OBLIQUE PROJECTION BASED STABILIZING FEEDBACK FOR NONAUTONOMOUS COUPLED PARABOLIC-ODE SYSTEMS

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Abstract. Global feedback stabilizability results are derived for nonautonomous coupled systems arising from the linearization around a given time-dependent trajectory of FitzHugh–Nagumo type systems. The feedback is explicit and is based on suitable oblique (nonorthogonal) projections in Hilbert spaces. The actuators are, typically, a finite number of indicator functions and act only in the parabolic equation. Subsequently, local feedback stabilizability to time-dependent trajectories results are derived for nonlinear coupled parabolic-ODE systems of the FitzHugh–Nagumo type.

Simulations are presented showing the stabilizing performance of the feedback control.

1. **Introduction.** In this work we focus on coupled systems consisting of a parabolic equation with controls, and an ordinary differential equation. This class of systems include well-known models describing electric excitations and the propagation of electric waves in nerve fibers and in heart tissue. In this context the system of equations is referred to as the *monodomain equations*. Given a bounded domain $\Omega \subset \mathbb{R}^{\mathbf{d}}$, where $\mathbf{d}$ is a positive integer, with smooth boundary $\Gamma = \partial\Omega$, the controlled monodomain equations read

$$\tfrac{\partial}{\partial t} v = \nu\Delta v - av^3 + bv^2 - cv - \mathfrak{M}_1(v,w) + f + \sum_{i=1}^{M} u_i 1_{\omega_i}, \quad v(0) = v_0, \tag{1a}$$

$$\tfrac{\partial}{\partial t} w = -\delta w - \mathfrak{M}_2(v,w), \quad w(0) = w_0, \tag{1b}$$

with Neumann boundary conditions

$$(\tfrac{\partial}{\partial \mathbf{n}} v)|_{\Gamma} = 0, \tag{1c}$$

where $f = f(x,t)$ is an external forcing, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are suitable functions coupling the two equations, $\mathbf{n}$ is the unit outward normal vector to Γ, and the constants in $\{\nu, \delta, a, b, c\}$ are all given and strictly positive. Further, $\{\omega_i \mid i \in \{1, 2, \dots, M\}\}$ is

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a family of open sets of Ω and our M actuators $\{1_{\omega_i} = 1_{\omega_i}(x) \mid i \in \{1, 2, \dots, M\}\} \subset L^2(\Omega)$, are the indicator functions of the subsets ω_i,

$$1_{\omega_i}(x) \coloneqq \begin{cases} 1 & \text{if } x \in \omega_i, \\ 0 & \text{if } x \in \Omega \setminus \overline{\omega}_i. \end{cases}$$

Finally $u = u(t)$, taking values in $\mathbb{R}^M$, is a control function at our disposal.

In electrophysiology, see [9, Section 12.3.3], the variable v models the transmembrane electric potential of the human heart and w is represents a gating variable. Typical models include the FitzHugh–Nagumo, the Rogers–McCulloch, and the Aliev–Panfilov model. See [5, 15, 18, 1, 6]. The coupling $\mathfrak{M}$ takes the form

$$\mathfrak{M}(w, v) \coloneqq \begin{bmatrix} \mathfrak{M}_1(w, v) \\ \mathfrak{M}_2(w, v) \end{bmatrix} \coloneqq \begin{bmatrix} dw + evw \\ -\gamma v + \rho v^2 \end{bmatrix} \tag{2a}$$

where the constants d, e, γ, ρ are also given, and are all strictly positive with the following exceptions defining the model:

$$\begin{array}{lll} e = \rho = 0, & \text{for the FitzHugh–Nagumo model,} & \text{[FN]}. \\ d = \rho = 0, & \text{for the Rogers–McCulloch model,} & \text{[RM]}. \\ d = 0, & \text{for the Aliev–Panfilov model,} & \text{[AP]}. \end{array} \tag{2b}$$

Assume that a desired heart rhythm is given as the solution of the (uncontrolled) system (1), corresponding to a suitable initial condition $(\overline{v}_0, \overline{w}_0) \in L^2(\Omega) \times L^2(\Omega)$,

$$\tfrac{\partial}{\partial t}\overline{v} = \nu\Delta\overline{v} - a\overline{v}^3 + b\overline{v}^2 - c\overline{v} - \mathfrak{M}_1(\overline{w}, \overline{v}) + f, \qquad \overline{v}(0) = \overline{v}_0, \tag{3a}$$

$$\tfrac{\partial}{\partial t}\overline{w} = -\delta\overline{w} - \mathfrak{M}_2(\overline{w}, \overline{v}), \qquad \overline{w}(0) = \overline{w}_0, \tag{3b}$$

$$(\tfrac{\partial}{\partial \mathbf{n}}\overline{v})\big|_\Gamma = 0, \tag{3c}$$

If $(v_0, w_0) \neq (\overline{v}_0, \overline{w}_0)$, then the behavior of (1) without control can be considerably different from the desired behavior of $(\overline{v}, \overline{w})$, even if $(v_0, w_0) - (\overline{v}_0, \overline{w}_0)$ is small. Here we look for a feedback control u, so that the solution of (1) goes to the desired heart rhythm $(\overline{v}, \overline{w})$, solving (3), provided the difference $(v_0, w_0) - (\overline{v}_0, \overline{w}_0)$ is small.

We will follow a standard idea: first we find a feedback operator which stabilizes *globally* the linearization of (1) around $(\overline{v}, \overline{w})$, then we use a suitable fixed point argument to conclude that the same feedback operator also stabilizes the nonlinear system locally.

More precisely, by direct computations, we can see that the (controlled) linearization around $(\overline{v}, \overline{w})$, satisfies

$$\tfrac{\partial}{\partial t}y + Ay + A_{\mathrm{r}}y + \widetilde{\mathcal{S}}z - \sum_{i=1}^{M} u_i 1_{\omega_i} = 0, \qquad y(0) = y_0 \tag{4a}$$

$$\tfrac{\partial}{\partial t}z + \mathcal{D}z + \mathcal{R}v = 0, \qquad z(0) = z_0. \tag{4b}$$

Moreover the difference $(y, z) \coloneqq (v, w) - (\overline{v}, \overline{w})$, to the targeted solution, solves the system

$$\tfrac{\partial}{\partial t}y + Ay + A_{\mathrm{r}}y + \widetilde{\mathcal{S}}z - \sum_{i=1}^{M} u_i 1_{\omega_i} = \mathcal{N}_1(y, z), \qquad y(0) = y_0 \tag{5a}$$

$$\tfrac{\partial}{\partial t}z + \mathcal{D}z + \mathcal{R}v = \mathcal{N}_2(y, z), \qquad z(0) = z_0, \tag{5b}$$

with the linear operators

$$y \mapsto Ay \coloneqq -\nu\Delta y + y, \tag{6a}$$
$$\text{with} \quad \langle Ay, v\rangle_{H^1(\Omega)', H^1(\Omega)} \coloneqq \nu(\nabla y, \nabla v)_{L^2(\Omega)^d} + (y, v)_{L^2(\Omega)},$$
$$z \mapsto \mathcal{D}z \coloneqq \delta z, \tag{6b}$$
$$y \mapsto A_{\mathrm{r}}y \coloneqq -y - (-3a\overline{v}^2 y + 2b\overline{v}y - cy - e\overline{w}y), \tag{6c}$$
$$z \mapsto \widetilde{\mathcal{S}}z \coloneqq dz + e\overline{v}z, \tag{6d}$$
$$y \mapsto \mathcal{R}y \coloneqq (-\gamma + 2\rho\overline{v})y, \tag{6e}$$

and the nonlinearities

$$(y, z) \mapsto \mathcal{N}_1(y, z) \coloneqq -ay^3 - (-b + 3a\overline{v})y^2 - eyz, \tag{6f}$$
$$(y, z) \mapsto \mathcal{N}_2(y, z) \coloneqq -\rho y^2. \tag{6g}$$

Without loss of generality we may suppose that the set of actuators $\{1_{\omega_i} \mid i \in \{1, 2, \dots, M\}\} \subset L^2(\Omega)$ is linearly independent. We also set $U_M \coloneqq \operatorname{span}\{1_{\omega_i} \mid i \in \{1, 2, \dots, M\}\}$.

Let α_i, $i \in \{1, 2, 3, \dots\}$, be the increasing sequence of repeated (Neumann) eigenvalues of $A = -\Delta + 1$ and let $E_M \coloneqq \operatorname{span}\{e_i \mid i \in \{1, 2, \dots, M\}\}$ be the space spanned by the first eigenfunctions of A in $L^2(\Omega)$,

$$0 < \alpha_1 \le \alpha_2 \le \dots \le \alpha_n \le \alpha_{n+1} \to +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n.$$

For simplicity let us denote $L^2 \coloneqq L^2(\Omega)$ and $H^1 \coloneqq H^1(\Omega)$. We will show that a sufficient condition for *global* stabilizability of system (4), and for *local* stabilizability of system (5), is given by

$$H = U_M \oplus E_M^\perp, \tag{7a}$$
$$\alpha_{M+1} > \left(6 + 4\left|P_{U_M}^{E_M^\perp}\right|^2_{\mathcal{L}(L^2)}\right)|A_{\mathrm{r}}|^2_{L^\infty(\mathbb{R}_0, \mathcal{L}(L^2, (H^1)'))}$$
$$+ \frac{\left|P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}\right|_{L^\infty(\mathbb{R}_0, \mathcal{L}(L^2))}|\mathcal{R}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(L^2))}}{\delta}, \tag{7b}$$

where $\mathcal{L}\left(L^2, (H^1)'\right)$ stands for the space of bounded linear functionals from L^2 into $(H^1)'$, and $\mathcal{L}\left(L^2\right)$ stands for the space of bounded linear functionals from L^2 into itself. Further, $P_{U_M}^{E_M^\perp} \colon L^2 \to U_M$ is the oblique projection in L^2 onto U_M along $E_M^\perp$, and $P_{E_M^\perp}^{U_M} = \mathrm{Id} - P_{U_M}^{E_M^\perp}$ is the complementary oblique projection in L^2 onto $E_M^\perp$ along U_M.

The following regularity condition on the desired trajectory will be used.

$$(\overline{v}, \overline{w}) \in L^\infty(\mathbb{R}_0 \times \Omega) \times L^\infty(\mathbb{R}_0 \times \Omega). \tag{8}$$

Recall that in [11] it is shown that when $\widetilde{\mathcal{S}} = 0$, then (7) is a sufficient stabilizability condition for the parabolic system (4a).

The following Theorems 1.1 and 1.2 are the main results of this paper.

Theorem 1.1 (Linearized monodomain equations)**.** *Let $\lambda > 0$ and $s_0 \ge 0$. If (7) and (8) holds true, then there are constants $C \ge 1$ and $\mu > 0$, independent of*

(s_0, y_0, z_0)*, such that the solution of the linear system*

$$\tfrac{\partial}{\partial t}y + Ay + A_{\rm r}y + \widetilde{\mathcal{S}}z - P_{U_M}^{E_M^\perp}(Ay + A_{\rm r}y + \widetilde{\mathcal{S}}z - \lambda y) = 0, \qquad y(s_0) = y_0, \tag{9a}$$

$$\tfrac{\partial}{\partial t}z + \mathcal{D}z + \mathcal{R}v = 0, \qquad z(s_0) = z_0, \tag{9b}$$

satisfies

$$|(y(t), z(t))|_{L^2\times L^2} \le C\mathrm{e}^{-\mu(t-s_0)} |(y_0, z_0)|_{L^2\times L^2}, \qquad t \ge s_0.$$

for all $(y_0, z_0) \in L^2 \times L^2$.

Theorem 1.2 (Monodomain equations)**.** *Let* $\lambda > 0$ *and* $s_0 \ge 0$*. If* (7) *and* (8) *holds true, then there are constants* $C \ge 1$*,* $\mu > 0$*, and* $\epsilon > 0$*, independent of* (s_0, y_0, z_0)*, such that the solution of the system*

$$\tfrac{\partial}{\partial t}y + Ay + A_{\rm r}y + \widetilde{\mathcal{S}}z - P_{U_M}^{E_M^\perp}(Ay + A_{\rm r}y + \widetilde{\mathcal{S}}z - \lambda y) = \mathcal{N}_1(y, z), \qquad y(s_0) = y_0, \tag{10a}$$

$$\tfrac{\partial}{\partial t}z + \mathcal{D}z + \mathcal{R}v = \mathcal{N}_2(y, z), \qquad z(s_0) = z_0, \tag{10b}$$

satisfies

$$|(y(t), z(t))|_{H^1\times L^2} \le C\mathrm{e}^{-\mu(t-s_0)} |(y_0, z_0)|_{H^1\times L^2}, \qquad t \ge s_0,$$

provided that $|(y_0, z_0)|_{H^1\times L^2} < \epsilon$.

Remark 1. Observe that (10) is exactly (5), with u given by

$$\sum_{i=1}^{M} u_i 1_{\omega_i} = P_{U_M}^{E_M^\perp}(Ay + A_{\rm r}y + \widetilde{\mathcal{S}}z - \lambda y). \tag{11}$$

Furthermore, by definition E_M is an M dimensional space, which together with (35a) implies that the (ordered) family of actuators $\mathcal{U}_M := (1_{\omega_1}, 1_{\omega_2}, \dots, 1_{\omega_M})$ is linearly independent (with $M \ge 1$). Thus, denoting the bijection $[\mathcal{U}_M]\colon \mathbb{R}^M \to U_M$, with $u \mapsto \sum_{i=1}^{M} u_i 1_{\omega_i}$, the control u as in (11) is uniquely defined, and we may write $u = [\mathcal{U}_M]^{-1} P_{U_M}^{E_M^\perp}(Ay + A_{\rm r}y + \widetilde{\mathcal{S}}z - \lambda y)$.

Remark 2. Observe that Theorem 1.2 provides a result on the stabilizability to trajectories, since it asserts that the solution $(v, w) = (\overline{v}, \overline{w}) + (y, z)$ of (1) goes exponentially to the targeted trajectory $(\overline{v}, \overline{w})$, solving (3), provided (7) holds true, and $|(v_0 - \overline{v}_0, w_0 - \overline{w}_0)|_{H^1\times L^2} < \epsilon$.

Concerning (7b), it is proven in [17] for 1D domains of the form $\Omega = (0, L) \subset \mathbb{R}$ that for any given $r \in (0, 1)$ and any given $M \ge 1$ we can place M actuators $1_{\omega_i} = 1_{\omega_i^M}$, with $\omega_i^M \subset (0, L)$, so that the operator norm $\left|P_{U_M}^{E_M^\perp}\right|_{\mathcal{L}(L^2)}$ remains bounded as M increases. Furthermore, the total volume covered by the actuators is equal rL, and thus independent of M. The extension of these results to multi-dimensional rectangles follows from the results in [11, Section 4.8], see also [11, Remark 3.9]. Note that, once the operator norm of the oblique projection remains bounded remains and $\alpha_{M+1} \to +\infty$, then we can set M large enough so that (7) is satisfied. Thus for an arbitrary "coupling" quadruple (d, e, γ, ρ), we can set M large enough so that system (4) is stable.

Recall also that in [4] the stabilizability of (1.1) is proven, with a Riccati based feedback, for the case $\rho = 0$. Comparing the results in this paper to those in [4], we can say that in [4] all the actuators are supported in an a-priori given subset of the physical domain Ω and stability of the Riccati based closed-loop system is guaranteed under a condition on the coupling triple (d, e, γ), see [4, Corollary 3.2 and Eq. (38)]. In this paper stability of the oblique projection based closed-loop system is obtained for an arbitrary coupling quadruple (d, e, γ, ρ) under a boundedness condition on the operator norm of the oblique projection $P_{U_M}^{E_M^\perp}$ onto the span of the actuators, and the placement of the actuators is supposed to be at our disposal.

Contents. The rest of the paper is organized as follows. In Sect. 2 we derive results on the stability of linear coupled systems in an abstract setting. These results are applied in Sect. 3 to obtain stabilizability conditions for general linear coupled parabolic-ODE systems. In Sect. 4 we derive the stabilizability result for the concrete example of the linearized monodomain equations, in particular, it contains the proof of Main Theorem 1.1 in Sect. 4.1 and that of Main Theorem 1.2 in Sect. 4.2. Numerical simulations are presented in Section 5, for both the linearized system (9) and the nonlinear system (10), confirming the theoretical results and showing the performance of the explicit oblique projection stabilizing feedback control. Finally the Appendix gathers comments on the existence and uniqueness of weak solutions for the systems involved in the main text.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_r \coloneqq (r, +\infty)$, for $r \in \mathbb{R}$, and $\mathbb{N}_0 \coloneqq \mathbb{N} \setminus \{0\}$.

We denote by $\Omega \subset \mathbb{R}^{\mathbf{d}}$ a bounded open connected subset, with $\mathbf{d} \in \mathbb{N}_0$.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm, by X' its dual, and by $\langle\cdot,\cdot\rangle_{X',X}$ the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} \coloneqq \sup\{\langle f, x\rangle_{X',X} \mid x \in X \text{ and } |x|_X = 1\}$. In case X is a Hilbert space we denote the inner product by $(\cdot,\cdot)_X$.

