order optimality conditions for (P)

We start this section by establishing the necessary optimality conditions satisfied by any local solution of (P).

Theorem 4.1. *Let $\bar{u}$ be a local solution of* (P) *in the $L^2(Q_\omega)$-sense. Then, there exist $\bar{y}, \bar{\varphi} \in W(0,\infty) \cap L^\infty(Q)$ such that*

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} - \Delta \bar{y} + a\bar{y} + f(\bar{y}) = g + \chi_\omega \bar{u} \text{ in } Q, \\ \partial_n \bar{y} = 0 \text{ on } \Sigma,\ \bar{y}(0) = y_0 \text{ in } \Omega, \end{cases} \tag{4.1}$$

$$\begin{cases} -\dfrac{\partial \bar{\varphi}}{\partial t} - \Delta \bar{\varphi} + a\bar{\varphi} + f'(\bar{y})\bar{\varphi} = \bar{y} - y_d \text{ in } Q, \\ \partial_n \bar{\varphi} = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \|\bar{\varphi}(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{4.2}$$

$$J'(\bar{u}) = \bar{\varphi}_{|Q_\omega} + \kappa \bar{u} = 0, \tag{4.3}$$

$$J''(\bar{u})v^2 = \int_Q [1 - \bar{\varphi} f''(\bar{y})] z_v^2 \,\mathrm{d}x\,\mathrm{d}t + \kappa \int_{Q_\omega} v^2 \,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall v \in L^2(Q_\omega), \tag{4.4}$$

where $z_v = G'(\bar{u})v$.

Proof. Theorem 3.3 yields that $\bar{u} \in \mathcal{U}_p$ and $\bar{u}$ is a local solution of (P) in the $\mathcal{U}_p$-sense. Hence, the statement of the theorem is a straightforward consequence of Corollary 2.4, Remark 2.5, and that the first and second order necessary conditions $J'(\bar{u})v = 0$ and $J''(\bar{u})v^2 \geq 0$ for all $v \in \mathcal{U}_p$ hold at any local minimizer. □

Next we address sufficient second order conditions for optimality.

Theorem 4.2. *Let $\bar{u} \in \mathcal{U}_p$ obey the first order necessary conditions* (4.1)–(4.3)*. Assume in addition that $J''(\bar{u})v^2 > 0$ holds for all $v \in L^2(Q_\omega) \setminus \{0\}$. Then, there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$J(\bar{u}) + \frac{\delta}{4}\|u - \bar{u}\|^2_{L^2(Q_\omega)} \leq J(u) \quad \forall u \in \mathcal{U}_p \text{ with } \|u - \bar{u}\|_{\mathcal{U}_p} \leq \varepsilon. \tag{4.5}$$

Proof. We split the proof into three steps.

Step I - $\delta = \inf_{\|v\|_{L^2(Q_\omega)}=1} J''(\bar{u})v^2 > 0$. We proceed by contradiction. If $\delta = 0$, there exists a minimizing sequence $\{v_k\}_{k=1}^\infty$ such that $\|v_k\|_{L^2(Q_\omega)} = 1$ for every $k \geq 1$ and $J''(\bar{u})v_k^2 \to 0$ as $k \to \infty$. By taking a subsequence, we can assume that $v_k \rightharpoonup v$ in $L^2(Q_\omega)$. In view of Remark 2.5 and our assumption, we have that the quadratic mapping $J''(\bar{u}) : L^2(Q_\omega) \longrightarrow \mathbb{R}$ is continuous and positive definite. Hence, it is also convex. Then, the inequality $J''(\bar{u})v^2 \leq \lim_{k\to\infty} J''(\bar{u})v_k^2 = 0$ follows. According to our assumption, this is possible only if $v = 0$. Using again Remark 2.5, we deduce the convergence $z_{v_k} = G'(\bar{u})v_k \rightharpoonup G'(\bar{u})v = 0$ in $W(0,\infty)$. In particular, we have that $z_{v_k} \to 0$ in $L^2(Q_T)$ for every $T < \infty$.

Let $\rho > 0$ be arbitrarily small. From (4.2) we infer the existence of $T_\rho < \infty$ such that

$$\|\bar{\varphi}(t)\|_{L^2(\Omega)} < \rho \quad \forall t > T_\rho.$$

Equation (4.1) implies that $\bar{y} \in L^\infty(Q)$ and, hence, $|f''(y(x,t))| \le C$ for almost every $(x,t) \in Q$. Exploiting these facts, we obtain

$$J''(\bar{u})v_k^2 = \int_Q [1 - \bar{\varphi} f''(\bar{y})] z_{v_k}^2 \,\mathrm{d}x\,\mathrm{d}t + \kappa \int_{Q_\omega} v_k^2 \,\mathrm{d}x\,\mathrm{d}t \ge -\int_Q \bar{\varphi} f''(\bar{y}) z_{v_k}^2 \,\mathrm{d}x\,\mathrm{d}t + \kappa$$
$$= -\int_0^{T_\rho}\int_\Omega \bar{\varphi} f''(\bar{y}) z_{v_k}^2 \,\mathrm{d}x\,\mathrm{d}t - \int_{T_\rho}^\infty \bar{\varphi} f''(\bar{y}) z_{v_k}^2 \,\mathrm{d}x\,\mathrm{d}t + \kappa \ge -C\|\bar{\varphi}\|_{L^\infty(Q)}\|z_{v_k}\|^2_{L^2(Q_{T_\rho})} - C\int_{T_\rho}^\infty \|\bar{\varphi}(t)\|_{L^2(\Omega)}\|z_{v_k}\|^2_{L^4(\Omega)}\,\mathrm{d}t + \kappa$$
$$\ge -C\|\bar{\varphi}\|_{L^\infty(Q)}\|z_{v_k}\|^2_{L^2(Q_{T_\rho})} - C'\|z_{v_k}\|^2_{L^2(0,\infty;H^1(\Omega))}\rho + \kappa \ge -C\|\bar{\varphi}\|_{L^\infty(Q)}\|z_{v_k}\|^2_{L^2(Q_{T_\rho})} - C'\|G'(\bar{u})\|^2\rho + \kappa,$$

where we have used the embedding $H^1(\Omega) \subset L^4(\Omega)$ and that $\|z_{v_k}\|_{W(0,\infty)} \le \|G'(\bar{u})\|\,\|v_k\|_{L^2(Q_\omega)} = \|G'(\bar{u})\|$ in the last inequality. This yields

$$\kappa \le C\|\bar{\varphi}\|_{L^\infty(Q)}\|z_{v_k}\|^2_{L^2(Q_{T_\rho})} + C'\|G'(\bar{u})\|^2\rho + J''(\bar{u})v_k^2 \to C'\|G'(\bar{u})\|^2\rho \quad \forall \rho > 0,$$

being a contradiction to the assumption $\kappa > 0$. Consequently, we have that $\delta > 0$.

