On non-autonomous parabolic equations with measure-valued right hand sides and applications to optimal control

Karl Kunisch[1] and Joachim Rehberg[2]

[1]Institute of Mathematics, University of Graz, Austria and Radon Institute, Austrian Academy of Sciences, Linz, Austria, karl.kunisch@uni-graz.at

[2]Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany, rehberg@wias-berlin.de

January 9, 2025

Abstract

The main aim of this paper is to develop a theory for non-autonomous parabolic equations with time-dependent measures on the spatial domain appearing as right hand sides. Restricting these measures to ones which have their supports on 'curves' or 'surfaces' – the latter understood in the sense of geometric measure theory – we succeed in interpreting them as distributional objects from a (negatively indexed) Sobolev-Slobodetskii space $W^{s,2}(\Omega)$ with s close to -1. For these indices s a tailor suited parabolic theory is established, based on results of [19] and [27]. The proposed frame work is well-suited for optimal control problems with controls acting on submanifolds.

Keywords: non-autonomous evolution equations, parabolic initial boundary value problems, maximal parabolic regularity, measure-valued right hand sides, optimal control

MSC codes:35B65, 35K20, 35B45, 28A75, 49J20

Contents

1 Introduction

The investigation of generally non-autonomous parabolic equations

$$\frac{\partial u}{\partial t} + \mathcal{A}(\cdot)u = \varrho, \quad u(0) = 0 \tag{1}$$

with right hand sides including measures has been carried out in the pioneering paper [2]. In that work the spatial geometry and the time-dependent coefficients are assumed to be smooth. Concerning the measure, in the literature typically the case is considered where the measure on the space time cylinder is of the kind

$$C_0(]0,T[\times\Omega) \ni f \mapsto \int_0^T \int_\Omega f(t,x)\, d\rho_t(x)\, dt, \tag{2}$$

$\{\rho_t\}_{t\in]0,T]}$ being a suitable family of Radon measures on Ω, which is – in its dependence of t – weak* measurable. The procedure how to treat such parabolic equations is widely common: embed $\mathcal{M}(\Omega)$, the space of bounded Radon measures on Ω, into a space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ and identify the r.h.s. in this manner with a function $f \in L^r(]0,T[, W^{-1,q}_{\mathfrak{D}}(\Omega))$ ($W^{1,q}_{\mathfrak{D}}(\Omega)$ denoting the usual Sobolev space which includes a trace-zero condition on $\mathfrak{D} \subset \partial\Omega$ and $W^{-1,q}_{\mathfrak{D}}(\Omega)$ being the space of continuous antilinear forms on $W^{1,q'}_{\mathfrak{D}}(\Omega)$). For such right hand sides one may – under mild conditions – apply maximal parabolic regularity of the second order divergence operators involved to get a solution which belongs to the maximal parabolic regularity space (see (26) below). It is almost clear that this is widely optimal for the solution. Unfortunately, this has two serious drawbacks: In order to catch *all* bounded Radon measures on Ω, one has to chose q's which ly definitely below $\frac{d}{d-1}$, d being the space dimension. Therefore the domain of the elliptic second order divergence operator can be at best $W^{1,q}_{\mathfrak{D}}(\Omega)$ – with this limitation of q. This is more irregular than $W^{1,2}_{\mathfrak{D}}(\Omega)$. But even worse: in general it is extremely delicate – in view of pathologies which were discovered already by Serrin in [45] – to give the divergence operators on $W^{-1,q}_{\mathfrak{D}}(\Omega)$ a precise meaning at all, if $q < 2$ is far from 2.
Secondly, one has for non-autonomous, second order parabolic equations no results on maximal parabolic regularity in the $W^{-1,q}$ scale at hand if the geometry of the spatial domain is really non-smooth and the time dependence of the coefficients is 'wild' – as long as q is not close to 2.

Consequently, in this paper we go another way: We consider sets of codimension 1 or 2 in the spatial domain Ω. If focused on dimensions 2 and 3 this results in measures which live on 'curves' or 'surfaces' and are absolutely continuous with respect to the induced Hausdorff measures. 'Curves ' and 'surfaces' are to be understood here in an extremely broad sense – based on the concept of l-sets from geometric measure theory, developped by Jonsson and Wallin (see [29]) in the seventies. These sets M are characterized by the condition

$$\mathfrak{c}_\bullet\, r^l \le \mathcal{H}_l(M \cap B(x,r)) \le \mathfrak{c}^\bullet\, r^l, \quad x \in M,\ r \in]0,1], \tag{3}$$

where $l \in \{1,\dots,d-1\}$ and $\mathcal{H}_l$ is the l-dimensional Hausdorff measure. So, M being an l-set, we consider measures $\sigma\mathcal{H}_l|_M$, with σ a function from $L^2(M;\mathcal{H}_l)$. Fortunately, the pioneering results of Jonsson/Wallin admit in our case embeddings

$$L^2(M;\mathcal{H}_l) \ni \sigma \mapsto \sigma\mathcal{H}_l|_M \in W^{-1\pm\epsilon,2}_{\mathfrak{D}}(\Omega), \tag{4}$$

where, in our context, $\epsilon > 0$ may be taken arbitrarily small. Even more: one gets uniform boundedness for norms of the mappings (4), if M runs through a class of subsets in Ω admitting a *uniform* upper l estimate, i.e. in case the constant $\mathfrak{c}^\bullet$ in (3) can be chosen uniformly for all sets under consideration.
All of this provides a constellation which is quite comfortable concerning the investigation of the parabolic equation in the context of (non-autonomous) maximal parabolic regularity, namely: in [27] an elliptic extrapolation theorem was established which asserts that

the operator

$$-\nabla \cdot \mu \nabla + 1 : W^{1\pm\epsilon,2}_{\mathfrak{D}}(\Omega) \to W^{-1\pm\epsilon,2}_{\mathfrak{D}}(\Omega), \tag{5}$$

as a topological isomorphism for $\epsilon = 0$ by Lax-Milgram, extends to a (consistent) isomorphism for small $\epsilon > 0$ under very general assumptions. Having this at hand, it is not too difficult to show that (the negative of) these extrapolated operators indeed generate analytic semigroups on the corresponding Hilbert spaces $W^{1\pm\epsilon,2}_{\mathfrak{D}}(\Omega)$, see Thm. 2.6 below. The next step is easy: it has been established since long that the negative of a generator of an analytic semigroup satisfies maximal parabolic regularity – if the underlying Banach space is topologically a Hilbert space. Knowing this, we deduce from these foregoing insights and the central result of [19], that the mapping

$$\begin{aligned}
&w \mapsto w' - div\, \hat{\mu}\, grad\, w \\
&\text{from } W^{1,q}_0(J; W^{-1\pm\epsilon,2}_{\mathfrak{D}}(\Omega)) \cap L^q(J; W^{1\pm\epsilon,2}_{\mathfrak{D}}(\Omega)) \text{ to } L^q(J; W^{-1\pm\epsilon,2}_{\mathfrak{D}}(\Omega)),
\end{aligned} \tag{6}$$

which is a topological isomorphism by the classical Lions' result ([13, Section XVIII.3, Remark 9]) for $\epsilon = 0$ and $q = 2$, extrapolates to an isomorphism for $q \sim 2$ and small ϵ. Fitting everything together: the embedding (4) – including the control over the embedding constants – with the non-autonomous parabolic regularity result, one gets as much regularity for the solution as one can realistically expect: maximal parabolic regularity. Astonishingly, q and $s = 1 \pm \epsilon$ in their inter-relation cleverly chosen, one can achieve that the space of solutions, namely the left hand side of (6), even embeds *compactly* in the usual trace space $C(\overline{J}; L^2(\Omega))$.
In recent years also the numerical analysis of such problems has been treated, see [30], [31], and also [44], [36], [11]. In the first paper it is reflected that discontinuous diffusion coefficients allow the treatment of moving interfaces – a property which is clearly required in applications. In [31] and [11] discuss real world problems where the – time dependent – measures on the right hand side of the parabolic equation are concentrated on hypersurfaces.

In the last section of this paper our analysis of (1) will be used in the context of optimal control problems. The analysis for these problems is typically carried out for the cases where the control acts on subdomains of Ω or $\partial\Omega$. The situation where the support of the control has no interior in Ω or $\partial\Omega$ has received surprisingly little attention. In [32] considers problems with point control in the interior of the domain. Optimal control problems with controls as measures were extensively investigated, see eg. [47, Chapter 4] and the literature cited there. It should, however, be noticed that formulating optimal control problems over the whole space of measures favors minimizers which are pointwise source functions, see eg. [9], [8], [42]. The optimal controls obtained in this manner are typically not concentrated on lower dimensional manifolds. The present paper aims at providing necessary prerequisites for optimal control on possibly time-dependent lower dimensional manifolds of codimension 1 and 2, and first steps towards exploiting these results are taken in section 4. There are only few other publications which also focus on such control problems. All of them consider the problem under investigation in a Banach (Sobolev-) space setting, while in the present paper we favor a Hilbert space frame-work, as much as this is possible. Our coefficient functions may depend *discontinuously* on space *and* time, and in fact *non-autonomous* parabolic equations are one of the main subjects of this paper. Likely the paper most closely related to ours is [39], where also a convection term is admitted in the equation. The cost functional there includes a gradient term which leads to an adjoint equation which is much less regular than in our case. In [34, 36] finite element approximation of the optimal control problems with controls on manifolds are investigated. Let us also mention [11] where approximate controllability of the heat equation by controls acting on a lower dimensional manifold is investigated.

Throughout this paper we denote by d the dimension of the domain Ω and by $\mathcal{H}_l$ the l-dimensional Hausdorff measure, where $l \in \{1, \ldots, d-1\}$. We recall that on smooth and Lipschitzian submanifolds of $\mathbb{R}^d$ the Hausdorff measure is identical with the measure defined by parametrizations on this manifold, see [23, Ch. 3.3/3.4]. Moreover, if $M \subset \Omega \subset \mathbb{R}^d$ then we abbreviate $L^p(M;\mathcal{H}_l|_M)$ by $L^p(M;\mathcal{H}_l)$ in all what follows. For $\Omega \subset \mathbb{R}^d$ a bounded domain, then we denote by $\mathcal{M}(\Omega)$ the space of finite Radon measures on Ω. Finally, for two Banach spaces X, Y, with Y continuously embedded into X, we denote by $(X,Y)_{\theta,r}$ the usual real interpolation space and by $[X,Y]_\theta$ the corresponding complex interpolation space (see [48, Ch. I]).

We generally admit *complex* coefficients in this paper. In the sequel we need the following measure theoretic notion of sets

Definition 1.1. For $l \in]1,\dots,d]$ the l-dimensional Hausdorff measure on $\mathbb{R}^d$ is denoted by $\mathcal{H}_l$. Let $M \subset \mathbb{R}^d$ be Borel set. We call M an l-set if (3) holds for positive constants $\mathfrak{c}_\bullet$, $\mathfrak{c}^\bullet$. In case $l = d-1$, it is said that M satisfies the Ahlfors-David condition.

In all what follows we assume that the following assumption is in power.

Assumption 1.2. $\Omega \subset \mathbb{R}^d$ is a bounded domain $d \geq 2$.

(a) $\mathfrak{D}$ is a closed subset of $\partial\Omega$ which satisfies the *Ahlfors–David condition.*

(b) For every $x \in \overline{\partial\Omega \setminus \mathfrak{D}}$ there exists an open neighbourhood U_x of x and a bi-Lipschitz map Φ_x from U_x onto the cube $K :=]-1,1[^d$, such that the following three conditions are satisfied:

$$\Phi_x(x) = 0,$$
$$\Phi_x(U_x \cap \Omega) = \{x \in K : x_d < 0\},$$
$$\Phi_x(U_x \cap \partial\Omega) = \{x \in K : x_d = 0\}.$$

(c)
$$|\Omega \cap B(x,r)| \geq c\, r^d, \quad x \in \Omega, \quad r \in]0,1] \tag{7}$$

for some constant c.

Above and in the sequel $B(x,r)$ denote the ball in $\mathbb{R}^d$ with centre x and radius r. Unless indicated otherwise, the the integrability index p is always assumed to be in $]1,\infty[$.

Definition 1.3 (Sobolev-Slobodetskii spaces)**.** $W^{1,p}(\mathbb{R}^d)$ is the usual Sobolev space. For $s \in]0, 1+\frac{1}{p}[\setminus\{1\}$ write $s = k+\sigma$ with $k \in \{0,1\}$ and $\sigma \in]0,1[$. Then the space $W^{s,p}(\mathbb{R}^d)$ is given by the normed vector space of functions $\psi \in L^2(\mathbb{R}^d)$ for which

$$\|\psi\|_{W^{s,p}(\mathbb{R}^d)} := \|\psi\|_{W^{k,p}(\mathbb{R}^d)} + \left(\sum_{i=1}^d \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\partial_i^k \psi(x) - \partial_i^k \psi(y)|^p}{|x-y|^{d+p\sigma}} dxdy \right)^{1/p} < \infty.$$

For further purpose we also need Sobolev-Slobodetskii spaces on l-sets.

Definition 1.4 (Sobolev-Slobodetskii spaces on singular sets)**.** Let $M \subset \mathbb{R}^d$ be an l-set, $s \in]0,1[$. Define, for $\psi \in L^p(M;\mathcal{H}_l)$

$$\|\psi\|_{W^{s,p}(M)} := \|\psi\|_{L^p(M;\mathcal{H}_l)} + \left(\iint_{M\times M} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{d+ps}}\, d\mathcal{H}_l(x)\, d\mathcal{H}_l(y) \right)^{1/p}, \tag{8}$$

finite or infinite. We introduce $W^{s,p}(M)$ as the space of functions on M, for which (8) is finite.

Proposition 1.5. *Let $E \subset \mathbb{R}^d$ be an l-set, $l \in \{d-2, d-1\}$. Assume $p \in]1,\infty[$ and $s \in]\frac{1}{p}, 1+\frac{1}{p}[$ such that $\beta = s - \frac{d-l}{p} > 0$. Then, for $\psi \in W^{s,p}(\mathbb{R}^d)$, the limit*

$$(\operatorname{tr}_E \psi)(x) := \lim_{r\searrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \psi, \quad x \in E \tag{9}$$

exists for $\mathcal{H}_l$-almost all $x \in E$ and the thus defined operator tr_E maps $W^{s,p}(\mathbb{R}^d)$ continuously onto $W^{s-\frac{d-l}{p},p}(E)$.
Conversely, for tr_E exists a continuous right inverse $\mathfrak{F}_E : W^{s-\frac{d-l}{p},p}(E) \to W^{s,p}(\mathbb{R}^d)$, such that every function $\mathfrak{F}_E u$ is smooth on $\mathbb{R}^d \setminus E$. Moreover, in case of $l = d-1$, $\mathfrak{F}_E$ maps the Lipschitzian functions on E into the set of Lipschitzian functions on $\mathbb{R}^d$. Finally, the extension operators are consistent *for all $s \in]\frac{1}{p}, 1+\frac{1}{p}, [$.*

Proof. For the first two statements see [29, Thm. VI.1]. The assertion on Lipschitz continuity of the extension is proved in [26]. □

Definition 1.6 (Function spaces with zero trace)**.** Let $E \subset \mathbb{R}^d$ be a $(d-1)$-set and let $s \in]\frac{1}{p}, 1+\frac{1}{p}[$. Then we define $W^{s,p}_E(\mathbb{R}^d) := \ker \mathrm{tr}_E$ in $W^{s,p}(\mathbb{R}^d)$.

The analogues of the spaces $W^{s,p}(\mathbb{R}^d)$ and $W^{s,p}_E(\mathbb{R}^d)$ on Ω are defined as quotient spaces corresponding to restriction to Ω of their $\mathbb{R}^d$ versions as follows:

Definition 1.7 (Function spaces on Ω)**.** Let $p \in]1, \infty[$ and $s \in]0, 1+\frac{1}{p}[$.

(i) We define $W^{s,p}(\Omega)$ to be the factor space of restrictions of $W^{s,p}(\mathbb{R}^d)$ to Ω, equipped with the natural quotient norm. Moreover, $W^{-s,p}(\Omega) := \big(W^{s,p'}(\Omega)\big)^\star$.

(ii) Let $E \subseteq \overline{\Omega}$ be a $(d-1)$-set. Then, as before, we define $W^{s,p}_E(\Omega)$ as the factor space of restrictions to Ω of $W^{s,p}_E(\mathbb{R}^d)$, equipped with the natural quotient norm. Moreover, $W^{-s,p}_E(\Omega) := \big(W^{s,p'}_E(\Omega)\big)^\star$.

Remark 1.8. i) The definition of the spaces $W^{s,p}(\Omega)$ as factor spaces of restrictions implies that these spaces inherit the usual Sobolev-type embeddings between them from their full-space analogues.

ii) Let $s \in]0, 1[$. Then it is well-known that – since by assumption Ω satisfies (7) – the factor space $W^{s,p}(\Omega)$ agrees with the space $W^{s,p}_*(\Omega)$ defined intrinsically by the set of all functions $u \in L^p(\Omega)$ such that

$$\|\psi\|_{W^{s,p}_*(\Omega)} := \|\psi\|_{L^p(\Omega)} + \left(\iint_{\Omega\times\Omega} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{d+ps}} dxdy\right)^{1/p} < \infty \qquad (10)$$

up to equivalent norms (see [29, Thm. V.1]). Moreover, very recently it was shown in [6] that, if $E \subseteq \partial\Omega$ is $(d-1)$-regular and Ω satisfies the interior thickness condition (7) for $x \in \partial\Omega \setminus E$, then $W^{s,p}_E(\Omega)$ coincides with the intrinsically given $W^{s,p}_*(\Omega) \cap L^p(\Omega, \mathrm{dist}_E^{-sp})$, also up to equivalent norms.

iii) The reader should carefully notice that, in case of $p = 2$ the so defined Sobolev-Slobodetskii spaces on $\mathbb{R}^d$ are identical with the corresponding Bessel potential spaces, i.e. $H^{s,2}(\mathbb{R}^d)$, see [48, Ch. 2.3.2]. We decided to maintain the notation $W^{s,2}/W^{s,2}_{\mathfrak{D}}$ by the following reason: at least for $s \in]0, 1[$ one has, due to ii), an explicit description of the spaces also on Ω and hence a more clear perception.

Since the domain Ω is fixed through the whole paper, we mostly omit the $'\Omega'$ from now on - writing e.g. $W^{1,2}_{\mathfrak{D}}$ instead of $W^{1,2}_{\mathfrak{D}}(\Omega)$.

Proposition 1.9. *i) One has the interpolation equality*

$$[L^2, W^{1,2}_{\mathfrak{D}}]_s = \begin{cases} W^{s,2}_{\mathfrak{D}} & \text{for } s \in]\frac{1}{2}, 1[\\ W^{s,2} & \text{for } s \in]0, \frac{1}{2}[. \end{cases} \qquad (11)$$

ii) Assume $s_0, s_1 \in]\frac{1}{2}, \frac{3}{2}[$ and put $s = (1-\theta)s_0 + \theta s_1$, $\theta \in]0, 1[$. Then

$$(W^{s_0,2}_{\mathfrak{D}}, W^{s_1,2}_{\mathfrak{D}})_{\theta,2} = [W^{s_0,2}_{\mathfrak{D}}, W^{s_1,2}_{\mathfrak{D}}]_\theta = W^{s,2}_{\mathfrak{D}}. \qquad (12)$$

Proof. The results are proved as Thm. 7.1 in [21]. □

Lemma 1.10. *Define, for* $s \in]0,1[\setminus\{\frac{1}{2}\}$, $\mathcal{W}_s$ *as the right hand side of* (11). *Then the embedding* $\mathcal{W}_s \hookrightarrow L^2$ *is compact.*

Proof. Under Assumption 1.2 b) one knows the existence of a linear, continous extension operator $\mathfrak{E} : W^{1,2}_{\mathfrak{D}} \to W^{1,2}(\mathbb{R}^d)$, see [3, Lemma 3.2]. Hence, the embedding $W^{1,2}_{\mathfrak{D}} \hookrightarrow L^2$ is compact. So, taking into account (11), the assertion follows from [48, Ch. 1.16.4]. □

For the following density result we require yet another definition.