For an open interval $I \subseteq \mathbb{R}$ and two Banach spaces X, Y, we write $W(I, X, Y) \coloneqq \{f \in L^2(I, X) \mid \frac{\partial}{\partial t} f \in L^2(I, Y)\}$, where the derivative $\frac{\partial}{\partial t} f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I,X,Y)} \coloneqq \bigl(|f|^2_{L^2(I,X)} + |\frac{\partial}{\partial t} f|^2_{L^2(I,Y)}\bigr)^{1/2}$. In case $X = Y$, we write $H^1(I, X) \coloneqq W(I, X, X)$.

The time derivative (in the distribution sense) of a vector function v taking values in a Banach space X will be denoted by $\dot v \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} v$.

If the inclusions $X \subseteq Z$ and $Y \subseteq Z$ are continuous, where Z is a Hausdorff topological space, then we can define the Banach spaces $X \times Y$, $X \cap Y$, and $X + Y$, endowed with the norms $|(a, b)|_{X\times Y} \coloneqq \bigl(|a|^2_X + |b|^2_Y\bigr)^{\frac12}$; $|a|_{X\cap Y} \coloneqq |(a, a)|_{X\times Y}$; and $|a|_{X+Y} \coloneqq \inf_{(a^X, a^Y)\in X\times Y}\bigl\{|(a^X, a^Y)|_{X\times Y} \mid a = a^X + a^Y\bigr\}$, respectively. We can show that, if X and Y are endowed with a scalar product, then also $X \times Y$, $X \cap Y$, and $X + Y$ are. In case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead.

Again, if X and Y are endowed with a scalar product, then also $W(I, X, Y)$ is. The space of continuous linear mappings from X into Y will be denoted by $\mathcal{L}(X, Y)$. When $X = Y$ we simply write $\mathcal{L}(X) \coloneqq \mathcal{L}(X, X)$.

If the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \xhookrightarrow{\mathrm{d}} Y$, respectively $X \xhookrightarrow{\mathrm{c}} Y$, if the inclusion is also dense, respectively compact.

$\overline{C}_{[a_1,\dots,a_k]}$ denotes a nonnegative function of nonnegative variables a_j that increases in each of its arguments.

Finally, C, C_i, $i = 0, 1, \dots$, stand for unessential positive constants.

2. **Stability of coupled evolutionary systems.** We shall present a stability result for a coupled abstract system in a form which is convenient for our purposes. We introduce the bounded time intervals

$$I \coloneqq (s_0, s_1), \quad 0 \le s_0 < s_1 < +\infty, \tag{12}$$

where the real numbers s_0, s_1 are given. We shall utilize a a separable Hilbert space H, which we consider as a pivot space, $H = H'$. Further we introduce a subspace $G \subseteq H$, and two evolutionary systems

$$\dot v = -\mathcal{A}v, \qquad v(s_0) = v_0 \in G, \tag{13}$$
$$\dot w = -\mathcal{D}w, \qquad w(s_0) = w_0 \in H, \tag{14}$$

in the Hilbert spaces G and H, where for suitable Hilbert spaces $\mathcal{V} \subset G$ and $\mathcal{W} \subset H$, we suppose that $\mathcal{A}(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ and $\mathcal{D}(t) \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$ are operators in G and H, respectively, with domains $\mathrm{D}(\mathcal{A}) = \mathrm{D}(\mathcal{A}(t)) \subseteq G$ and $\mathrm{D}(\mathcal{D}) = \mathrm{D}(\mathcal{D}(t)) \subseteq H$, independent of t.

Furthermore, we assume that

$$\mathrm{D}(\mathcal{A}) \xhookrightarrow{\mathrm{d}} \mathcal{V} \xhookrightarrow{\mathrm{d}} G \xhookrightarrow{\mathrm{d}} \mathcal{V}' \xhookrightarrow{\mathrm{d}} \mathrm{D}(\mathcal{A})' \qquad \text{and} \qquad \mathrm{D}(\mathcal{D}) \xhookrightarrow{\mathrm{d}} \mathcal{W} \xhookrightarrow{\mathrm{d}} H \xhookrightarrow{\mathrm{d}} \mathcal{W}' \xhookrightarrow{\mathrm{d}} \mathrm{D}(\mathcal{D})'. \tag{15}$$

Given linear operators $\mathcal{S} = \mathcal{S}(t) \in \mathcal{L}(H, G)$ and $\mathcal{R} = \mathcal{R}(t) \in \mathcal{L}(G, H)$, we will consider the coupled system

$$\dot v = -\mathcal{A}v - \mathcal{S}w, \qquad v(s_0) = v_0, \tag{16a}$$
$$\dot w = -\mathcal{D}w - \mathcal{R}v, \qquad w(s_0) = w_0, \tag{16b}$$

whose stability properties are the main focus in this section. Namely, we look for a suitable condition, so that the stability of (16) follows from the stability of each of the systems (13) and (14) separately.

Let us set

$$\mathbf{H} \coloneqq G \times H, \quad \mathbf{V} \coloneqq \mathcal{V} \times \mathcal{W}, \quad \text{and} \quad \mathcal{M} \coloneqq \begin{bmatrix} -\mathcal{A} & -\mathcal{S} \\ -\mathcal{R} & -\mathcal{D} \end{bmatrix}. \tag{17}$$

Note that, since $G \subseteq H$ is a closed subspace we may write

$$H' = H, \quad G' = G, \quad \text{and} \quad \mathbf{H} = \mathbf{H}',$$

and we have

$$\mathcal{M} \in \mathcal{L}(\mathbf{V}, \mathbf{V}'), \quad \mathrm{D}(\mathcal{M}) = \mathrm{D}(\mathcal{A}) \times \mathrm{D}(\mathcal{D}), \quad \mathrm{D}(\mathcal{M}) \xhookrightarrow{\mathrm{d}} \mathbf{V} \xhookrightarrow{\mathrm{d}} \mathbf{H} \xhookrightarrow{\mathrm{d}} \mathbf{V}' \xhookrightarrow{\mathrm{d}} \mathrm{D}(\mathcal{M})'.$$

Assumption 1. For every bounded time interval $J \subseteq \mathbb{R}_0$, we have the existence and uniqueness of:

- a $(G, \mathcal{V})$-weak solution v for system (13), for any given $v_0 \in G$,
- a $(H, \mathcal{W})$-weak solution w for system (14), for any given $w_0 \in H$,
- a $(\mathbf{H}, \mathbf{V})$-weak solution $z = (v, w)$ for system (16) for any given $z_0 \in \mathbf{H}$.

Thus, since the existence and uniqueness of solutions is not the focus of this paper, it is simply assumed by us. In the appendix, we briefly recall the standard procedure on the construction of weak solutions as a weak limit of suitable Galerkin approximations. In particular we will see that the above assumption is satisfied for a general class of operators $\mathcal{A}$, $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{R}$. Applications to the linearized monodomain equations will be given in Sect. 4.

Assumption 2. The operators $\mathcal{S}$ and $\mathcal{R}$ are essentially bounded:

$$\mathcal{S} \in L^\infty(\mathbb{R}_{s_0}, \mathcal{L}(H,G)) \quad \text{and} \quad \mathcal{R} \in L^\infty(\mathbb{R}_{s_0}, \mathcal{L}(G,H)).$$

Assumption 3. There are pairs $(C_\mathcal{A}, \mu_\mathcal{A})$ and $(C_\mathcal{D}, \mu_\mathcal{D})$, both in $[1,+\infty)\times(0,+\infty)$, so that

$$\left|U_{(t,s_0)}^{-\mathcal{A}} v_0\right|_G \le C_\mathcal{A} \mathrm{e}^{-\mu_\mathcal{A}(t-s_0)} \left|v_0\right|_G \quad \text{and} \quad \left|U_{(t,s_0)}^{-\mathcal{D}} w_0\right|_H \le C_\mathcal{D} \mathrm{e}^{-\mu_\mathcal{D}(t-s_0)} \left|w_0\right|_H \tag{18}$$

for all $t \ge s_0 \ge 0$.

Theorem 2.1. *Under Assumptions 3 and 2, and if the inequality*

$$\xi := \frac{C_\mathcal{A} C_\mathcal{D} \|\mathcal{S}\| \|\mathcal{R}\|}{\mu_\mathcal{A} \mu_\mathcal{D}} < 1 \tag{19}$$

holds true, then for all ε sufficiently small there exists a constant $D_\mathrm{c} \ge 1$ such that the weak solution of system (16) satisfies

$$\left|(v(t), w(t))\right|_{G\times H} \le D_\mathrm{c} \mathrm{e}^{-\varepsilon(t-s_0)} \left|(v_0, w_0)\right|_{G\times H} \tag{20}$$

independently of (v_0, w_0).

The proof is given below. Before we derive some auxiliary results, where Assumptions 2 and 3 are assumed to hold true.

Lemma 2.2. *The weak solution of system (16) satisfies the estimate*

$$\left|v\right|_{L^1(\mathbb{R}_{s_0}, G)} \le \frac{C_\mathcal{A}}{\mu_\mathcal{A}} \left|v_0\right|_G + \frac{C_\mathcal{A} C_\mathcal{D} \|\mathcal{S}\|}{\mu_\mathcal{A} \mu_\mathcal{D}} \left|w_0\right|_H + \frac{C_\mathcal{A} C_\mathcal{D} \|\mathcal{S}\| \|\mathcal{R}\|}{\mu_\mathcal{A} \mu_\mathcal{D}} \left|v\right|_{L^1(\mathbb{R}_{s_0}, G)}. \tag{21}$$

where $\|\mathcal{S}\| := \left|\mathcal{S}\right|_{L^\infty(\mathbb{R}_{s_0}, \mathcal{L}(H,G))}$ and $\|\mathcal{R}\| := \left|\mathcal{R}\right|_{L^\infty(\mathbb{R}_{s_0}, \mathcal{L}(G,H))}$.

Proof. Using Duhamel formula, we integrate the equation in (16b) and obtain, for $t \ge s_0$,

$$w(t) = U_{(t,s_0)}^{-\mathcal{D}} w_0 - \int_{s_0}^t U_{(t,s)}^{-\mathcal{D}} (\mathcal{R}(s) v(s)) \, \mathrm{d}s,$$

and

$$\left|w(t)\right|_H \le C_\mathcal{D} \mathrm{e}^{-\mu_\mathcal{D}(t-s_0)} \left|w_0\right|_H + C_\mathcal{D} \int_{s_0}^t \mathrm{e}^{-\mu_\mathcal{D}(t-s)} \left|\mathcal{R}(s) v(s)\right|_H \, \mathrm{d}s, \tag{22}$$

Therefore, for $t \ge s_0$, we obtain

$$\begin{aligned} \left|v(t)\right|_G &= \left| U_{(t,s_0)}^{-\mathcal{A}} v_0 - \int_{s_0}^t U_{(t,s)}^{-\mathcal{A}} (\mathcal{S}(s) w(s)) \mathrm{d}s \right|_G \\ &\le C_\mathcal{A} \mathrm{e}^{-\mu_\mathcal{A}(t-s_0)} \left|v_0\right|_G + C_\mathcal{A} \|\mathcal{S}\| \int_{s_0}^t \mathrm{e}^{-\mu_\mathcal{A}(t-s)} |w(s)|_H \, \mathrm{d}s \qquad (23) \\ &\le C_\mathcal{A} \mathrm{e}^{-\mu_\mathcal{A}(t-s_0)} \left|v_0\right|_G + C_\mathcal{A} C_\mathcal{D} \|\mathcal{S}\| \int_{s_0}^t \mathrm{e}^{-\mu_\mathcal{A}(t-s)} \mathrm{e}^{-\mu_\mathcal{D}(s-s_0)} \left|w_0\right|_H \, \mathrm{d}s \\ &\quad + C_\mathcal{A} C_\mathcal{D} \|\mathcal{S}\| \|\mathcal{R}\| \int_{s_0}^t \mathrm{e}^{-\mu_\mathcal{A}(t-s)} \int_{s_0}^s \mathrm{e}^{-\mu_\mathcal{D}(s-\tau)} \left|v(\tau)\right|_G \, \mathrm{d}\tau \, \mathrm{d}s. \qquad (24) \end{aligned}$$

Let us consider first the case $\mu_\mathcal{A} \ne \mu_\mathcal{D}$. In this case we obtain

$$\begin{aligned}
&\int_{s_0}^{t} \mathrm{e}^{-\mu_{\mathcal{A}}(t-s)}\mathrm{e}^{-\mu_{\mathcal{D}}(s-s_0)} \left|w_0\right|_H \,\mathrm{d}s = \left|w_0\right|_H \mathrm{e}^{-\mu_{\mathcal{A}}t+\mu_{\mathcal{D}}s_0} \int_{s_0}^{t} \mathrm{e}^{(\mu_{\mathcal{A}}-\mu_{\mathcal{D}})s}\,\mathrm{d}s\\
&= \frac{\left|w_0\right|_H}{\mu_{\mathcal{A}}-\mu_{\mathcal{D}}}\mathrm{e}^{-\mu_{\mathcal{A}}t+\mu_{\mathcal{D}}s_0}\left(\mathrm{e}^{(\mu_{\mathcal{A}}-\mu_{\mathcal{D}})t}-\mathrm{e}^{(\mu_{\mathcal{A}}-\mu_{\mathcal{D}})s_0}\right)\\
&= \frac{\left|w_0\right|_H}{\mu_{\mathcal{A}}-\mu_{\mathcal{D}}}\left(\mathrm{e}^{-\mu_{\mathcal{D}}(t-s_0)}-\mathrm{e}^{-\mu_{\mathcal{A}}(t-s_0)}\right) \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{s_0}^{t} \mathrm{e}^{-\mu_{\mathcal{A}}(t-s)} \int_{s_0}^{s} \mathrm{e}^{-\mu_{\mathcal{D}}(s-\tau)} \left|v(\tau)\right|_G \,\mathrm{d}\tau\,\mathrm{d}s\\
&= \int_{s_0}^{t} \mathrm{e}^{-\mu_{\mathcal{A}}t+\mu_{\mathcal{D}}\tau} \left|v(\tau)\right|_G \int_{\tau}^{t} \mathrm{e}^{(\mu_{\mathcal{A}}-\mu_{\mathcal{D}})s}\,\mathrm{d}s\,\mathrm{d}\tau\\
&= \frac{1}{\mu_{\mathcal{A}}-\mu_{\mathcal{D}}}\int_{s_0}^{t} \mathrm{e}^{-\mu_{\mathcal{A}}t+\mu_{\mathcal{D}}\tau}\left(\mathrm{e}^{(\mu_{\mathcal{A}}-\mu_{\mathcal{D}})t}-\mathrm{e}^{(\mu_{\mathcal{A}}-\mu_{\mathcal{D}})\tau}\right)\left|v(\tau)\right|_G\,\mathrm{d}\tau\\
&= \frac{1}{\mu_{\mathcal{A}}-\mu_{\mathcal{D}}}\int_{s_0}^{t}\left(\mathrm{e}^{-\mu_{\mathcal{D}}(t-\tau)}-\mathrm{e}^{-\mu_{\mathcal{A}}(t-\tau)}\right)\left|v(\tau)\right|_G\,\mathrm{d}\tau \qquad (26)
\end{aligned}$$

Now, from (24), (25), and (26), it follows

$$\begin{aligned}
\left|v\right|_{L^1(\mathbb{R}_{s_0},G)} &\le \frac{C_{\mathcal{A}}}{\mu_{\mathcal{A}}}\left|v_0\right|_G + \frac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|}{\mu_{\mathcal{A}}-\mu_{\mathcal{D}}}\left(\frac{1}{\mu_{\mathcal{D}}}-\frac{1}{\mu_{\mathcal{A}}}\right)\left|w_0\right|_H\\
&\quad + \frac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|\|\mathcal{R}\|}{\mu_{\mathcal{A}}-\mu_{\mathcal{D}}}\int_{s_0}^{+\infty}\int_{s_0}^{t}\left(\mathrm{e}^{-\mu_{\mathcal{D}}(t-\tau)}-\mathrm{e}^{-\mu_{\mathcal{A}}(t-\tau)}\right)\left|v(\tau)\right|_G\,\mathrm{d}\tau\,\mathrm{d}t.\\
&= \frac{C_{\mathcal{A}}}{\mu_{\mathcal{A}}}\left|v_0\right|_G + \frac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|}{\mu_{\mathcal{A}}\mu_{\mathcal{D}}}\left|w_0\right|_H\\
&\quad + \frac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|\|\mathcal{R}\|}{\mu_{\mathcal{A}}-\mu_{\mathcal{D}}}\int_{s_0}^{+\infty}\left|v(\tau)\right|_G\int_{\tau}^{+\infty}\left(\mathrm{e}^{-\mu_{\mathcal{D}}(t-\tau)}-\mathrm{e}^{-\mu_{\mathcal{A}}(t-\tau)}\right)\mathrm{d}t\,\mathrm{d}\tau,
\end{aligned}$$

that is, in the case $\mu_{\mathcal{A}} \ne \mu_{\mathcal{D}}$ we have

$$\left|v\right|_{L^1(\mathbb{R}_{s_0},G)} \le \tfrac{C_{\mathcal{A}}}{\mu_{\mathcal{A}}}\left|v_0\right|_G + \tfrac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|}{\mu_{\mathcal{A}}\mu_{\mathcal{D}}}\left|w_0\right|_H + \tfrac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|\|\mathcal{R}\|}{\mu_{\mathcal{A}}\mu_{\mathcal{D}}}\left|v\right|_{L^1(\mathbb{R}_{s_0},G)}. \qquad (27)$$

It remains to prove that the last inequality also holds in the case $\mu_{\mathcal{A}} = \mu_{\mathcal{D}}$. Thus, let $\mu_{\mathcal{A}} = \mu_{\mathcal{D}}$ and notice that we have $\left|U_{(t,s_0)}^{-\mathcal{D}}w_0\right|_H \le C_{\mathcal{D}}\mathrm{e}^{-(\mu_{\mathcal{D}}-\rho)(t-s_0)}\left|w_0\right|_H$ for any $\rho \in (0,\mu_{\mathcal{D}})$, $t \ge s_0 \ge 0$. We know, from (27), that

$$\begin{aligned}
\left|v\right|_{L^1(\mathbb{R}_{s_0},G)} &= \lim_{\rho\to 0}\left|v\right|_{L^1(\mathbb{R}_{s_0},G)}\\
&\le \lim_{\rho\to 0}\left(\tfrac{C_{\mathcal{A}}}{\mu_{\mathcal{A}}}\left|v_0\right|_G + \tfrac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|}{\mu_{\mathcal{A}}(\mu_{\mathcal{D}}-\rho)}\left|w_0\right|_H + \tfrac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|\|\mathcal{R}\|}{\mu_{\mathcal{A}}(\mu_{\mathcal{D}}-\rho)}\left|v\right|_{L^1(\mathbb{R}_{s_0},G)}\right),
\end{aligned}$$

from which we conclude that (27) also holds in the case $\mu_{\mathcal{A}} = \mu_{\mathcal{D}}$. □

Corollary 1. *If inequality (19) holds true, then the weak solution of system (16) satisfies*

$$\left|(v,w)\right|_{L^\infty(\mathbb{R}_{s_0},G\times H)} \le D_{\mathrm{c}}\left|(v_0,w_0)\right|_{G\times H} \qquad (28)$$

for a suitable constant $D_{\mathrm{c}} \ge 1$, independent of (v_0, w_0).