Step II - $\exists \varepsilon > 0$ such that $|[J''(u) - J''(\bar{u})]v^2| \le \frac{\delta}{2}\|v\|^2_{L^2(Q_\omega)}$ $\forall v \in L^2(Q_\omega)$, if $\|u - \bar{u}\|_{\mathcal{U}_p} \le \varepsilon$. We denote by $B(\bar{u})$ the closed ball of $\mathcal{U}_p$ centered at $\bar{u}$ with radius 1. For every $u \in B(\bar{u})$ we have that $\|u\|_{\mathcal{U}_p} \le 1 + \|\bar{u}\|_{\mathcal{U}_p}$. Hence, Theorem 2.2 implies the existence of C_1 such that

$$\|y_u\|_{W(0,\infty)} + \|y_u\|_{L^\infty(Q)} \le C_1 \quad \forall u \in B(\bar{u}). \tag{4.6}$$

Applying [7, Theorem A.4] to the adjoint equation (2.15) and using (4.6), we obtain a constant C_2 such that

$$\|\varphi_u\|_{W(0,\infty)} + \|\varphi_u\|_{L^\infty(Q)} \le C_2 \quad \forall u \in B(\bar{u}). \tag{4.7}$$

Given an arbitrary element $u \in B(\bar{u})$ we set $y = y_u - \bar{y}$, then

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f'(\bar{y} + \theta(y_u - \bar{y}))y = \chi_\omega(u - \bar{u}) & \text{in } Q,\\ \partial_n y = 0 \quad \text{on } \Sigma,\ y(0) = 0 & \text{in } \Omega.\end{cases}$$

Invoking [7, Theorem TA.3] and (4.6), we get

$$\|y_u - \bar{y}\|_{W(0,\infty)} + \|y_u - \bar{y}\|_{L^\infty(Q)} \le C_3\|u - \bar{u}\|_{\mathcal{U}_p} \quad \forall u \in B(\bar{u}). \tag{4.8}$$

This yields the property

$$\forall \rho > 0\ \exists \varepsilon \in (0,1] \text{ such that } \|f'(y_u) - f'(\bar{y})\|_{L^\infty(Q)} + \|f''(y_u) - f''(\bar{y})\|_{L^\infty(Q)} \le \rho, \quad \text{if } \|u - \bar{u}\|_{\mathcal{U}_p} < \varepsilon. \tag{4.9}$$

Now, we set $\varphi = \varphi_u - \bar{\varphi}$, then we have

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} - \Delta\varphi + a\varphi + f'(\bar{y})\varphi = y_u - \bar{y} + [f'(\bar{y}) - f'(y_u)]\varphi_u \text{ in } Q,\\ \partial_n\varphi = 0 \text{ on } \Sigma,\ \lim_{t\to\infty}\|\varphi(t)\|_{L^2(\Omega)} = 0.\end{cases}$$

Applying [7, Theorem TA.4] to this equation and using (4.7) and (4.9) we arrive at

$$\forall \rho > \exists \varepsilon \in (0,1] \text{ such that } \|\varphi_u - \bar{\varphi}\|_{W(0,\infty)} + \|\varphi_u - \bar{\varphi}\|_{L^\infty(Q)} \le \rho, \quad \text{if } \|u - \bar{u}\|_{\mathcal{U}_p} < \varepsilon. \tag{4.10}$$

Putting $z_{u,v} = G'(u)v$, we deduce from [7, Theorem TA.3] and (4.6)

$$\|z_{u,v}\|_{W(0,\infty)} \le C_4\|v\|_{L^2(Q_\omega)} \quad \forall u \in B(\bar{u}) \text{ and } \forall v \in L^2(Q_\omega). \tag{4.11}$$

Denoting $z_v = G'(\bar{u})v$ and $z = z_{u,v} - z_v$ we have

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(\bar{y})z = [f'(\bar{y}) - f'(y_u)]z_{u,v} \text{ in } Q,\\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega.\end{cases}$$

Applying again [7, Theorem TA.3] and taking into account (4.9) and (4.11), we deduce the following property

$$\forall \rho > \exists \varepsilon \in (0,1] \text{ such that } \|z_{u,v} - z_v\|_{W(0,\infty)} \le \rho \|v\|_{L^2(Q_\omega)} \ \ \forall v \in L^2(Q_\omega), \quad \text{if } \|u - \bar u\|_{\mathcal{U}_p} < \varepsilon. \tag{4.12}$$

Next we estimate $|[J''(u) - J''(\bar u)]v^2|$:

$$\begin{aligned} |[J''(u) - J''(\bar u)]v^2| &= \left| \int_Q [1 - \varphi_u f''(y_u)] z_{u,v}^2 \,\mathrm{d}x\,\mathrm{d}t - \int_Q [1 - \bar\varphi f''(\bar y)] z_v^2 \,\mathrm{d}x\,\mathrm{d}t \right| \\ &\le \int_Q |z_{u,v} - z_v|\,|z_{u,v} + z_v| \,\mathrm{d}x\,\mathrm{d}t + \int_Q |\varphi_u - \bar\varphi||f''(\bar y_u)| z_{u,v}^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q |\bar\varphi|\,|f''(y_u) - f''(\bar y)| z_{u,v}^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_Q |\bar\varphi|\,|f''(\bar y)|\,|z_{u,v} - z_v|\,|z_{u,v} + z_v| \,\mathrm{d}x\,\mathrm{d}t = \sum_{i=1}^4 I_i. \end{aligned}$$

Taking $\rho = \frac{\delta}{16C_4}$ in (4.12) and using (4.11), we deduce the existence of $\varepsilon_1 \in (0,1)$ such that $I_1 \le \frac{\delta}{8}\|v\|^2$ if $\|u - \bar u\|_{\mathcal{U}_p} \le \varepsilon_1$. With (4.6), (4.10), and (4.11) we infer the existence of $\varepsilon_2 \in (0,1)$ such that $I_2 \le \frac{\delta}{8}\|v\|^2$ if $\|u - \bar u\|_{\mathcal{U}_p} \le \varepsilon_2$. The same estimate is obtained for I_3 for some $\varepsilon_3 \in (0,1)$ with the aid of (4.7), (4.9), and (4.11). Finally, I_4 is estimated similarly to I_1 and using (4.7) and (4.9). Taking $\varepsilon = \min_{1 \le i \le 4} \varepsilon_i$, the proof of Step II is concluded.