Definition 1.11. The space of infinitely differentiable functions with bounded gradient on Ω is denoted by $C^\infty_b(\Omega)$.

Lemma 1.12. *Let* $\Omega \subset \mathbb{R}^d$ *be bounded and* $E \subset \partial\Omega$ *be a closed* $(d-1)$*-set.*

i) Assume $s \in]\frac{1}{2}, 1]$. *Then* $C^\infty_E(\Omega) \subseteq W^{s,2}_E(\Omega) \cap C^\infty(\Omega)$ *is dense in* $W^{s,2}_E(\Omega)$.

ii) Assume $s \in]1, \frac{3}{2}[$. *Then* $W^{s,2}_E(\Omega) \cap C^\infty_b(\Omega)$ *is dense in* $W^{s,2}_E(\Omega)$.

Proof. i) is proved in [21, Prop. 3.7]. ii) We first prove that the statement is correct if Ω is replaced by $\mathbb{R}^d$. Let $\tilde{\psi} \in W^{s,2}_E(\mathbb{R}^d) \subset W^{s,2}(\mathbb{R}^d)$. Then there is a sequence $\{\psi_n\}$ in $C^\infty_0(\mathbb{R}^d)$ converging towards $\tilde{\psi}$ in the $W^{s,2}$ topology, see [48, Ch. 2.3.2]. Let tr_E and $\mathfrak{F}_E$ be the restriction/extension operators from Proposition 1.5. Since tr_E is a left inverse of $\mathfrak{F}_E$, the operator $\mathfrak{P} := 1 - \mathfrak{F}_E tr_E$ is a continuous projection in $W^{s,2}(\mathbb{R}^d)$, called the Jonsson/Wallin projection. Recall that, for $\phi \in C^\infty_0(\mathbb{R}^d)$, $\mathfrak{P}\phi$ is smooth on $\mathbb{R}^d \setminus E$. Moreover, it is also Lipschitzian on $\mathbb{R}^d$, so that the gradient is globally bounded, in particular bounded on Ω (cf. Proposition 1.5). Applying the projector $\mathfrak{P}$ and taking into account $\mathfrak{P}\tilde{\psi} = \tilde{\psi}$, one gets $\lim_{n\mapsto\infty} \mathfrak{P}\psi_n = \mathfrak{P}\tilde{\psi} = \tilde{\psi}$. But $\mathfrak{P}\psi_n$ is C^∞ in $\mathbb{R}^d \setminus E$, and this is in particular true in Ω. Now assume $\psi \in W^{s,2}_E(\Omega)$. Take $\tilde{\psi} \in W^{s,2}_E(\mathbb{R}^d)$ as any extension of ψ. Then it is clear that $\psi - \mathfrak{P}\psi_n|_\Omega = (\tilde{\psi} - \mathfrak{P}\psi_n)|_\Omega$ converges to zero in $W^{s,2}_E(\Omega)$ in the factor topology. □

2 Elliptic and parabolic regularity in the $W^{-s,2}_{\mathfrak{D}}$ scale

2.1 Elliptic operators

Definition 2.1. For $\mu \in L^\infty(\Omega; \mathbb{C}^{d\times d})$, we define the operator

$$-\nabla \cdot \mu\nabla + 1 : W^{1,2}_{\mathfrak{D}} \to W^{-1,2}_{\mathfrak{D}} \tag{13}$$

by

$$\langle -\nabla \cdot \mu\nabla\psi + \psi, \varphi\rangle = \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi} + \psi\overline{\varphi}, \quad \psi, \varphi \in W^{1,2}_{\mathfrak{D}}. \tag{14}$$

If μ satisfies the strong ellipticity condition

$$\Re(\mu(x)\xi, \xi)_{\mathbb{C}^d} \geq m|\xi|^2, \quad \xi \in \mathbb{C}^d \tag{15}$$

uniformly for almost all $x \in \Omega$, then (2.1) is a topological isomorphism by the Lax-Milgram theorem.

Definition 2.2 (Multiplier)**.** Let X be a Banach space of functions $\Omega \to \mathbb{C}$. A bounded function $\zeta : \Omega \to \mathbb{C}$ is a *multiplier on* X if the multiplication operator M_ζ defined by $(M_\zeta f)(x) := \zeta(x)f(x)$ maps X continuously into itself. We write $\zeta \in \mathfrak{M}(X)$ and the multiplier norm is given by $\|\zeta\|_{\mathfrak{M}(X)} := \|M_\zeta\|_{X\to X}$.

Assumption 2.3. There exits $\delta \in]0, \frac{1}{2}[$, such that all components $\mu_{i,j}$ are multipliers on the space $W^{s,2}$, $s \in]0, \delta]$

Remark 2.4. i) Trivially, any $\omega \in L^\infty$ is a multiplier on L^2. So, if ζ is multiplier on $W^{\epsilon,2}$ and, additionally, $\zeta \in L^\infty(\Omega)$, then one deduces from Prop. 1.9 and interpolation that ζ is a also multiplier for all $W^{s,2}$ for $s \in]0, \epsilon[$.

ii) If $s \in]0, \frac{1}{2}[$ and $\sigma > s$, then every $\zeta \in C^\sigma(\Omega)$ is a multiplier on $W^{s,2}$.

iii) The class of Hölder functions does not exhausts the set of multipliers. On the contrary, every indicator function χ_Λ is a multiplier for each space $W^{s,2}$, $s \in]0, \frac{1}{2}[$ as long as $\Lambda \subset \Omega$ is a set of locally finite perimeter ([43, p. 214ff]). An advanced criterion for 'set of finite perimeter' is given in [23, Thm. 5.23].

iv) Further remarks on the multipliers can be found in [27, Ch. 5] and references therein.

Proposition 2.5. *Let the coefficient function μ be bounded and strongly elliptic, and let Assumption 2.3 hold. Then there is $\iota > 0$ such that* (13) *consistently extrapolates for $s \in] - \iota, \iota[$ to a topological isomorphism*

$$-\nabla \cdot \mu \nabla + 1 : W^{1+s,2}_{\mathfrak{D}} \to W^{s-1,2}_{\mathfrak{D}}. \tag{16}$$

Further, both ι and the norms of the inverse operators $(-\nabla \cdot \mu\nabla + 1)^{-1}$ between $W^{s-1,2}_D$ and $W^{1+s,2}_D$ for $s \in]-\iota, \iota[$ can be estimated uniformly in the norm of the multiplier norms $\|\mu_{i,j}\|_{\mathfrak{M}}$ and the ellipticity constant m in (15)*.*

Proof. see [27, Thm. 1] □

In all what follows the letter ι has the meaning of characterizing the interval of numbers s, such that (16) is a topological isomorphim.

In the context of this proposition we always use the same symbol $-\nabla \cdot \mu \nabla$ irrespective what s is. Unfortunately, one cannot expect in general that ι significantly differs from zero as is known since long (see [38], compare also [22, Ch. 4]). The following result asserts that $\nabla \cdot \mu \nabla - 1$ generates an analytic semigroup on $W^{-s}_{\mathfrak{D}}$ for $s \in] - \frac{3}{2}, -\frac{1}{2}[$.

Theorem 2.6. *i) For $s \in]\frac{1}{2}, 1[$ the part of*

$$\nabla \cdot \mu \nabla - 1 : W^{1,2}_{\mathfrak{D}} \to W^{-1,2}_{\mathfrak{D}}$$

in $W^{-s,2}_{\mathfrak{D}}$ generates an analytic semigroup on that space. In particular, $-(\nabla \cdot \mu \nabla - 1)$ is a positive operator in the sense of Triebel, see [48, Ch. 1.14].

ii) Assume $s \in]0, \iota[$. Then $\nabla \cdot \mu \nabla - 1$ admits the following resolvent estimate

$$\|(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1}\|_{\mathcal{L}(W^{-1-s,2}_{\mathfrak{D}})} \le \frac{c}{1 + |\lambda|}, \quad \Re\lambda \ge 0, \tag{17}$$

and, hence, it generates an analytic semigroup on $W^{-1-s,2}_{\mathfrak{D}}(\Omega)$. In particular, $-\nabla \cdot \mu \nabla + 1$ is a positive *operator on $W^{-1-s,2}_{\mathfrak{D}}$ in the sense of Triebel.*

Proof. i) $\nabla \cdot \mu \nabla - 1$ generates analytic semigroups on both, L^2 and $W^{-1,2}_{\mathfrak{D}}$, see [40, Ch. 1.4.2]. In particular, it satisfies resolvent estimates like

$$\|(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1}\|_{\mathcal{L}(L^2)} \le \frac{c}{1 + |\lambda|}, \quad \Re\lambda \ge 0 \tag{18}$$

and

$$\|(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1}\|_{\mathcal{L}(W^{-1,2}_{\mathfrak{D}})} \le \frac{c}{1 + |\lambda|}, \quad \Re\lambda \ge 0. \tag{19}$$

Applying the well-known characterization of an analytic generator property via the resolvent decay, it is clear that $\nabla \cdot \mu \nabla - 1$ generates an analytic semigroup also on every (real or complex) interpolation space between L^2 and $W^{-1,2}_{\mathfrak{D}}$. Taking into account (11)

and applying the duality formula for complex interpolation (see [48, Ch. 1.11.3]) one gets $[L^2, W_{\mathfrak{D}}^{-1,2}]_s = W^{-s,2}$.
ii) It is well-known that the L^2 realization of $-\nabla \cdot \mu \nabla + 1$ admits a bounded holomorphic calculus (see [25, Cor. 7.1.17], compare also [14, Ch. 2.3]) and, hence, bounded purely imaginary powers (see [14, Ch. 2.3]).
Secondly, the positive resolved Kato square root problem in [21] tells us

$$\mathrm{Dom}\Big(\big((-\nabla \cdot \mu \nabla + 1)|_{L^2} \big)^{\frac{1}{2}} \Big) = W_{\mathfrak{D}}^{1,2}. \tag{20}$$

Together, this gives, according to [48, Ch. 1.15.3],

$$\begin{aligned} \mathrm{Dom}\Big(\big((-\nabla \cdot \mu \nabla + 1)|_{L^2} \big)^{\frac{1-s}{2}} \Big) = \big[L^2, \mathrm{Dom}\Big(\big((-\nabla \cdot \mu \nabla + 1)|_{L^2} \big)^{\frac{1}{2}} \Big) \big]_{1-s} = \\ = [L^2, W_{\mathfrak{D}}^{1,2}]_{1-s} = W_{\mathfrak{D}}^{1-s,2}, \end{aligned} \tag{21}$$

thanks to (11). In other words,

$$\big(-\nabla \cdot \mu \nabla + 1 \big)^{\frac{1-s}{2}} : W_{\mathfrak{D}}^{1-s,2} \to L^2$$

is a topological isomorphism. Combining this with (16) one also gets the isomorphism property

$$\big(-\nabla \cdot \mu \nabla + 1 \big)^{\frac{1-s}{2}} (-\nabla \cdot \mu \nabla + 1)^{-1} : W_{\mathfrak{D}}^{-1-s,2} \to L^2 \tag{22}$$

Let us now prove that the – well-known – resolvent decay on L^2 implies the asserted resolvent decay on $W_{\mathfrak{D}}^{-1-s,2}$: due to the consistency of $-\nabla \cdot \mu \nabla + 1$ on different spaces under consideration, one has for $\psi \in L^2$ and $\lambda \in \mathbb{C}$ with $\Re \lambda \geq 0$

$$\begin{aligned} \|(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \psi\|_{W_{\mathfrak{D}}^{-1-s,2}} = \\ \|\big(-\nabla \cdot \mu \nabla + 1 \big)^{\frac{1+s}{2}} (-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \big(-\nabla \cdot \mu \nabla + 1 \big)^{-\frac{1+s}{2}} \psi\|_{W_{\mathfrak{D}}^{-1-s,2}} \leq \\ \|(-\nabla \cdot \mu \nabla + 1) \big(-\nabla \cdot \mu \nabla + 1 \big)^{\frac{s-1}{2}} \|_{\mathcal{L}(L^2; W_{\mathfrak{D}}^{-1-s,2})} \quad \|(-\nabla \cdot \mu \nabla + 1 + \lambda)\|_{\mathcal{L}(L^2)} \\ \|\big(-\nabla \cdot \mu \nabla + 1 \big)^{\frac{1-s}{2}} (-\nabla \cdot \mu \nabla + 1)^{-1} \|_{\mathcal{L}(W_{\mathfrak{D}}^{-1-s,2}; L^2)} \|\psi\|_{W_{\mathfrak{D}}^{-1-s,2}}. \end{aligned}$$

Since L^2 is dense in $W_{\mathfrak{D}}^{-1-s,2}$ this resolvent estimate extends to *all* $\psi \in W_{\mathfrak{D}}^{-1-s,2}$, and the theorem is proved. □

Remark 2.7. The reader should carefully notice that now, *knowing* that $-\nabla \cdot \mu \nabla + 1$ in fact is a positive operator on $W_{\mathfrak{D}}^{-1-s,2}$, one indeed has the operator identity

$$\big(-\nabla \cdot \mu \nabla + 1 \big)^{\frac{1-s}{2}} (-\nabla \cdot \mu \nabla + 1)^{-1} = \big(-\nabla \cdot \mu \nabla + 1 \big)^{-\frac{1+s}{2}} \tag{23}$$

as an operator equality on $W_{\mathfrak{D}}^{-1-s,2}$. So (22) can be read as

$$\mathrm{Dom}_{W_{\mathfrak{D}}^{-1-s,2}} \big((-\nabla \cdot \mu \nabla + 1)^{\frac{s+1}{2}} \big) = L^2. \tag{24}$$

2.2 Maximal parabolic regularity: Definition and results

Throughout the rest of this paper let $T > 0$ and set $J =]0, T[$. Let us start by recalling the following (standard) definition.

Definition 2.8. If X is a Banach space and $q \in]1, \infty[$, then we denote by $L^q(J; X)$ the space of X-valued functions f on J which are Bochner-measurable and for which $\int_J \|f(t)\|^q \, dt$ is finite. We define $W^{1,q}(J; X) := \{u \in L^q(J; X) : \frac{\partial u}{\partial t} \in L^q(J; X)\}$, where $\frac{\partial u}{\partial t}$ is to be understood as the time derivative of u in the sense of X-valued distributions (cf. [1, Section III.1]). Moreover, we introduce the subspace

$$W_0^{1,q}(J; X) := \{u \in W^{1,q}(J; X) : u(0) = 0\}.$$

We equip this subspace always with the norm $u \mapsto \|\frac{\partial u}{\partial t}\|_{L^q(J;X)}$.

Definition 2.9. Let X, D be Banach spaces with D densely and continuously embedded in X. Let $J \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D; X)$ be a bounded and strongly measurable map and suppose that the operator $\mathcal{A}(t)$ is closed in X for all $t \in J$. Let $q \in]1, \infty[$. Then we say that the family $\{\mathcal{A}(t)\}_{t \in J}$ satisfies **(non-autonomous) maximal parabolic $L^q(J; X)$-regularity**, if for each $f \in L^q(J; X)$ there is a unique function $u \in L^q(J; D) \cap W_0^{1,q}(J; X)$ which satisfies

$$\frac{\partial u}{\partial t} + \mathcal{A}(t)u(t) = f(t) \tag{25}$$

for almost all $t \in J$. We write

$$\mathrm{MR}_0^q(J; D, X) := L^q(J; D) \cap W_0^{1,q}(J; X) \tag{26}$$

for the space of maximal parabolic regularity. Introducing the norm

$$\|u\|_{\mathrm{MR}_0^q(J;D,X)} = \|u\|_{L^q(J;D)} + \|\frac{\partial u}{\partial t}\|_{L^q(J;X)},$$

makes $\mathrm{MR}_0^q(J; D, X)$ a Banach space.

We emphasize that $\mathrm{Dom}(\mathcal{A}(t)) = D$ for all $t \in J$ in Definition 2.9. In particular, all operators $\mathcal{A}(t)$ have the same domain. If all operators $\mathcal{A}(t)$ are equal to one (fixed) operator $\mathcal{A}_0$, and there exists an $q_0 \in]1, \infty[$ such that $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^{q_0}(J; X)$-regularity, then $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^q(J; X)$-regularity for all $q \in]1, \infty[$ and we say that $\mathcal{A}_0$ satisfies **maximal parabolic regularity on X**.

Let us recall some abstract embedding properties of the space of maximal parabolic regularity which we will need later.

Proposition 2.10. *Let X, Y be Banach spaces with Y is continuously embedded into X. Let $T > 0$ and set $J =]0, T[$.*

i) If $q \in]1, \infty[$, then

$$W^{1,q}(J; X) \cap L^q(J; Y) \hookrightarrow C(\overline{J}; (X, Y)_{1-\frac{1}{q},q}), \tag{27}$$

(see [1, Ch. III Thm. 4.10].

ii) If $q \in]1, \infty[$ and $\theta \in]0, 1 - \frac{1}{q}[$, then

$$W^{1,q}(J; X) \cap L^q(J; Y) \hookrightarrow C^\beta(J; (X, Y)_{\theta,1}), \tag{28}$$

where $\beta = 1 - \frac{1}{q} - \theta$, (see [19, Lemma 2.11]).

Our next aim is to show that the second order divergence operators indeed satisfy maximal parabolic regularity on the spaces $W_{\mathfrak{D}}^{-s,2}$ – starting with the *autonomous* case. We recall that Assumptions 1.2 and 2.3 are supposed to hold throughout.

Theorem 2.11. *i) For all $s \in [-\iota, \iota]$, the domain of the operator $-\nabla \cdot \mu \nabla + 1$, when considered in $W_{\mathfrak{D}}^{-1+s}$, is $W_{\mathfrak{D}}^{1+s}$.*

ii) *Put* $\epsilon = min(\iota, \frac{1}{2})$*. For* $s \in [-\epsilon, \frac{1}{2}[$*,* $-\nabla\cdot\mu\nabla+1$ *satisfies maximal parabolic regularity in* $W_{\mathfrak{D}}^{-1+s}$*.*

iii) *If a family of coefficient functions* $\{\mu_\tau\}_\tau$ *admits a uniform in* τ *ellipticity constants and uniform in* τ *multiplier norms, then* ϵ *may be taken uniformly in* τ*.*

Proof. Property i) follows directly from Theorem 2.5. ii) It is well-known that every negative of a generator of an analytic semigroup on a Hilbert space satisfies maximal parabolic regularity there. So, for $s \in [0, \frac{1}{2}[$ the claim follows from Thm. 2.6 1. For $s \in [-\epsilon, 0[$ the result is obtained by Thm. 2.6 2. iii) the uniformity in τ for i) is contained in Theorem 2.5. From this the uniformity in ii) follows by the uniformity of i) and Theorem 2.6. □

Now we pass to *non-autonomous* parabolic operators on the $W_{\mathfrak{D}}^{-s,2}$ scale.