Proof. From (21) and (19) we obtain

$$(1-\xi)\,|v|_{L^1(\mathbb{R}_{s_0},\,G)} < \frac{C_{\mathcal{A}}}{\mu_{\mathcal{A}}}\,|v_0|_G + \frac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|}{\mu_{\mathcal{A}}\mu_{\mathcal{D}}}\,|w_0|_H\,.$$

Then, from (22), we arrive at

$$\begin{aligned}|w|_{L^\infty(\mathbb{R}_{s_0},\,H)} &\le C_{\mathcal{D}}\,|w_0|_H + C_{\mathcal{D}}\|\mathcal{R}\|\,|v|_{L^1(\mathbb{R}_{s_0},\,G)}\\ &\le C_{\mathcal{D}}\,|w_0|_H + (1-\xi)^{-1}C_{\mathcal{D}}\|\mathcal{R}\|\left(\frac{C_{\mathcal{A}}}{\mu_{\mathcal{A}}}\,|v_0|_G + \frac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|}{\mu_{\mathcal{A}}\mu_{\mathcal{D}}}\,|w_0|_H\right)\end{aligned}$$

Finally, from (23), we derive

$$|v|_{L^\infty(\mathbb{R}_{s_0},\,G)} \le C_{\mathcal{A}}\,|v_0|_G + \frac{C_{\mathcal{A}}\|\mathcal{S}\|\,|w|_{L^\infty(\mathbb{R}_{s_0},\,H)}}{\mu_{\mathcal{A}}},$$

which ends the proof. □

Proof of Theorem 2.1. If (v,w) solves (16), then $(\overline{v}(t),\overline{w}(t)) \coloneqq \mathrm{e}^{\varepsilon(t-s_0)}(v(t),w(t))$ solves

$$\dot{\overline{v}} = -(\mathcal{A}-\varepsilon)\overline{v} - \mathcal{S}\overline{w}, \qquad \overline{v}(s_0) = v_0, \tag{29a}$$

$$\dot{\overline{w}} = -(\mathcal{D}-\varepsilon)\overline{w} - \mathcal{R}\overline{v}, \qquad \overline{w}(s_0) = w_0. \tag{29b}$$

We also have that z solves $\dot{z} = -\mathcal{A}z, \quad z(s_0) = z_0$ if, and only if, $\overline{z}(t) \coloneqq \mathrm{e}^{\varepsilon(t-s_0)}z(t)$ satisfies

$$\dot{\overline{z}} = -(\mathcal{A}-\varepsilon)\overline{z}, \quad \overline{z}(s_0) = z_0, \quad \text{and} \quad |\overline{z}|_G \le C_{\mathcal{A}}\mathrm{e}^{-(\mu_{\mathcal{A}}-\varepsilon)(t-s_0)}\,|z_0|_G\,.$$

Similarly u solves $\dot{u} = -\mathcal{D}u, \quad u(s_0) = u_0$ if, and only if, $\overline{u}(t) \coloneqq \mathrm{e}^{\varepsilon(t-s_0)}u(t)$ satisfies

$$\dot{\overline{u}} = -(\mathcal{D}-\varepsilon)\overline{u}, \quad \overline{u}(s_0) = u_0, \quad \text{and} \quad |\overline{u}|_H \le C_{\mathcal{D}}\mathrm{e}^{-(\mu_{\mathcal{D}}-\varepsilon)(t-s_0)}\,|u_0|_H\,.$$

Now if (19) holds true, then it also holds

$$\xi_\varepsilon \coloneqq \frac{C_{\mathcal{A}}C_{\mathcal{D}}\|\mathcal{S}\|\|\mathcal{R}\|}{(\mu_{\mathcal{A}}-\varepsilon)(\mu_{\mathcal{D}}-\varepsilon)} < 1$$

for a small enough $\varepsilon \in (0,\,\min\{\mu_{\mathcal{A}},\mu_{\mathcal{D}}\})$. From Corollary 1 it follows that

$$|(\overline{w},\overline{v})|_{L^\infty(\mathbb{R}_{s_0},\,G\times H)} \le D_{\mathrm{c}}\,|(v_0,w_0)|_{G\times H} \tag{30}$$

for a suitable constant $D_{\mathrm{c}} \ge 1$ independent of (v_0,w_0), from which we can conclude (20). □

Let us now consider a perturbation of system (16) as follows, where $(\eta^\flat,\eta^\sharp) \in L^2(\mathbb{R}_0, G\times H)$,

$$\dot{v} = -\mathcal{A}v - \mathcal{S}w + \eta^\flat, \qquad v(s_0) = v_0, \tag{31a}$$

$$\dot{w} = -\mathcal{D}w - \mathcal{R}v + \eta^\sharp, \qquad w(s_0) = w_0. \tag{31b}$$

Corollary 2. *Under the assumptions of Theorem 2.1 the weak solution of system (31) satisfies*

$$|(v(t),w(t))|_{G\times H} \le D_{\mathrm{c}}\mathrm{e}^{-\varepsilon(t-s_0)}\left(|(v_0,w_0)|_{G\times H} + (2\varepsilon)^{-\frac{1}{2}}\left|(\eta^\flat,\eta^\sharp)\right|_{L^2(\mathbb{R}_0,G\times H)}\right), \tag{32}$$

independently of (s_0,v_0,w_0).

Proof. With $\mathcal{M}$ as in (17), by Duhamel formula we find that

$$|(v(t), w(t))|_{G\times H} \le \left|U^{\mathcal{M}}_{(t,s_0)}(v_0, w_0)\right|_{G\times H} + \left|\int_{s_0}^{t} U^{\mathcal{M}}_{(t,s)}(\eta^\flat, \eta^\sharp)(s)\,\mathrm{d}s\right|_{G\times H}$$

and, by Theorem 2.1,

$$\begin{aligned}
&|(v(t), w(t))|_{G\times H}\\
&\le D_{\mathrm{c}}\mathrm{e}^{-\varepsilon(t-s_0)}\,|(v_0, w_0)|_{G\times H} + D_{\mathrm{c}}\int_{s_0}^{t} \mathrm{e}^{-\varepsilon(t-s)}\left|(\eta^\flat, \eta^\sharp)(s)\right|_{G\times H}\,\mathrm{d}s\\
&\le D_{\mathrm{c}}\mathrm{e}^{-\varepsilon(t-s_0)}\,|(v_0, w_0)|_{G\times H} + D_{\mathrm{c}}\left(\frac{\mathrm{e}^{-2\varepsilon(t-s_0)}}{2\varepsilon}\right)^{\frac12}\left(\int_{s_0}^{t}\left|(\eta^\flat, \eta^\sharp)(s)\right|^2_{G\times H}\,\mathrm{d}s\right)^{\frac12},
\end{aligned}$$

from which we obtain (32). □

3. Stability of the coupled parabolic-ODE closed-loop system.

Here we show that the explicit oblique projections based feedback control proposed in [11] for stabilization of nonautonomous linear parabolic equations is also able to stabilize a general class of nonautonomous linear coupled parabolic-ODE systems, where the control acts (only) in the parabolic component. We will prove that the coupled system

$$\dot y(t) + Ay(t) + A_{\mathrm{r}}(t)y(t) + \widetilde{\mathcal{S}}(t)w(t) - \mathcal{K}_{U_M}(t)(y(t), w(t)) = 0, \quad y(s_0) = y_0, \tag{33a}$$

$$\dot w(t) + \mathcal{D}(t)w(t) + \mathcal{R}(t)y(t) = 0, \quad w(s_0) = w_0, \tag{33b}$$

with the explicit feedback

$$(y, w) \to \mathcal{K}_{U_M}(y, w) := P^{E_M^\perp}_{U_M}\left(Ay + A_{\mathrm{r}}y - \lambda y + \widetilde{\mathcal{S}}w\right), \tag{33c}$$

is stable, under suitable assumptions on the linear span of the actuators $U_M = \mathrm{span}\{1_{\omega_1}, 1_{\omega_2}, \dots, 1_{\omega_M}\}$ and on the operators in (33).

3.1. Assumptions.

We look at system (33) as an evolutionary system in $H \times H$, where H is our pivot separable Hilbert space, $H' = H$.

First we present our assumptions on the operators in (33). They will guarantee that the assumptions in the abstract setting of Section 2 are fulfilled. Most of the assumptions are standard, and concern the existence and uniqueness of weak solutions for auxiliary systems. The only nonstandard assumption is the main stabilizability condition given below in Assumption 10. It can be seen as a generalization of the condition presented in [11] for the case of parabolic equations.

Let V be another Hilbert space with $V \subset H$.

Assumption 4. $A \in \mathcal{L}(V \to V')$ is symmetric and $(y, z) \mapsto \langle Ay, z\rangle_{V',V}$ is a complete scalar product in V.

From now we will suppose that V is endowed with the scalar product $(y, z)_V := \langle Ay, z\rangle_{V',V}$, which still makes V a Hilbert space. Necessarily, $A\colon V \to V'$ is an isometry.

Assumption 5. The inclusion $V \subseteq H$ is dense, continuous, and compact.

Necessarily, we have that

$$\langle y,\, z\rangle_{V',\,V} = (y,\, z)_H, \quad \text{for all } (y,\, z) \in H \times V,$$

and also that the operator A is densely defined in H, with domain $\mathrm{D}(A)$ satisfying

$$\mathrm{D}(A) \xhookrightarrow{\mathrm{d,\,c}} V \xhookrightarrow{\mathrm{d,\,c}} H \xhookrightarrow{\mathrm{d,\,c}} V' \xhookrightarrow{\mathrm{d,\,c}} \mathrm{D}(A)'.$$

Further, A has a compact inverse $A^{-1}\colon H \to \mathrm{D}(A)$, and we can find a nondecreasing system of (repeated) eigenvalues $(\alpha_n)_{n\in\mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_n)_{n\in\mathbb{N}_0}$:

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \alpha_{n+1} \to +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n.$$

We can define, for every $\beta \in \mathbb{R}$, the fractional powers A^β, of A, by

$$A^\beta \sum_{n=1}^{+\infty} y_n e_n \coloneqq \sum_{n=1}^{+\infty} \alpha_n^\beta y_n e_n,$$

and the corresponding domains $\mathrm{D}(A^{|\beta|}) \coloneqq \{y \in H \mid A^{|\beta|} y \in H\}$, and $\mathrm{D}(A^{-|\beta|}) \coloneqq \mathrm{D}(A^{|\beta|})'$. We have that $\mathrm{D}(A^\beta) \xhookrightarrow{\mathrm{d,\,c}} \mathrm{D}(A^{\beta_1})$, for all $\beta > \beta_1$, and we can see that $\mathrm{D}(A^0) = H$, $\mathrm{D}(A^1) = \mathrm{D}(A)$, $\mathrm{D}(A^{\frac{1}{2}}) = V$.

For the time-dependent operators we assume the following:

Assumption 6. For almost every $t > 0$ we have $A_{\mathrm{r}}(t) \in \mathcal{L}(H, V')$, and we have a uniform bound, that is, $A_{\mathrm{r}} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H, V'))$.

Assumption 7. We have $\mathcal{W} \xhookrightarrow{d} H \xhookrightarrow{d} \mathcal{W}'$, and $\mathcal{D} \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$ as in (15). For any $s_0 \ge 0$ and $w_0 \in H$ there is one, and only one, $(H, \mathcal{W})$-weak solution for

$$\dot{w}(t) + \mathcal{D}(t)w(t) = 0, \quad w(s_0) = w_0,$$

and there are constants $C_{\mathcal{D}} \ge 1$ and $\mu_{\mathcal{D}} > 0$, independent of (s_0, w_0), such that the solution w satisfies $|w(t)|_H \le C_{\mathcal{D}} \mathrm{e}^{-\mu_{\mathcal{D}}(t-s)} |w(s)|_H$, for all $t \ge s \ge s_0 \ge 0$.

Assumption 8. The coupling operators $\widetilde{\mathcal{S}}$ and $\mathcal{R}$ are both in $L^\infty(\mathbb{R}_0, \mathcal{L}(H))$.

Assumption 9. There exists one, and only one, solution $(v, w) \in W_{\mathrm{loc}}(\mathbb{R}_{s_0}, V \times \mathcal{W}, V' \times \mathcal{W}')$ for the system

$$\dot{v} + P_{E_M^\perp}^{U_M} Av + P_{E_M^\perp}^{U_M} A_{\mathrm{r}} v + P_{E_M^\perp}^{U_M} \widetilde{\mathcal{S}} w = 0, \qquad v(s_0) = v_0 \in E_M^\perp, \tag{34a}$$

$$\dot{w} + \mathcal{D}w + \mathcal{R}v = 0, \qquad w(s_0) = w_0 \in H, \tag{34b}$$

Note that in case $\mathcal{K}_{U_M}(t) = 0$, we have that (33) is in the form of system (16), with $G = H$ and $\mathcal{V} = V$. Note that (33) is in the form of system (16), with $G = E_M^\perp = P_{E_M^\perp} H$ and $\mathcal{V} = E_M^\perp \bigcap V$.

Let us denote (cf. Lemma 2.2)

$$\|P_{E_M^\perp}^{U_M} \widetilde{\mathcal{S}}\| \coloneqq \left| P_{E_M^\perp}^{U_M} \widetilde{\mathcal{S}} \right|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H, E_M^\perp))} \qquad \text{and} \qquad \|\mathcal{R}\| \coloneqq |\mathcal{R}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(E_M^\perp, H))}$$

Assumption 10. The linear span U_M of the actuators satisfies

$$H = U_M \oplus E_M^\perp, \tag{35a}$$

$$\alpha_{M+1} > \inf_{\substack{\gamma\in\mathbb{R}_0^2,\\ (2-\gamma_1-\gamma_2)>0}} \left(\frac{\gamma_1^{-1}\Xi_1 + \gamma_2^{-1}\left(2+2\left|P_{U_M}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right)\Xi_2}{(2-\gamma_1-\gamma_2)} + \frac{C_{\mathcal{D}}\|P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}\|\|\mathcal{R}\|}{(2-\gamma_1-\gamma_2)\mu_{\mathcal{D}}} \right), \tag{35b}$$

where

$$\Xi_1 \coloneqq \sup_{(t,Y)\in\mathbb{R}_0\times(E_M^\perp\cap V)} \frac{|\langle A_{\rm r}(t)Y,Y\rangle_{V',V}|_{\mathbb{R}}^2}{|Y|_H^2\,|Y|_V^2} \le \left|P_{E_M^\perp}A_{\rm r}P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2, \tag{36a}$$

$$\Xi_2 \coloneqq \left|P_{E_M}A_{\rm r}P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2. \tag{36b}$$

Remark 3. Note that (7b) is a particular case of (35b), with $\gamma_1=\gamma_2=\frac{1}{2}$.

Remark 4. Recall that from [11] (by taking $y_0\in E_M^\perp$ in [11, Theorem 3.6]) we know that when $\widetilde{\mathcal{S}}=0$, then the system $\dot v + P_{E_M^\perp}^{U_M}Av + P_{E_M^\perp}^{U_M}A_{\rm r}v = 0$ is stable provided (35) holds true, with $\mathcal{R}=0$. In particular we observe that Assumptions 1 and 2, and estimates (18) hold true for system (16) with $\mathcal{A} = A + P_{E_M^\perp}^{U_M}A_{\rm r}$ and $\mathcal{S} = P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}$, that is, they hold true for system (34).

3.2. **The stability result.** The main result of this section, which will lead to main Theorems 1.1 and 1.2, is the following.