Step III - Proof of (4.5). We perform a Taylor expansion and use the results of the Steps I, II, along with formula (4.3) to infer

$$\begin{aligned} J(u) &= J(\bar u) + J'(\bar u)(u - \bar u) + \frac{1}{2} J''(\bar u + \theta(u - \bar u))(u - \bar u)^2 \\ &= J(\bar u) + \frac{1}{2} J''(\bar u)(u - \bar u)^2 + \frac{1}{2}[J''(\bar u + \theta(u - \bar u)) - J''(\bar u)](u - \bar u)^2 \\ &\ge J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(Q_\omega)} - \frac{\delta}{4}\|u - \bar u\|^2_{L^2(Q_\omega)} = J(\bar u) + \frac{\delta}{4}\|u - \bar u\|^2_{L^2(Q_\omega)} \text{ for all } u \text{ with } \|u - \bar u\|_{\mathcal{U}_p} \le \varepsilon. \end{aligned}$$

□

Remark 4.3. *As proved in Step I of the above demonstration, the second order condition $J''(\bar u)v^2 > 0$ for all $v \in L^2(Q_\omega) \setminus \{0\}$ implies that*

$$\exists \delta > 0 \ \text{ such that } \ J''(\bar u)v^2 \ge \delta \|v\|^2_{L^2(Q_\omega)} \ \ \forall v \in L^2(Q_\omega). \tag{4.13}$$

Using this fact and the statement proved in Step II, we infer in addition that

$$\exists \bar r > 0 \ \text{ such that } \ J''(u)v^2 \ge \frac{\delta}{2}\|v\|^2_{L^2(Q_\omega)} \ \ \forall v \in L^2(Q_\omega) \ \text{ and } \ \forall u \in \bar B_{\bar r}(\bar u), \tag{4.14}$$

where δ is given by (4.13) *and $\bar B_{\bar r}(\bar u)$ is the $\mathcal{U}_p$-closed ball centered at $\bar u$ with radius $\bar r$.*

5. The local value function

In this section, $\bar u \in L^\infty(Q_\omega) \cap \mathcal{U}_p$ denotes a local solution of (P) satisfying the second order condition $J''(\bar u)v^2 > 0$ for all $v \in L^2(Q_\omega) \setminus \{0\}$. This control $\bar u$ will be our reference control for the rest of the paper.

For initial data $\eta \in L^\infty(\Omega)$ we define the control problem

$$(\mathrm{P}_\eta) \qquad \min_{u \in L^2(Q_\omega)} J_\eta(u) := \frac{1}{2}\int_Q (y_{\eta,u} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{Q_\omega} u^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where $y_{\eta,u} \in W(0,\infty)$ is the solution of the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) = g + \chi_\omega u & \text{in } Q, \\ \partial_n y = 0 \quad \text{on } \Sigma, \ y(0) = \eta & \text{in } \Omega. \end{cases} \tag{5.1}$$

In view of our previous analysis, it should be clear, why we do not consider the value function for initial data from $L^2(Q_\omega)$. The restriction to essentially bounded functions η causes the main difficulties of our paper.

Analogously to Theorem 3.1, (P_η) has at least one solution. Moreover, every local solution in the $L^2(Q_\omega)$-sense is an element of $L^\infty(Q_\omega) \cap \mathcal{U}_p$; see Theorem 3.2. Further, according to Theorem 3.3, u_η is a local solution of (P_η) in the $L^2(Q_\omega)$-sense if and only if it is a local solution in the $\mathcal{U}_p$-sense.

We define the value function $V : L^\infty(\Omega) \longrightarrow \mathbb{R}$ by $V(\eta) = \inf\,(\mathrm{P}_\eta)$. Since the problems (P_η) can have several global solutions, this function V can be even discontinuous. For this reason we introduce a *local* value function. We proceed as follows. Given $r > 0$, we define the control problems

$$(\mathrm{P}_{r,\eta}) \qquad \min_{u\in L^2(Q_\omega),\, \|u-\bar u\|_{L^2(Q_\omega)}\le r} J_\eta(u) := \frac{1}{2}\int_Q (y_{\eta,u} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{Q_\omega} u^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where $y_{\eta,u} \in W(0,\infty)$ is the solution of the equation (5.1). We need the following technical result.

Theorem 5.1. *For every $\alpha > 0$ there exist r_α and M_α, monotone increasing with respect to α, such that for $\|\eta - y_0\|_{L^\infty(\Omega)} \le \alpha$ and $r \le r_\alpha$ any solution u_η of $(\mathrm{P}_{r,\eta})$ satisfies $u_\eta \in \mathcal{U}_p \cap L^\infty(Q_\omega)$ and*

$$\|u_\eta\|_{L^\infty(Q_\omega)} \le M_\alpha \text{ and } \|u_\eta - \bar u\|_{\mathcal{U}_p} \le \bar r, \tag{5.2}$$

where $\bar r > 0$ was introduced in (4.14). Finally, there exists $\bar\alpha$ such that, for any $\eta \in L^\infty(\Omega)$ with $\|\eta - y_0\|_{L^\infty(\Omega)} < \bar\alpha$ and any $r \le r_{\bar\alpha}$, problem $(\mathrm{P}_{r,\eta})$ has a unique solution.

Proof. In view of $\|\eta\|_{L^\infty(\Omega)} \le \|y_0\|_{L^\infty(\Omega)} + \alpha$, we can argue as in the proof of Lemma 3.8 to deduce the existence of M_α, monotone increasing with respect to α, such that the estimate $\|u_\eta\|_{L^\infty(Q_\omega)} \le M_\alpha$ holds. Moreover, we have

$$\begin{aligned}\|u_\eta - \bar u\|_{\mathcal{U}_p} &= \|u_\eta - \bar u\|_{L^2(Q_\omega)} + \|u_\eta - \bar u\|_{L^p(0,\infty;L^2(\omega))} \le r + |\omega|^{\frac{p-2}{2p}} (M_\alpha + \|\bar u\|_{L^\infty(Q_\omega)})^{\frac{p-2}{2}} \|u_\eta - \bar u\|^{\frac{2}{p}}_{L^2(Q_\omega)}\\ &\le r_\alpha + |\omega|^{\frac{p-2}{2p}} (M_\alpha + \|\bar u\|_{L^\infty(Q_\omega)})^{\frac{p-2}{2}} r_\alpha^{\frac{2}{p}} \le \bar r\end{aligned}$$

for $r_\alpha > 0$ small enough. To conclude the proof we establish the following stability result for (4.14) with respect to η: there exists $\bar\alpha > 0$ such that

$$J''_\eta(u)v^2 \ge \frac{\delta}{4}\|v\|^2_{L^2(Q_\omega)} \quad \forall v \in L^2(Q_\omega),\ \|\eta - y_0\|_{L^\infty(\Omega)} < \bar\alpha, \text{ and } \|u - \bar u\|_{\mathcal{U}_p} \le \bar r. \tag{5.3}$$