Assumption 2.12. i) Let $\hat{\mu}\colon J\times\Omega \to \mathbb{C}^{d\times d}$ be a bounded mapping, such that

$$J \in t \mapsto \mu(t,\cdot) \in L^1(\Omega, \mathbb{C}^{d\times d}), \tag{29}$$

is measurable.

ii) The coeffcient functions $\mu(t,\cdot)$ are elliptic, and the ellipticity constants may be taken uniform with respect to $t \in J$, see (15).

iii) For every $t \in J$ the coefficient function $\mu(t,\cdot) =: \mu$ satisfies Assumption 2.3 The corresponding norms as multipliers are uniformly (in $t \in J$ and $s \in]0,\delta]$) bounded.

We consider the coefficient function $\hat{\mu}$ satisfying this assumption as fixed.

Note that the set of points in Ω where $\mu(t,\cdot)$ is discontinuous may depend on t. Consequently we admit the situation that the mapping $t \mapsto \mu(t,\cdot)$ from J into $L^\infty(\Omega;\mathbb{C}^{d\times d})$ is discontinuous at every time point t. In this case it cannot be measurable, see the example in [18, Ch. 7.1]. For this reason we only demand L^1-measurability in Assumption 2.12.

Lemma 2.13. *Let* $q \in]1,\infty[$ *and* $s \in]-\delta,\delta[$*, with* δ *as in Assumption 2.12. Then the following properties hold.*

i) *For every* $\psi \in W_{\mathfrak{D}}^{1+s,2}$*, the mapping*

$$t \mapsto -\nabla\cdot\hat{\mu}(t,\cdot)\nabla\psi \tag{30}$$

is (strongly) measurable from J *into* $W_{\mathfrak{D}}^{-1+s,2}$*.*

ii) *Define the mapping* $\mathcal{A} : L^q(J;W_{\mathfrak{D}}^{1+s,2}) \to L^q(J, W_{\mathfrak{D}}^{-1+s,2})$ *by*

$$\big(\mathcal{A}u\big)(t) = -\nabla\cdot\hat{\mu}(t,\cdot)\nabla(u(t)), \quad u \in L^q(J;W_{\mathfrak{D}}^{1+s,2}). \tag{31}$$

Then

$$\mathrm{MR}_0^q(J;W_{\mathfrak{D}}^{1+s,2}, W_{\mathfrak{D}}^{-1+s,2}) \ni u \mapsto \frac{\partial u}{\partial t} - \mathcal{A}\,u$$

is a bounded linear map into $L^q(J;W_{\mathfrak{D}}^{-1+s,2})$ *with a norm which depends on the multiplier norms of the coefficient functions* $\hat{\mu}(t,\cdot)$*.*

Proof. i) We start with the case $s > 0$. One first considers

$$J \ni t \mapsto \langle -\nabla\cdot\mu(t,\cdot)\nabla\psi, \varphi\rangle_{(W_{\mathfrak{D}}^{-1+s,2};W_{\mathfrak{D}}^{1-s,2})} = \langle \mu(t,\cdot)\nabla\psi, \nabla\overline{\varphi}\rangle_{(W_{\mathfrak{D}}^{s,2};W_{\mathfrak{D}}^{-s,2})}. \tag{32}$$

Taking $\psi \in W_{\mathfrak{D}}^{1+s,2} \cap C_b^\infty$ (see Definition 1.11) and φ from the set $C_{\mathfrak{D}}^\infty(\Omega)$, the right hand side in (32) equals $\int_\Omega \mu(t,\cdot)\nabla\psi\cdot\nabla\overline{\varphi}$. Since $\nabla\psi, \nabla\varphi \in L^\infty(\Omega)$, the measurability in t follows from the asserted measurability for $\hat{\mu}$. But $W_{\mathfrak{D}}^{1+s,2} \cap C_b^\infty(\Omega)$ is dense in

$W_{\mathfrak{D}}^{1+s,2}$ and $C_{\mathfrak{D}}^{\infty}(\Omega)$ is dense in $W_{\mathfrak{D}}^{1-s,2}$ by Lemma 1.12. So the measurability for general $\psi \in W_{\mathfrak{D}}^{1+s}$ and $\varphi \in W_{\mathfrak{D}}^{1-s,2}$ follows by taking the limit in (32). Thus, we have proved *weak* measurability of (30). But this implies strong measurability since $W_{\mathfrak{D}}^{1-s,2}$ is separable and reflexive – and so is $W_{\mathfrak{D}}^{-1+s,2}$. The case $s < 0$ can be treated analogously, identifying $\langle -\nabla \cdot \mu(t,\cdot)\nabla\psi, \varphi\rangle_{(W_{\mathfrak{D}}^{-1+s,2};W_{\mathfrak{D}}^{1-s,2})}$ as $\langle \nabla\psi, \big(\mu(t,\cdot)\big)^*\nabla\varphi\rangle_{(W_{\mathfrak{D}}^{s,2};W_{\mathfrak{D}}^{-s,2})}$ (see [27, p. 1237] for more details) and then arguing as before.
Property ii) is easily deduced from Proposition 2.5. □

Proposition 2.14. *There is an open interval $\mathfrak{I} \ni 2$, such that for each $q \in \mathfrak{I}$*

$$\frac{\partial}{\partial t} - \mathcal{A} : \mathrm{MR}_0^q(J; W_{\mathfrak{D}}^{1,2}, W_{\mathfrak{D}}^{-1,2}) \to L^q(J; W_{\mathfrak{D}}^{-1,2}) \tag{33}$$

is a topological isomorphism i.e. $-\mathcal{A}$ satisfies non-autonomous maximal parabolic regularity on $L^q(J, W_{\mathfrak{D}}^{-1,2})$.

Proof. The result is proved in [18, Thm. 7.2]. The reader should notice that here *no* geometric suppositions on Ω are required and that for $\mathfrak{D}$ (relative) closedness in $\partial\Omega$ suffices. □

Theorem 2.15. *For every $q \in \mathfrak{I}$, there is an open interval $\mathfrak{J}(q) \ni 0$ so that for $s \in \mathfrak{J}(q)$*

$$\frac{\partial}{\partial t} - \mathcal{A} : \mathrm{MR}_0^q(J; W_{\mathfrak{D}}^{1+s,2}, W_{\mathfrak{D}}^{s-1,2}) \to L^q(J; W_{\mathfrak{D}}^{s-1,2}) \tag{34}$$

is a topological isomorphism. Thus $-\mathcal{A}$ satisfies non-autonomous maximal parabolic regularity on $L^q(J, W_{\mathfrak{D}}^{s-1,2})$.

Proof. Let $q \in \mathfrak{I}$. Assumption 2.12 implies the existence of an $\epsilon \in]0, \frac{1}{2}[$ such that, for every $s \in [-\epsilon, \epsilon]$

$$-\nabla \cdot \mu(t,\cdot)\nabla + 1 : W_{\mathfrak{D}}^{1+s,2} \to W_{\mathfrak{D}}^{-1+s,2}, \qquad t \in J$$

is a toplogical isomorphisms (see Theorem 2.5) and, additionally, each of these operators satisfies maximal (autonomous) maximal parabolic regularity, see Theorem 2.15. Moreover, it is known that (33) is a topological isomorphism, which is (34) for $s = 0$.
Thanks to (12), one may write, for $\epsilon \in]-\frac{1}{2}, \frac{1}{2}[$,

$$W_{\mathfrak{D}}^{1,2} = [W_{\mathfrak{D}}^{1+\epsilon,2}, W_{\mathfrak{D}}^{1-\epsilon,2}]_{\frac{1}{2}},$$

and, by duality (see [48, Ch. 1.11.3])

$$W_{\mathfrak{D}}^{-1,2} = [W_{\mathfrak{D}}^{-1-\epsilon,2}, W_{\mathfrak{D}}^{-1+\epsilon,2}]_{\frac{1}{2}} = [W_{\mathfrak{D}}^{-1+\epsilon,2}, W_{\mathfrak{D}}^{-1-\epsilon,2}]_{\frac{1}{2}}$$

Hence, (33) can be interpreted in the sense that

$$\frac{\partial}{\partial t} - \mathcal{A} : \mathrm{MR}_0^q(J; [W_{\mathfrak{D}}^{1+\epsilon,2}, W_{\mathfrak{D}}^{1-\epsilon,2}]_{\frac{1}{2}}, [W_{\mathfrak{D}}^{-1+\epsilon,2}, W_{\mathfrak{D}}^{-1-\epsilon,2}]_{\frac{1}{2}}) \mapsto$$
$$\mapsto L^q(J; [W_{\mathfrak{D}}^{-1+\epsilon,2}, W_{\mathfrak{D}}^{-1-\epsilon,2}]_{\frac{1}{2}}) \tag{35}$$

provides a topological isomorphism. But then [18, Thm. 3.4] tells us that this remains a topological isomorphism, if the interpolation index $\frac{1}{2}$ is replaced by indices θ sufficiently close to $\frac{1}{2}$. Re-identifying, by (12), $[W_{\mathfrak{D}}^{1+\epsilon,2}, W_{\mathfrak{D}}^{1-\epsilon,2}]_\theta$ as $W_{\mathfrak{D}}^{1+\epsilon(1-2\theta),2}$ and, by duality,

$$[W_{\mathfrak{D}}^{-1+\epsilon,2}, W_{\mathfrak{D}}^{-1-\epsilon,2}]_\theta = [W_{\mathfrak{D}}^{1-\epsilon,2}, W_{\mathfrak{D}}^{1+\epsilon,2}]_\theta^* = \big(W_{\mathfrak{D}}^{1-\epsilon(1-2\theta),2}\big)^* = W_{\mathfrak{D}}^{-1+\epsilon(1-2\theta),2},$$

one obtains the assertion. □

Up to now we considered the initial value problem with initial value 0. We will now pass to initial values $u_0 \neq 0$.

Lemma 2.16. *Let X be a Banach space and B the generator of an analytic semigroup on X. Then the following identity of sets holds*

$$(X, D(B))_{1-\frac{1}{p},p} = \{x \in X : Be^{-tB}x \in L^p(]0,1[;X)\}. \tag{36}$$

Moreover, both spaces in (36) also coincide topologically.

Proof. See [35, Prop. 2.2.2] □

Theorem 2.17. *Choose $q \in \mathfrak{I}$, and $s \in \mathfrak{J}(q)$ as established in Theorem 2.15. Then for every $u_0 \in (W^{-1+s,2}_{\mathfrak{D}}, W^{1+s,2}_{\mathfrak{D}})_{1-\frac{1}{q},q}$ and $f \in L^q(J; W^{-1+s,2}_{\mathfrak{D}})$ there exists a unique solution to*

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = f, \quad u(0) = u_0. \tag{37}$$

This solution belongs to the maximal parabolic space

$$W^{1,q}(J; W^{-1+s,2}_{\mathfrak{D}}) \cap L^q(J; W^{1+s,2}_{\mathfrak{D}}) =: X$$

and admits the estimate

$$\|u\|_X \leq C(\|u_0\|_{(W^{-1+s,2}_{\mathfrak{D}}, W^{1+s,2}_{\mathfrak{D}})_{1-\frac{1}{q},q}} + \|f\|_{L^2(J; W^{-1+s,2}_{\mathfrak{D}})}), \tag{38}$$

for a constant C independent of u_0 and f.

Proof. *Step 1* As in Theorem 2.15 we shall utilize that

$$-\nabla \cdot \mu(t, \cdot)\nabla + 1 : W^{1+s,2}_{\mathfrak{D}} \to W^{-1+s,2}_{\mathfrak{D}}$$

are topological isomorphisms, uniformly in $t \in J$. Further $A_0 = -\nabla \cdot \mu(0, \cdot)\nabla + 1$ generates an analytic semigroup on $W^{-1+s,2}_{\mathfrak{D}}$. Let us set $\breve{u} = e^{-tA_0}u_0$, and argue that

$$\frac{\partial \breve{u}}{\partial t} - \mathcal{A}\, \breve{u} \in L^q(J; W^{-1+s,2}_{\mathfrak{D}})$$

and

$$\|\frac{\partial \breve{u}}{\partial t} - \mathcal{A}\, \breve{u}\|_{L^q(J; W^{-1+s,2}_{\mathfrak{D}})} \leq c_1 \|u_0\|_{(W^{-1+s,2}_{\mathfrak{D}}, W^{1+s,2}_{\mathfrak{D}})_{1-\frac{1}{q},q}} \tag{39}$$

for a constant c_1 independent of u_0. Indeed, for almost every $t \in J$ we have

$$\tfrac{\partial \breve{u}}{\partial t} - \nabla \cdot \mu(t, \cdot)\nabla \breve{u}(t) \quad = \big(-1 - \nabla \cdot \mu(t, \cdot)\nabla A_0^{-1}\big) A_0 \breve{u}(t).$$

Thanks to Lemma 2.16, we know that $A_0\breve{u}(t) \in L^q(]0,1]; W^{-1+s,2}_{\mathfrak{D}})$ and hence $A_0\breve{u}(t) \in L^q(J; W^{-1+s,2}_{\mathfrak{D}})$. By supposition the operator family $\{1 - \nabla \cdot \mu(t, \cdot)\nabla A_0^{-1}\}_{t \in J}$ is a bounded in $\mathcal{L}(W^{1-s,2}_{\mathfrak{D}})$ uniformly in $t \in J$, thus also $\frac{\partial \breve{u}}{\partial t} - \mathcal{A}\, \breve{u}$ belongs to $L^q(J; W^{-1+s,2}_{\mathfrak{D}})$. Estimate (39) again follows from Lemma 2.16.

Step 2 We make the ansatz $u = \breve{u} + v$, $v \in W^{1,q}(J; W^{-1+s,2}_{\mathfrak{D}}) \cap L^2(J; W^{1+s,2}_{\mathfrak{D}})$ for the solution of (37). So we are looking for v as the solution of

$$\frac{\partial v}{\partial t} - \mathcal{A}\, v = f - \frac{\partial \breve{u}}{\partial t} + \mathcal{A}\, \breve{u}, \text{ with } v(0) = 0. \tag{40}$$

Evidently, u then satisfies (37). By Theorem 2.15 equation (40) admits a unique solution satisfying

$$\begin{aligned} &\|\tfrac{\partial v}{\partial t}\|_{L^q(J; W^{-1+s,2}_{\mathfrak{D}})} + \|v\|_{L^q(J; W^{1+s,2}_{\mathfrak{D}})} \\ &\quad \leq c_2\|f\|_{L^q(J; W^{-1+s,2}_{\mathfrak{D}})} + c_2\|\tfrac{\partial \breve{u}}{\partial t} - \mathcal{A}\, \breve{u}\|_{L^q(J; W^{-1+s,2}_{\mathfrak{D}})}, \end{aligned} \tag{41}$$

for a constant c_2 independent of f and the expression containing $\check{u}$. By (39) this implies that

$$\|\frac{\partial v}{\partial t}\|_{L^q(J;W_{\mathfrak{D}}^{-1+s,2})}+\|v\|_{L^q(J;W_{\mathfrak{D}}^{1+s,2})} \le c_2(\|f\|_{L^q(J;W_{\mathfrak{D}}^{-1+s,2})}+c_1\|u_0\|_{(W_{\mathfrak{D}}^{-1+s,2},W_{\mathfrak{D}}^{1+s,2})_{1-\frac{1}{q},q}}). \tag{42}$$

For $\check{u}$ we have $\|\check{u}\|_{L^2(J;W_{\mathfrak{D}}^{1+s,2})} \sim \|A_0\check{u}\|_{L^2(J;W_{\mathfrak{D}}^{-1+s,2})} \le c\|u_0\|_{(W_{\mathfrak{D}}^{-1+s,2},W_{\mathfrak{D}}^{1+s,2})_{\frac{1}{2},2}}$. The estimate for $\|\frac{\partial \check{u}}{\partial t}\|_{L^q(J;W_{\mathfrak{D}}^{-1+s,2})}$ follows from the latter and the equation $\frac{\partial \check{u}}{\partial t} = A_0\check{u}$. Combined with (42) these considerations imply (38). □

Having now the inclusion of the solution in the maximal parabolic regularity space at hand, our next aim are some embedding results for the space of maximal parabolic regularity in the $W_{\mathfrak{D}}^{-s,2}$ scale, based on Proposition 2.10. For this we require interpolation results which we turn to next.

Proposition 2.18. *Let V be a reflexive Banach space and H a Hilbert space with continuous, dense injection $V \hookrightarrow H$. Then one has the (complex) interpolation identity $[V,V^*]_{\frac{1}{2}} = H$.*

Proof. The result goes back to [41], compare also [49] and [12]. □

Lemma 2.19. *Assume that $s \in]0,\frac{1}{2}[$. If $\theta \in]\frac{1+s}{2},1[$, then one has the interpolation identity*

$$[W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1-s,2}]_\theta = W^{2\theta-1-s,2}, \tag{43}$$

and, hence, the embedding

$$(W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1-s,2})_{\theta,1} \hookrightarrow W^{2\theta-1-s,2}. \tag{44}$$

Proof. Proposition 2.18 implies that $L^2 = [W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]_{\frac{1}{2}}$, from which we make repeated use. By re-iteration we have

$$W_{\mathfrak{D}}^{1-s,2} = [L^2,W_{\mathfrak{D}}^{1+s,2}]_{\frac{1-s}{1+s}} = \big[[W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]_{\frac{1}{2}},W_{\mathfrak{D}}^{1+s,2}\big]_{\frac{1-s}{1+s}} = \\ = [W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]_{\frac{1}{1+s}}, \tag{45}$$

and thus

$$[W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1-s,2}]_\theta = \big[W_{\mathfrak{D}}^{-1-s,2},[W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]_{\frac{1}{1+s}}\big]_\theta = \\ = [W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]_{\frac{\theta}{1+s}}. \tag{46}$$

Since $\frac{\theta}{1+s} > \frac{1}{2}$, (46) may be written by re-iteration as

$$\big[[W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]_{\frac{1}{2}},W_{\mathfrak{D}}^{1+s,2}\big]_{\frac{2\theta-1-s}{1+s}} = [L^2,W_{\mathfrak{D}}^{1+s,2}]_{\frac{2\theta-1-s}{1+s}} = W^{2\theta-1-s,2},$$

which provides the desired result. □

Theorem 2.20. *For $s \in]0,\frac{1}{2}[$ the space $L^2(J;W_{\mathfrak{D}}^{1+s,2}) \cap W^{1,2}(J;W_{\mathfrak{D}}^{-1+s,2}) =: X$ embeds compactly into $C(\overline{J};L^2)$.*

Proof. According to Proposition 2.10 the embedding $X \hookrightarrow C^\beta(J;(W_{\mathfrak{D}}^{-1+s,2},W_{\mathfrak{D}}^{1+s,2})_{\theta,1})$ holds for $\theta \in]0,\frac{1}{2}[$, where $\beta = \frac{1}{2}-\theta$. Thus, by Arzela/Ascoli it is sufficient to show that, θ clever chosen, the space $(W_{\mathfrak{D}}^{-1+s,2},W_{\mathfrak{D}}^{1+s,2})_{\theta,1}$ compactly embeds into L^2. In this spirit, we take $\theta \in]\frac{1-s}{2},\frac{1}{2}[$. From (45) and duality (see [48, Ch. 1.11.3]) we obtain

$$W_{\mathfrak{D}}^{-1+s,2} = \big(W_{\mathfrak{D}}^{1-s,2}\big)^* = [W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]^*_{\frac{1}{1+s}} = [W_{\mathfrak{D}}^{1+s,2},W_{\mathfrak{D}}^{-1-s,2}]_{\frac{1}{1+s}} = \\ = [W_{\mathfrak{D}}^{-1-s,2},W_{\mathfrak{D}}^{1+s,2}]_{\frac{s}{1+s}}.$$