Theorem 3.1. *Under Assumptions 4–10, the coupled system (33) is stable: there are constants $C_{\rm c}\ge 1$ and $\mu_{\rm c}>0$ such that*

$$|(y(t),w(t))|_{H\times H} \le C_{\rm c}{\rm e}^{-\mu_{\rm c}(t-s_0)}\,|(y_0,w_0)|_{H\times H}\,, \qquad t\ge s_0.$$

with $(C_{\rm c},\mu_{\rm c})$ independent of (s_0,y_0,w_0).

The proof is presented below, in section 3.2.2.

3.2.1. *Auxiliary results.* Here we derive some auxiliary results we will use in the proof of Theorem 3.1. We suppose that the Assumptions 4–10 do hold true. Recall also the interval I in (12).

Notice that $G=E_M^\perp$ is a closed subspace of H (with G endowed with the norm inherited from H). Let us set $\mathcal{V}=G\cap V$, which we endow with the norm inherited from V. Observe that $\mathcal{V}$ is a closed subspace of V, and A maps $\mathcal{V}$ onto $\mathcal{V}'$.

Lemma 3.2. *There exists a unique $(E_M^\perp, V\cap E_M^\perp)$-weak solution for the system*

$$\dot v + P_{E_M^\perp}^{U_M}Av + P_{E_M^\perp}^{U_M}A_{\rm r}v = 0, \qquad v(s_0)=v_0\in E_M^\perp. \tag{37}$$

Moreover, we have

$$|v(t)|_H^2 \le {\rm e}^{-\mu(t-s_0)}\,|v_0|_H^2\,, \qquad \textit{for a suitable} \qquad \mu > \frac{C_{\mathcal{D}}\|P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}\|\|\mathcal{R}\|}{\mu_{\mathcal{D}}}. \tag{38}$$

Proof. The existence and uniqueness is proven in [11, Section 3.1]. Note that we have $P_{E_M^\perp}^{U_M} A v = A P_{E_M^\perp}^{U_M} v = Av$ for any $v \in E_M^\perp$. Now, proceeding as in [11, Section 3.1], by multiplying the equation by $2v$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|v\right|_H^2 &= -2\left|v\right|_V^2 - 2\langle P_{E_M^\perp}^{U_M} A_{\mathrm{r}} v, v\rangle_{V',V}\\ &= -2\left|v\right|_V^2 - 2\langle P_{E_M^\perp} A_{\mathrm{r}} v, v\rangle_{V',V} - 2\langle P_{E_M^\perp}^{U_M} P_{E_M} A_{\mathrm{r}} v, v\rangle_{V',V}\end{aligned}$$

and for any given positive constants γ_1 and γ_2,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|v\right|_H^2 \le -(2-\gamma_1-\gamma_2)\left|v\right|_V^2 + \gamma_1^{-1}\Xi_1\left|v\right|_H^2 + \gamma_2^{-1}\left|P_{E_M^\perp}^{U_M} P_{E_M}\right|_{\mathcal{L}(V')}^2 \Xi_2\left|v\right|_H^2$$

with Ξ_1 and Ξ_2 as in (36). Now from $\left|v\right|_V^2 \ge \alpha_{M+1}\left|v\right|_H^2$, because the solution v takes its values in $E_M^\perp$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|v\right|_H^2 \le -\left((2-\gamma_1-\gamma_2)\alpha_{M+1} - \gamma_1^{-1}\Xi_1 - \gamma_2^{-1}\left|P_{E_M^\perp}^{U_M} P_{E_M}\right|_{\mathcal{L}(V')}^2 \Xi_2\right)\left|v\right|_H^2, \quad (39)$$

and from (35b), it follows that we can choose γ_1 and γ_2 such that

$$\mu := (2-\gamma_1-\gamma_2)\alpha_{M+1} - \gamma_1^{-1}\Xi_1 - \gamma_2^{-1}\left|P_{E_M^\perp}^{U_M} P_{E_M}\right|_{\mathcal{L}(V')}^2 \Xi_2 \;>\; \frac{C_{\mathcal{D}}\|P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}\|\|\mathcal{R}\|}{\mu_{\mathcal{D}}},$$

which implies (38). □

Corollary 3. *Given $q \in H^1(\mathbb{R}_0, E_M)$ and $(v_0, w_0) \in E_M^\perp \times H$, there exists a unique solution for system*

$$\dot v + P_{E_M^\perp}^{U_M} A(v+q) + P_{E_M^\perp}^{U_M} A_{\mathrm{r}}(v+q) + P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}} w + P_{E_M^\perp}^{U_M}\dot q = 0, \quad v(s_0) = v_0, \quad \text{(40a)}$$

$$\dot w + \mathcal{D}w + \mathcal{R}(v+q) = 0, \quad w(s_0) = w_0. \quad \text{(40b)}$$

This solution satisfies

$$\begin{aligned}&\left|(v(t), w(t))\right|_{H\times H}\\ &\le D_{\mathrm{c}}\mathrm{e}^{-\varepsilon(t-s_0)}\left(\left|(v_0,w_0)\right|_{H\times H} + (2\varepsilon)^{-\frac12}\|\mathcal{T}\|\left|(3^{\frac12}q,\dot q)\right|_{L^2(\mathbb{R}_0, H\times H)}\right),\end{aligned} \quad (41)$$

with $\|\mathcal{T}\| = \max\{|\mathcal{T}P_{E_M}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \mid \mathcal{T} \in \{P_{E_M^\perp}^{U_M} A, P_{E_M^\perp}^{U_M} A_{\mathrm{r}}, P_{E_M^\perp}^{U_M}, \mathcal{R}\}\}$, *and with suitable constants* $D_{\mathrm{c}} \ge 1$ *and* $\varepsilon > 0$ *independent of* (s_0, v_0, w_0). *Furthermore,* $\varepsilon \in (0, \min\{\mu, \mu_{\mathcal{D}}\})$, *where* μ *is as in* (38).

Proof. When $q = 0$, the existence and uniqueness of the solution, in the space $W_{\mathrm{loc}}(\mathbb{R}_{s_0}, (E_M^\perp \bigcap V) \times \mathcal{W}, (E_M^\perp \bigcap V') \times \mathcal{W}')$, is given by Assumption 9. Setting $\mathcal{A} = P_{E_M^\perp}^{U_M} A + P_{E_M^\perp}^{U_M} A_{\mathrm{r}}$, from Lemma 3.2 we have that the solution of

$$\dot v + \mathcal{A}v = 0, \qquad v(s_0) = v_0 \in E_M^\perp$$

satisfies $|v(t)|_{E_M^\perp} \le C_{\mathcal{A}}\mathrm{e}^{-\mu_{\mathcal{A}}(t-s_0)}|v_0|_{E_M^\perp}$ with $(C_{\mathcal{A}}, \mu_{\mathcal{A}}) = (1, \mu)$, where μ is as in (38). Then, we observe that

$$\xi = \frac{C_{\mathcal{A}} C_{\mathcal{D}}\|P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}\|\|\mathcal{R}\|}{\mu_{\mathcal{A}}\mu_{\mathcal{D}}} < 1.$$

Observe also that system (40) is in the form of system (31), by setting $\eta^\sharp = \mathcal{R}q$ and $\eta^\flat = P_{E_M^\perp}^{U_M} Aq + P_{E_M^\perp}^{U_M} A_{\rm r} q + P_{E_M^\perp}^{U_M} \dot q$. Then, from Corollary 2, it follows

$$\begin{aligned}&|(v(t), w(t))|_{E_M^\perp\times H}\\ &\le D_{\rm c}{\rm e}^{-\varepsilon(t-s_0)}\left(|(v_0,w_0)|_{E_M^\perp\times H} + (2\varepsilon)^{-\frac12}\left|(\eta^\flat,\eta^\sharp)\right|_{L^2(\mathbb{R}_0,E_M^\perp\times H)}\right),\end{aligned}$$

which implies (41). □

3.2.2. *Proof of Theorem 3.1.* Let us set $q_0 \coloneqq P_{E_M} y_0$, $v_0 \coloneqq P_{E_M^\perp} y_0$, and $q(t) \coloneqq {\rm e}^{-\lambda(t-s_0)}q_0$. Then we also set the solution (v,w) for system (40), given by Corollary 3, with the initial condition $(v,w)(s_0) = (v_0,w_0)$. We start by showing that $(y,w) \coloneqq (q+v,w)$ solves (33). Indeed, with $y = q+v$, we can rewrite (40a) as

$$0 = \dot y - \dot q + P_{E_M^\perp}^{U_M} Ay + P_{E_M^\perp}^{U_M} A_{\rm r} y + P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}w + P_{E_M^\perp}^{U_M}\dot q$$

that is,

$$\begin{aligned}0 &= \dot y + Ay + A_{\rm r}y + \widetilde{\mathcal{S}}w - P_{U_M}^{E_M^\perp}Ay - P_{U_M}^{E_M^\perp}A_{\rm r}y - P_{U_M}^{E_M^\perp}\widetilde{\mathcal{S}}w - P_{U_M}^{E_M^\perp}\dot q\\ &= \dot y + Ay + A_{\rm r}y + \widetilde{\mathcal{S}}w - P_{U_M}^{E_M^\perp}\left(Ay + A_{\rm r}y + \widetilde{\mathcal{S}}w - \lambda y\right),\end{aligned}$$

because $\dot q = -\lambda q$ and $P_{U_M}^{E_M^\perp} q = P_{U_M}^{E_M^\perp} P_{E_M} y = P_{U_M}^{E_M^\perp} y$. Therefore, we conclude that (y,w) solves (33), with $\mathcal{K}$ as in (33c). Finally, we notice that $(v(t),w(t))$ is orthogonal to $(q(t),0)$ in $H\times H$, because $v(t)\in E_M^\perp$ is orthogonal to $q(t)\in E_M$ in H. Therefore $|(y(t),w(t))|_{H\times H} = |(v(t),w(t))|_{H\times H} + |(q(t),0)|_{H\times H}$ and, using (41) in Corollary 3, we arrive at

$$\begin{aligned}&|(y(t),w(t))|_{H\times H}\\ &\le D_{\rm c}{\rm e}^{-\varepsilon(t-s_0)}\left(|(v_0,w_0)|_{H\times H} + (2\varepsilon)^{-\frac12}\|\mathcal{T}\|\left|(3^{\frac12}q,\lambda q)\right|_{L^2(\mathbb{R}_0,H^2)}\right) + {\rm e}^{-\lambda(t-s_0)}|q_0|_H\\ &\le D_{\rm c}{\rm e}^{-\varepsilon(t-s_0)}|(v_0,w_0)|_{H\times H} + \left(D_{\rm c}{\rm e}^{-\varepsilon(t-s_0)}(\tfrac{3+\lambda^2}{4\lambda\varepsilon})^{\frac12}\|\mathcal{T}\| + {\rm e}^{-\lambda(t-s_0)}\right)|q_0|_H\\ &\le \overline{D}_{\rm c}{\rm e}^{-\mu_{\rm c}(t-s_0)}|(y_0,w_0)|_{H\times H}\end{aligned}$$

with $\mu_{\rm c} = \min\{\varepsilon,\lambda\}$ and $\overline{D}_{\rm c} = \max\{D_{\rm c}, 1 + D_{\rm c}(\frac{3+\lambda^2}{4\lambda\varepsilon})^{\frac12}\|\mathcal{T}\|\}$. The proof of Theorem 3.1 is finished. □

4. Applications. Stabilization of the monodomain equations.

Assume that the desired trajectory $(\overline{v},\overline{w})$ is given as the solution of the uncontrolled system (1). As mentioned in the Introduction, the difference $(y,z)\coloneqq(v,w)-(\overline{v},\overline{w})$ satisfies system (10), with the operators defined as in (6). In this section we prove Theorems 1.1 and 1.2 which concern the stabilization of systems (9) and (10), respectively.

4.1. Proof of main Theorem 1.1.

Theorem 1.1 will follow from Theorem 3.1. For this purpose we discuss Assumptions 4–10, for the choice

$$H = L^2(\Omega),\qquad V = H^1(\Omega),\qquad\text{and}\qquad {\rm D}(A) = \left\{h\in H^2(\Omega)\mid (\tfrac{\partial}{\partial\mathbf{n}}h)|_\Gamma = 0\right\}.$$

Assumptions 4 and 5 are satisfied and due to (8) Assumptions 6 and 8 hold true. It is easy to see that Assumption 7 holds with $\mathcal{W} = H$ and $(C_{\mathcal{D}},\mu_{\mathcal{D}}) = (1,\delta)$.

Proposition 1. *If* (8) *and Assumption 10 are satisfied, then Assumption 9 holds.*

The proof follows from standard arguments. A few details are given in the Appendix. Thus, under condition (8) and Assumption 10, all the assumptions in Section 3.1 are satisfied. By Theorem 3.1 this implies the following Corollary, which in turn implies Theorem 1.1. From Remark 3 we recall that condition (7), which is assumed in Theorem 1.1, implies that Assumption 10 is satisfied.

Corollary 4. *Let $\lambda > 0$ and $s_0 \geq 0$. If (8) and Assumption 10 hold true, then there are constants $C \geq 1$ and $\mu > 0$, such that the solution of the linear system (9) satisfies*

$$|(y(t), z(t))|_{H\times H} \leq C\mathrm{e}^{-\mu(t-s_0)} \,|(y_0, z_0)|_{H\times H}\,, \qquad t \geq s_0.$$

for all $(y_0, z_0) \in H \times H$. Here the constants C and μ are independent of the triple (s_0, y_0, z_0).

4.2. **Proof of main Theorem 1.2.** To deal with the nonlinear systems, we will need strong solutions. We can prove that such solutions exist due to the fact that the reaction operator A_{r} defined as in (6) satisfies

$$A_{\mathrm{r}} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V, H)), \tag{42}$$

if (8) holds.

Proposition 2. *Let (8) and Assumption 10 be satisfied, let μ be as in Corollary 4, and $(y_0, z_0) \in V \times H$, with $s_0 \geq 0$. Then the solution for (9) is strong, that is, $(y, z) \in W_{\mathrm{loc}}(\mathbb{R}_{s_0}, \mathrm{D}(A) \times H, H \times H)$. Moreover, we have*

$$\sup_{s\geq s_0} \left|\mathrm{e}^{\mu(\cdot - s_0)}(y, z)\right|_{W((s,s+1),\mathrm{D}(A)\times H,H\times H)} \leq C\,|(y_0, z_0)|_{V\times H}\,.$$

The proof, which relies in part on known arguments, is sketched in the Appendix.

The next lemma gathers estimates on our nonlinearity $\mathcal{N} = \mathcal{N}(y) = \mathcal{N}(t, y)$. Let us write, for simplicity,

$$\mathbf{H} = H \times H, \qquad \mathbf{V} = V \times H, \qquad \text{and} \qquad \mathbf{D}(A) = \mathrm{D}(A) \times H.$$

Lemma 4.1. *There exists a constant $\widehat{C}_1 \geq 0$ such that for all pairs $\mathbf{p} = (y, z)$ and $\tilde{\mathbf{p}} = (\tilde{y}, \tilde{z})$ in $\mathrm{D}(A) \times H$,*

$$\begin{aligned}|\mathcal{N}(\mathbf{p}) - \mathcal{N}(\tilde{\mathbf{p}})|^2_{\mathbf{H}} \leq{}& \widehat{C}_1|\mathbf{p} - \tilde{\mathbf{p}}|^2_{\mathbf{V}}\,(1 + |\mathbf{p}|^{\varepsilon_1}_{\mathbf{V}} + |\tilde{\mathbf{p}}|^{\varepsilon_2}_{\mathbf{V}})\left(|\mathbf{p}|^2_{\mathbf{D}(A)} + |\tilde{\mathbf{p}}|^2_{\mathbf{D}(A)}\right)\\ &+ \widehat{C}_1|\mathbf{p} - \tilde{\mathbf{p}}|^2_{\mathbf{D}(A)}\,(|\mathbf{p}|^{\varepsilon_3}_{\mathbf{V}} + |\tilde{\mathbf{p}}|^{\varepsilon_4}_{\mathbf{V}})\,.\end{aligned} \tag{43a}$$

Further for any given $\varsigma > 0$ there exists $\widehat{C}_2 \geq 0$ such that

$$\begin{aligned}(\mathcal{N}(\mathbf{p}) - \mathcal{N}(\tilde{\mathbf{p}}), \mathbf{p} - \tilde{\mathbf{p}})_{\mathbf{H}} \leq{}& \varsigma\,|\mathbf{p} - \tilde{\mathbf{p}}|^2_{\mathbf{V}} + \widehat{C}_2(1 + |\mathbf{p}|^{\varepsilon_5}_{\mathbf{V}}\\ &+ |\tilde{\mathbf{p}}|^{\varepsilon_6}_{\mathbf{V}})(1 + |\mathbf{p}|^2_{\mathbf{D}(A)} + |\tilde{\mathbf{p}}|^2_{\mathbf{D}(A)})\,|\mathbf{p} - \tilde{\mathbf{p}}|^2_{\mathbf{H}}\,,\end{aligned} \tag{43b}$$

with

$$\{\varepsilon_1, \varepsilon_2\} \in [0, +\infty) \quad \text{and} \quad \{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} \in [2, +\infty). \tag{43c}$$

Proof. Recall that from (6), we have $\mathcal{N}(y, z) = (-ay^3 + (b - 3a\overline{v})y^2 - eyz, -\rho y^2)$, which we write as

$$\mathcal{N} = \sum_{j=1}^{4} \mathcal{G}_j, \quad \text{with} \quad \begin{cases} \mathcal{G}_1(y, z) \coloneqq (-ay^3, 0), & \mathcal{G}_2(y, z) \coloneqq ((b - 3a\overline{v})y^2, 0),\\ \mathcal{G}_3(y, z) \coloneqq (0, -\rho y^2), & \mathcal{G}_4(y, z) \coloneqq (eyz, 0).\end{cases}$$

We will argue that (43) holds with $\mathcal{N}$ replaced by $\mathcal{G}_j$ for $j = 1, \ldots, 4$. This implies that (43) holds for $\mathcal{N} = \sum_{j=1}^{4} \mathcal{G}_j$. From [16, Section 5, Examples 1 and 2], we can conclude that (43) is satisfied with $\mathcal{N}$ replaced by $\mathcal{G}_1$ and $\mathcal{G}_2$ involving only the PDE component y of the solution and the PDE coordinate of the nonlinearity. Recall that the component y of the solution lives in $\mathrm{D}(A)$, for (almost) every time $t\ge 0$. This regularity is not necessarily satisfied by the ODE component z, so we must check the details with the monomial term yz in $\mathcal{G}_4$. On the other hand $\mathcal{G}_3$ involves the ODE component of the nonlinearity, so also in this case we will check the details.