Indeed, this inequality proves that J_η is strictly convex at the ball $\bar B_{\bar r}(\bar u)$ if $\|\eta - y_0\|_{L^\infty(\Omega)} < \bar\alpha$. Since (5.2) implies that any solution of $(\mathrm{P}_{r,\eta})$ belongs to the ball $\bar B_{\bar r}(\bar u)$ for $r \le r_{\bar\alpha}$, the uniqueness follows.

In order to prove (5.3), we compare $J''_\eta(u)v^2$ with $J''(u)v^2$ for $u \in \bar B_{\bar r}(\bar u)$ and use (4.14). We will see that this difference is as small as needed if we take $\bar\alpha$ sufficiently small. We denote by $y_{\eta,u}, z_{\eta,u,v}, \varphi_{\eta,u}$ the solutions of (5.1), (2.11) and (2.15) with y_u replaced by y_η. We also denote by $y_u, z_{u,v}, \varphi_u$ the solutions of (1.1), (2.11), and (2.15). Then, we have

$$\begin{aligned}|[J''_\eta(u) - J''(u)]v^2| &= \Big|\int_Q [1 - \varphi_{\eta,u} f''(y_{\eta,u})] z^2_{\eta,u,v} \,\mathrm{d}x\,\mathrm{d}t - \int_Q [1 - \varphi_u f''(y_u)] z^2_{u,v} \,\mathrm{d}x\,\mathrm{d}t\Big|\\ &\le \int_Q |z^2_{\eta,u,v} - z^2_{u,v}| \,\mathrm{d}x\,\mathrm{d}t + \int_Q |\varphi_{\eta,u} - \varphi_u||f''(y_{\eta,u})| z^2_{\eta,u,v} \,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_Q |\varphi_u||f''(y_{\eta,u}) - f''(y_u)| z^2_{\eta,u,v} \,\mathrm{d}x\,\mathrm{d}t + \int_Q |\varphi_u||f''(y_u)||z^2_{\eta,u,v} - z^2_{u,v}| \,\mathrm{d}x\,\mathrm{d}t = \sum_{i=1}^4 I_i.\end{aligned} \tag{5.4}$$

We have to estimate the terms $y_{\eta,u} - y_u$, $z_{\eta,u,v} - z_{u,v}$, and $\varphi_{\eta,u} - \varphi_u$. Setting $w = y_{\eta,u} - y_u$ and using the mean value theorem we get

$$\left\{\begin{array}{l} \dfrac{\partial w}{\partial t} - \Delta w + aw + f'(y_\theta)w = 0 \text{ in } Q,\\ \partial_n w = 0 \text{ on } \Sigma,\ w(0) = \eta - y_0 \text{ in } \Omega \end{array}\right.$$

for some measurable function $\theta : Q \longrightarrow [0,1]$ and $y_\theta = y_u + \theta(y_{\eta,u} - y_u)$. From Theorem 2.2 we know that $y_{\eta,u}$ and y_u are uniformly bounded in $L^\infty(Q)$ if $\|\eta - y_0\|_{L^\infty(\Omega)} \le \bar\alpha$. From [7, Theorem A.3] we deduce that

$$\|y_{\eta,u} - y_u\|_{L^2(Q)} + \|y_{\eta,u} - y_u\|_{L^\infty(Q)} \le C_1\|\eta - y_0\|_{L^\infty(\Omega)} \le C_1\bar\alpha. \tag{5.5}$$

Now, we put $z = z_{\eta,v} - z_{u,v}$ and apply again the mean value theorem to get

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z = f''(y_\vartheta)(y_u - y_{\eta,u})z_{\eta,u,v} \text{ in } Q,\\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega. \end{cases}$$

Once again from [7, Theorem A.3] and (5.5) we infer

$$\|z_{\eta,u,v}\|_{L^2(Q)} + \|z_{u,v}\|_{L^2(Q)} \le C_2\|v\|_{L^2(Q_\omega)} \quad \text{and} \quad \|z_{\eta,u,v} - z_{u,v}\|_{L^2(Q)} \le C_3\bar\alpha\|v\|_{L^2(Q_\omega)}. \tag{5.6}$$

For the last estimate we set $\phi = \varphi_{\eta,u} - \varphi_u$ and once again use the mean value theorem to obtain

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - \Delta\phi + a\phi + f'(y_u)\phi = [f''(y_\vartheta)\varphi_\eta - 1](y_u - y_{\eta,u}) \text{ in } Q,\\ \partial_n\phi = 0 \text{ on } \Sigma,\ \lim_{t\to\infty}\|\phi(t)\|_{L^2(\Omega)} = 0. \end{cases}$$

With [7, Theorem A.4] and (5.5) we get

$$\|\varphi_{\eta,u} - \varphi_u\|_{L^\infty(Q)} = \|\phi\|_{L^\infty(Q)} \le C_4\bar\alpha. \tag{5.7}$$

Finally, inserting the estimates (5.5)-(5.7) in (5.4) we conclude that

$$|[J''_\eta(u) - J''(u)]v^2| \le \frac{\delta}{4}\|v\|^2_{L^2(Q_\omega)}$$

if $\bar\alpha$ is small enough. This completes the proof with (4.14). □

Definition 5.2. *Let $\bar\alpha > 0$ and $r_{\bar\alpha}$ be as in Theorem 5.1. The open ball of $L^\infty(\Omega)$ centered at y_0 with radius $\bar\alpha$ is denoted by $B_{\bar\alpha}(y_0)$. Given $r \in (0, r_{\bar\alpha}]$, we define the local value function $V_r : B_{\bar\alpha}(y_0) \longrightarrow \mathbb{R}$ by $V_r(\eta) = \inf(\mathrm{P}_{r,\eta})$.*

Theorem 5.3. *The local value function V_r is of class C^1 and its derivative is given by*

$$V'_r(\eta)\xi = \int_\Omega \varphi_\eta(x,0)\xi(x)\,\mathrm{d}x \quad \forall\eta \in B_{\bar\alpha}(y_0) \text{ and } \forall\xi \in L^\infty(\Omega), \tag{5.8}$$

where φ_η is the adjoint state corresponding to the unique solution u_η of $(\mathrm{P}_{r,\eta})$. Furthermore, if a sequence $\{\eta_k\}_{k=1}^\infty \subset B_{\bar\alpha}(y_0)$ converges to $\eta \in B_{\bar\alpha}(y_0)$ in $L^\infty(\Omega)$, then $V'_r(\eta_k) = \varphi_{\eta_k}(\cdot,0)$ converges to $V'_r(\eta) = \varphi_\eta(\cdot,0)$ in $C(\bar\Omega)$.