So we get, exploiting again re-iteration

$$(W_{\mathfrak{D}}^{-1+s,2}, W_{\mathfrak{D}}^{1+s,2})_{\theta,1} \hookrightarrow [W_{\mathfrak{D}}^{-1+s,2}, W_{\mathfrak{D}}^{1+s,2}]_\theta = \big[[W_{\mathfrak{D}}^{-1-s,2}, W_{\mathfrak{D}}^{1+s,2}]_{\frac{s}{1+s}}, W_{\mathfrak{D}}^{1+s,2}\big]_\theta$$

$$= [W_{\mathfrak{D}}^{-1-s,2}, W_{\mathfrak{D}}^{1+s,2}]_\kappa \quad \text{with } \kappa = (1-\theta)\frac{s}{1+s} + \theta. \tag{47}$$

Our choice of θ implies $\kappa > \frac{1}{2}$. So we may again apply re-iteration in order to write

$$[W_{\mathfrak{D}}^{-1-s,2}, W_{\mathfrak{D}}^{1+s,2}]_\kappa = \big[[W_{\mathfrak{D}}^{-1-s,2}, W_{\mathfrak{D}}^{1+s,2}]_{\frac{1}{2}}, W_{\mathfrak{D}}^{1+s,2}\big]_{2\kappa-1} = \big[L^2, W_{\mathfrak{D}}^{1+s,2}\big]_{2\kappa-1}. \tag{48}$$

The space $W_{\mathfrak{D}}^{1+s,2}$ continuously embeds into $W_{\mathfrak{D}}^{1,2}$. Since $W_{\mathfrak{D}}^{1,2}(\Omega)$ admits a linear, continuous extension operator into $W_{\mathfrak{D}}^{1,2}(\mathbb{R}^d)$, the embedding $W_{\mathfrak{D}}^{1,2} \hookrightarrow L^2(\Omega)$ is compact. So the r.h.s. of (48) compactly embeds into L^2, see [48, Ch. 1.16.4]. The claim follows from these facts. $\square$

Theorem 2.21. *Let $q > 2$, $s \in]0, \frac{1}{2}[$ such that the interval $]\frac{1+s}{2}, 1 - \frac{1}{q}[$ is non-empty and $\theta \in]\frac{1+s}{2}, 1 - \frac{1}{q}[$. Put $\beta = 1 - \frac{1}{q} - \theta$. Then one has the embedding*

$$W^{1,q}(J; W_{\mathfrak{D}}^{-1-s,2}) \cap L^q(J; W_{\mathfrak{D}}^{1-s,2}) \hookrightarrow C^\beta(J; W^{2\theta-1-s,2}), \tag{49}$$

and the maximal parabolic space $\mathrm{MR}_0^q(J; W_{\mathfrak{D}}^{1-s,2}, W_{\mathfrak{D}}^{-1-s,2})$ embeds compactly into $C(\overline{J}; L^2)$.

Proof. The first assertion is implied by a combination of (28) and (44). The second follows from Lemma 1.10, the compactness of the embedding $W^{2\theta-1-s,2} \hookrightarrow L^2$ and the Arzela/Ascoli theorem. $\square$

3 Non-autonomous problems with measure-valued functions as right hand sides

3.1 Generalities

In this chapter we investigate non-autonomous parabolic problems like

$$\frac{\partial w}{\partial t} - \mathcal{A}\, w = \varrho, \quad w(0) = 0, \tag{50}$$

with ϱ a function on J, taking values as bounded Radon measures ρ_t on Ω at every $t \in J$. It is important to consider mappings $J \ni t \mapsto \rho_t \in \mathcal{M}$, which are only weak* measurable, this means: mappings for which

$$J \ni t \mapsto \langle \rho_t, \psi \rangle, \quad \text{for all } \psi \in C(\overline{\Omega}) \tag{51}$$

are measurable (compare the discussion in [8, Ch. 2.1]). Otherwise one would exclude examples as the following:
Let $J \ni t \mapsto x(t)$ be an injective curve in Ω. If one defines $\rho_t := \delta_{x(t)}$ (the Dirac measure in the point $x(t)$), then the mapping $J \ni t \mapsto \delta_{x(t)}$ is in *every* point discontinuous, if one equipps the space of (bounded) measures with the strong topology. Hence, it is not measurable if one defines the structure of measurability via this strong topology. On the contrary, if one considers the weak* topology and the induced concept of measurability, then the mapping $J \ni t \mapsto \delta_{x(t)}$ is at least measurable if the mapping $J \ni t \mapsto x(t)$ is measurable itself.
If $\mathcal{N}$ is a space of measures for which one knows an embedding $\mathcal{N} \hookrightarrow W_{\mathfrak{D}}^{-s,2}$, then the measurability of (51) is in particular true for functions $\psi \in C_b^\infty(\Omega) \cap W_{\mathfrak{D}}^{s,2}$, which are dense in $W_{\mathfrak{D}}^{s,2}$ (see Lemma 1.12). Hence, the measurability carries over to all functions $\psi \in W_{\mathfrak{D}}^{s,2}$ by density. But this means: the mapping $J \ni t \mapsto \rho_t \in W_{\mathfrak{D}}^{-s,2}$ is weakly

measurable in this case. Then the separability of $W_{\mathfrak{D}}^{-s,2}$ implies, quite in contrast to the situation in $\mathcal{M}(\Omega)$, even the strong measurability. Thus one is, via embeddding, in a situation in which rather general mappings $J \ni t \mapsto \rho_t$ are admissable and, additionally, suit in the context of maximal parabolic regularity – even in the non-autonomous case. However, the reader should carefully notice: weak* limits of measures, these being possibly concentrated on sets of lower Hausdorff dimension, can be of entirely different nature. E.g. *every* Radon measure on Ω is the weak* limit of linear combinations of Dirac measures on Ω. In other words: the affiliation of a measure to a class of measures, concentrated on lower dimensional objects, is by no means necessarily preserved for the weak* limit.

3.2 Interpretation of singular measures as elements from $W_{\mathfrak{D}}^{-s,2}$

Up to now we have established a parabolic theory for second order operators in the $W_{\mathfrak{D}}^{-s,2}$ scale. In order to treat parabolic second order equations with measure valued right hand side it is, in consequence, necessary to allow an interpretation for these objects as elements from $W_{\mathfrak{D}}^{-s,2}$. This will be delivered next.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$, $\mathfrak{D}$ be a closed subset of $\partial\Omega$. Then the space of bounded Radon measures on Ω continuously embeds into any space $W_{\mathfrak{D}}^{-1-\epsilon,2}$ if $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. For every $u \in W_{\mathfrak{D}}^{1+\epsilon,2}$ there is a $\tilde{u} \in W^{1+\epsilon,2}(\mathbb{R}^d)$ the restriction of which is u and, additionally, $\|\tilde{u}\|_{W^{1+\epsilon,2}(\mathbb{R}^d)} \le 2\|u\|_{W_{\mathfrak{D}}^{1+\epsilon,2}}$. So, for every $\epsilon > 0$ one has $W_{\mathfrak{D}}^{1+\epsilon,2} \hookrightarrow C(\overline{\Omega})$. Thus, one gets for every bounded Radon measure $\mathfrak{m}$ on Ω

$$\|\mathfrak{m}\|_{W_{\mathfrak{D}}^{-1-\epsilon,2}} = \sup_{\|\psi\|_{W_{\mathfrak{D}}^{1+\epsilon,2}}=1} \Big| \int_\Omega \psi \, d\mathfrak{m} \Big| \le \sup_{\|\psi\|_{W_{\mathfrak{D}}^{1+\epsilon,2}}=1} \sup_{x \in \Omega} |\psi(x)| \, \|\mathfrak{m}\| \le c \, \|\mathfrak{m}\|.$$

□

So far, this affects *general* bounded Radon measures on Ω irrespective of their singularity – even Dirac measures are admitted, compare e.g. [32], [44], [11].

In the sequel we restrict the class of measures which are admitted. The reason is twofold: first the classes which we will consider are the most relevant ones in view of applications. Secondly, as we have seen, optimal elliptic and parabolic regularity are only available for second order divergence operators if the differentiability index s of the corresponding Hilbert space is close to -1, see Theorems 2.15 and 2.17. Thus, concerning parabolic equations, one is restricted to measures which can be considered as elements of $W_{\mathfrak{D}}^{-1\pm\epsilon,2}$ with $\epsilon \sim 0$. In two space dimensions it turns out that – besides the class of all bounded Radon measures – the measures situated on sets of Hausdorff dimension 1 deserve special attention. In three space dimensions this affects the measures concentrated on 'surfaces' and 'curves' – in fact: sets of Hausdorff dimensions 1 or 2. In order to make this precise, we need some preparation. Recall first the definition of an l-set from the introduction.

Lemma 3.2. *If the closed set $M \subset \mathbb{R}^d$ is an l-set satisfying* (3), *and one defines the measure ϖ on $\mathbb{R}^d$ by $\varpi(N) = \mathcal{H}_l(N \cap M)$ for every Borel set $N \subset \mathbb{R}^d$, then ϖ satisfies $\varpi(B(x,r)) \le 2^l \mathfrak{c}^\bullet r^l$ for $r \le 1/2$. Hence, $\varpi(M)$ is finite.*

Proof. For all $x \in \mathbb{R}^d$ with $\operatorname{dist}(x,M) > 1/2$ one has $B(x,r) \cap M = \emptyset$ for $r \le 1/2$, so that $\varpi(B(x,r)) = 0$ for these r. If $\operatorname{dist}(x,M) = r \le 1/2$, then exists a $y \in M$ with $|x-y| = r \le 1/2$. But then $B(x,r) \subseteq B(y,2r)$ and the assertion follows. □

Proposition 3.3. *If $M \subset \Omega$ is a Borel set of finite Hausdorff measure $\mathcal{H}_l$, then the mapping*

$$C_0(\Omega) \ni v \mapsto \int_M v \, d\mathcal{H}_l$$

is a bounded Radon measure on Ω.

Proof. Since $\mathcal{H}_l$ is a Borel measure on $\mathbb{R}^d$ (see [23, Ch. 2 Thm. 2.1]) and $\mathcal{H}_l(M)$ is finite, the restriction of $\mathcal{H}_l$ to M is a (bounded) Radon measure on $\mathbb{R}^d$ (see [23, Ch. 2 Thm. 2.1]). It is clear that the restriction of this to Ω remains a (bounded) Radon measure. □

From the previous two results we conclude that if $M \subset \Omega$ is a Borelian l-set, then $\mathcal{H}_l|_M$ is a bounded Radon measure on Ω. Moreover, the total mass of $\overline{M} \supset M$ with respect to $\mathcal{H}_l$ can be estimated by $\mathfrak{c} \times \tau$, where τ is the number of (shifted) unit balls required for a covering of $\overline{M}$.

Proposition 3.4. *Let $M \subset \mathbb{R}^d$ be an l-set with $0 < l \in \{d-2, d-1\}$. If $l = d-2$, let $\alpha \in]1, \frac{3}{2}[$; and if $l = d-1$ let $\alpha \in]\frac{1}{2}, 1]$.*

i) For $f \in L^2(\mathbb{R}^d)$ and $u = G_\alpha \star f$ one has

$$\|u\|_{L^2(M;\mathcal{H}_l)} \leq c\|f\|_{L^2(\mathbb{R}^d)}, \tag{52}$$

G_α being the corresponding Bessel potential (see [46, Ch. V.3]). The constant c can be chosen independent from f.

ii) The constant c may be taken even uniform for sets M obeying the right estimate (3) with a uniform $\mathfrak{c}^\bullet$.

Proof. i) is a special case of [29, Ch. VI. Lemma 6]. ii) follows by a careful inspection of that proof. For the convenience of the reader we give some comments in the appendix how to read the proof of [29, Ch. VI. Lemma 6] in the special case under consideration here. □

Corollary 3.5. *Let $M \subset \mathbb{R}^d$ be a closed l-set with $0 < l \in \{d-2, d-1\}$. Let α be as in Prop. 3.4. Then one has a continuous embedding $W^{\alpha,2}(\mathbb{R}^d) = H_2^\alpha(\mathbb{R}^d)$ into $L^2(M; \mathcal{H}_l)$, $H_p^\alpha(\mathbb{R}^d)$ being the well-known Bessel potential space (see [48, Ch. 2.3.3], compare also [46, Ch. V.3]).*
The embedding constants are uniformly bounded for different sets M obeying the right estimate (3) with a uniform $\mathfrak{c}^\bullet$.

Proof. As is well-known, the space $H_2^\alpha(\mathbb{R}^d)$ can be defined as the set $\{G_\alpha \star f : f \in L^2(\mathbb{R}^d)\}$, equipped with the corresponding graph norm (see [48, Ch. 2.3.4]). So (52) can be interpreted as

$$\|u\|_{L^2(M;\mathcal{H}_l)} \leq c\|u\|_{H^\alpha(\mathbb{R}^d)}, \tag{53}$$

and the assertions follow. □

Theorem 3.6. *i) Suppose that M is a closed subset of Ω, which is a $(d-1)$-set. Then, for $s \in]\frac{1}{2}, 1[$, $W_{\mathfrak{D}}^{s,2}$ continuously embeds into $L^2(M; \mathcal{H}_{d-1})$.*
The embedding constants for the mapping $W_{\mathfrak{D}}^{s,2} \hookrightarrow L^2(M; \mathcal{H}_{d-1})$ are uniformly bounded for different sets M obeying the right estimate (3) with a uniform $\mathfrak{c}^\bullet$.

ii) Let now $d = 3$. Suppose that M is a closed subset of Ω, which is an 1-set. Then, for $s \in]1, \frac{3}{2}[$, $W_{\mathfrak{D}}^{s,2}$ continuously embeds into $L^2(M; \mathcal{H}_1)$.
The embedding constants for the mapping $W_{\mathfrak{D}}^{s,2} \hookrightarrow L^2(M; \mathcal{H}_1)$ are uniformly bounded for different sets M obeying the right estimate (3) with a uniform $\mathfrak{c}^\bullet$.

Proof. Recall that $W_{\mathfrak{D}}^{s,2}$ is the space of restrictions of $W_{\mathfrak{D}}^{s,2}(\mathbb{R}^d)$ functions, equipped with the factor topology. i) In this spirit, let, for $u \in W_{\mathfrak{D}}^{s,2}$, let $\tilde{u} \in W_{\mathfrak{D}}^{s,2}(\mathbb{R}^d)$ be an extension of u with $\|\tilde{u}\|_{W_{\mathfrak{D}}^{s,2}(\mathbb{R}^d)} \leq 2\|u\|_{W_{\mathfrak{D}}^{s,2}}$. Now one applies Corollary 3.5, which implies $tr_{\overline{M}}\tilde{u} \in L^2(\overline{M}; \mathcal{H}_{d-1})$– inclusively a corresponding estimate. Evidently, then also

$$tr_M \tilde{u} = tr_{\overline{M}}\tilde{u}|_M \in L^2(M; \mathcal{H}_{d-1}).$$

Finally, one takes into account that forming the trace on M is the same for the extended function $\tilde{u}$ and the function u on Ω since $M \subset \Omega$ and Ω is open. ii) The proof proceeds along the same lines, again fundamentally resting on Cor. 3.5. □

Having this at hand, our next aim is to show, by duality, that (suitable) measures $\sigma\mathcal{H}_l|_M$ may be considered in a natural manner as elements from (suitable) spaces $W^{-s,2}_{\mathfrak{D}}$.

Lemma 3.7. *Let Ω be a bounded domain in $\mathbb{R}^d$. Let M be a closed subset of Ω of finite $\mathcal{H}_l$ measure, $l \in \{1, \dots, d\}$. Suppose that $W^{s,2}_{\mathfrak{D}}$ continuously embeds into $L^2(M; \mathcal{H}_l)$ with embedding constant $\mathfrak{c}$. Then, for all $\sigma \in L^2(M; \mathcal{H}_l)$, the measure $\sigma\mathcal{H}_l|_M$ belongs to $W^{-s,2}_{\mathfrak{D}}$ and the mapping*

$$L^2(M; \mathcal{H}_l) \ni \sigma \mapsto \sigma\mathcal{H}_l|_M =: \Psi \in W^{-s,2}_{\mathfrak{D}} \tag{54}$$

is well defined and has a norm not larger than $\mathfrak{c}$.

Proof. One has

$$|\langle \sigma\mathcal{H}_l|_M, \psi\rangle| \le \int_M |\psi||\sigma| d\mathcal{H}_l \le \|\sigma\|_{L^2(M;\mathcal{H}_l)} \|\psi\|_{L^2(M;\mathcal{H}_l)} \le$$

$$\mathfrak{c}\ \|\sigma\|_{L^2(M;\mathcal{H}_l)} \|\psi\|_{W^{s,2}_{\mathfrak{D}}}, \quad \psi \in W^{s,2}_{\mathfrak{D}}. \tag{55}$$

□

Theorem 3.8. *i) Adopt the assumptions of Theorem 3.6 i). Then, for $s \in]\frac{1}{2}, 1[$, $L^2(M; \mathcal{H}_{d-1})$ continuously embeds into $W^{-s,2}_{\mathfrak{D}}$.*

ii) Adopt the assumptions of Theorem 3.6 ii). Then, for $s \in]1, \frac{3}{2}[$, $L^2(M; \mathcal{H}_{d-2})$ continuously embeds into $W^{-s,2}_{\mathfrak{D}}$.

The embedding constants for the two previous mappings are uniformly bounded for different sets M obeying the right estimate (3) with uniform $\mathfrak{c}^\bullet$.

Proof. The claims follow from Theorem 3.6 and Lemma 3.7. □

Let us have a closer look at what kind of restriction the uniformity of the constant $\mathfrak{c}$ means in a simple example:
Consider a bounded domain $\Omega \subset \mathbb{R}^2$ which includes $0 \in R^2$ and a closed ball $\overline{B(0, r_0)}$ around it. Take a sequence $\{\alpha_k\}_n$ from the interval $[0, \pi]$, which converges to zero. From this we form the set

$$\mathcal{N}_N := \cup_{k \le N} \{x \in \mathbb{R}^2 : x = re^{i\alpha_k}, r \in [0, r_0]\}.$$

Then condition (3), with $l = 1$, obviously gives a bound for the admissible N. In any case, the union over *all* k is not admissible.
However: if one changes the above set to

$$\cup_k \{x \in \mathbb{R}^2 : x = re^{i\alpha_k}, r \in [0, r_k]\}$$

and chooses the r_k suitably, then (3) can indeed be satisfied in this case. Very roughly speaking, one can say: only finitely many 'curves' of constant length are admissible, but if the lengths may shrink to zero, then infinitely many may be admissible and still satisfy (3).
Let us now take a function $\eta \in C^\infty_0(\Omega)$ which is identical 1 on $\overline{B(0, r_0)}$. Then $\|\eta\|_{L^2(\mathcal{N}_N)} = Nr_0^{1/2}$. This clearly shows that, in order to delimitate the embedding constant of $W^{1,2}_{\mathfrak{D}}(\Omega) \hookrightarrow L^2(\mathcal{N}_N)$ one must delimitate N. So an inequality like (3) seems not to be too far from a necessary one for the required embedding.
Clearly, one can construct analogous examples also in higher dimensions.