Concerning $\mathcal{G}_4$ we proceed as follows: estimate (43) is trivially satisfied for $e=0$, thus we consider only the case $e\neq 0$, then with $(\zeta_y,\zeta_z)\coloneqq \mathbf{p}-\tilde{\mathbf{p}}=(y-\tilde y, z-\tilde z)\in \mathrm{D}(A)\times H$, we find

$$\begin{aligned}
\tfrac{1}{e^2}|\mathcal{G}_4(\mathbf{p})-\mathcal{G}_4(\tilde{\mathbf{p}})|_{\mathbf{H}}^2 &= |yz-\tilde y\tilde z|_H^2\le 2\left(|\zeta_y z|_H^2+|\tilde y\zeta_z|_H^2\right)\\
&\le 2\left(|\zeta_y|_{L^\infty}^2|z|_H^2+|\zeta_z|_H^2|\tilde y|_{L^\infty}^2\right)\\
\tfrac{1}{e}\left(\mathcal{G}_4(\mathbf{p})-\mathcal{G}_4(\tilde{\mathbf{p}}),\mathbf{p}-\tilde{\mathbf{p}}\right)_{\mathbf{H}} &= (yz-\tilde y\tilde z, y-\tilde y)_H=(\zeta_y z+\tilde y\zeta_z,\zeta_y)_H\\
&\le |\zeta_y|_{L^4}^2|z|_{L^2}+|\zeta_y|_{L^2}|\zeta_z|_{L^2}|\tilde y|_{L^\infty}.
\end{aligned}$$

Next we consider the case $\mathbf{d}=3$, that is, $\Omega\subset\mathbb{R}^3$. An analogous argument can be followed for $\mathbf{d}\in\{1,2\}$.

From suitable Sobolev embeddings, the Agmon inequality, and suitable interpolation inequalities, we obtain

$$\begin{aligned}
\tfrac{1}{e^2}|\mathcal{G}_4(\mathbf{p})-\mathcal{G}_4(\tilde{\mathbf{p}})|_{\mathbf{H}}^2 &\le C_1\left(|\zeta_y|_{\mathrm{D}(A)}^2|z|_H^2+|\zeta_z|_H^2|\tilde y|_{\mathrm{D}(A)}^2\right)\\
&\le C_1\left(|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{D}(A)}^2|\mathbf{p}|_{\mathbf{H}}^2+|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{H}}^2|\tilde{\mathbf{p}}|_{\mathbf{D}(A)}^2\right)\\
&\le C_2\left(|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{D}(A)}^2|\mathbf{p}|_{\mathbf{V}}^2+|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{V}}^2|\tilde{\mathbf{p}}|_{\mathbf{D}(A)}^2\right),\\
\tfrac{1}{e}\left(\mathcal{G}_4(\mathbf{p})-\mathcal{G}_4(\tilde{\mathbf{p}}),\mathbf{p}-\tilde{\mathbf{p}}\right)_{\mathbf{H}} &\le C_1|\zeta_y|_H^{\frac12}|\zeta_y|_V^{\frac32}|z|_H+C_1|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{H}}^2|\tilde y|_{\mathrm{D}(A)}\\
&\le C_3|\zeta_y|_H^2|z|_H^4+\varsigma|\zeta_y|_V^2+\tfrac12 C_1|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{H}}^2\left(1+|\tilde y|_{\mathrm{D}(A)}^2\right)
\end{aligned}$$

for any given $\varsigma>0$, and for suitable constants $C_1>0$, $C_2>0$ and $C_3=C_3(\varsigma)>0$. Therefore we can conclude that the inequalities in (43) also hold true for $\mathcal{G}_4$.

Finally, concerning $\mathcal{G}_3$ we proceed as follows: First

$$|\mathcal{G}_3(\mathbf{p})-\mathcal{G}_3(\tilde{\mathbf{p}})|_{\mathbf{H}}^2\le|y-\tilde y|_H^2\,|y+\tilde y|_{L^\infty}^2\le C_1(|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{H}}^2(|\mathbf{p}|_{\mathbf{D}(A)}^2+|\tilde{\mathbf{p}}|_{\mathbf{D}(A)}^2),$$

for $C_1>0$ independent of $\mathbf{p}$ and $\tilde{\mathbf{p}}$ in $\mathbf{D}(A)\times H$, second we find that

$$\begin{aligned}
\tfrac{1}{\rho}\left(\mathcal{G}_3(\mathbf{p})-\mathcal{G}_3(\tilde{\mathbf{p}}),\mathbf{p}-\tilde{\mathbf{p}}\right)_{\mathbf{H}} &= \left(y^2-\tilde y^2, z-\tilde z\right)_H=(\zeta_y(y+\tilde y),\zeta_z)_H\\
&\le |y+\tilde y|_{L^\infty}|\zeta_y|_{L^2}|\zeta_z|_{L^2}\le C_1|y+\tilde y|_{\mathrm{D}(A)}|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{H}}^2\\
&\le C_1(1+|y|_{\mathrm{D}(A)}^2+|\tilde y|_{\mathrm{D}(A)}^2)|\mathbf{p}-\tilde{\mathbf{p}}|_{\mathbf{H}}^2.
\end{aligned}$$

Thus (43) holds true also for $\mathcal{N}$ replaced by $\mathcal{G}_3$. □

Lemma 4.2. *If (8) holds, then the feedback operator in (33c) is bounded:* $\mathcal{K}_{U_M}\in L^\infty(\mathbb{R}_0,\mathcal{L}(H\times H,H))$.

Proof. Indeed, proceeding as in [11, Proof of Theorem 3.7] we find

$$\begin{aligned}
&|\mathcal{K}_{U_M}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H\times H,H))}\\
&\le\left|P_{U_M}^{E_M^\perp}P_{E_M}\right|_{\mathcal{L}(V',H)}\left(\alpha_M^{\frac12}+|A_{\mathrm{r}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}+\left|\widetilde{\mathcal{S}}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}+\lambda\alpha_1^{-\frac12}\right)
\end{aligned}$$

and from (8) and (6) it follows that $A_{\mathrm{r}} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H, V'))$ and $\widetilde{\mathcal{S}} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H))$. Therefore we have the following Corollary, which in turn implies Theorem 1.2, because (35), assumed in Theorem 1.2, imply Assumption 10. See Remark 3. ☐

Remark 5. We are now prepared to provide the proof for Theorem 1.2, which relies on fixed point arguments. With two modifications, we can rely on the arguments utilized in [4, Section 3.2] (see also [16, Section 3]). First, the feedback operator in these references is obtained as the solution of a suitable differential Riccati equation, in place of $\mathcal{K}_{U_M}$ as in (33c). However, thanks to the estimates in Corollary 4 and Proposition 2 we can replace the Riccati-based feedback operator in [4, Section 3.2] by $\mathcal{K}_{U_M}$. Second the conditions on the nonlinearity $\mathcal{N}$ in [16, Section 3] are slightly stronger than those available here.

In fact, in [16, Section 3], instead of property (43b), the following stronger assumption is assumed:

$$\begin{aligned}&(\mathcal{G}_j(\mathbf{p}) - \mathcal{G}_j(\tilde{\mathbf{p}}), \mathbf{p} - \tilde{\mathbf{p}})_{\mathbf{H}} \\ &\le \widetilde{C}_2(1 + |\mathbf{p}|_{\mathbf{V}}^{\varepsilon_5} + |\tilde{\mathbf{p}}|_{\mathbf{V}}^{\varepsilon_6})^{\frac{1}{2}}(1 + |\mathbf{p}|_{\mathbf{D}(A)}^2 + |\tilde{\mathbf{p}}|_{\mathbf{D}(A)}^2)^{\frac{1}{2}} |\mathbf{p} - \tilde{\mathbf{p}}|_{\mathbf{V}} |\mathbf{p} - \tilde{\mathbf{p}}|_{\mathbf{H}} \\ &\quad + \widetilde{C}_2(1 + |\mathbf{p}|_{\mathbf{V}}^{\varepsilon_5} + |\tilde{\mathbf{p}}|_{\mathbf{V}}^{\varepsilon_6})(1 + |\mathbf{p}|_{\mathbf{D}(A)}^2 + |\tilde{\mathbf{p}}|_{\mathbf{D}(A)}^2) |\mathbf{p} - \tilde{\mathbf{p}}|_{\mathbf{H}}^2 .\end{aligned} \tag{44}$$

It is clear that that the last estimate implies (43b), because $2 |\mathbf{p} - \tilde{\mathbf{p}}|_{\mathbf{V}} |\mathbf{p} - \tilde{\mathbf{p}}|_{\mathbf{H}} \le \beta^{-1} |\mathbf{p} - \tilde{\mathbf{p}}|_{\mathbf{H}} + \beta |\mathbf{p} - \tilde{\mathbf{p}}|_{\mathbf{V}}^2$, for arbitrary $\beta > 0$.

Since Assumption 10 is implied by (7), Theorem 1.2 follows from the following.

Theorem 4.3. *Let (8) and Assumption 10 hold true. Then there are constants $C \ge 1$ and $\epsilon > 0$, independent of (s_0, y_0, z_0), such that the solution of the system (10) satisfies*

$$|(y(t), z(t))|_{\mathbf{V}} \le C\mathrm{e}^{-\mu(t-s_0)} |(y_0, z_0)|_{\mathbf{V}}, \qquad t \ge s_0, \tag{45}$$

provided that $|(y_0, z_0)|_{\mathbf{V}} < \epsilon$, where $\mu > 0$ is as in Corollary 4.

Proof. We follow the main argument sketched in [16, Section 3] and [2, Section 4]. Let us consider the system

$$\dot{y} = -Ay - A_{\mathrm{r}}y - \widetilde{\mathcal{S}}z + \mathcal{K}_{U_M}(y, z) + g_1, \qquad y(s_0) = y_0 \in V, \tag{46a}$$

$$\dot{z} = -\delta z - \mathcal{R}y + g_2, \qquad z(s_0) = z_0 \in H. \tag{46b}$$

with $g \in L^2((s_0, s_0 + T), \mathbf{H})$. Multiplying the dynamics by $\mathfrak{L}z$, with $\mathfrak{L} = \begin{bmatrix} A & 0 \\ 0 & \delta \end{bmatrix}$ and $\mathrm{D}(\mathfrak{L}) = \mathbf{D}(A)$, gives us

$$\begin{aligned}&|(y, z)|_{L^\infty((s_0, s_0+T), \mathbf{V})}^2 + |(y, z)|_{L^2((s_0, s_0+T), \mathbf{D}(A))}^2 \\ &\le D_3(T)\left(|(y, z)(s_0)|_{\mathbf{V}}^2 + |g|_{L^2((s_0, s_0+T), \mathbf{H})}^2\right),\end{aligned} \tag{47}$$

with the constant $D_3(T)$ depending on $|A_r|_{L^\infty(\mathbb{R}_0, \mathcal{L}(V, H))}$, $|\mathcal{K}_{U_M}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H\times H, H))}$, and T, and independent of s_0 and $(y, z)(s_0)$. Here we use Lemma 4.2.

For simplicity we next denote

$$\mathbf{H} = H \times H, \qquad \mathbf{V} = V \times H, \qquad \text{and} \qquad \mathbf{D}(A) = \mathrm{D}(A) \times H,$$

and we write (46) as

$$\dot{\mathbf{v}} + \mathfrak{L}\mathbf{v} + \mathfrak{C}\mathbf{v} - \mathfrak{F}\mathbf{v} = g, \qquad \mathbf{v}(s_0) = \mathbf{v}_0 \in \mathbf{V} \tag{48}$$

with $\mathbf{v}_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$, $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, $\mathfrak{C} = \begin{bmatrix} A_{\rm rc} & \widetilde{\mathcal{S}} \\ 0 & \mathcal{R} \end{bmatrix}$, and $\mathfrak{F} = \begin{bmatrix} \mathfrak{F}_{(1,1)} & \mathfrak{F}_{(1,2)} \\ 0 & 0 \end{bmatrix}$. In particular, notice that $|\mathfrak{F}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(\mathbf{H}))} = |\mathcal{K}_{U_M}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(\mathbf{H},H))}$. Here $\mathfrak{F}_{(1,1)}y := \mathcal{K}_{U_M}(y,0)$ and $\mathfrak{F}_{(1,2)}z := \mathcal{K}_{U_M}(0,z)$.

We will look for a solution of the nonlinear system in a subset $\mathcal{Z}^\mu_\varrho \subset \mathcal{Z}^\mu$ of the Banach space

$$\mathcal{Z}^\mu := \left\{ \mathbf{v} \in L^2_{\rm loc}\left(\mathbb{R}_{s_0}, \mathbf{H}\right) \;\middle|\; |\mathbf{v}|_{\mathcal{Z}^\mu} < \infty \right\}$$

endowed with the norm $|\mathbf{v}|_{\mathcal{Z}^\mu} := \sup_{r\ge s_0} \left| \mathrm{e}^{\mu(\cdot - s_0)}\mathbf{v} \right|_{W((r,r+1),\mathbf{D}(\Delta),\mathbf{H})}$. We also set

$$\mathcal{Z}^\mu_{\rm loc} := \left\{ \mathbf{v} \in L^2_{\rm loc}\left(\mathbb{R}_{s_0}, \mathbf{H}\right)\right) \;\middle|\; \left| \mathrm{e}^{\mu(\cdot - s_0)}\mathbf{v} \right|_{W((r,r+1),\mathbf{D}(\Delta),\mathbf{H})} < \infty, \text{ for all } r \ge s_0 \right\}.$$

For a given constant $\varrho > 0$ we define the subset $\mathcal{Z}^\mu_\varrho$ as follows.

$$\mathcal{Z}^\mu_\varrho := \left\{ \mathbf{v} \in \mathcal{Z}^\mu \mid |\mathbf{v}|^2_{\mathcal{Z}^\mu} \le \varrho |\mathbf{v}_0|^2_{\mathbf{V}} \right\}.$$

Further we define the mapping $\Psi\colon \mathcal{Z}^\mu_\varrho \to \mathcal{Z}^\mu_{\rm loc}$, $\bar{\mathbf{v}} \mapsto \mathbf{v}$, taking a given vector $\bar{\mathbf{v}}$ to the solution $\mathbf{v}$ of

$$\dot{\mathbf{v}} + \mathfrak{L}\mathbf{v} + \mathfrak{C}\mathbf{v} - \mathfrak{F}\mathbf{v} = \mathcal{N}\left(\bar{\mathbf{v}}\right), \qquad \mathbf{v}(s_0) = \mathbf{v}_0. \tag{49}$$

Ⓢ Step [**i**]: *a preliminary estimate.* Proceeding as in [2, Section 4.2] we can conclude that the solution of the system (48) satisfies, for a suitable constant $\overline{C}$,

$$\sup_{r\ge 0} |\mathrm{e}^{\mu(\cdot - s_0)}\mathbf{z}(\cdot)|^2_{W((r,r+1),\mathbf{D}(\Delta),\mathbf{H})} \le \overline{C} \left(|\mathbf{z}_0|^2_{\mathbf{V}} + \sup_{k\in\mathbb{N}} \int_k^{k+1} \mathrm{e}^{4\mu(s-s_0)} |g(s)|^2_{\mathbf{H}} \, \mathrm{d}s \right). \tag{50}$$

Ⓢ Step [**ii**]: *Existence. The fixed point argument.* Proceed as in [16, Section 3], by using (50), with $\mathcal{N}\left(\bar{\mathbf{v}}\right)$ in the place of f, and using (43a), we can conclude that

- Ψ maps $\mathcal{Z}^\mu_\varrho$ into itself, if $|\mathbf{z}_0|_{\mathbf{V}} \le \epsilon_1$, where ϵ is small enough, and
- Ψ is a contraction, if $|\mathbf{z}_0|_{\mathbf{V}} \le \epsilon$, where $\epsilon \le \epsilon_1$ is small enough.

Therefore, we can conclude that if $\mathbf{z}_0 \in \mathbf{V}$ is sufficiently small, $|\mathbf{z}_0|^2_V < \epsilon$, then there exists a unique fixed point $\mathbf{z} = \Psi(\bar{\mathbf{z}}) = \bar{\mathbf{z}} \in \mathcal{Z}^\epsilon_\varrho$ for Ψ. That is,

$$\dot{\mathbf{v}} + \mathfrak{L}\mathbf{v} + \mathfrak{C}\mathbf{v} - \mathfrak{F}\mathbf{v} = \mathcal{N}\left(\mathbf{v}\right), \qquad \mathbf{v}(0) = \mathbf{v}_0. \tag{51}$$

Necessarily $\begin{bmatrix} y \\ z \end{bmatrix} := \mathbf{v}$ solves (10). Further, notice that (45) follows from $\mathbf{z} \in \mathcal{Z}^\mu_\rho$.