Remark 5.4. *The above theorem states that $V_r : B_{\bar\alpha}(y_0) \subset L^\infty(\Omega) \longrightarrow \mathbb{R}$ is a function of class C^1. So we have that $V'_r : B_{\bar\alpha}(y_0) \longrightarrow L^\infty(\Omega)^*$ is a continuous function. Formula (5.8) allows to identify $V'_r(\eta)$ with $\varphi_\eta(\cdot,0)$, which is an element of $C(\bar\Omega)$; see Corollary 2.4.*

Proof. First we prove that V_r is Gâteaux differentiable. Given $(\eta,\xi) \in B_{\bar\alpha}(y_0) \times L^\infty(\Omega) \setminus \{0\}$ and $0 < \rho < \frac{\bar\alpha - \|\eta - y_0\|_{L^\infty(\Omega)}}{\|\xi\|_{L^\infty(\Omega)}}$, we set $\eta_\rho = \eta + \rho\xi \in B_{\bar\alpha}(y_0)$, u_ρ is the solution of the control problem $(\mathrm{P}_{r,\eta_\rho})$, y_ρ is the state associated with (η_ρ, u_ρ), and φ_ρ the corresponding adjoint state. We also denote by u_η, y_η, and φ_η the solution of $(\mathrm{P}_{r,\eta})$, its associated state, and the corresponding adjoint state, respectively. Now, we define

$$v_\rho = \frac{u_\rho - u_\eta}{\rho},\quad z_\rho = \frac{y_\rho - y_\eta}{\rho},\quad \text{and}\quad \psi_\rho = \frac{\varphi_\rho - \varphi_\eta}{\rho}.$$

Subtracting the optimality systems for u_ρ and u_η we infer

$$\begin{cases} \dfrac{\partial z_\rho}{\partial t} - \Delta z_\rho + a z_\rho + f'(\hat{y}_\rho) z_\rho = \chi_\omega v_\rho \text{ in } Q, \\ \partial_n z_\rho = 0 \text{ on } \Sigma,\ z_\rho(0) = \xi \text{ in } \Omega, \end{cases} \tag{5.9}$$

$$\begin{cases} -\dfrac{\partial \psi_\rho}{\partial t} - \Delta \psi_\rho + a\psi_\rho + f'(y_\eta)\psi_\rho = (1 - f''(\tilde{y}_\rho)\varphi_\rho) z_\rho \text{ in } Q, \\ \partial_n \psi_\rho = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \|\psi_\rho(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{5.10}$$

$$\psi_\rho + \kappa v_\rho = 0 \quad \text{in} \quad Q_\omega, \tag{5.11}$$

where $\hat{y}_\rho = y_\eta + \theta_\rho(y_\rho - y_\eta)$ and $\tilde{y}_\rho = y_\eta + \vartheta_\rho(y_\rho - y_\eta)$ for some measurable functions $\theta_\rho, \vartheta_\rho : Q \longrightarrow [0,1]$. The rest of this proof is split into four steps.

Step I.- $\exists C > 0$ such that $\|v_\rho\|_{L^2(Q_\omega)} \le C\|\xi\|_{L^\infty(\Omega)}$ for all $0 < \rho < \frac{\bar{\alpha} - \|\eta - y_0\|_{L^\infty(\Omega)}}{\|\xi\|_{L^\infty(\Omega)}}$.

Using the optimality of u_ρ and u_η and the fact that $\|u_\rho - \bar{u}\|_{L^2(Q_\omega)} \le r$ and $\|u_\eta - \bar{u}\|_{L^2(Q_\omega)} \le r$ we infer

$$J'_\eta(u_\eta)(u_\rho - u_\eta) \ge 0 \quad \text{and} \quad J'_{\eta_\rho}(u_\rho)(u_\eta - u_\rho) \ge 0.$$

This yields $[J'_\eta(u_\rho) - J'_\eta(u_\eta)](u_\rho - u_\eta) \le [J'_{\eta_\rho}(u_\rho) - J'_\eta(u_\rho)](u_\eta - u_\rho)$. Using the mean value theorem we obtain

$$J''_\eta(u_\eta + \sigma_\rho(u_\rho - u_\eta))(u_\rho - u_\eta)^2 \le \|\varphi_\rho - \varphi_{\eta,u_\rho}\|_{L^2(Q_\omega)} \|u_\eta - u_\rho\|_{L^2(Q_\omega)}$$

for some $\sigma_\rho \in [0,1]$. Here, φ_{η,u_ρ} is the adjoint state satisfying the equation

$$\begin{cases} -\dfrac{\partial \varphi_{\eta,u_\rho}}{\partial t} - \Delta \varphi_{\eta,u_\rho} + a\varphi_{\eta,u_\rho} + f'(y_{\eta,u_\rho})\varphi_{\eta,u_\rho} = y_{\eta,u_\rho} - y_d \text{ in } Q, \\ \partial_n \varphi_{\eta,u_\rho} = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \|\varphi_{\eta,u_\rho}(t)\|_{L^2(\Omega)} = 0, \end{cases}$$

where y_{η,u_ρ} is the solution of (5.1) for $u = u_\rho$. From Theorem 5.1 we deduce that u_η and u_ρ belong to the $\mathcal{U}_p$-ball centered at $\bar{u}$ and radius $\bar{r}$. Hence, $u_\eta + \sigma_\rho(u_\rho - u_\eta)$ also belongs to that ball. Since $\|\eta - y_0\|_{L^\infty(\Omega)} \le \bar{\alpha}$, we deduce from the above inequality and (5.3) that $\frac{\delta}{4}\|u_\eta - u_\rho\|_{L^2(Q_\omega)} \le \|\varphi_\rho - \varphi_{\eta,u_\rho}\|_{L^2(Q_\omega)}$. Dividing this inequality by ρ we obtain $\|v_\rho\|_{L^2(Q_\omega)} \le \frac{4}{\delta}\|\phi_\rho\|_{L^2(Q_\omega)}$, where $\phi_\rho = \frac{\varphi_\rho - \varphi_{\eta,u_\rho}}{\rho}$. Hence, we need to estimate ϕ_ρ. To this end, we first estimate $w_\rho = \frac{y_\rho - y_{\eta,u_\rho}}{\rho}$. Noting that both states y_ρ and y_{η,u_ρ} are associated to the same control and $w_\rho(0) = \xi$, we can proceed as in the estimate (5.5) to deduce that

$$\|w_\rho\|_{L^2(Q)} + \|w_\rho\|_{L^\infty(Q)} \le C_1\|\xi\|_{L^\infty(\Omega)}.$$