Lemma 3.9. *Assume $s \in]\frac{1}{2}, \frac{3}{2}[$. For every $t \in J$, let ρ_t be a bounded Borel measure on Ω, such that the mapping $J \ni t \mapsto \rho_t$ is weak* measurable. Suppose further that*

$$\sup_{\psi \in W^{s,2}_{\mathfrak{D}} \cap C^\infty(\Omega), \|\psi\|_{W^{s,2}_{\mathfrak{D}}}=1} \Big| \int_\Omega \psi \, d\rho_t \Big| < \infty \text{ for every } t \in J. \tag{56}$$

Then the mapping

$$W^{s,2}_{\mathfrak{D}} \cap C^\infty(\Omega) \ni \psi \mapsto \int_\Omega \overline{\psi} \, d\rho_t \tag{57}$$

extends by density to an element $\Psi_t \in W^{-s,2}_{\mathfrak{D}}$. Moreover, the mapping $J \ni t \mapsto \Psi_t \in W^{-s,2}_{\mathfrak{D}}$ is strongly *measurable.*

Proof. First, recall Lemma 1.12. Assumption (56) implies that (57) is a continuous (anti)linear functional on $W^{s,2}_{\mathfrak{D}} \cap C^\infty(\Omega)$ with respect to the induced $W^{s,2}_{\mathfrak{D}}$ topology. By density of $W^{s,2}_{\mathfrak{D}} \cap C^\infty(\Omega)$ in $W^{s,2}_{\mathfrak{D}}$ this can be extended to a continuous antilinear functional on the whole space $W^{s,2}_{\mathfrak{D}}$.
Secondly, from the supposed weak* measurability it follows that $J \ni t \mapsto \int_\Omega \overline{\psi} \, d\rho_t = \langle \Psi_t, \psi \rangle$ is measurable as long as $\psi \in W^{s,2}_{\mathfrak{D}} \cap C^\infty(\Omega)$. But this latter set is dense in $W^{s,2}_{\mathfrak{D}}$, so the measurability for general $\psi \in W^{s,2}_{\mathfrak{D}}$ follows. This implies *weak* measurability of $J \ni t \mapsto \Psi_t \in W^{-s,2}_{\mathfrak{D}}$. Since $W^{s,2}_{\mathfrak{D}}$ is separable and reflexive, the asserted strong measurability follows. □

Remark 3.10. It is not by accident that we consider (57) first only on $W^{s,2}_{\mathfrak{D}} \cap C^\infty(\Omega)$ since it is *not* a priori clear that all elements of $W^{s,2}_{\mathfrak{D}}$ are measurable with respect to ρ_t – and, hence, that the mapping (57) is well defined for all $\psi \in W^{s,2}_{\mathfrak{D}}$.

Up to now we were primarily interested in individual measures $\sigma \mathcal{H}_l|_M$. Having parabolic equations with *varying in time* measures as right hand sides in our general focus, we must find a concept which allows to identify the *time dependent*, measure-valued function as one with values in the Bessel potential space $W^{-s,2}_{\mathfrak{D}}$ – including suitable measurability and integrability properties. This is achieved in the next

Theorem 3.11. *Let Ω be a bounded domain in $\mathbb{R}^d$ and $\mathfrak{D}$ be a closed portion of $\partial\Omega$. Suppose $0 < l \in \{d-2, d-1\}$. For every $t \in J$, let M_t be a closed subset of Ω which is an l-set, with uniform $\mathfrak{c}^\bullet$ in* (3)*.*

Assume that for every $t \in J$ there exists $\sigma_t \in L^2(M_t; \mathcal{H}_l)$ such that
a) the mapping

$$J \ni t \mapsto \sigma_t \mathcal{H}_l|_{M_t} \in \mathcal{M}(\Omega) \tag{58}$$

is weak measurable*
and
b) the upper integral (see [17, Ch. 13.5]) $\int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \, dt$ is finite.
Let $s \in]\frac{1}{2}, \frac{3}{2}[$ be as in Theorem 3.8 (adapted to the points i) and ii)) and $\Psi_t \in W^{-s,2}_{\mathfrak{D}}$ be the element which is associated to the measure $\sigma_t \mathcal{H}_l|_{M_t}$ by Theorem 3.8 .

Then the mapping $J \ni t \mapsto \Psi_t \in W^{-s,2}_{\mathfrak{D}}$ is strongly measurable and one has

$$\int_J \|\Psi_t\|^q_{W^{-s,2}_{\mathfrak{D}}} \, dt \le \mathfrak{k} \int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \tag{59}$$

for some constant $\mathfrak{k}$. Moreover, the constant $\mathfrak{k}$ is uniform with respect to all families $\{\sigma_t\}_{t \in J}$ for which $\int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \, dt < \infty$.

Proof. Applying Lemma 3.9 – the assumptions of which are fulfilled according to Theorems 3.6 and 3.7 the asserted measurability follows. Moreover thanks to Theorems 3.6 and 3.7 the uniform upper l-estimate implies $\|\Psi_t\|_{W^{-s,2}_{\mathfrak{D}}} \le \mathfrak{l} \|\sigma_t\|_{L^2(M_t,\mathcal{H}_l)}$ with a *uniform*

in t constant $\mathfrak{l}$. This proves (59).
We come back to this point in Subsection 4.1, see Thm. 3.13 and Thm. 3.15.

□

Remark 3.12. Reconsidering the assumptions of the previous theorem, the reader should carefully notice that – besides the weak* measurablity of the function (58) – *no* measurability condition is supposed for the function $t \mapsto \sigma_t$ and even not for $t \mapsto \|\sigma_t\|_{L^2(M_t,\mathcal{H}_l)}$. To make such a measurability precise would be a challing task – and not easy to control in examples. For the finiteness of the upper integral a uniform boundedness condition for the functions σ_t, for example, is a sufficient one since the function $J \ni t \mapsto \mathcal{H}_l(M_t)$ is bounded by the (supposed) uniform (upper) l-property of the sets M_t.

3.3 Regularity for non-autonomous parabolic equations with measure-valued right hand sides

In this chapter we arrive at one of the final aims of this paper: namely to prove parabolic regularity results for equations with measure valued right hand sides. The crucial point is two-fold: on one hand, the results of the foregoing chapter allow to interpret suitable measures as elements from $W^{-s,2}_{\mathfrak{D}}$. Here in the two dimensional case there are no restrictions concerning the measures under consideration: all bounded Radon measures are admissable. In the higher dimensional cases one is restricted in this concept to measures which live on sets with Hausdorff dimension one or two and are, additionally, absolutely continuous with respect to the corresponding Hausdorff measure there. Subsequently we are in the position to apply the results of maximal *non-autonomous* parabolic regularity from section 2.

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^2$ and assume that $\varrho : J \to \mathcal{M}(\Omega)$ is weakly* measurable with $\int_J^* \|\rho_t\|^{q_0}_{\mathcal{M}} dt < \infty$ fo some $q_0 > 2$.*
Then, there exists a $q > 2$ such that, for sufficiently small $\epsilon > 0$ the solution of the problem (50) lies in the space $W^{1,q}_0(J; W^{-1-\epsilon,2}_{\mathfrak{D}}) \cap L^q(J; W^{1-\epsilon,2}_{\mathfrak{D}}) = \mathrm{MR}^q_0(J, W^{1-\epsilon,2}_{\mathfrak{D}}, W^{-1-\epsilon,2}_{\mathfrak{D}})$ – inclusively the appropriate estimate for the solution.

Proof. In case of two space dimensions, the space of bounded Radon measures on Ω continuously embeds into *every* space $W^{-1-\epsilon,2}_{\mathfrak{D}}$ with $\epsilon \in]0, \frac{1}{2}[$ – see Lemma 3.1. Associating therefore to the measure ρ_t an element $\Psi(t) \in W^{-1-\epsilon}_{\mathfrak{D}}$, one can prove as in Theorem 3.11 the measurability of the mapping $J \ni t \mapsto \Psi(t) \in W^{-1-s,q}_{\mathfrak{D}}$ inclusively the estimate $\int_J \|\Psi(t)\|^{q_0} dt \le c \int_J^* \|\rho_t\|^{q_0}_{\mathcal{M}} dt$. Now the claim follows from Theorem 2.15. □

Corollary 3.14. *Under the assumptions of the previous theorem, the solution belongs to $C^\beta(J; W^{\kappa,2})$, for some $\beta > 0$ and $\kappa > 0$.*

Proof. Given $q > 2$, there exists$\epsilon > 0$ sufficiently small such that the interval $]\frac{1+\epsilon}{2}, 1 - \frac{1}{q}[$ is non-empty. The claim then follows with Theorem 2.21. □

This result should not be far from optimal. Unfortunately, the range of admissable integrability exponents q with respect to time is restricted to be close to 2.
The next result shows that the solution is more regular with respect to the spatial variable, if one restricts the admissible measures to those living on lower dimensional sets. In particular, in dimension 2 the function $u(t, \cdot)$ is Hölderian on Ω for almost all $t \in J$. This is as special case of the following result which concentrates on subsets of codimension 1 in $\mathbb{R}^2$, respectively $\mathbb{R}^3$.

For the following theorems recall the definition of $\mathfrak{J}$ from Proposition 2.14.

Theorem 3.15. *Let $l = d - 1$, and suppose the following:*

(a) *For every $t \in J$, let $M_t \subset \Omega$ be a closed subset of Ω, and assume that*

$$\mathcal{H}_{d-1}(\overline{M}_t \cap B(x,r)) \le \mathfrak{c}\, r^{d-1}, \quad x \in \overline{M}_t, \quad r \in]0,1] \tag{60}$$

for a constant $\mathfrak{c}$ independent of $t \in J$.

(b) *For every $t \in J$ there is a $\sigma_t \in L^2(M_t; \mathcal{H}_{d-1})$, such that the mapping*

$$J \ni t \mapsto \sigma_t \mathcal{H}_{d-1}|_{M_t} =: \rho(t) \in \mathcal{M}(\Omega) \tag{61}$$

is weakly continuous, and $\int_J^* \|\sigma_t\|^{q_0}_{L^2(M_t;\mathcal{H}_{d-1})} dt < \infty$ holds for some $q_0 \ge 2$.*

Then, for $s \in]0, \frac{1}{2}[$, each $\rho(t)$ can be understood as an element in $W^{-1+s,2}_{\mathfrak{D}}$, and by Theorem 3.11 the function ρ is in $L^{q_2}(J; W^{-1+s,2}_{\mathfrak{D}})$. Further the solution u of

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = \rho_1(\cdot), \quad u(0) = 0 \tag{62}$$

belongs to the space of maximal parabolic regularity $\mathrm{MR}^q_0(J; W^{1+s,2}_{\mathfrak{D}}, W^{-1+s,2}_{\mathfrak{D}})$, for $s > 0$ sufficiently small, and $q = 2$, if $q_0 = 2$, else $2 < q \in \mathfrak{I}$.

Proof. Thanks to Theorem 3.11, the function $t \mapsto \rho(t)$ can be interpreted as a measurable with values in $W^{-1+s,2}_{\mathfrak{D}}$ with an integrability exponent $q \le q_0$ in time, as long as $s \in]0, \frac{1}{2}[$. Possibly diminishing s and q, one may now apply Theorem 2.15. □

We next address the situation of codimension 2 in $\mathbb{R}^d$, $d > 2$.

Theorem 3.16. *Let $l = d - 2$, and suppose the following:*

(a) *For every $t \in J$, let $M_t \subset \Omega$ be a closed subset of Ω and assume*

$$\mathcal{H}_1(\overline{M}_t \cap B(x,r)) \le \mathfrak{c}_1\, r, \quad x \in \overline{M}_t, \quad r \in]0,1] \tag{63}$$

for a constant $\mathfrak{c}_1$ independent of $t \in J$.

(b) *For every $t \in J$ there is a $\sigma_t \in L^2(M_t; \mathcal{H}_1)$, such that the mapping*

$$J \ni t \mapsto \sigma_t \mathcal{H}_1|_{M_t} =: \rho_1(t) \in \mathcal{M}(\Omega) \tag{64}$$

is weakly continuous and $\int_J^* \|\sigma_t\|^{q_1}_{L^2(M_t;\mathcal{H}_1)} dt < \infty$ holds for some $q_1 \ge 2$.*

Then, for $s \in]0, \frac{1}{2}[$ each $\rho_1(t)$ can be understood as an element in $W^{-1-s,2}_{\mathfrak{D}}$ and by Theorem 3.11 the function ρ_1 is in $L^{q_1}(J; W^{-1-s,2}_{\mathfrak{D}})$. Further the solution u of

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = \rho_1(\cdot), \quad u(0) = 0 \tag{65}$$

belongs to the space of maximal parabolic regularity $\mathrm{MR}^q_0(J; W^{1-s,2}_{\mathfrak{D}}, W^{-1-s,2}_{\mathfrak{D}})$, for $s > 0$ sufficiently small, and $q = 2$, if $q_1 = 2$, else $2 < q \in \mathfrak{I}$.

Proof. The proof again follows from Theorems 2.15 and 3.11. □

4 Optimal control

4.1 Measure theoretic preliminaries

Up to now we considered a parabolic equation with prescribed right hand side $\sigma_t \mathcal{H}_l|_{M_t}$ only demanding the finiteness of the upper integral

$$\int_J^* \|\sigma_t\|^q_{L^2(M_t;\mathcal{H}_l)} dt < \infty \tag{66}$$

and, secondly, the measurability of the mappings

$$J \ni t \mapsto \langle \sigma_t \mathcal{H}_l, \psi \rangle_{W_{\mathfrak{D}}^{-s,2} \times W_{\mathfrak{D}}^{s,2}} = \int_{M_t} \sigma_t \, \overline{\psi}|_{M_t} \, d\mathcal{H}_l, \quad \text{for all } \psi \in W_{\mathfrak{D}}^{s,2}. \tag{67}$$

Within this subsection we will investigate this second condition, and describe a specific construction for the choice of σ. In order to illustrate the problem, consider first the case where all sets M_t are identical, i.e. $M_t = M$ for a fixed l-set M. Then it is clear that the weak*-measurability amounts to the measurability of the function $J \ni t \mapsto \sigma_t$. But, if the sets M_t evolve in time, then this aspect becomes a non-trivial one. We investigate this in some particular setting by making the following assumption which is supposed to hold throughout the rest of the paper.

Assumption 4.1. Let M be a closed subset of $\mathbb{R}^d$ and an l-set, such that, for all $t \in J$, there is a bi-Lipschitz diffeomorphism ϕ_t from M onto M_t. The Lipschitz constants l_t of the ϕ_t's and their inverses ϕ_t^{-1}, l_t^-, are uniformly bounded in $t \in J$.

Proposition 4.2. *Suppose $M_\bullet, M_\blacktriangle \subset \mathbb{R}^d$ and let ϕ be bi-Lipschitzian from $M_\bullet$ onto $M_\blacktriangle$. Then*

$$\gamma \mathcal{H}_l(\phi(M_\bullet)) \le \mathcal{H}_l(M_\bullet)) \le \gamma^{-1} \mathcal{H}_l(\phi(M_\bullet)), \tag{68}$$

γ depending only on the Lipschitz constants of ϕ and ϕ^{-1}.

Proof. The left hand side of (68) is proved in [23, Thm. 2.8] for the case when ϕ is defined on whole $\mathbb{R}^d$. In our case this is not fulfilled, but ϕ may be extended to $\mathbb{R}^d$ as a Lipschitzian function into $\mathbb{R}^d$ with the *same* Lipschitz constant, see [23, Ch. 3.1.1]. To this extended function the above quoted theorem applies. The right hand side of (68) is proved by the same arguments, this time applied to ϕ^{-1}. □

Lemma 4.3. *If M is an l-set, then the M_t's are l-sets as well. The corresponding constants $\mathfrak{c}_t^\bullet$ in* (3) *may be taken uniform in t.*

Proof. Note first, that every ϕ_t extends to a bi-Lipschitzian mapping from $\overline{M}$ onto $\overline{M_t}$ – under preservation of the Lipschitz constants. The Lipschitz continuity of ϕ_t implies , for all $x \in \overline{M_t}$,

$$\overline{M_t} \cap B(x; r)) = \phi_t(\overline{M}) \cap B(x; r) \subseteq \phi_t\big(\overline{M} \cap B(\phi_t^{-1} x, \lambda r)\big), \tag{69}$$

where λ may be chosen uniform in t. Take now in particular $M_\bullet = \overline{M} \cap B(\phi_t^{-1} x, \lambda r)$ for an $x \in \overline{M_t}$ and apply (68), to conclude the proof. □

Lemma 4.4. *Suppose that M is an l-set. Consider the image, named ϖ_t, of the Hausdorff measure $\mathcal{H}_l$ on M under ϕ_t on M_t. Then ϖ_t is of the form $\varpi_t = \varsigma_t \mathcal{H}_l$, where ς_t is $\mathcal{H}_l$-measurable and is bounded from above and below by constants, uniform in t.*

Proof. For each $\mathcal{H}_l$-measurable subset $M_\bullet \subset M$ inequality (68), with ϕ replaced by ϕ_t holds. The constant γ may be taken uniform in t since the Lipschitz constants of ϕ_t, ϕ_t^{-1} are uniform by supposition. This implies that ϖ_t is absolutely continuous with respect to $\mathcal{H}_l$ on M_t and, hence, admits a density ς_t by the Radon-Nikodym theorem. It is clear that (68) implies the (uniform in t) boundedness of the ς_t's from above and below by positive constants. □

Lemma 4.5. *Let ψ be uniformly continuous on Ω and assume that the mappings $J \ni t \mapsto \phi_t(x) \in \Omega$ are measurable for everyx $x \in M$. Then*

$$J \ni t \mapsto \psi(\phi_t(\cdot)) =: f_t \in L^2(M; \mathcal{H}_l) \tag{70}$$

is measurable.

Proof. First one observes that the system of functions $\{f_t\}_t$ is equicontinuous on M according to the uniform continuity of ψ and the (uniform) Lipschitz properties of the mappings ϕ_t. Let $\{x_j\}_j$ be a countable, dense subset of M. Standard arguments (see [17, Ch. 13.9, 13.9.6]) tell us that, for every $x \in M$, the function $J \ni t \mapsto f_t(x)$ is measurable. Let $\epsilon > 0$ be arbitrary. So, by Lusin's theorem, for every j there is a compact set $\mathcal{K}^j_\epsilon \subset J$, such that $|J \setminus \mathcal{K}^j_\epsilon| \leq \epsilon 2^{-j-1}$ and the mapping

$$\mathcal{K}^j_\epsilon \ni t \mapsto f_t(x_j)$$

is continuous (see [17, Ch. 13.9, 13.9.4]). Define $\mathcal{K} = \cap_j \mathcal{K}^j_\epsilon$. We show:
For *every* $x \in M$, the mapping

$$\mathcal{K} \ni t \mapsto f_t(x) \tag{71}$$

is continuous. One has

$$|f_t(x) - f_s(x)| \leq |f_t(x) - f_t(x_j)| + |f_t(x_j) - f_s(x_j)| + |f_s(x) - f_s(x_j)|,$$

and all three addends can be made arbitrarily small by taking x_j close enough to x. Let $\varphi \in L^2(M; \mathcal{H}_l)$.