Ⓢ Step [**iii**]: *Uniqueness.* We show now that a solution for (10) in the space $Z := L^2_{\rm loc}(\mathbb{R}_0, \mathbf{D}(\Delta)) \bigcap C([0,+\infty), \mathbf{V}) \supset \mathcal{Z}^\mu_\varrho$ is unique. Indeed, let $\mathbf{z}_1$ and $\mathbf{z}_2$ be two solutions in Z, for (51). Then $e := \mathbf{z}_1 - \mathbf{z}_2$ solves (49) with $g = \mathcal{N}(\mathbf{z}_1) - \mathcal{N}(\mathbf{z}_2)$ in the place of $\mathcal{N}(\bar{\mathbf{z}})$. Using (43b), and following standard arguments, we can obtain

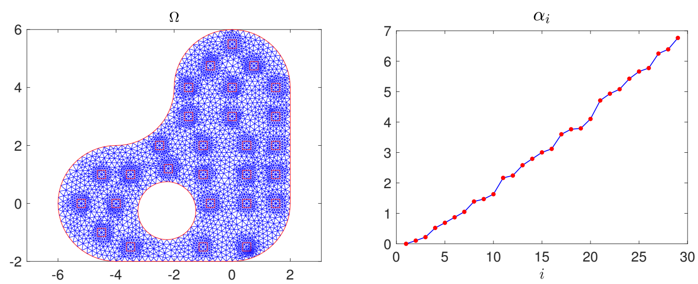
$$\frac{\mathrm{d}}{\mathrm{dt}} |e|^2_{\mathbf{H}} \le C_4 (1 + |\mathbf{z}_1|^{\varepsilon_5}_{\mathbf{V}} + |\mathbf{z}_2|^{\varepsilon_6}_{\mathbf{V}})(1 + |\mathbf{z}_1|^2_{\mathbf{D}(\Delta)} + |\mathbf{z}_2|^2_{\mathbf{D}(\Delta)})\, |e|^2_{\mathbf{H}}\,,$$

which implies $|e(t)|^2_{\mathbf{H}} \le \mathrm{e}^{\int_{s_0}^t \mathcal{G}(s)\,\mathrm{d}s} |e(s_0)|^2_{\mathbf{H}}$, for all $t \ge s_0$, with

$$\mathcal{G}(s) := C_4 \left(1 + |\mathbf{z}_1(s)|^{\varepsilon_5}_{\mathbf{V}} + |\mathbf{z}_2(s)|^{\varepsilon_6}_{\mathbf{V}}\right) \left(1 + |\mathbf{z}_1(s)|^2_{\mathbf{D}(\Delta)} + |\mathbf{z}_2(s)|^2_{\mathbf{D}(\Delta)}\right).$$

Using $e(s_0) = 0$ we find $z_2(t) - z_1(t) = e(t) = 0$, for $t \ge s_0$. This ends the proof. □

5. **Numerical simulations.** Here we present simulation results for (9) and (10). As actuators we will take suitable (piecewise constant) indicator functions $\{1_{\omega_i} \mid i \in \{1, 2, \dots, M\}\} \subset H$. In this section $H \coloneqq L^2(\Omega)$. In Figure 1 we can see the placement of (rectangular) actuators' regions. Recall also that the possibly repeated eigenvalues α_i of the Laplacian, under Neumann boundary conditions increase likely linearly, as we would expect from Weyl asymptotic formula [21]. See also [10, 8].



(a) The (triangulation of the) domain Ω.

(b) First 29 Neumann eigenvalues of $-\Delta$.

FIGURE 1. The domain Ω, and the (computed) first Neumann Laplacian eigenvalues.

The following simulations were carried out using a finite elements based spatial approximation of our equations, with a Crank–Nicolson time discretization.

We will set the parameters in the FitzHugh–Nagumo, Rogers–McCulloch, and Aliev–Panfilov models so that we will have 3 constant steady states $(\widehat{v}^i, \widehat{w}^i)$, $i \in \{1, 2, 3\}$ and so that the first (PDE) components are $\widehat{v}_1 = 0$, $\widehat{v}_2 = 1$, and $\widehat{v}_3 = 2$. As we will see:

— for $\mathfrak{M}$ as in (2)-([FN] or [RM]):

$$(\widehat{v}_1, \widehat{w}_1) = (0, 0), \quad (\widehat{v}_2, \widehat{w}_2) = (1, 10), \quad (\widehat{v}_3, \widehat{w}_3) = (2, 20). \tag{52a}$$

— for $\mathfrak{M}$ as in (2)-[AP]:

$$(\widehat{v}_1, \widehat{w}_1) = (0, 0), \quad (\widehat{v}_2, \widehat{w}_2) = (1, 5), \quad (\widehat{v}_3, \widehat{w}_3) = (2, 0). \tag{52b}$$

We will argue that for all 3 models:

- the linearizations around the steady state $(\widehat{v}_1, \widehat{w}_1)$ and around the steady state $(\widehat{v}_3, \widehat{w}_3)$ are stable,
- the linearization around the steady states $(\widehat{v}_2, \widehat{w}_2)$ is unstable.

We will consider the systems (9) and (10) with two choices as reference trajectories. The first one is the unstable steady state

$$(\overline{v}(t), \overline{w}(t)) = (\widehat{v}_2, \widehat{w}_2), \tag{53a}$$

the second one is spatially and temporally dependent and is defined as follows: we consider the time-dependent vector function $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) \colon \mathbb{R}_0 \to \mathbb{R}^2$,

$$\mathcal{T}_1(t) = \begin{cases} \widehat{v}_2 + (t - 2\lfloor \frac{t}{2} \rfloor)(\widehat{v}_3 - \widehat{v}_2) & \text{if } 0 \le t - 2\lfloor \frac{t}{2} \rfloor \le 1, \\ \widehat{v}_3 + (t - 2\lfloor \frac{t}{2} \rfloor - 1)(\widehat{v}_2 - \widehat{v}_3) & \text{if } 1 < t - 2\lfloor \frac{t}{2} \rfloor \le 2, \end{cases}$$

$$\mathcal{T}_2(t) = \begin{cases} \widehat{w}_2 + (t - 2\lfloor \frac{t}{2} \rfloor)(\widehat{w}_3 - \widehat{w}_2) & \text{if } 0 \le t - 2\lfloor \frac{t}{2} \rfloor \le 1, \\ \widehat{w}_3 + (t - 2\lfloor \frac{t}{2} \rfloor - 1)(\widehat{w}_2 - \widehat{w}_3) & \text{if } 1 < t - 2\lfloor \frac{t}{2} \rfloor \le 2, \end{cases}$$

where $\lfloor s \rfloor \in \mathbb{N}$ stands for the smallest integer which is smaller than $s \le \lfloor s \rfloor < s+1$. Then we define our reference trajectory $(\overline{v}(t,x), \overline{w}(t,x))$ as

$$(\overline{v}(t,x_1,x_2), \overline{w}(t,x_1,x_2)) = (\mathcal{T}_1(t) + \mathfrak{T}_1(t,x_1,x_2), \mathcal{T}_2(t) + \mathfrak{T}_2(t,x_1,x_2)), \tag{53b}$$

with

$$\mathfrak{T}_1(t,x_1,x_2) \coloneqq 0.001 \left|\sin(\pi t)\Big(\cos(\pi t + 0.1(x_1 - x_2)) + (x_2 - 1)\sin(0.2x_1)\Big)\right|_{\mathbb{R}},$$
$$\mathfrak{T}_2(t,x_1,x_2) \coloneqq 0.1 \left|\sin(\pi t)\Big(\sin(\pi t + 0.1(x_1 - x_2)) + 0.1(x_2^2 - 3x_2)\Big)\right|_{\mathbb{R}}.$$

Notice that $(\overline{v}, \overline{w})$ is periodic in time with period 2, $(\overline{v}(t+2,x), \overline{w}(t+2,x)) = (\overline{v}(t,x), \overline{w}(t,x))$ for all $t \ge 0$. Furthermore, for any given nonnegative integer $m \in \mathbb{N}$, $(\overline{v}(2m,x), \overline{w}(2m,x)) = (\widehat{v}_2, \widehat{w}_2)$ and $(\overline{v}(2m+1,x), \overline{w}(2m+1,x)) = (\widehat{v}_3, \widehat{w}_3)$. That is, the trajectory $(\overline{v}, \overline{w})$ "oscillates" between the unstable steady state $(\widehat{v}_2, \widehat{w}_2)$ and the stable steady state $(\widehat{v}_3, \widehat{w}_3)$.

The initial condition in all the simulations, unless otherwise explicitly stated, is the constant

$$(v_0, w_0) = (-1, -1). \tag{54}$$

In the feedback (33c), we have set the parameter

$$\lambda = 1. \tag{55}$$

5.1. **The linearized FitzHugh–Nagumo model.** Here we consider the linear system (9), with the parameters in (6) set as

$$\begin{array}{llll} \nu = 1, & \delta = 0.01, & & \\ a = 1, & b = 3, & c = 1, & \\ d = 0.1, & e = 0, & \gamma = 0.1, & \rho = 0. \end{array} \tag{56}$$

We can see that the vectors as in (52a) are constant steady states of the uncontrolled and unforced *nonlinear* coupled system (1) (i.e., with $u = 0$ and $f = 0$). Indeed we can see that for a constant vector $(\widehat{v}, \widehat{w})$ we have $\Delta\widehat{v} = 0$ and that

$$\xi(\widehat{v}, \widehat{w}) = \begin{bmatrix} \xi_1(\widehat{v}, \widehat{w}) \\ \xi_2(\widehat{v}, \widehat{w}) \end{bmatrix} \coloneqq \begin{bmatrix} a\widehat{v}^3 - b\widehat{v}^2 + c\widehat{v} + d\widehat{w} + e\widehat{w}\widehat{v} \\ \delta\widehat{w} - \gamma\widehat{v} + \rho\widehat{v}^2 \end{bmatrix}$$

satisfies, for $\widehat{v} \in \{0, 1, 2\}$,

$$\xi(0, \widehat{w}) = \begin{bmatrix} d\widehat{w} \\ \delta\widehat{w} \end{bmatrix}, \qquad \xi(1, \widehat{w}) = \begin{bmatrix} a - b + c + d\widehat{w} + e\widehat{w} \\ \delta\widehat{w} - \gamma + \rho \end{bmatrix}, \tag{57a}$$

$$\xi(2, \widehat{w}) = \begin{bmatrix} 8a - 4b + 2c + d\widehat{w} + 2e\widehat{w} \\ \delta\widehat{w} - 2\gamma + 4\rho \end{bmatrix}. \tag{57b}$$

Therefore with the parameters as in (56) we find that

$$\xi(0,0) = \xi(1,10) = \xi(2,20) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which shows that the vectors as in (52a) are constant steady states for system (1).

Observe also that the linearization (4) is stable around $(0, 0)$ and $(2, 20)$, and is unstable around $(1, 10)$. Indeed, the linearization around a constant vector $(\widehat{v}, \widehat{w})$ reads

$$\tfrac{\partial}{\partial t} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} -3a\widehat{v}^2 + 2b\widehat{v} - c - e\widehat{w} & -d - e\widehat{v} \\ \gamma - 2\rho\widehat{v} & -\delta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, \tag{58}$$

and we find that with the parameters as in (56):

- For $(\widehat{v}, \widehat{w}) = (0,0)$, the eigenvalues of the matrix $\begin{bmatrix} -c & -d \\ \gamma & -\delta \end{bmatrix}$ are characterized by $(-1-\lambda)(-\delta-\lambda)+d\gamma = 0$, that is, $\lambda^2 + (1+\delta)\lambda + \delta + d\gamma = 0$. Therefore, $2\lambda = -(1+\delta) \pm \sqrt{(1+\delta)^2 - 4(\delta + d\gamma)}$. It is clear that both eigenvalues have strictly negative real part, because $(1+\delta)^2 - 4(\delta+d\gamma) < (1+\delta)^2$.
- For $(\widehat{v}, \widehat{w}) = (2,20)$, the eigenvalues of the matrix $\begin{bmatrix} -12a+4b-c & -d \\ \gamma & -\delta \end{bmatrix}$ are again characterized by $(-1-\lambda)(-\delta-\lambda)+d\gamma = 0$, that is, they both have strictly negative real part.
- For $(\widehat{v}, \widehat{w}) = (1,10)$, the eigenvalues of the matrix $\begin{bmatrix} -3a+2b-c & -d \\ \gamma & -\delta \end{bmatrix}$ are characterized by $(2-\lambda)(-\delta-\lambda)+d\gamma = 0$, that is, $\lambda^2 + (\delta-2)\lambda - 2\delta + d\gamma = 0$. Thus $2\lambda = (2-\delta) \pm \sqrt{(2-\delta)^2 - 4(-2\delta+d\gamma)} = (2-\delta) \pm \sqrt{(2+\delta)^2 + 4(\delta+d\gamma)}$. It is clear that the largest eigenvalue have strictly positive real part, because $(2+\delta)^2 + 4(\delta+d\gamma) > 0$ and $(2-\delta) > 0$.

5.1.1. *The case of the constant unstable steady state.* Here we consider the time-independent trajectory $(\overline{v}, \overline{w}) = (1,10)$, as in (53a), which is the constant unstable steady state for the FitzHugh–Nagumo model.

In the figures below the annotation "dyn=feed" refers to the dynamics under the feedback control and "dyn=free" to the free dynamics (without control). The annotation "model=..." specifies the type of the monodomain model defined by the selection of parameters (cf. (2)).

In Figure 2 we see the performance of our feedback control, which is able to stabilize system (9). The slope of $\log(|(y,z)|^2_{H\times H})$ is approximately $\frac{3.3-3.8}{30-5} = -0.02$ for time $t \geq 5$, which means that $|(y,z)|_{H\times H}$ decreases exponentially with rate approximately $-\delta = -0.01$.

Recall that the explicit feedback has been constructed to stabilize the PDE component, and as a corollary the stability of the ODE component follows as well. Consequently we cannot expect to obtain a rate of decay for the ODE component which is better (smaller) than $-\delta$.

In Figure 2 we can see also the curve illustrating the squared norm $|y|^2_H$ of the PDE component versus the squared norm $|z|^2_H$ of the ODE component. The configuration at initial time $(t=0)$ is the starting point of the curve located at $(|y_0|^2_H, |z_0|^2_H) \approx (42, 42)$ (the red dot). The configuration at final time ($t = 30$ in case of Figure 2) is the end point of the curve (the green dot).

Further notice that the free dynamics is exponentially unstable, the norm is increasing exponentially with rate approximately $\frac{1}{2}\frac{120}{30} = 2$.

5.1.2. *The case of a time-dependent trajectory.* We consider the time-dependent trajectory $(\overline{v}, \overline{w})$ as in (53b). In Figure 3 we observe that after time $t \geq 5$ the norm of the solution of the linear system (9) decreases exponentially with rate approximately $-\delta = -0.01$. The free dynamics is exponentially unstable with norm increasing exponentially with rate approximately 1.

5.2. **The nonlinear FitzHugh–Nagumo model.** We consider again the time-dependent reference trajectory as in (53b), but now for the nonlinear system (10). In Figure 4 we see that the norm of the controlled nonlinear system also decreases exponentially to zero with rate -0.01. The free dynamics is not stable, the norm of the solution does not decrease as time increases.

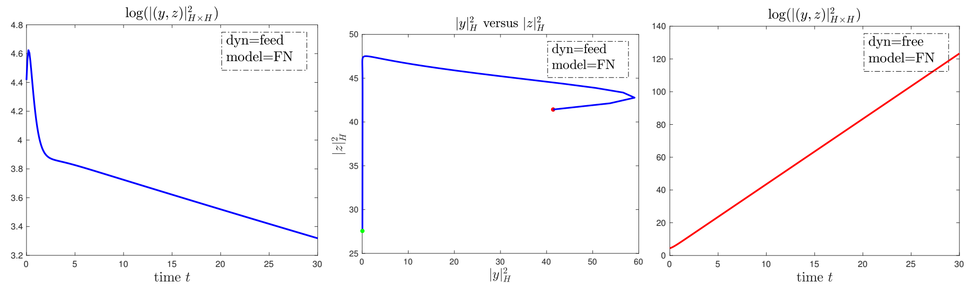


FIGURE 2. Norm of solution. Linearization around the steady state.

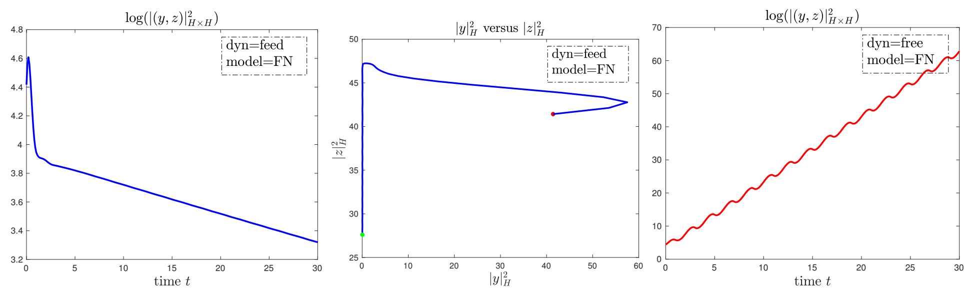


FIGURE 3. Norm of solution. Linearization around the time-dependent trajectory.