Using this fact, the estimate $\|\phi_\rho\|_{L^2(Q)} \le C_2\|\xi\|_{L^\infty(\Omega)}$ follows similarly as the one of ϕ in (5.7). This completes the proof of Step I.

Step II.- Passing to the limit in the system (5.9)–(5.11). First, we observe that the boundedness of v_ρ implies the convergence $\|u_\rho - u_\eta\|_{L^2(Q_\omega)} \to 0$ as $\rho \to 0$. This convergence, the fact that $\eta_\rho \to \eta$ in $L^\infty(\Omega)$, and the boundedness of $\{u_\rho\}_\rho$ and u_η in $\mathcal{U}_p$ implies that $\{y_\rho\}_\rho$ is bounded in $L^\infty(Q)$ and additionally $y_\rho \to y_\eta$ in $W(0,\infty)$. Using the boundedness of v_ρ and $\hat{y}_\rho$ in (5.9) we deduce the boundedness of $\{z_\rho\}_\rho$ in $W(0,\infty)$. Now, from (5.10) the boundedness of ψ_ρ in $W(0,\infty)$ also follows. Therefore, taking subsequences and using the mentioned properties it is easy to pass to the limit in (5.9)–(5.11) and to obtain that $(\bar{z}, \bar{\psi}, \bar{v})$ is a solution of the system

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_\eta) z = \chi_\omega v \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = \xi \text{ in } \Omega, \end{cases} \tag{5.12}$$

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta \psi + a\psi + f'(y_\eta)\psi = (1 - f''(y_\eta)\varphi_\eta) z \text{ in } Q, \\ \partial_n \psi = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \|\psi(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{5.13}$$

$$\psi + \kappa v = 0 \quad \text{in} \quad Q_\omega, \tag{5.14}$$

where $(z_{\rho_k}, \psi_{\rho_k}, v_{\rho_k}) \rightharpoonup (\bar{z}, \bar{\psi}, \bar{v})$ in $W(0,\infty) \times W(0,\infty) \times L^2(Q_\omega))$ and $\rho_k \to 0$ as $k \to \infty$. Next we prove that this system has a unique solution, which implies the weak convergence of the whole family: $(z_\rho, \psi_\rho, v_\rho) \rightharpoonup (\bar{z}, \bar{\psi}, \bar{v})$ in $W(0,\infty) \times W(0,\infty) \times L^2(Q_\omega))$ as $\rho \to 0$.

To prove the uniqueness of a solution of (5.12)–(5.14) we observe that this is the optimality system for the following linear-quadratic control problem:

$$(Q) \qquad \min_{v\in L^2(Q_\omega)} \mathcal{J}(v) := \frac{1}{2}\int_Q \Big[1 - f''(y_\eta)\varphi_\eta\Big] z_v^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{Q_\omega} v^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where z_v is the solution of (5.12). It is enough to prove that $\mathcal{J}$ is strictly convex to conclude the result. To this end we write $z_v = z_{u_\eta,v} + z_\xi$, where $z_{u_\eta,v}$ is the solution of (2.11) with y_u replaced by y_η, and z_ξ satisfies the partial differential equation of (2.11) with $v = 0$ and y_u substituted by y_η, and the initial condition $z_\xi(0) = \xi$. Then, we have

$$\begin{aligned}
\mathcal{J}(v) &= \frac{1}{2}\int_Q \Big[1 - f''(y_\eta)\varphi_\eta\Big] z_{u_\eta,v}^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{Q_\omega} v^2 \,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_Q \Big[1 - f''(y_\eta)\varphi_\eta\Big] z_{u_\eta,v}\xi \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q \Big[1 - f''(y_\eta)\varphi_\eta\Big] z_\xi^2 \,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{1}{2}J_\eta''(u_\eta)v^2 + \int_Q \Big[1 - f''(y_\eta)\varphi_\eta\Big] z_{u_\eta,v}\xi \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q \Big[1 - f''(y_\eta)\varphi_\eta\Big] z_\xi^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Then, it is enough to use (5.3) to deduce the strict convexity of $\mathcal{J}$. Notice that the second term is linear in v and the third one is constant.

Step III.- Gâteaux differentiability of V_r. With the above notations we have

$$\begin{aligned}
&\frac{V_r(\eta + \rho\xi) - V_r(\eta)}{\rho} = \frac{J_{\eta_\rho}(u_\rho) - J_\eta(u_\eta)}{\rho}\\
&= \frac{1}{2}\int_Q z_\rho(y_\rho + y_\eta - 2y_d)\,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_{Q_\omega} v_\rho(u_\rho + u_\eta)\,\mathrm{d}x\,\mathrm{d}t \stackrel{\rho\to 0}{\longrightarrow} \int_Q \bar z(y_\eta - y_d)\,\mathrm{d}x\,\mathrm{d}t + \kappa\int_{Q_\omega} \bar v u_\eta \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

With the adjoint state φ_η, integrating by parts, and (5.12) we get

$$\int_Q \bar z(y_\eta - y_d)\,\mathrm{d}x\,\mathrm{d}t + \kappa\int_{Q_\omega} \bar v u_\eta\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varphi_\eta(x,0)\xi(x)\,\mathrm{d}x + \int_{Q_\omega}(\varphi_\eta + \kappa u_\eta)\bar v\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varphi_\eta(x,0)\xi(x)\,\mathrm{d}x,$$

which proves (5.8).