Knowing the continuity of (71), Lebesgue dominance tells us that

$$\mathcal{K} \ni t \mapsto \int_M f_t \, \varphi \, d\mathcal{H}_l \tag{72}$$

is continuous, see [17, Ch. 13.8, 13.8.6]. But the measure of $J \setminus \mathcal{K}$ is at most ϵ. So Lusin's theorem again applies and tells us that

$$J \ni t \mapsto \int_M f_t \, \varphi \, d\mathcal{H}_l \tag{73}$$

is measurable. So (70) is weakly, and, hence, strongly measurable. □

Lastly, if one only aims at measurability of the mapping (67), then Assumption 4.1 can be relaxed as follows: divide the interval into intervals $J_1, J_2, \ldots$ and demand for every subinterval $J = J_k$ again Assu. 4.1. Subsequently on each of the subintervals J_k the same calculus can be done as for J now, and can subsequently be concatenated to again obtain functions on J.

Consider now, for every t, the mapping $V_t : L^2(M; \mathcal{H}_l) \to L^2(M_t; \mathcal{H}_l)$ defined by

$$\big(V_t(\varphi)\big)(x) = \varsigma_t(x)\varphi(\phi_t^{-1}x), \quad \varphi \in L^2(M; \mathcal{H}_l), \, x \in M_t. \tag{74}$$

Then the definition of the image of a measure together with Lemma 4.4 show that, for every $t \in J$, V_t is a bounded linear mapping from $L^2(M; \mathcal{H}_l)$ onto $L^2(M_t; \mathcal{H}_l)$, the norms of which together with their inverses are uniformly bounded in $t \in J$. Our choice for σ_t in (67) will be

$$\sigma_t = V_t(\varphi), \text{ for } \varphi \in L^2(M; \mathcal{H}_l).$$

With reference to Theorem 3.8 we introduce the embedding operators

$$\begin{aligned} \mathcal{I}_t : L^2(M_t; \mathcal{H}_l) \ni \sigma \to \sigma\mathcal{H} \in W^{-1+\tau,2}_{\mathfrak{D}}, \text{ for } \sigma \in L^2(M_t; \mathcal{H}_l), \\ \text{with } \tau \in (0, \tfrac{1}{2}) \text{ if } l = d-1, \text{ and } \tau \in (-\tfrac{1}{2}, 0) \text{ if } l = d-2. \end{aligned} \tag{75}$$

Here and throughout the remainder the case $l = d-2$ *is only considered for* $d \geq 3$.

For $v \in L^q(J; L^2(M; \mathcal{H}_l))$ and $l \in \{d-1, d-2\}$ the crucial point now is the measurability – or not – of the mappings

$$\begin{aligned} J \ni t \mapsto \langle \mathcal{I}_t V_t(v(t)), \psi \rangle_{W^{-1+\tau,2}_{\mathfrak{D}} \times W^{1-\tau,2}_{\mathfrak{D}}} \\ = \langle V_t(v(t))\mathcal{H}_l, \psi \rangle_{W^{-1+\tau,2}_{\mathfrak{D}} \times W^{1-\tau,2}_{\mathfrak{D}}} = \textstyle\int_{M_t} V_t(v(t)) \, \overline{\psi}|_{M_t} \, d\mathcal{H}_l, \end{aligned} \tag{76}$$

for every $\psi \in W^{1-\tau,2}$. In this respect we have the following

Lemma 4.6. *With Assumption 4.1 holding and $V, \mathcal{I}$ as defined in* (74) *and* (75)*, the function*

$$J \ni t \mapsto \mathcal{I}_t V_t v(t) \in W_{\mathfrak{D}}^{-1+\tau,2} \tag{77}$$

is measurable for every $v \in L^q(M; \mathcal{H}_l)$, with $q \in]1, \infty[$, $l \in \{d-1, d-2\}$, if and only if

$$J \ni t \mapsto \psi(\phi_t(\cdot)) \in L^2(M; \mathcal{H}_l) \tag{78}$$

is measurable for every function $\psi \in C_b^\infty(\Omega) \cap W_{\mathfrak{D}}^{1-\tau,2}$ in case $l = d-2$ and $\psi \in C^\infty \cap W_{\mathfrak{D}}^{1-\tau,2}$ in case $l = d-1$.

Proof. We utilize that by a well-known theorem of Pettis the measurability of $t \mapsto \mathcal{I}_t V_t v(t)$ follows from its weak measurability, thus from (67) with $s = 1 - \tau$. We now turn to the case $l = d-2$. According to Lemma 1.12 we may restrict ourselves to $\psi \in C_b^\infty(\Omega) \cap W_{\mathfrak{D}}^{1-\tau,2}$. One calculates for $v \in L^q(J; L^2(M; \mathcal{H}_l))$

$$\begin{aligned} \langle \mathcal{I}_t V_t(v(t)), \psi \rangle_{W_{\mathfrak{D}}^{-1+\tau,2} \times W_{\mathfrak{D}}^{1+\tau,2}} &= \int_{M_t} V_t(v(t))\, \overline{\psi}|_{M_t}\, d\mathcal{H}_l \\ = \int_{M_t} \varsigma_t\, v_t(\phi_t^{-1}(\cdot))\, \overline{\psi}|_{M_t}\, d\mathcal{H}_l &= \int_{M_t} v_t(\phi_t^{-1}(\cdot))\, \overline{\psi}|_{M_t}\, d\varpi_t \\ = \int_M v_t\, \overline{\psi}(\phi_t(\cdot))\, d\mathcal{H}_l, & \end{aligned} \tag{79}$$

where we used that ϖ_t was the image of the measure $\mathcal{H}_l|_M$ under ϕ_t.
The reader should notice that the function $M \ni x \mapsto \psi(\phi_t(x)) \to \mathbb{C}$ is bounded and continuous – hence measurable with respect to $\mathcal{H}_l$. Since $\mathcal{H}_l(M)$ is finite, the function $v_t\, \overline{\psi}(\phi_t(\cdot))$, consequently, belongs to $L^2(M; \mathcal{H}_l)$, and the last term in (79) is well defined – irrespective of the Hausdorff dimension of M.
Since the functions v run through the whole space $L^q(J; L^2(M; \mathcal{H}_l))$, it is straight forward that the measurability of (79) is equivalent to the measurability of the function $t \mapsto \psi(\phi_t(\cdot)) \in L^2(M; \mathcal{H}_l)$ for every function $\psi \in C_b^\infty(\Omega)$, which is (78). For the case $l = d-1$ one chooses $\psi \in C^\infty \cap W_{\mathfrak{D}}^{1-\tau,2}$ and proceeds in the same manner. □

This lemma guarantees the measurability of the right hand side of equation (84) below. The latter is assured by the measurability of (78), which is addressed in Lemma 4.5 under a natural and extremely general condition.

4.2 An optimal control problem

We start the analysis of an optimal control problem. For this purpose it is convenient to summarize the conditions which be assumed henceforth.

Assumption 4.7. We assume that Assumptions 1.2, 2.12 and 4.1 are satisfied and that for every $x \in M$, the mapping $J \ni t \mapsto \phi_t(x) \in \mathbb{R}^d$ is measurable.

Thus the prerequisites established in Section 4.1 are at our disposal. We further recall the definition of the spaces $X = W^{1,q}(J; W_{\mathfrak{D}}^{-1+\tau,2}) \cap L^q(J; W_{\mathfrak{D}}^{1+\tau,2})$, where τ will be chosen positive respectively negative as asked for in (80) – (82) below. We shall consider combinations of (d, l) corresponding to the cases covered in Theorems 3.15 and 3.16, in particular

$$l = d-1 \text{ with } \tau \in]0, \epsilon[, \text{ and } q = 2, \tag{80}$$

or

$$l = d-2 \text{ with } \tau \in]-\epsilon, 0[, \text{ and } q = 2, \alpha = 0, \tag{81}$$

or

$$l = d-2 \text{ with } \tau \in]-\epsilon, 0[, \text{ and } \mathfrak{I} \ni q > 2, \tag{82}$$

with $\epsilon > 0$ sufficiently small, α to be introduced below, and $\mathfrak{I}$ from Proposition 2.14.

We proceed by setting the control-space to be $L^2(M;\mathcal{H}_l)$ and define the time dependent control operators for a.e. t by

$$B(t) : L^2(M;\mathcal{H}_l) \to W^{-1+\tau,2}_{\mathfrak{D}} \text{ with } B(t) = \mathcal{I}_t V_t,$$

with

$$V_t : L^2(M;\mathcal{H}_l) \to L^2(M_t;\mathcal{H}_l), \text{ and } \mathcal{I}_t : L^2(M_t;\mathcal{H}_l) \to W^{-1+\tau,2}_{\mathfrak{D}}.$$

Here V_t is as defined in (74) and $\mathcal{I}_t$ as in (75). Recall that the V_t's are uniformly (in t) *bounded*, Thus there exists $k_1 > 0$ such that for all $t \in J$:

$$\|V_t\|_{\mathcal{L}(L^2(M;\mathcal{H}_l),L^2(M_t;\mathcal{H}_l))} \le k_1,$$

and there exist a constant a k_2 such that for a.e. t

$$\|\mathcal{I}_t\|_{\mathcal{L}(L^2(M_t;\mathcal{H}_l)),W^{-1+\tau,2}_{\mathfrak{D}}} \le k_2,$$

for τ as in 75, see Theorem 3.8. Consequently $\|B(t)\|_{\mathcal{L}(L^2(M;\mathcal{H}_l),W^{-1+\tau,2}_{\mathfrak{D}}} \le k_1 k_2$ for a.a. $t \in J$ and $B(t)$ induces a mapping $B \in \mathcal{L}(L^q(J;L^2(M;\mathcal{H}_l)), L^q(J;W^{-1+\tau,2}_{\mathfrak{D}}))$, satisfying

$$\|B\|_{\mathcal{L}(L^q(J;L^2(M;\mathcal{H}_l)),L^q(J;W^{-1+\tau,2}_{\mathfrak{D}}))} \le k_1 k_2. \tag{83}$$

Let $u_0 \in (W^{-1+\tau,2}_{\mathfrak{D}}, W^{1+\tau,2}_{\mathfrak{D}})_{1-\frac{1}{q},q}$ be fixed. By Theorem 2.17 and (83) for every $\xi \in L^q(J;L^2(M;\mathcal{H}_l)$ the control system

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = B\xi, \quad u(0) = u_0 \tag{84}$$

has a unique solution $u = u(\xi) \in X$ satisfying

$$\|u(\xi)\|_X \le c(\|\xi\|_{L^q(J;L^2(M;\mathcal{H}_l))} + \|u_0\|_{(W^{-1+\tau,2}_{\mathfrak{D}},W^{1+\tau,2}_{\mathfrak{D}})_{1-\frac{1}{q},q}}), \tag{85}$$

with c independent of u_0 and ξ. Here ϵ is assumed to be sufficiently small so that Theorem 2.17 is applicable for $\tau \in]0,\epsilon[$, respectively $\tau \in]-\epsilon, 0[$.

For $u_d \in L^2(J;L^2)$ and $u_T \in L^2(\Omega)$ we consider the optimal control problem

$$\begin{cases} \min_{\xi \in L^q(J;L^2(M;\mathcal{H}_l))} & \mathcal{J}(\xi) \\ \text{subject to (84)}, \end{cases} \tag{$\mathcal{P}$}$$

where

$$\mathcal{J}(\xi) = \frac{1}{2}\int_0^T \|u(\xi)(t) - u_d(t)\|^2_{L^2}\, dt + \frac{\alpha}{2}\|u(\xi)(T) - u_T\|^2_{L^2} + \frac{\beta}{q}\int_0^T \|\xi\|^q_{L^q(M;\mathcal{H}_l)}\, dt,$$

and $\alpha \ge 0$, $\beta > 0$.

Theorem 4.8. *Let Assumption 4.7 hold and let ϵ be sufficiently small. Then for l and q as in cases* (80)-(82) *problem* ($\mathcal{P}$) *admits a unique solution $\xi^* \in L^2(J;L^2(M;\mathcal{H}_l))$.*

Proof. We start by arguing the well-posedness of the cost-functional for each of the three cases. For each of them existence of a solution $u(\xi) \in X$ to (84) for all $\xi \in L^2(J;\mathcal{H}_l)$ is guaranteed, see (85). For (80) moreover $u(\xi) \in C(\bar{J};L^2)$, see Theorem 2.20, and hence $\mathcal{J}$ is welldefined. For (81) well-posedness of $\mathcal{J}$ already follows from $u(\xi) \in X$. Turning to (82) we utilize Theorem 2.21 with $q \in \mathfrak{I}, q > 2$, and τ such that $0 < -\tau < \frac{1}{2} - \frac{1}{q}$, then $\theta = \frac{1}{2} - \tau$, to get $\beta = \frac{1}{2} + \tau - \frac{1}{q} > 0$, and $X \subset C^\beta(J;W^{-\tau,2})$. In particular $u(\xi) \in C(\bar{J};L^2)$ and $\mathcal{J}$ is also well-defined for the case described by (82).

Next, let $\{\xi_n\}$ be a minimizing sequence for $(\mathcal{P})$. Then $\mathcal{J}(\xi_n) \le \mathcal{J}(0) + 1$ for all sufficiently large n. Hence $\{\xi_n\}$ is bounded in $L^q(J; L^2(M; \mathcal{H}_l))$. Consequently $\{\xi_n\}$ admits a weakly convergent subsequence, which is denoted by the same symbols, and $\xi^* \in L^q(J; L^2(M; \mathcal{H}_l))$ with $\xi_n \rightharpoonup \xi^*$, see e.g. [16, Ch. IV.1 Cor. 2]. By the uniform boundedness of $B(t)$ with respect to t, it is simple to argue that $B\xi_n \rightharpoonup B\xi^*$ in $L^2(J; W_{\mathfrak{D}}^{-1+\tau,2})$.

For each ξ_n the solution $u(\xi_n)$ to (84) can be decomposed as $u(\xi_n) = u_1(\xi_n) + u_h$ where $u_1(\xi_n)$ is the solution to (84) with $\xi = \xi_n, u_0 = 0$, and u_h is the solution to (84) with $\xi = 0, u(0) = u_0$. By (85) the sequence $u_1(\xi_n)$ is bounded in X and thus there exists a subsequence, again denoted by the same indices, and $u_1^* \in X$ such that $u_1(\xi_n) \rightharpoonup u_1^*$ in X. Lemma 2.13 implies that $\mathcal{A}\, u_1^*(\xi_n) \rightharpoonup \mathcal{A} u_1$ in $L^q(J; W_{\mathfrak{D}}^{-1+\tau,2})$. Thus we can take the weak limit in $L^q(J; W_{\mathfrak{D}}^{-1+\tau,2})$ in the equation

$$\frac{\partial u_1(\xi_n)}{\partial t} - \mathcal{A}\, u_1(\xi_n) = B\xi_n, \quad u(0) = 0,$$

to arrive at

$$\frac{\partial u_1^*}{\partial t} - \mathcal{A}\, u_1^* = B\xi^*, \quad u(0) = 0.$$

Uniqueness of the solution to this equation imply that $u_1^* = u_1(\xi^*)$. It follows that $u(\xi_n) \rightharpoonup u(\xi^*) = u_1(\xi^*) + u_h$ in X, with $u(\xi^*)$, the solution to (84) setting $\xi = \xi^*$. Consequently ξ^* is an admissible control for $(\mathcal{P})$.

To pass to the limit in the cost functional we use Theorems 2.20 and 2.21 to assert that $\lim_{n\to\infty} u(\xi_n) = u(\xi^*)$ in $C(\bar{J}; L^2)$ for cases (80) and (82), and we use the compact embedding of $W^{1,2}(J; W_{\mathfrak{D}}^{-1+\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1+\tau,2})$ into $L^2(J; L^2)$ to obtain $\lim_{n\to\infty} u(\xi_n) = u(\xi^*)$ in $L^2(\bar{J}; L^2)$ for case (81). These convergence properties, and recalling that $\alpha = 0$ in case (81) justify the following inequalities:

$$\begin{aligned}
\mathcal{J}(\xi^*) &= \tfrac{1}{2} \textstyle\int_0^T \|u(\bar{\xi})(t) - u_d(t)\|_{L^2}^2\, dt + \tfrac{\alpha}{2} \|u(\xi^*)(T) - u_T\|_{L^2}^2 \\
&\quad + \tfrac{\beta}{2} \textstyle\int_0^T \|\xi^*\|_{L^2(M;\mathcal{H}_l)}^2\, dt \\
&\le \lim_{n\to\infty} \tfrac{1}{2} \textstyle\int_0^T \|u(\xi_n)(t) - u_d(t)\|_{L^2}^2\, dt + \lim_{n\to\infty} \tfrac{\alpha}{2} \|u(\xi_n)(T) - u_T\|_{L^2}^2 \\
&\quad + \liminf_{n\to\infty} \tfrac{\beta}{2} \textstyle\int_0^T \|\xi_n\|_{L^2(M;\mathcal{H}_l)}^2\, dt \\
&\le \liminf_{n\to\infty} \mathcal{J}(\xi_n) = \inf_{\xi \in L^2(J; L^2(M;\mathcal{H}_l))} \mathcal{J}(\xi),
\end{aligned}$$

and thus ξ^* is an optimal solution. Uniqueness of this solution follows from the strict convexity of the cost-functional. □

4.3 Optimality condition

Now we present the optimality condition associated to the unique solution ξ^* of $(\mathcal{P})$. The analysis builds upon the adjoint equation associated to ξ^* which is given by

$$-\frac{\partial \varphi}{\partial t} + \widehat{\mathcal{A}}\, \varphi(t) = -(u(\xi^*)(t) - u_d(t)) \text{ on } J, \quad \varphi(T) = -\alpha(u(\xi^*)(T) - u_T) \tag{86}$$

$\widehat{\mathcal{A}}$ defined as in (31), with the coefficient function replaced by its adjoint. We have the following regularity results for the adjoint state:

Lemma 4.9. *Concerning the regularity of the adjoint state the following properties hold:* $\varphi \in W^{1,2}(J; W_{\mathfrak{D}}^{-1,2}) \cap L^2(J; W_{\mathfrak{D}}^{1,2})$ *for* (80), $\varphi \in W^{1,2}(J; W_{\mathfrak{D}}^{-1-\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1-\tau,2})$ *for* (81), *and* $\varphi \in W^{1,2}(J; W_{\mathfrak{D}}^{-1-\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1-\tau,2})$ *for* (82) *provided that* $u_T \in W^{-\tau,2}$.

Proof. The solution to (86) can be decomposed as $\varphi = \varphi_1 + \varphi_2$ with φ_1 the solution (86) with $\varphi_1(T) = 0$ and φ_2 the solution to the differential equation in (86) with right hand side equal 0.

For case (80) we have $\varphi_2 \in W^{1,2}(J; W_{\mathfrak{D}}^{-1,2}) \cap L^2(J; W_{\mathfrak{D}}^{1,2})$ and $\varphi_1 \in W^{1,2}(J; W_{\mathfrak{D}}^{-1+\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1+\tau,2})$, and thus $\varphi \in W^{1,2}(J; W_{\mathfrak{D}}^{-1,2}) \cap L^2(J; W_{\mathfrak{D}}^{1,2})$. As a side remark we observe that by Proposition 2.10 we have that $X \subset C(\bar{J}; (W_{\mathfrak{D}}^{-1+\tau,2}, W_{\mathfrak{D}}^{1+\tau,2})_{\frac{1}{2},2})$, and consequently $u(\xi^*)(T) \in (W_{\mathfrak{D}}^{-1+\tau,2}, W_{\mathfrak{D}}^{1+\tau,2})_{\frac{1}{2},2}$. Thus if $u_T \in (W_{\mathfrak{D}}^{-1+\tau,2}, W_{\mathfrak{D}}^{1+\tau,2})_{\frac{1}{2},2}$, then $\varphi_2 \in W^{1,2}(J; W_{\mathfrak{D}}^{-1+\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1+\tau,2})$ and subsequently $\varphi \in W^{1,2}(J; W_{\mathfrak{D}}^{-1+\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1+\tau,2})$.