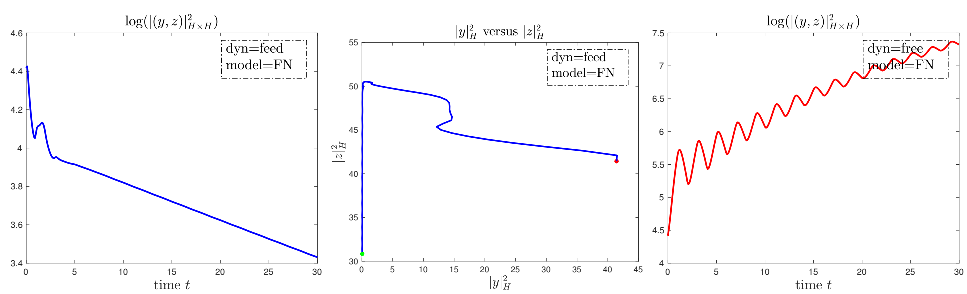


FIGURE 4. Norm of solution. Around the time-dependent trajectory.

5.3. **The linearized Rogers–McCulloch model.** Here, in system (9), we take the parameters

$$\begin{array}{llll} \nu = 1, & \delta = 0.01, & & \\ a = 1, & b = 4, & c = 2, & \\ d = 0, & e = 0.1, & \gamma = 0.1, & \rho = 0. \end{array} \tag{59}$$

Note that from (57), with the parameters as in (59), we find

$$\xi(0,0) = \xi(1,10) = \xi(2,20) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which shows that the vectors as in (52a) are constant steady states for system (1). For the linearization around a constant steady state (58), we find that with the parameters as in (59):

- For $(\widehat{v}, \widehat{w}) = (0, 0)$, the eigenvalues of the matrix $\begin{bmatrix} -c & -d \\ \gamma & -\delta \end{bmatrix}$ with $d = 0$ are given by $\{-c, -\delta\}$. Thus both eigenvalues are strictly negative.
- For $(\widehat{v}, \widehat{w}) = (2, 20)$, the eigenvalues of $\begin{bmatrix} -12a + 4b - c - 2 & -d - 2e \\ \gamma & -\delta \end{bmatrix}$ are characterized by $(0 - \lambda)(-\delta - \lambda) + 2e\gamma = 0$, that is, $\lambda^2 + \delta\lambda + 2e\gamma = 0$. Therefore, $2\lambda = -\delta \pm \sqrt{\delta^2 - 8e\gamma}$, and we can conclude that both eigenvalues have a strictly negative real part because $\delta^2 - 8e\gamma < \delta^2$.
- For $(\widehat{v}, \widehat{w}) = (1, 10)$, the eigenvalues of $\begin{bmatrix} -3a + 2b - c - 1 & -d - e \\ \gamma & -\delta \end{bmatrix}$ are characterized by $(2-\lambda)(-\delta-\lambda) + e\gamma = 0$, that is, $\lambda^2 + (\delta-2)\lambda - 2\delta + e\gamma = 0$. Thus, $2\lambda = (2-\delta) \pm \sqrt{(2-\delta)^2 + 4\delta}$, because from (59) we have $e\gamma = \delta$. It is clear that the largest eigenvalue has a strictly positive real part.

5.3.1. *The case of the constant unstable steady state.* Here the reference trajectory $(\overline{v}, \overline{w}) = (1, 10)$ is as in (53a). In Figure 5 we can see that the explicit feedback is able to stabilize the system (9) exponentially with rate approximately $-\delta = -0.01$, while the free dynamics is exponentially unstable with rate approximately 2.

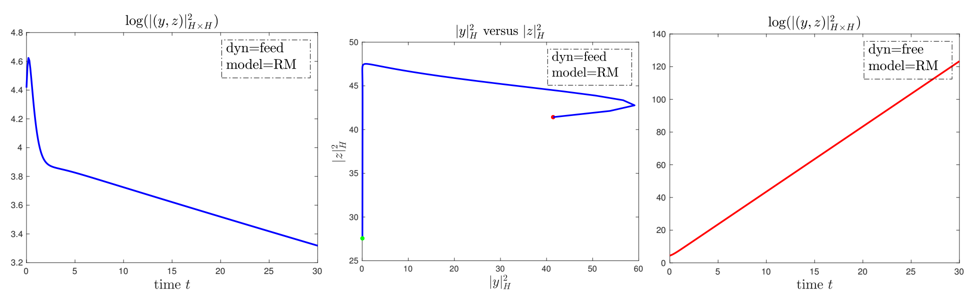


FIGURE 5. Norm of solution. Linearization around the steady state.

5.3.2. *The case of a time-dependent trajectory.* We consider the time-dependent trajectory $(\overline{v}, \overline{w})$ as in (53b). In Figure 6 we can see that the explicit feedback is able to stabilize the system (9) exponentially with rate approximately $-\delta = -0.01$, while the free dynamics is exponentially unstable with rate approximately 1.5.

5.4. **The nonlinear Rogers–McCulloch model.** Here, we consider the time-dependent trajectory $(\overline{v}, \overline{w})$ as in (53b), and consider the corresponding nonlinear system (10) with the parameters as in (59). In Figure 7 we see that our explicit feedback is able to exponentially stabilize system (10). The norm of the controlled system decreases exponentially with rate approximately $-\delta = -0.01$. Notice also that the free dynamics is unstable.

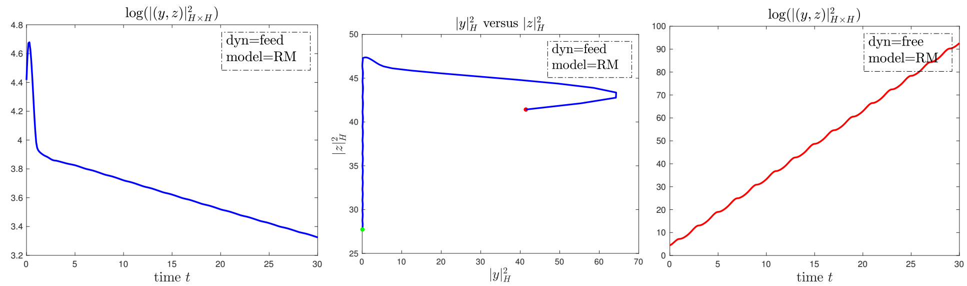


FIGURE 6. Norm of solution. Linearization around the time-dependent trajectory.

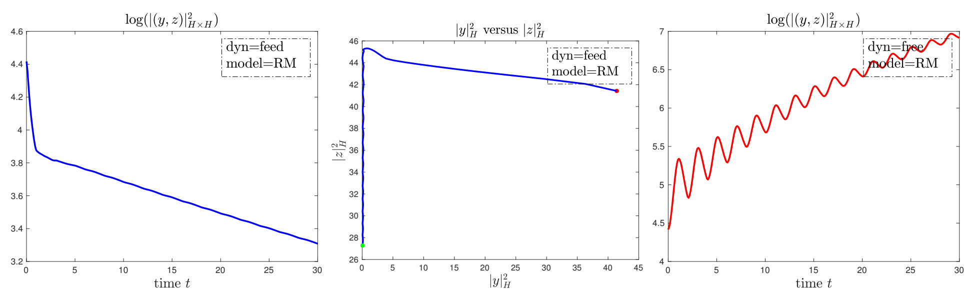


FIGURE 7. Norm of solution. Around the time-dependent trajectory.

5.5. **The linearized Aliev–Panfilov model.** Here we consider the linear system (9), with the parameters

$$\begin{aligned} &\nu = 1, && \delta = 0.01, && && \\ &a = 1, && b = 2.5, && c = 1, && \\ &d = 0, && e = 0.1, && \gamma = 0.1, && \rho = 0.05. \end{aligned} \tag{60}$$

Note that from (57), with the parameters as in (60), we find

$$\xi(0,0) = \xi(1,5) = \xi(2,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which shows that the vectors as in (52b) are constant steady states for system (1). For the linearization around a constant steady state (58), we find that with the parameters as in (59):

- For $(\widehat{v}, \widehat{w}) = (0,0)$, the eigenvalues of the matrix $\begin{bmatrix} -c & -d \\ \gamma & -\delta \end{bmatrix}$ are $\{-c, -\delta\}$. They are both strictly negative.
- For $(\widehat{v}, \widehat{w}) = (2,0)$, the eigenvalues of $\begin{bmatrix} -12a + 4b - c & -d - 2e \\ \gamma - 0.2 & -\delta \end{bmatrix}$ are characterized by $(-3-\lambda)(-\delta-\lambda) + 2e(\gamma - 0.2) = 0$, that is, $\lambda^2 + (3+\delta)\lambda + \delta = 0$. Therefore, $2\lambda = -(3+\delta) \pm \sqrt{(3+\delta)^2 - 4\delta}$, and we have that both eigenvalues have a strictly negative real part.

- For $(\widehat{v},\widehat{w}) = (1,5)$, the eigenvalues of $\begin{bmatrix} -3a+2b-c-\frac{1}{2} & -d-d_2 \\ 0 & -\delta \end{bmatrix}$ are given by $\lambda \in \{-\delta, \frac{1}{2}\}$, that is, the largest eigenvalue is strictly positive.

5.5.1. *The case of the constant unstable steady state.* Here we consider the time-independent trajectory $(\overline{v}(t),\overline{w}(t)) = (1,5)$, is an in (53a). In Figure 8 we see that our explicit feedback is able to stabilize system (9). The norm of the solution decreases exponentially with rate approximately $-\delta = -0.01$. The norm of the free dynamics is exponentially increasing with rate approximately 0.5.

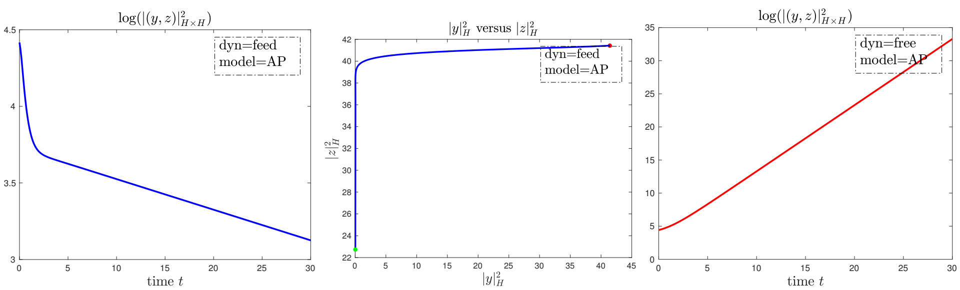


FIGURE 8. Norm of solution. Linearization around the steady state.

5.5.2. *The case of a time-dependent trajectory.* We consider again the linear system (9), with the time-dependent trajectory $(\overline{v},\overline{w})$ as in (53b). In Figure 9 we confirm that also in this case the feedback is able to stabilize system (9). The norm decreases exponentially with rate approximately $-\delta = -0.01$. The free dynamics is not stable. The norm of the solution is not converging to zero, and is "close" to a periodic behavior.

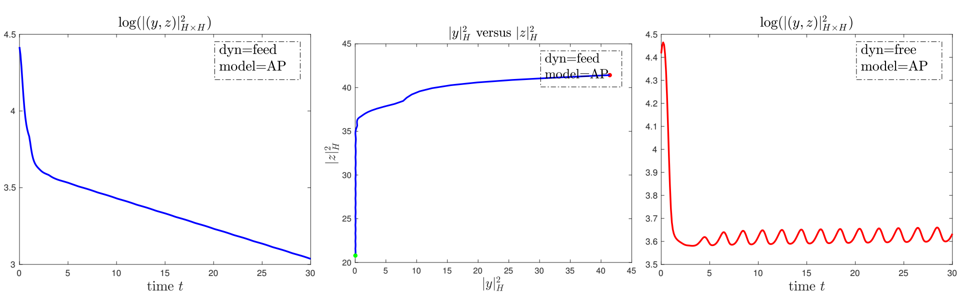


FIGURE 9. Norm of solution. Linearization around the time-dependent trajectory.

5.6. **The nonlinear Aliev–Panfilov model.** We take the time-dependent trajectory $(\overline{v},\overline{w})$ as in (53b), and the corresponding nonlinear system (10), again with the parameters as in (60). For Figure 10 we ran the simulation for time $t \in [0,30]$ as in previous examples. In this situation the results do not allow us to conclude that the feedback is stabilizing. The norm seems to decrease but, we cannot yet clearly see a rate of stability.

Therefore we ran the simulation for a larger time interval and the results are shown in Figure 11. Again we observe that an exponentially decreasing rate is achieved. In fact, it is approximately $-\delta = -0.01$ (the slope for time $t \geq 100$ is approximately -0.02).

We also observe that the norm corresponding to the free dynamics is not converging to zero, that is the free dynamics is not stable.

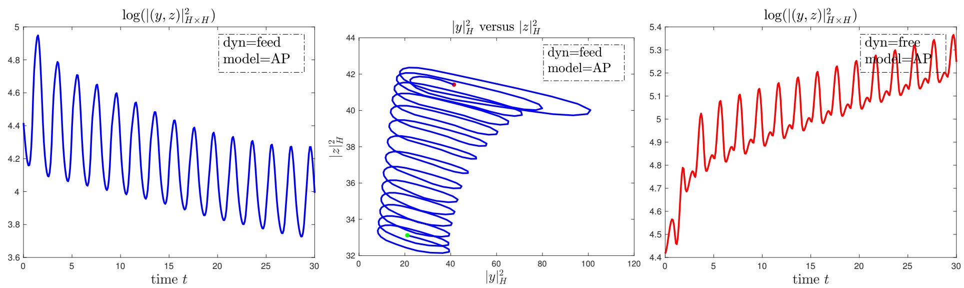


FIGURE 10. Norm of solution. Around the time-dependent trajectory.

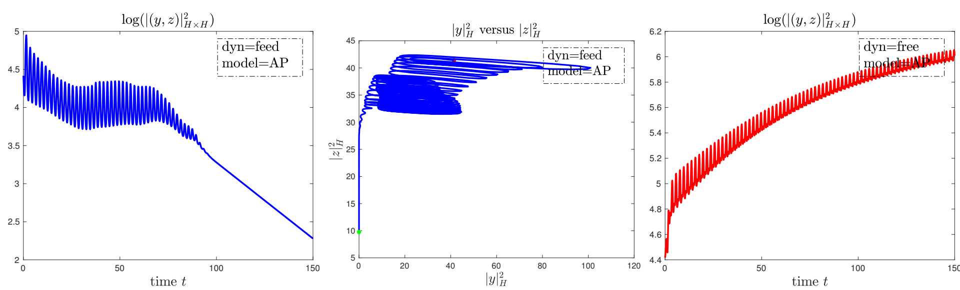


FIGURE 11. Norm of solution. Around the time-dependent trajectory.

5.7. **On the sign of the leading coefficient a.** In the above examples the linearization based feedback control is able to stabilize the nonlinear system for the initial condition $(-1,-1)$, as in (54).

Notice that in the models above, the sign of the leading coefficient $a = 1$ of $av^3 - b^2v^2 + cv$ is positive, and in this case we may say that is has the good sign, in the sense that such sign somehow favors stability of the nonlinear system, for example it may guarantee the existence of global solutions (cf. [7, Chapter 5, Remark 5.1], for the uncoupled parabolic equation). The positive sign of a may explain why the feedback still works for the nonlinear system for such a "large" initial condition.

Recall that simulations in [16], for a single uncoupled parabolic equation, show that the feedback is able to stabilize the nonlinear system only if the initial condition is small enough. This is also the statement of Theorem 4.3, concerning the coupled system.

Below we will consider the "$a-$" model where we take a negative leading coefficient a. Notice that our results are independent of the sign of a. We will consider the following parameters

$$\begin{aligned} &\nu = 1, && \delta = 0.01, && && \\ &a = -1, && b = 0, && c = -1, && \\ &d = 0, && d_2 = 0.1, && \gamma = 0.1, && \rho = 0.05. \end{aligned} \qquad [a-] \qquad (61)$$

which roughly can be seen a variant of the Aliev–Panfilov model with the negative sign for a and c and with $b = 0$. In such situation we may expect the uncontrolled solution of the uncoupled corresponding parabolic system to blow up in finite time. See [7, Chapter 5], [12], and references therein. Therefore we may expect the solution of the corresponding coupled system to explode as well.

5.7.1. *The linearized "a−" model.* We consider the time-dependent trajectory $(\overline{v}, \overline{w})$ as in (53b), with $(1, 1)$ in the place of the steady states: that is with $(\widehat{v}_2, \widehat{w}_2) \coloneqq (1, 1) \eqqcolon (\widehat{v}_3, \widehat{w}_3)$.

In Figure 12 we see that the norm of the controlled solution of system (9) decreases exponentially with rate approximately $-\delta = -0.01$. The norm of the free dynamics increases exponentially with rate approximately 4.