Step IV.- V_r is of class C^1. Now we prove that $V_r' : B_{\bar\alpha}(y_0) \longrightarrow L^\infty(\Omega)^*$ is continuous. This continuity and the Gâteaux differentiability imply that V_r is of class C^1. Since $V_r'(\eta)$ is identified with $\varphi_\eta(0)$ by the formula (5.8), the continuity follows if we prove that $V_r'(\eta_k) \to V_r'(\eta)$ in $C(\bar\Omega)$ if $\eta_k \to \eta$ in $L^\infty(\Omega)$. Let $\{\eta_k\}_{k=1}^\infty \subset B_{\bar\alpha}(y_0)$ such that $\eta_k \to \eta$ in $L^\infty(\Omega)$ as $k \to \infty$ and $\eta \in B_{\bar\alpha}(y_0)$. We define $(v_k, z_k, \psi_k) = (u_{\eta_k} - u_\eta, y_{\eta_k} - y_\eta, \varphi_{\eta_k} - \varphi_\eta)$. Subtracting the optimality systems for u_{η_k} and u_η we infer

$$\left\{\begin{array}{l} \dfrac{\partial z_k}{\partial t} - \Delta z_k + a z_k + f'(\hat y_k) z_k = \chi_\omega v_k \text{ in } Q,\\ \partial_n z_k = 0 \text{ on } \Sigma,\ z_k(0) = \eta_k - \eta \text{ in } \Omega,\end{array}\right. \tag{5.15}$$

$$\left\{\begin{array}{l} -\dfrac{\partial \psi_k}{\partial t} - \Delta\psi_k + a\psi_k + f'(y_\eta)\psi_k = (1 - f''(\tilde y_k)\varphi_{\eta_k}) z_k \text{ in } Q,\\ \partial_n\psi_k = 0 \text{ on } \Sigma,\ \lim_{t\to\infty}\|\psi_k(t)\|_{L^2(\Omega)} = 0,\end{array}\right. \tag{5.16}$$

$$\psi_k + \kappa v_k = 0 \quad \text{in } \ Q_\omega, \tag{5.17}$$

where $\hat y_k = y_\eta + \theta_k(y_{\eta_k} - y_\eta)$ and $\tilde y_k = y_\eta + \vartheta_k(y_{\eta_k} - y_\eta)$ for some measurable functions $\theta_k, \vartheta_k : Q \longrightarrow [0,1]$. Arguing similarly to Step I we infer that $\|v_k\|_{L^2(Q_\omega)} \le C\|\eta_k - \eta\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$. Moreover, Theorem 5.1 implies that $\{u_{\eta_k}\}_{k=1}^\infty$ is bounded in $L^\infty(Q_\omega)$. This yields $\|u_{\eta_k} - u_\eta\|_{\mathcal{U}_p} \to 0$ as $k \to \infty$. From Theorem 2.2 we deduce that $\{\hat y_k\}_{k=1}^\infty$ and $\{\tilde y_k\}_{k=1}^\infty$ are bounded in $L^\infty(Q)$. Then, applying [7, Theorems A.3 and A.4] to equations (5.15) and (5.16) we get that $y_{\eta_k} \to y_\eta$ and $\varphi_{\eta_k} \to \varphi_\eta$ in $W(0,\infty) \cap L^\infty(Q)$ as $k \to \infty$. The last convergence along with Corollary 2.4 implies that $\varphi_{\eta_k} \to \varphi_\eta$ in $C(\bar\Omega \times [0,\infty))$ and, consequently, $V_r'(\eta_k) = \varphi_{\eta_k}(0) \to \varphi_\eta(0) = V_r'(\eta)$ in $C(\bar\Omega)$. □

6. The Hamilton-Jacobi-Bellman equation

The goal of this section is to derive the Hamilton-Jacobi-Bellman equation satisfied by the value function V_r. To this end we make the following assumption

$$y_d = g = 0 \quad \text{and} \quad a \in L^\infty(\Omega) \text{ with } 0 \le a \not\equiv 0.$$

Following 5.2 and Theorem 5.3, we have that the value function $V_r : B_{\bar\alpha}(y_0) \longrightarrow \mathbb{R}$ defined by $V_r(\eta) = \inf\,(\mathrm{P}_{r,\eta})$ is of class C^1 and $V_r'(\eta) = \varphi_\eta(0) \in L^\infty(\Omega)$ for every $\eta \in B_{\bar\alpha}(y_0)$, where $B_{\bar\alpha}(y_0)$ denotes the open ball of $L^\infty(\Omega)$ centered at y_0 and radius $\bar\alpha$. In this section, u_η denotes the solution of $(\mathrm{P}_{r,\eta})$ and y_η and φ_η are the associated state and adjoint state.

Now, we introduce the operator $A : D(A) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by $Ay = \Delta y - ay$, where

$$D(A) = \{y \in H^1(\Omega) : Ay \in L^2(\Omega) \text{ and } \partial_n y = 0\}.$$

As usual we consider the following norm in $D(A)$: $\|y\|_{D(A)} = \|y\|_{L^2(\Omega)} + \|Ay\|_{L^2(\Omega)}$. It is obvious that $D(A)$ is a Hilbert space. Moreover, from the classical results for elliptic equations we infer that $D(A)$ is continuously embedded in $C(\bar\Omega)$; see, for instance, [15].

The rest of this section is dedicated to prove the following theorem.

Theorem 6.1. *The following Hamilton-Jacobi-Bellman equation is satisfied by V_r:*

$$\frac{1}{2}\|\eta\|^2_{L^2(\Omega)} - \frac{1}{2\kappa}\|V_r'(\eta)_{|\omega}\|^2_{L^2(\omega)} + (V_r'(\eta), \Delta\eta - a\eta - f(\eta))_{L^2(\Omega)} = 0 \quad \forall \eta \in B_{\bar\alpha}(y_0) \cap D(A). \tag{6.1}$$

Proof. Let $\eta \in B_{\bar\alpha}(y_0) \cap D(A)$ be chosen arbitrarily. Since $D(A) \subset C(\bar\Omega)$ we infer that $y_\eta \in C(\bar Q)$; see [9]. Using this continuity we infer the existence of $t_0 > 0$ such that $\{y_\eta(t) : t \in [0, t_0]\} \subset B_{\bar\alpha}(y_0)$. Moreover, since $D(A) \subset H^1(\Omega)$ we also obtain that $y_\eta \in C([0, t_0]; H^1(\Omega))$; see [16, Page 114].

As established in the proof of Theorem 5.3, we also have that $\varphi_\eta \in C(\bar\Omega \times [0, T])$ for every $T < \infty$. Then, from (4.3) we deduce that $u_\eta \in C(\bar\omega \times [0, T])$ for all $T < \infty$.

We split the proof into three steps.