For case (81) we have $\alpha = 0$ and hence $\varphi_2 = 0$, and $u(\xi^*) \in W^{1,2}(J; W_{\mathfrak{D}}^{-1+\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1+\tau,2}) \subset L^2(J; W_{\mathfrak{D}}^{-1-\tau,2})$ and $\varphi = \varphi_2 \in W^{1,2}(J; W_{\mathfrak{D}}^{-1-\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1-\tau,2})$.

For case (82) we have $u(\xi^*) \in W^{1,q}(J; W_{\mathfrak{D}}^{-1+\tau,2}) \cap L^q(J; W_{\mathfrak{D}}^{1+\tau,2}) \subset C^\beta(\bar{J}; W^{-\tau,2})$, for some $\beta > 0$, see the proof of Theorem 4.8. Hence $u(\xi^*)(T) \in W_{\mathfrak{D}}^{-\tau,2} = (W_{\mathfrak{D}}^{-1-\tau,2}, W_{\mathfrak{D}}^{1-\tau,2})_{\frac{1}{2},2}$, see Lemma 4.10 below, and consequently $\alpha(u(\xi^*)(T) - u_T) \in W_{\mathfrak{D}}^{-\tau,2}$. Moreover we have that $u(\xi^*) \in L^q(J; W_{\mathfrak{D}}^{1+\tau,2}) \subset L^2(J; W_{\mathfrak{D}}^{-1-\tau,2})$ and consequently $\varphi \in W^{1,2}(J; W_{\mathfrak{D}}^{-1-\tau,2}) \cap L^2(J; W_{\mathfrak{D}}^{1-\tau,2})$. □

Lemma 4.10. *For $\tau \in]0, \frac{1}{2}[$ we have $(W_{\mathfrak{D}}^{-1+\tau,2}, W_{\mathfrak{D}}^{1+\tau,2})_{\frac{1}{2},2} = W_{\mathfrak{D}}^{\tau,2}$.*

Proof. One writes, by employing Prop.2.18 and re-iteration,

$$W_{\mathfrak{D}}^{-1+\tau,2} = [W_{\mathfrak{D}}^{-1-\tau,2}, L^2]_{\frac{2\tau}{1+\tau}} = \big[W_{\mathfrak{D}}^{-1-\tau,2}, [W_{\mathfrak{D}}^{-1-\tau,2}, W_{\mathfrak{D}}^{1+\tau,2}]_{\frac{1}{2}}\big]_{\frac{2\tau}{1+\tau}} =$$

$$= [W_{\mathfrak{D}}^{-1-\tau,2}, W_{\mathfrak{D}}^{1+\tau,2}]_{\frac{\tau}{1+\tau}}.$$

Consequently, we get, again employing Prop.2.18 and re-iteration,

$$(W_{\mathfrak{D}}^{-1+\tau,2}, W_{\mathfrak{D}}^{1+\tau,2})_{\frac{1}{2},2} = ([W_{\mathfrak{D}}^{-1-\tau,2}, W_{\mathfrak{D}}^{1+\tau,2}]_{\frac{\tau}{1+\tau}}, W_{\mathfrak{D}}^{1+\tau,2})_{\frac{1}{2},2} =$$

$$(W_{\mathfrak{D}}^{-1-\tau,2}, W_{\mathfrak{D}}^{1+\tau,2})_{\frac{2\tau+1}{2(1+\tau)},2} = \big([W_{\mathfrak{D}}^{-1-\tau,2}, W_{\mathfrak{D}}^{1+\tau,2}]_{\frac{1}{2}}, W_{\mathfrak{D}}^{1+\tau,2}\big)_{\frac{\tau}{1+\tau},2} =$$

$$= \big(L^2, W_{\mathfrak{D}}^{1+\tau,2}\big)_{\frac{\tau}{1+\tau},2} = W^{\tau,2}.$$

□

The optimality condition is presented next.

Theorem 4.11. *Suppose that Assumption 4.7 holds and that $u_T \in W^{-\tau,2}$ in case* (82). *Then the necessary and sufficient optimality condition for ξ^* to be a minimizer of $(\mathcal{P})$ is given by*

$$\beta \|\xi^*(t)\|_{L^2(M;\mathcal{H}_l)}^{q-2} \xi^*(t) = B^*(t)\varphi(t) \text{ for a.a. } t \in J, \tag{87}$$

with φ given by (86).

Proof. Since $\xi \to u(\xi)$ is affine the linearization of this mapping in direction $\delta\xi \in L^q(J; (M, \mathcal{H}_l))$ is given as the solution $u' = u'(\delta\xi)$ to

$$\frac{\partial u'(t)}{\partial t} - \mathcal{A}\, u'(t) = B(t)\delta\xi(t) \text{ on } J, \quad u'(0) = 0.$$

Step 1 We first turn to the cases (80), for which $\tau > 0$. In this situation $u' \in \mathrm{MR}_0^2(J; W_{\mathfrak{D}}^{1+\tau,2}, W_{\mathfrak{D}}^{-1+\tau,2}) \subset C(\bar{J}; L^2)$, by Theorem 2.15 with $\tau > 0$. This will justify the following identities, where we use (86) with $q=2$:

$$\begin{aligned}
&\tfrac{d}{d\xi}\mathcal{J}(\xi^*)\,\delta\xi = \textstyle\int_0^T (u(\bar\xi)(t)-u_d(t), u'(\delta\xi)(t))_{L^2}dt \\
&\quad +\alpha(u(\bar\xi)(T)-u_T, u'(\delta\xi)(T))_{L^2} + \beta \textstyle\int_0^T (\xi(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\frac{\partial}{dt}\varphi(t) - \widehat{\mathcal{A}}\,\varphi(t), u'(\delta\xi)(t))_{W^{-1,2}_{\mathfrak{D}}, W^{1,2}_{\mathfrak{D}}}\,dt - (\varphi(T), u'(\delta\xi)(T))_{L^2} \\
&\qquad +\beta \textstyle\int_0^T (\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\varphi(t), -\frac{\delta}{dt}u'(\delta\xi)(t) - \mathcal{A}\,u'(\delta\xi)(t))_{W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}}}\,dt \\
&\qquad +\beta \textstyle\int_0^T (\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\varphi(t), -B(t)\delta\xi(t))_{W^{1-\tau,2}_{\mathfrak{D}}, W^{-1+\tau,2}_{\mathfrak{D}}}\,dt + \beta\textstyle\int_0^T (\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt = 0.
\end{aligned}$$

The above identities hold for all $\delta\xi \in L^2(J;L^2(M;\mathcal{H}_l))$ and for a.a. $t\in J$, and consequently $B^*(t)\varphi(t) = \beta\xi^*(t)$, as desired.

Step 2 Now we consider case (81) where $\tau < 0$ and $\alpha = 0$. In this case $u' \in \mathrm{MR}^2_0(J; W^{1+\tau,2}_{\mathfrak{D}}, W^{-1+\tau,2}_{\mathfrak{D}}) \subset C(\bar J; W^{\tau,2}_{\mathfrak{D}}))$, where we use Lemma 4.10, and $\varphi \in W^{1,2}(J; W^{-1-\tau,2}_{\mathfrak{D}}) \cap L^2(J; W^{1-\tau,2}_{\mathfrak{D}}) \subset C(\bar J; W^{-\tau,2}_{\mathfrak{D}})$ and $\varphi \in W^{1,2}(J; W^{-1-\tau,2}_{\mathfrak{D}}) \cap L^2(J;(J; W^{-1-\tau,2}_{\mathfrak{D}}) \subset C(\bar J; W^{-\tau,2})$. We can now follow the computation of $\frac{d}{d\xi}\mathcal{J}(\xi^*)\,\delta\xi$ as in Step 1, making appropriate changes in the duality pairings:

$$\begin{aligned}
&\tfrac{d}{d\xi}\mathcal{J}(\xi^*)\,\delta\xi = \textstyle\int_0^T (u(\bar\xi)(t)-u_d(t), u'(\delta\xi)(t))_{L^2}dt + \beta\textstyle\int_0^T (\xi(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\frac{\partial}{dt}\varphi(t) - \widehat{\mathcal{A}}\,\varphi(t), u'(\delta\xi)(t))_{W^{-1-\tau,2}_{\mathfrak{D}}, W^{1+\tau,2}_{\mathfrak{D}}}\,dt \\
&\quad +\beta \textstyle\int_0^T (\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\varphi(t), -\frac{\delta}{dt}u'(\delta\xi)(t) - \mathcal{A}\,u'(\delta\xi)(t))_{W^{1-\tau,2}_{\mathfrak{D}}, W^{-1+\tau,2}_{\mathfrak{D}}}\,dt \\
&\qquad +\beta \textstyle\int_0^T (\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\varphi(t), -B(t)\delta\xi(t))_{W^{1-\tau,2}_{\mathfrak{D}}, W^{-1+\tau,2}_{\mathfrak{D}}}\,dt + \beta\textstyle\int_0^T (\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt = 0,
\end{aligned}$$

where the temporal integration by parts in third equality above can be verified with a density argument. Again we obtain the desired equality (87).

Step 3 In this case we use that $u(\xi^*) \in C(\bar J; W^{-\tau,2}_{\mathfrak{D}})$, (see proof of Theorem 4.8, $\varphi \in W^{1,q}(J; W^{-1-\tau,2}_{\mathfrak{D}}) \cap L^q(J; W^{1-\tau,2}_{\mathfrak{D}}) \subset C(\bar J; W^{-\tau,2}_{\mathfrak{D}})$, (see Lemma 4.9 and Proposition 2.10), and $u' \in W^{1,q}(J; W^{-1+\tau,2}_{\mathfrak{D}}) \cap L^q(J; W^{1+\tau,2}_{\mathfrak{D}}) \subset C(\bar J; W^{-\tau,2}_{\mathfrak{D}})$, (see Theorem 2.21). We now equate

$$\begin{aligned}
&\tfrac{d}{d\xi}\mathcal{J}(\xi^*)\,\delta\xi = \textstyle\int_0^T (u(\bar\xi)(t)-u_d(t), u'(\delta\xi)(t))_{L^2}dt \\
&\quad +\alpha(u(\bar\xi)(T)-u_T, u'(\delta\xi)(T))_{L^2} + \beta \textstyle\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\frac{\partial}{dt}\varphi(t) - \widehat{\mathcal{A}}\,\varphi(t), u'(\delta\xi)(t))_{W^{-1-\tau,2}_{\mathfrak{D}}, W^{1+\tau,2}_{\mathfrak{D}}}\,dt - (\varphi(T), u'(\delta\xi)(T))_{L^2} \\
&\qquad +\beta \textstyle\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\varphi(t), -\frac{\delta}{dt}u'(\delta\xi)(t) - \mathcal{A}\,u'(\delta\xi)(t))_{W^{1-\tau,2}_{\mathfrak{D}}, W^{-1+\tau,2}_{\mathfrak{D}}}\,dt \\
&\qquad +\beta \textstyle\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt \\
&\quad = \textstyle\int_0^T (\varphi(t), -B(t)\delta\xi(t))_{W^{1-\tau,2}_{\mathfrak{D}}, W^{-1+\tau,2}_{\mathfrak{D}}}\,dt \\
&\qquad +\beta \textstyle\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}dt = 0,
\end{aligned}$$

and again (87) follows.

Step 4 Since the cost-functional $\xi \to \mathcal{J}(\xi)$ is strictly convex, the necessary optimality condition is also sufficient. □

Utilizing the structure of the control operator $B(t) = V_t \mathcal{I}_t$ it will be shown next that the optimal solution ξ^* exhibits extra regularity. This property of increased regularity of the minimizer arises frequently in optimal control, see e.g. [33, pg. 52]

4.4 Extra regularity of the optimal control

To obtain extra regularity of the optimal control ξ^* additional regularity properties of the problem data are required. Throughout this section we restrict ourselves to the case $d \in \{2, 3\}$.

Assumption 4.12. (a) For some $\alpha > \frac{1}{2}$ the coefficient function satisfies

$$\|\mu(t_1, \cdot) - \mu(t_2, \cdot)\|_{L^\infty(\Omega;\mathbb{C}^{d\times d})} \le c|t_1 - t_2|^\alpha \quad \text{for all } t_1, t_2 \in J. \tag{88}$$

(b) In case $d = 3$ the operator

$$-\nabla \cdot \mu(t, \cdot)\nabla + 1 : W^{1,3}_{\mathfrak{D}} \to W^{-1,3}_{\mathfrak{D}} \tag{89}$$

is a topological isomorphism for each $t \in J$, and that the norms of their inverses are uniformly in t bounded.
Moreover for every $t \in J$ all components of the coefficient function $\mu_t = \hat{\mu}(t, \cdot)$ are multipliers on the spaces $W^{s,3}$ at least for small $s \in]0, \delta_\bullet]$. The corresponding norms as multipliers on these spaces are uniformly in t bounded for $s \in]0, \delta_\bullet]$.
Finally, the set $\mathfrak{D} \cap \overline{\partial\Omega \setminus \mathfrak{D}}$ – where the Dirichlet boundary part meets the Neumann part – is a 1-set.

Remark 4.13. We are aware that the condition (88) restricts the class of admissable coefficients in comparison to Ass. 2.12 considerably. Prototypically, in the latter a (scalar) coefficient function $\hat{\mu}$ is allowed which is identically 1 up to a time point $t_0 \in J$, and from t_0 on it is identical 1 on a subdomain $\Omega_\bullet$ and 2 on $\Omega \setminus \Omega_\bullet$. Obviously, such a $\hat{\mu}$ does not satisfy (88). But the following is allowed: Let $\Omega_\bullet \subset \Omega$ be a subdomain and χ its indicator function. If one defines

$$\mu_t = \begin{cases} 1, \text{ if } t \le t_0 \\ 1 + (t - t_0)\chi, \end{cases}$$

then this coefficient function is admissible.

The following theorem, which is proved at the end of this section, refers to the equation with homogenous initial condition:

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = f, \quad u(0) = 0. \tag{90}$$

Theorem 4.14. *Let Assumptions 2.12, 2.3, and 4.12 be satisfied.*

(a) *For dimension $d = 2$, there exist $\beta > 0$ and $r \in]0, \delta[$ such that the solution u to (90) belongs to $C^\beta(J, W^{1+r,2}_{\mathfrak{D}})$, provided that $f \in L^{\tilde{q}}(J; L^2)$ for $\tilde{q} > 2$ sufficiently large.*

(b) *For dimension $d = 3$, there exist $\beta > 0$ and $r \in]0, \delta_\bullet[$ such that the solution u to (90) belongs to $C^\beta(J, W^{1+r,3}_{\mathfrak{D}})$ provided that $f \in L^{\tilde{q}}(J; L^2)$ for $\tilde{q} > 2$ sufficiently large.*

This result allows us to draw conclusions on the regularity of the optimal control ξ^*. Henceforth all the Assumptions 1.2, 2.3, 2.12, 4.1, 4.7, and 4.12 are supposed to hold.

Theorem 4.15. *Let the assumptions just mentioned hold, let $d \in \{2, 3\}$, $\alpha = 0$, and $u_d \in L^\infty(J; L^2)$. Then the optimal solution to $(\mathcal{P})$, satisfies*

(i) $\xi^* \in L^\infty(J; W^{1-\frac{1}{d+\kappa},d+\kappa}(M; \mathcal{H}_{d-1}))$ for case (80),

(ii) $\xi^* \in L^2(J; W^{-\tau,2}(M; \mathcal{H}_1))$ *for case* (81),

(iii) $\xi^*(t) \in W^{1-\frac{2}{3+\kappa},3+\kappa}(M; \mathcal{H}_1))$ *for a.e.* $t \in J$ *and* $\xi^* \in L^\infty(J; L^2(M; \mathcal{H}_1))$ *for case* (82),

for some $\kappa > 0$.

Proof. Let us recall the adjoint equation (86). As established at the beginning of the proof of Theorem 4.8 we have that $u(\xi^*) \in C(J; L^2)$ in the cases (80) and (82). In view of and the assumption on u_d we have that the right hand side of the adjoint equation satisfies $u(\xi^*)(t) - u_d(t) \in L^\infty(J; L^2)$. Next we observe that after time reversal the adjoint equation is a special case of (90), and from Theorem 4.14 we deduce that $\varphi \in C^\beta(J, W^{1+r,2}_{\mathfrak{D}}) \hookrightarrow C(\overline{J}, W^{1,2+\kappa}_{\mathfrak{D}})$ for $d = 2$ and $\varphi \in C^\beta(J, W^{1+r,3}_{\mathfrak{D}}) \hookrightarrow C(\overline{J}, W^{1,3+\kappa}_{\mathfrak{D}})$ for $d = 3$, for some $\beta > 0, r > 2$ and $\kappa > 0$.

Now we recall from Theorem 4.11 that

$$\beta \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)} \xi^*(t) = B^*(t)\varphi(t) = V_t^* \mathcal{I}_t^* \varphi(t) \text{ for a.a. } t \in J. \tag{91}$$

A straight forward computation shows that $V_t^* : L^2(M_t, \mathcal{H}_l) \to L^2(M, \mathcal{H}_l)$ is given by

$$(V_t^* \psi)(x) = \psi(\phi_t(x)),\ x \in M. \tag{92}$$

Now we continue with case (80) and obtain

$$\beta\, \xi^*(t) = V_t^* \varphi(t)|_{M_t} = \varphi(t, \phi_t(\cdot))|_M \text{ for a.a. t in } J, \tag{93}$$

where for the first equality we used (91) with $q = 2$, and Theorem 3.6 and (92) for the second. Let us now temporarily fix t and denote the function $\varphi(t, \cdot)$ by ψ. We point out that the right hand side of (93) is to be read as $\big(tr_{M_t} \varphi(t, \cdot)\big)(\phi_t(x)) = \big(tr_{M_t} \psi\big)(\phi_t(x))$. Recall that $\psi \in W^{1,d+\kappa} = W^{1,d+\kappa}(\Omega)$ and consider an extension operator $\mathfrak{E} : W^{1,d+\kappa}(\Omega) \to W^{1,d+\kappa}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. Further, let $\widehat{\phi_t}$ be a Lipschitzian extension of $\phi_t : M \to M_t$, to $\mathbb{R}^d \to \mathbb{R}^d$, having the same Lipschitz constant as ϕ_t. Each function $\mathfrak{E}\psi$ is continuous on $\mathbb{R}^d$, so, for every $x \in M$, $\big(tr_{M_t} \psi\big)(\phi_t(x)$ is obtained as the *pointwise evaluation* of the function $\mathfrak{E}\psi \circ \widehat{\phi_t}$ in x, i.e. equals $\big(tr_M(\mathfrak{E}\psi \circ \widehat{\phi_t})\big)(x)$. By construction, it is not hard to see that the family $\mathcal{F} = \{\mathfrak{E}\varphi(t, \widehat{\phi_t}(\cdot))\}_{t \in J}$ is a bounded one in $W^{1,d+\kappa}(\mathbb{R}^d)$. Hence, the family of traces, $\{\beta\xi^*(t)\}_{t\in J} = tr_M \mathcal{F}$ on M, is bounded in $W^{1-\frac{d-l}{d+\kappa},d+\kappa}(M)$, thanks to Prop. 1.5. It remains to assertain the measurability of ξ^* with values in $W^{1-\frac{d-l}{d+\kappa},d+\kappa}(M; \mathcal{H}_l)$. This follows from the fact that $t \to \xi^*(t) \in L^2(M; \mathcal{H}_l)$ is measurable, that $W^{1-\frac{d-l}{d+\kappa},d+\kappa}(M)$ is reflexive and separable, and the Pettis measurability theorem.

The case (82) can be treated with the same techniques as (80) except that now (91) needs to be considered with $q > 2$. Consequently we obtain $\beta\|\xi^*\|^{q-2}_{L^2(M;\mathcal{H}_1)}\xi^* \in L^\infty(J; W^{1-\frac{2}{3+\kappa},3+\kappa}(M; \mathcal{H}_1))$ and thus $\xi^*(t) \in W^{1-\frac{2}{3+\kappa},3+\kappa}(M; \mathcal{H}_1)$ for a.e. $t \in J$. Moreover from (91) we deduce that $\xi^* \in L^\infty(J; L^2(M; \mathcal{H}_1))$.

Finally we turn to case (81). In this situation the adjoint variable satisfies $\varphi \in L^2(J; W^{1-\tau,2}_{\mathfrak{D}})$, see the proof of Lemma 4.9, and recall that $\tau < 0$. We now follow the steps of case(80) above. Equation 93 is satisfied with $q = 2$. The existence of a continuous extension operator $\mathfrak{E} : W^{1-\tau,2}(\Omega) \to W^{1-\tau,2}(\mathbb{R}^3)$ is guaranteed by [29, Ch. V.1]. The trace mapping $tr_M : W^{1-\tau,2}(\mathbb{R}^3) \to W^{-\tau,2}(M; \mathcal{H}_1)$ is bounded by Proposition 1.5 in the pointwise $\mathcal{H}_1$ a.a. sense.

We thus have $\beta\xi^*(t) = tr_M \mathfrak{E}\varphi(t, \widehat{\phi_t}(\cdot)) \in W^{-\tau,2}(M; \mathcal{H}_1)$ for a.a. $t \in J$ in case (81). The measurability of $J \ni t \to \xi^*(t) \in W^{-\tau,2}(M; \mathcal{H}_1)$ again follows from the measurability of that mapping with range in $L^2(M; \mathcal{H}_1)$. Finally $\xi^* \in L^2(J; W^{-\tau,2}(M; \mathcal{H}_1))$ is implied by the boundedness of $\mathfrak{E}$ and the trace operator.

□

Now we turn to the proof of Thm. 4.14 and start with two preparatory lemmata.

Lemma 4.16. *Assume $d = 2$ and let $s \in]0, \frac{1}{2}[$. Then $[L^2, W^{1+s,2}_{\mathfrak{D}}]_\theta$ continuously embeds into $W^{1+r,2}_{\mathfrak{D}}$ for some $r > 0$ if θ is sufficiently close to 1.*

Proof. Take $\frac{1}{2} < \kappa \sim \frac{1}{2}$, and write

$$[L^2, W^{1+s,2}_{\mathfrak{D}}]_\theta = \big[[L^2, W^{1+s,2}_{\mathfrak{D}}]_\kappa, W^{1+s,2}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}} \hookrightarrow \big[[L^2, W^{1,2}_{\mathfrak{D}}]_\kappa, W^{1+s,2}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}} =$$

$$= [W^{\kappa,2}_{\mathfrak{D}}, W^{1+s,2}_{\mathfrak{D}}]_{\frac{\theta-\kappa}{1-\kappa}} = W^{\tau,2}_{\mathfrak{D}}, \quad \text{with } \tau = \theta + s\frac{\theta-\kappa}{1-\kappa}.$$

Obviously, τ exceeds 1 if θ is sufficiently close to 1. □

Lemma 4.17. *Assume $d = 3$, and let $s \in]0, \frac{1}{3}[$. Then $[L^2, W^{1+s,3}_{\mathfrak{D}}]_\theta$ continuously embeds into $W^{1+r,3}_{\mathfrak{D}}$ for some $r > 0$, if θ is sufficiently close to 1.*

Proof. First of all, let us state that, under the additional Assu. 4.12 (b), one has $[W^{-1,q}_{\mathfrak{D}}, W^{1,q}_{\mathfrak{D}}]_{\frac{1}{2}} = L^q$ ($q \in]1, \infty[$), see [4].

We fix a ζ such that L^2 continuously embeds into $W^{-\zeta,3}_{\mathfrak{D}}$. Clearly, then $[L^2, W^{1+s,3}_{\mathfrak{D}}]_\theta \hookrightarrow [W^{-\zeta,3}_{\mathfrak{D}}, W^{1+s,3}_{\mathfrak{D}}]_\theta$. We may write

$$W^{-\zeta,3}_{\mathfrak{D}} = [L^3, W^{-1,3}_{\mathfrak{D}}]_\zeta = [W^{-1,3}_{\mathfrak{D}}, L^3]_{1-\zeta} =$$
$$= \big[W^{-1,3}_{\mathfrak{D}}, [W^{-1,3}_{\mathfrak{D}}, W^{1,3}_{\mathfrak{D}}]_{\frac{1}{2}}\big]_{1-\zeta} = [W^{-1,3}_{\mathfrak{D}}, W^{1,3}_{\mathfrak{D}}]_{\frac{1-\zeta}{2}} \tag{94}$$

Let κ be a fixed number from $]\frac{\frac{1}{3}+\zeta}{1+\zeta}, \theta[$. Now using (94), we may continue

$$[W^{-\zeta,3}_{\mathfrak{D}}, W^{1+s,3}_{\mathfrak{D}}]_\theta = \big[[W^{-\zeta,3}_{\mathfrak{D}}, W^{1+s,3}_{\mathfrak{D}}]_\kappa, W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}} \hookrightarrow \big[[W^{-\zeta,3}_{\mathfrak{D}}, W^{1,3}_{\mathfrak{D}}]_\kappa, W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}}$$

$$= \Big[\big[[W^{-1,3}_{\mathfrak{D}}, W^{1,3}_{\mathfrak{D}}]_{\frac{1-\zeta}{2}}, W^{1,3}_{\mathfrak{D}}\big]_\kappa, W^{1+s,3}_{\mathfrak{D}}\Big]_{\frac{\theta-\kappa}{1-\kappa}} = \big[[W^{-1,3}_{\mathfrak{D}}, W^{1,3}_{\mathfrak{D}}]_\tau, W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}}$$

with $\tau = \frac{1}{2}(1-\zeta) + \frac{\kappa}{2}(1+\zeta)$. Observe that our condition on κ implies $\tau > \frac{1}{2}$. So we may continue

$$= \Big[\big[[W^{-1,3}_{\mathfrak{D}}, W^{1,3}_{\mathfrak{D}}]_{\frac{1}{2}}, W^{1,3}_{\mathfrak{D}}\big]_{2\tau-1}, W^{1+s,3}_{\mathfrak{D}}\Big]_{\frac{\theta-\kappa}{1-\kappa}} = \big[[L^3, W^{1,3}_{\mathfrak{D}}]_{2\tau-1}, W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}} =$$

$$= [W^{2\tau-1,3}_{\mathfrak{D}}, W^{1+s,3}_{\mathfrak{D}}]_{\frac{\theta-\kappa}{1-\kappa}}. \tag{95}$$

Observe that, due to our supposition on κ, we have $2\tau - 1 > \frac{1}{3}$. So the results in [4] allow to identify (95) with a space $W^{1+r,3}_{\mathfrak{D}}$, ($r > 0$), if θ is close to 1. □

Theorem 4.18. *(see [24]) Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triple of Hilbert spaces with dense embeddings. Assume that we are given, for each $t \in J$, a continuous, coercive sesquilinear form $\mathfrak{s}_t$ on V which altogether admit a common coercivity constant. Moreover, suppose that*

$$\sup_{\|\varphi\|_V = \|\psi\|_V = 1} \big|\mathfrak{s}_t(\varphi,\psi) - \mathfrak{s}_s(\varphi,\psi)\big| \le c|s-t|^\alpha, \quad s,t \in J \tag{96}$$

for an $\alpha > \frac{1}{2}$.
Let A_t be the sectorial operator which is induced by $\mathfrak{s}_t$ on H and $q \in]1, \infty[$.
Then, for every $f \in L^q(J, H) \hookrightarrow L^q(J; V^*)$ the solution of the equation

$$\frac{\partial u}{\partial t} + A(\cdot)u = f, \quad u(0) = 0 \tag{97}$$

exists, is unique and satisfies $u \in W^{1,q}(J, H)$, and, consequently,

$$J \ni t \mapsto A_t u(t) \in L^q(J, H). \tag{98}$$

Theorem 4.19. *Let $V = W^{1,2}_{\mathfrak{D}}$, $\hat{\mu}$ is a – time dependent – coefficient function, bounded and elliptic with a uniform in t ellipticity constant. Additionally, concerning the dependence on t, we suppose the condition $\mathfrak{s}_t$ is as in (2.1), there μ taken as a the coefficient function $\mu(t,\cdot)$. Let A_t denote the operator which is induced by the form $\mathfrak{s}_t$ on L^2. Finally, suppose the existence of a reflexive, separable Banach space with dense embedding $X \hookrightarrow L^2$ such that the X, X^* duality extends the L^2-self duality and*

$$\|\psi\|_X \le c\|A_t\psi\|_{L^2}, \quad \psi \in dom(A_t),\ t \in J \tag{99}$$

c being independt from t. Then, for every $L^q(J, L^2)$, the solution u of

$$\frac{\partial u}{\partial t} + A(\cdot)u = f, \quad u(0) = 0 \tag{100}$$

exists and is unique. It belongs to the space $\mathrm{MR}^q_0(J; X, L^2)$.

Proof. First of all, it is straight forward that condition (88) implies condition (96) in Theorem 4.18. So existence and uniqueness follow immediately from Theorem 4.18. Moreover, condition (99) shows that, for almost every t, $u(t)$ indeed belongs to X. Let us show that the function u is measurable when considered as X-valued. Since it is measurable in L^2 also the function $J \ni t \mapsto (u(t), v))_{L^2} = \langle u(t), v\rangle_{X\times X^*}$ is measurable for every $v \in L^2$. But L^2 is dense in X^* so $J \ni t \mapsto \langle u(t), v\rangle_{X\times X^*}$ is measurable even for all $v \in X^*$. Hence, u is weakly measurable when considered in X what implies also strong measurability in our case. Knowing this, (99) shows in combination with Theorem 4.18 that the assertion is true. □

Having this at hand, we may apply Prop. 2.10. This, in combination with the Lemmata 4.16/4.17 finishes the proof of Thm. 4.14.

Remark 4.20. It is not trivial to single out geometries of Ω and $\mathfrak{D}$ and/or coefficient functions μ such that (89) indeed is a topological isomorphism. Fortunately, a broad zoo of geometries and coefficient functions μ which implies this isomorphism property is established in [20] and discussed there in great detail.

5 Concluding remarks

(a) The assignment

$$C_0(J \times \Omega) \ni f \mapsto \int_J \int_\Omega f(t,x) d\rho_t(x) dt \tag{101}$$

defines a measure ϱ on $J \times \Omega$ if the mapping $t \mapsto \rho_t \in \mathcal{M}$ is weakly measurable and some integrability condition

$$\int_J \|\rho_t\|^q_{\mathcal{M}}\, dt < \infty, \quad q \ge 1 \tag{102}$$

holds.

Conversely, if ϱ is a measure on $J \times \Omega$, then it always admits a disintegration of type

$$C_0(J \times \Omega) \ni f \mapsto \int_J \int_\Omega f(t,x) d\rho_t(x) d\varpi(t), \tag{103}$$

with ρ_t a measure on Ω and ϖ a measure on $\overline{J}$, see [28].

Thus, our result is proved for measures ϱ on $J \times \Omega$ for which the measure ϖ is the Lebesgue measure on J and the measures ρ_t are of the form $\sigma_t \mathcal{H}_l|_{M_t}$, satisfying the integrability condition (102).

(b) Condition (102) appears to be reasonable for applications, see [8] and [44].

(c) For the central embedding results Corollary 3.5 and Theorem 3.8 the lower estimate in (3) is in fact not necessary. In this paper we focussed on l sets in the sense of Jonsson/Wallin, thus demanding also this lower estimate, for the following reason: if only demanding the upper estimate, one includes pathological cases like this: consider in $3d$ a subinterval I of the x-axis. This is negligible with respect to the two-dimensional Hausdorff measure $\mathcal{H}_2$. Thus it, trivially, satisfies the upper estimate in (3) for $l = 2$. Then Corollary 3.5 tells us that the trace operator is well defined and continuous as operator from $W_{\mathfrak{D}}^{\frac{3}{4},2}$ to $L^2(I, \mathcal{H}_2)$. But: since I is negligible with respect to the Hausdorff measure $\mathcal{H}_2$, the space $L^2(I, \mathcal{H}_2)$ only contains zero, and, consequently, the trace operator is the zero operator. This we consider as 'pathological' and excluded it by also demanding the lower estimate in (3).

(d) We restricted to the case where the measures live on subsets of *integer* dimension only for technical simplicity. The basis in geometric measure theory on which our results rest is established in [29] for the general case as well. Everything can then be proved quite analogously. Since we are not aware of any applications of this we did not carry out this here but restricted to integral dimensions.

(e) The elliptic result in Proposition 2.5, borrowed from [27], is proved even for systems in that paper. Also in the case of systems the Kato square root problem is solved in the affirmative in an extremely wide range of geometries, see [5]. Moreover, the (elliptic) system operator has a bounded holomorphic calculus on L^2, since it is an accretive one. So the above arguments should also work for systems.

(f) As the title of [31] suggests, it can happen that distributional objects are of interest which are *not* necessarily measures. Consider the following situation: Take $\Omega \subset \mathbb{R}^2$ as a Lipschitz domain which contains a subinterval $]-a, a[$ of the x-axis. Define the distribution Ψ on Ω as the PV distribution on $]-a, a[$ as follows:

$$\langle \Psi, v \rangle = \lim_{\epsilon \mapsto 0} \int_{-a}^{-\epsilon} \frac{\overline{v}(x)}{x} dx + \int_{-a}^{-\epsilon} \frac{\overline{v}(x)}{x} dx, \quad v \in W_{\mathfrak{D}}^{1,q},\ q > 2. \tag{104}$$

It is not hard to see that the mapping in (104) is well-defined and continuous on Hölder spaces, hence on $W_{\mathfrak{D}}^{1+\epsilon,2}$ with $\epsilon > 0$ arbitrary. Consequently, the so defined Ψ – *not* being a measure – belongs to any $W_{\mathfrak{D}}^{-1-\epsilon,2}$ and lives on a one-dimensional manifold. We expect that such distributional objects, entering in the parabolic equations as right hand sides, can be treated to a large degree in the same manner as the measures that we have considered above.

We suspect that similar constructions can be found also in higher dimensions, but do not expatiate this further here.

6 Appendix

As announced we give some explanations to the proof of **Prop. 3.4**.
The expression in question which one has to estimate is

$$\|G_\alpha \star f\|^2_{L^2(M;\mathcal{H}_l)} = \int_M \Big| \int_{\mathbb{R}^d} G_\alpha(x-y) f(y)\, dy \Big|^2 d\mathcal{H}_l(x) \tag{105}$$

We follow widely Jonsson/Wallin with the exception to determine the constant a explicitly here – what should allow an easier reading.
We define the number a via

$$\big(d - \frac{d-l}{2}\big)(1-a)2 = \frac{d+l}{2}(1-a)2 = d. \tag{106}$$

Re-arranging terms, one obtains

$$\big(d+l\big)a = \big(d - \frac{d-l}{2}\big)a\,2 = l. \tag{107}$$

Clearly, this gives $a = \frac{l}{d+l} \in]0,1[$. Evidently, (106) yields

$$\big(d-\alpha\big)(1-a)2 = \big(d - \frac{d-l}{2}\big)(1-a)2 - (\alpha - \frac{d-l}{2}\big)(1-a)2 =$$

$$= d - (\alpha - \frac{d-l}{2}\big)(1-a)2 < d \tag{108}$$

and (107) provides

$$(d-\alpha)2a = (d - \frac{d-l}{2})2a + (\frac{d-l}{2} - \alpha)2a = l - (\alpha - \frac{d-l}{2})2a < l, \tag{109}$$

thanks to the supposition $\alpha > \frac{d-l}{2}$.

One estimates the r.h.s of (105) by

$$\int_M \Big(\int_{\mathbb{R}^d} \big|G_\alpha(x-y)\big|^{1-a}\big|G_\alpha(x-y)\big|^a f(y)\,dy\Big)^2\, d\mathcal{H}_l(x).$$

Applying Hölder's inequality, one further estimates

$$\leq \int_M \Big(\int_{\mathbb{R}^d} \big|G_\alpha(x-y)\big|^{2a}|f(y)|^2 dy \cdot \big(\int_{\mathbb{R}^d} |G_\alpha(x-y)|^{2(1-a)} dy\big)\Big) d\mathcal{H}_l(x).$$

The crucial point is to show that the terms

$$\int_{\mathbb{R}^d} |G_\alpha(x-y)|^{2(1-a)} dy = \int_{\mathbb{R}^d} |G_\alpha(y)|^{2(1-a)} dy, \tag{110}$$

and

$$\int_M |G_\alpha(x-y)|^{2a} d\mathcal{H}_l(x), \quad y \in \mathbb{R}^d, \tag{111}$$

may be estimated uniformly for sets M admitting the same constant $\mathfrak{c}^\bullet$. Investing the exponential decay of the Bessel kernel at ∞ (see [46, Ch. V.3]) one can observe that (110) makes no difficulties at ∞. In a neighborhood of zero (110) also converges, thanks to

$$|G_\alpha(z)| \leq \gamma\, |z|^{\alpha-d}, \tag{112}$$

(see [46, Ch. V.3]) in combination with (108).

Finally (111) can be written as

$$\int_{M\cap\{x:|x-y|>1\}} |G_\alpha(x-y)|^{2a} d\mathcal{H}_l(x) + \int_{M\cap\{x:|x-y|\leq 1\}} |G_\alpha(x-y)|^{2a} d\mathcal{H}_l(x).$$

According to (112), the first integral is not larger than $\gamma^{2a}\, \mathcal{H}_l(M)$, and $\mathcal{H}_l(M)$ is not larger than $\mathfrak{c}^\bullet \times \tau - \tau$ being the number of(shifted) unit balls $B(z,1)$ required for a covering of M. The second integral is estimated by again employing (112) in combination with (109). This yields $|G_\alpha(x-y)|^{2a} \leq \gamma^{2a}|x-y|^{-\sigma}$ with $\sigma < l$. Afterwards one applies [29, Ch. V.1.2 Lemma 1]. This shows, first, that (111) is indeed finite – and may be estimated uniformly with respect to $y \in \mathbb{R}^d$. But even more, one observes that the constant $\mathfrak{c}^\bullet$ enters *linearly* in this estimate.

Acknowledgements The authors wish to thank Moritz Egert (Darmstadt) for helpfull discussions on the subject.

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