Step I - Computation of $y_\eta'(0^+)$. Next we prove that

$$\lim_{t\to 0^+} \frac{1}{t}(y_\eta(t) - \eta) = \Delta\eta - a\eta - f(\eta) + \chi_\omega u_\eta(0), \tag{6.2}$$

where the limit is taken in the weak topology of $L^2(\Omega)$. First we establish this limit in the weak topology of $H^1(\Omega)^*$. To this end we take $w \in H^1(\Omega)$ arbitrary. Testing the equation satisfied by y_η with w, integrating between 0 and t with $0 < t < t_0$, and dividing the resulting expression by t we get

$$\int_\Omega \frac{1}{t}(y_\eta(t) - \eta) w \,\mathrm{d}x = \frac{1}{t}\int_0^t \int_\Omega \{-\nabla y_\eta(s)\nabla w - [a y_\eta(s) + f(y_\eta(s)) - \chi_\omega u_\eta(s)]w\} \,\mathrm{d}x\,\mathrm{d}s.$$

Since $y_\eta \in C(\bar\Omega \times [0, T])$, $u_\eta \in C(\bar\omega \times [0, T])$, $\nabla y_\eta : [0, t_0] \longrightarrow L^2(\Omega)^n$ is continuous, and $y_\eta(0) = \eta$ we deduce from the above identity that

$$\lim_{t\to 0} \int_\omega \frac{1}{t}(y_\eta(t) - \eta) w \,\mathrm{d}x = \int_\Omega (-\nabla\eta\nabla w - [a\eta + f(\eta) - \chi_\omega u_\eta(0)]w) \,\mathrm{d}x\,\mathrm{d}s \quad \forall w \in H^1(\Omega).$$

This implies that the limit (6.2) holds in the $H^1(\Omega)^*$ weak topology. Now, if we prove the boundedness of $\left\{\frac{1}{t}(y_\eta(t) - \eta)\right\}_{t\in[0,t_0]}$ in $L^2(\Omega)$ we conclude that (6.2) holds also in the weak topology of $L^2(\Omega)$. To do this we consider the representation formula for y_η given by the semigroup $\{S(t)\}_{t>0}$ generated by the operator A:

$$y_\eta(t) = S(t)\eta + \int_0^t S(t - s)(-f(y_\eta(s)) + \chi_\omega u_\eta(s)) \,\mathrm{d}s.$$

This identity implies that

$$\frac{1}{t}(y_\eta(t)-\eta)=\frac{1}{t}(S(t)\eta-\eta)+\frac{1}{t}\int_0^t S(t-s)(-f(y_\eta(s))+\chi_\omega u_\eta(s))\,\mathrm{d}s. \tag{6.3}$$

Since $\eta\in D(A)$ we have

$$\lim_{t\to 0^+}\frac{1}{t}(S(t)\eta-\eta)=A\eta=\Delta\eta-a\eta \text{ in } L^2(\Omega). \tag{6.4}$$

Further, the contractivity of the semigroup $\{S(t)\}_{t>0}$ implies that

$$\begin{aligned}&\Big\|\frac{1}{t}\int_0^t S(t-s)(-f(y_\eta(s))+\chi_\omega u_\eta(s))\,\mathrm{d}s\Big\|_{L^2(\Omega)}\le\frac{1}{t}\int_0^t\|S(t-s)(-f(y_\eta(s))+\chi_\omega u_\eta(s))\|_{L^2(\Omega)}\,\mathrm{d}s\\&\le\frac{1}{t}\int_0^t\|-f(y_\eta(s))+\chi_\omega u_\eta(s)\|_{L^2(\Omega)}\,\mathrm{d}s\le\|f(y_\eta)\|_{C([0,t_0];L^2(\Omega))}+\|u_\eta\|_{C([0,t_0];L^2(\omega))}.\end{aligned}$$

This inequality along with (6.3) and (6.4) prove the boundedness of $\left\{\frac{1}{t}(y_\eta(t)-\eta)\right\}_{t\in[0,t_0]}$ in $L^2(\Omega)$.

Step II - Computation of $\lim_{t\to 0^+}\frac{1}{t}(V_r(y_\eta(t))-V_r(\eta))$. Using the mean value theorem we get

$$\frac{1}{t}(V_r(y_\eta(t))-V_r(\eta))=V_r'(\eta+\theta(t)(y_\eta(t)-\eta))\frac{1}{t}(y_\eta(t)-\eta), \tag{6.5}$$

where $\theta:[0,t_0]\longrightarrow[0,1]$ is a measurable function. Since $\eta+\theta(t)(y_\eta(t)-\eta)\to\eta$ in $L^\infty(\Omega)$ as $t\to 0^+$ we deduce from Theorem 5.3 that $V_r'(\eta+\theta(t)(y_\eta(t)-\eta))\to V_r'(\eta)$ strongly in $C(\bar\Omega)$. Using this fact and (6.2) we deduce from (6.5) that

$$\frac{1}{t}(V_r(y_\eta(t))-V_r(\eta))\to V_r'(\eta)[\Delta\eta-a\eta-f(\eta)+\chi_\omega u_\eta(0)] \text{ weakly in } L^2(\Omega). \tag{6.6}$$

Step III - Usage of Bellman's principle. By Bellman's principle we have for $t\in[0,t_0]$

$$V_r(\eta)=\frac{1}{2}\int_0^t\left(\|y_\eta(s)\|^2_{L^2(\Omega)}+\kappa\|u_\eta(s)\|^2_{L^2(\omega)}\right)\mathrm{d}s+V_r(y_\eta(t)).$$

This yields

$$\frac{1}{t}(V_r(y_\eta(t))-V_r(\eta))+\frac{1}{2t}\int_0^t\left(\|y_\eta(s)\|^2_{L^2(\Omega)}+\kappa\|u_\eta(s)\|^2_{L^2(\omega)}\right)\mathrm{d}s=0.$$

Passing to the limit as $t\to 0^+$ and using (6.6) and the continuity of the functions y_η and u_η we get

$$V_r'(\eta)[\Delta\eta-a\eta-f(\eta)+\chi_\omega u_\eta(0)]+\frac{1}{2}\|\eta\|^2_{L^2(\Omega)}+\frac{\kappa}{2}\|u_\eta(0)\|^2_{L^2(\omega)}=0.$$

Finally, using (4.3) and (5.8) we obtain that $u_\eta(0)=-\frac{1}{\kappa}\varphi_\eta(0)_{|\omega}=-\frac{1}{\kappa}V_r'(\eta)$. Inserting this twice in the above identity we derive (6.1). □

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