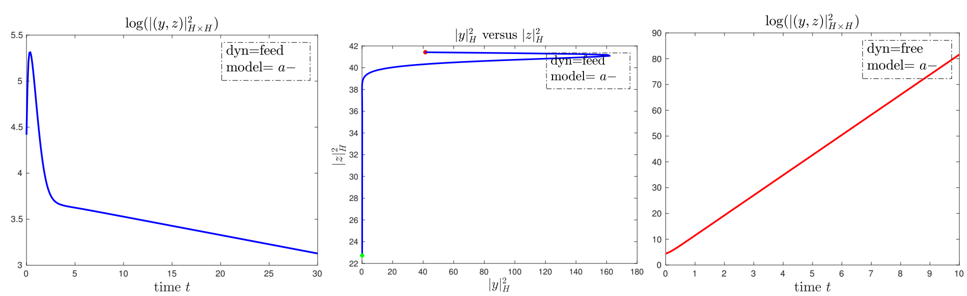


FIGURE 12. Norm of solution. Linearization around the time-dependent trajectory.

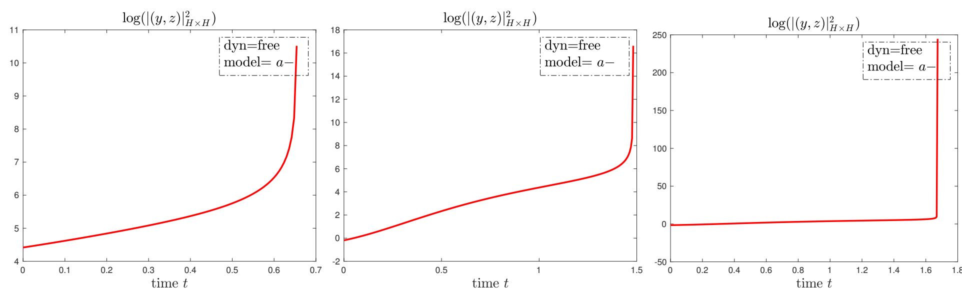
5.7.2. *The nonlinear "a−" model.* Here we consider the nonlinear system (10). We will test with the resized initial conditions

$$(v_0, w_0)_\varepsilon = \varepsilon(v_0, w_0), \qquad \text{with} \qquad \varepsilon \in \{1, 0.1, 0.05\}. \qquad (62)$$

First in Figure 13 we consider the free dynamics for the nonlinear system (10). We observe that the norm of the solution blows up in finite time, for all initial conditions as in (62).

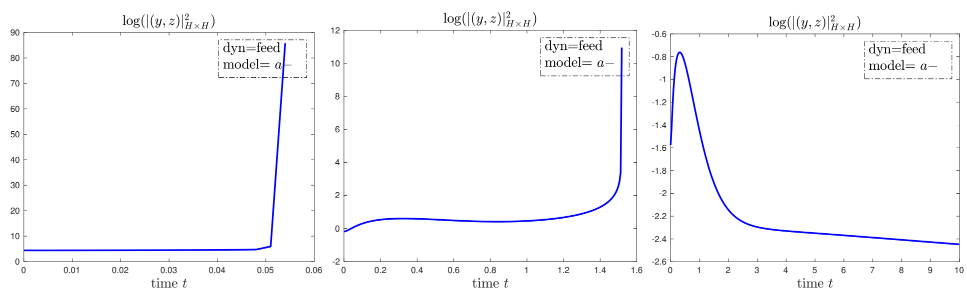
Next we consider the corresponding systems under the action of the feedback control. In this case, in Figure 14, we can see that our feedback is not able to stabilize the system for the initial conditions $(-1, -1)$ and $(-0.1, -0.1)$. But, it is able to stabilize the system if we take the smaller initial condition $(-0.05, -0.05)$ which is consistent with our local stabilization result for the nonlinear system, as stated in Theorem 1.2.

6. **Final Remarks.**



(a) Initial condition $(-1, -1)$ (b) Initial condition $(-0.1, -0.1)$ (c) Initial condition $(-0.05, -0.05)$

FIGURE 13. Norm of solution. Around the time-dependent trajectory.



(a) Initial condition $(-1, -1)$ (b) Initial condition $(-0.1, -0.1)$ (c) Initial condition $(-0.05, -0.05)$

FIGURE 14. Norm of solution. Around the time-dependent trajectory.

6.1. **On the number and placement of the actuators.** Here we consider the same setting as in Section 5.7.1. Recall that in Figure 12, we have the results corresponding to the case of 28 actuators as in Figure 1.

Now in Figure 15, we present the corresponding results in the case we take only 4 actuators as in Figure 15. We constructed the actuators so that the volume covered by the 4 actuators in Figure 15 is equal to the total volume covered by the 28 actuators in Figure 1. In either case the total volume covered by the actuators is approximately 5.4% of the volume of Ω. Though the volume covered by the actuators is the same, we conclude that the explicit feedback corresponding to the 4 actuators as in Figure 15 is not able to stabilize system (9), while the explicit feedback corresponding to the 28 actuators as in Figure 1 is, as shown by Figure 12.

From the results in [11] we can conclude that once we know that $\left|P_{U_M}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ remains bounded with respect to M, then increasing both λ and the number of actuators allows to achieve any prescribed rate of exponential stability in the case of a single uncoupled parabolic equation. In Figure 16 we observe that 4 actuators chosen as in Figure 15 are not able to stabilize the uncoupled parabolic equation, while 28 actuators as in Figure 1 are able to stabilize the uncoupled parabolic equation with rate approximately 1, which is the best rate we can get, independently of M, because we have chosen $\lambda = 1$ in our feedback law, see (55).

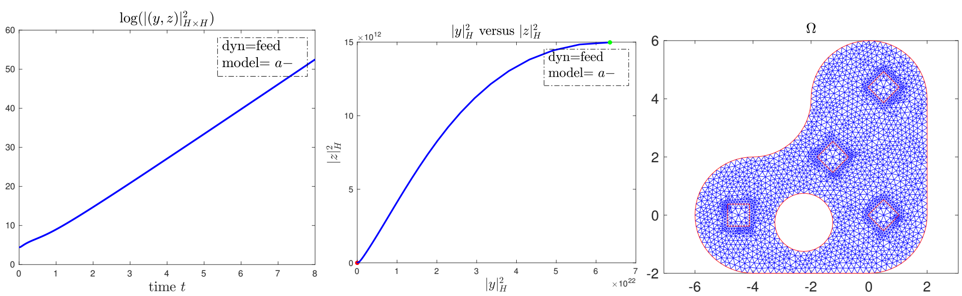
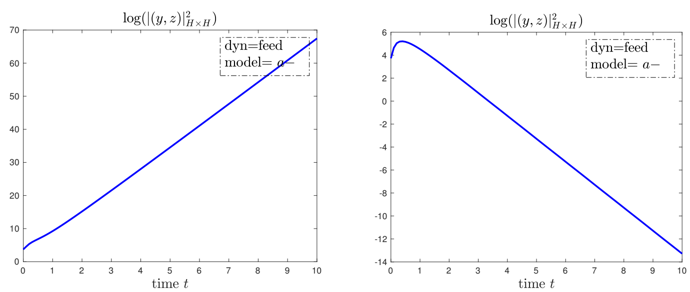


FIGURE 15. $M = 4$. Norm of solution. Linearization around the time-dependent trajectory.



(a) $M = 4$. (b) $M = 28$.

FIGURE 16. PDE only. With $d = d_2 = \gamma = \rho = 0$ (uncoupled system) and $w_0 = 0$. Solution norm $|(v, w)|^2_{H\times H} = |v|^2_H$. Linearization around the time-dependent trajectory.

Since the norm of the projection $\left|P_{U_M}^{E_M^\perp}\right|_{\mathcal{L}(H)}$ plays a role in the stabilizability condition (35b), it would be interesting to know whether we can place the actuators so that it remains bounded as M increases. This has been proven recently in 1D in [17] for parabolic equations with both Dirichlet and Neumann boundary conditions. Then from [11, Sect. 4.8.1] we can also construct suitable actuators in rectangular domains so that the operator norm of the corresponding projections remain bounded as the number of actuators increases.

Another important question concerns the optimal placement of the actuators. In Figure 1 and Figure 15 we simply tried to "spread" the actuators over Ω as "uniformly" as possible. However, the results in [17], show that the best way of "spreading" the actuators is a nontrivial question already in the 1D case.

By curiosity, the norm (computed numerically, see [11]) of the projection with the 4 actuators as in Figure 15 is given by $\left|P_{U_M}^{E_M^\perp}\right|^2_{\mathcal{L}(H)} \approx (0.0327)^{-1} \approx 30$, while the norm of the projection with 28 actuators as in Figure 1 is given by $\left|P_{U_M}^{E_M^\perp}\right|^2_{\mathcal{L}(H)} \approx (1.9367 \cdot 10^{-4}) \approx 5000$, which is considerably bigger.

From [11, Sections 4.2 and 5.1], we know that there are examples of parabolic equations where the number of necessary actuators is bounded from below, regardless of the location of the actuators. On the other hand the stability condition (35) (or its particular version (7)) tells us that it is preferable to have a set of actuators so that the operator norm of the oblique projection projection $P_{U_M}^{E_M^\perp}$ is as small as possible. Therefore the optimal placement of a given set of actuators, minimizing $\left|P_{U_M}^{E_M^\perp}\right|_{\mathcal{L}(H)}$, is an important problem whose solution may lead to better performance of the feedback control, see [11, Section 5.1].

In [11, 17], when looking for the placement of the set of M actuators $\{1_{\omega_i^M} \mid i \in \{1, 2, \dots, M\}\}$, $\omega_i^M \subset \Omega$, (indicator functions) spanning U_M and so that $P_{U_M}^{E_M^\perp}$ remains bounded, it is also required that the volume covered by the actuators is independent of M, that is, $\mathrm{vol}(\bigcup_{i=1}^M \omega_i^M) = C_{\mathrm{v}}$. Of course, by increasing M we are not just adding more actuators, we are instead taking different ones covering the same volume.

6.2. **On the stabilization to periodic solutions.** While our results apply for general bounded trajectories, in our numerical examples we have taken periodic trajectories as targeted solutions. We will show here that for this case our feedback law is also periodic. Indeed, suppose that our targeted solution $(\overline{v}, \overline{w})$ for system (3) is time-periodic: for a given period $T > 0$

$$(\overline{v}(s+T), \overline{w}(s+T)) = (\overline{v}(s), \overline{w}(s)), \quad \text{for all} \quad s \geq 0. \tag{63}$$

In that case we observe that the feedback law is also periodic

$$\mathcal{K}_{U_M}(s+T) = \mathcal{K}_{U_M}(s). \tag{64}$$

Indeed, for all $y, z \in H \times H$,

$$\begin{aligned}\mathcal{K}_{U_M}(s+T)(y,z) &= P_{U_M}^{E_M^\perp}\left(Ay + A_{\mathrm{r}}(s+T)y - \lambda y + \widetilde{\mathcal{S}}(s+T)z\right)\\ &= P_{U_M}^{E_M^\perp}\left(Ay + A_{\mathrm{r}}(s)y - \lambda y + \widetilde{\mathcal{S}}(s)z\right) = \mathcal{K}_{U_M}(s)(y,z),\end{aligned}$$

because, from (6), we see that $A_{\mathrm{r}}(s+T)y = A_{\mathrm{r}}(s)y$ and $\widetilde{\mathcal{S}}(s+T) = \widetilde{\mathcal{S}}(s)$, when $(\overline{v}, \overline{w})$ is T-periodic, as in (63).

Recalling that the monodomain equations provide a model for the heart rhythm we expect the healthy solution to be periodic, that is, it makes sense to consider at the very beginning that $(\overline{v}, \overline{w})$ satisfies (63).

Then it is natural to ask whether we can take advantage of the time periodicity of $A + A_{\mathrm{r}} + \widetilde{\mathcal{S}}$.

In [3, 14] the stabilization of time-periodic parabolic systems is proven under a suitable conditions on the set of actuators. Such conditions take advantage of the fact that the operators in the parabolic equation are time-periodic, which allows to derive suitable properties of the asymptotic behavior of the solutions from the spectral properties of the *periodic map* or *Floquet map*.

The feedback law proposed in [3, 14] is, however, not explicit. It is given implicitly and involves the solution of a suitable periodic Riccati equation. The computation of such a solution can be an expensive numerical task.

It would be interesting to know whether we can construct an ad-hoc, still explicit, feedback operator in this situation. Recall that in [11, Section 6.5], an ad-hoc explicit feedback is proposed for the case of time-independent targeted trajectories (i.e., steady states).

— Appendix —

In Assumptions 1, 7, and 9, the existence of weak solutions was assumed for abstract nonautonomous systems. As announced, we now give some comments on this subject. We also give the proofs of Propositions 1 and 2.

On the existence of weak solutions. Let $\mathfrak{H}$ be a separable Hilbert space, which we consider as a pivot space $\mathfrak{H} = \mathfrak{H}'$. Given another Hilbert space $\mathfrak{V} \overset{\mathrm{d}}{\hookrightarrow} \mathfrak{H}$, and a family of linear mappings $\mathfrak{L}(t) \in \mathcal{L}(\mathfrak{V}, \mathfrak{V}')$ it follows that the domain of $\mathfrak{L}(t)$, defined as $\mathrm{D}(\mathfrak{L}(t)) \coloneqq \{h \in \mathfrak{H} \mid \mathfrak{L}(t)h \in \mathfrak{H}\}$, satisfies

$$\mathrm{D}(\mathfrak{L}(t)) \overset{\mathrm{d}}{\hookrightarrow} \mathfrak{V} \overset{\mathrm{d}}{\hookrightarrow} \mathfrak{H} \overset{\mathrm{d}}{\hookrightarrow} \mathfrak{V}' \overset{\mathrm{d}}{\hookrightarrow} \mathrm{D}(\mathfrak{L}(t))'.$$

The domain $\mathrm{D}(\mathfrak{L}(t))$ is assumed to be endowed with the natural graph norm $|h|_{\mathrm{D}(\mathfrak{L}(t))} \coloneqq \left(|h|_{\mathfrak{H}}^2 + |\mathfrak{L}(t)h|_{\mathfrak{H}}^2\right)^{\frac{1}{2}}$. Further, for time dependent operators, we assume that the domain $\mathrm{D}(\mathfrak{L}(t))$ is independent of time:

$$\mathrm{D}(\mathfrak{L}(t)) = \mathrm{D}(\mathfrak{L}(0)), \quad \text{for all} \quad t \geq 0.$$

Therefore we simply write $\mathrm{D}(\mathfrak{L}) \coloneqq \mathrm{D}(\mathfrak{L}(t))$. Recall also the time interval $I \coloneqq (s_0, s_1)$, as in (12).

Definition A.1. We say that $z(t)$ is a $(\mathfrak{V}, \mathfrak{H})$-weak solution, in $I \times \mathfrak{H}$, for the evolutionary system

$$\dot{z} = -\mathfrak{L}z, \qquad z(s_0) = z_0 \in \mathfrak{H}, \tag{A.1}$$

if $z \in W^1(I, \mathfrak{V}, \mathfrak{V}')$, $z(s_0) = z_0$, and $\langle \dot{z} + \mathfrak{L}z, \phi \rangle_{\mathfrak{V}', \mathfrak{V}} = -(z, \dot{\phi})_{\mathfrak{H}} + \langle \mathfrak{L}z, \phi \rangle_{\mathfrak{V}', \mathfrak{V}} = 0$ for all $\phi \in \{f \in C^\infty(I, \mathfrak{V}) \mid \operatorname{supp} f \subset I\}$.

Recall that $W^1(I, \mathfrak{V}, \mathfrak{V}') \hookrightarrow C([s_0, s_1], \mathfrak{H})$. In case that for any nonempty interval $J = (t_1, t_2) \subseteq I$, the $(\mathfrak{V}, \mathfrak{H})$-weak solution, in $J \times \mathfrak{H}$, for the linear system (A.1) exists and is unique, for all initial conditions $z(t_1) = h \in \mathfrak{H}$, then we denote it by $z(t) = U_{(t,t_1)}^{-\mathfrak{L}} h$, for $t \in J$. Necessarily, for all $h \in \mathfrak{H}$, $z_0 \in \mathfrak{H}$, and $s_1 \geq t \geq t_1 \geq s_0$, we have that

$$U_{(t_1,t_1)}^{-\mathfrak{L}} h = h \quad \text{and} \quad U_{(t,s_0)}^{-\mathfrak{L}} z_0 = U_{(t,t_1)}^{-\mathfrak{L}} U_{(t_1,s_0)}^{-\mathfrak{L}} z_0.$$

Definition A.2. We say that $(\mathfrak{L}, \mathfrak{V}, \mathfrak{H})$ has the Galerkin approximation property if there exists a complete orthonormal basis $\{\psi_i \mid i \in \mathbb{N}_0\} \subset \mathfrak{V}$ so that

1. for all $h \in \mathfrak{H}$, $h = \sum_{i=1}^{+\infty} (h, \psi_i)_{\mathfrak{H}} \psi_i \coloneqq \lim_{N \to +\infty} \sum_{i=1}^{N} (h, \psi_i)_{\mathfrak{H}} \psi_i$,
2. for all $z_0 \in \mathfrak{H}$ and $N \in \mathbb{N}_0$, by setting $z_0^N \coloneqq Q^N h \coloneqq \sum_{i=1}^{N} (z_0, \psi_i)_{\mathfrak{H}} \psi_i$ and considering the solution z^N of
$$\dot{z}^N = -Q^N \mathfrak{L} z^N, \qquad z^N(s_0) = z_0^N \in \mathfrak{H}, \tag{A.2}$$
the sequence $(z^N)_{N \in \mathbb{N}_0}$ remains bounded: $\left|z^N\right|_{W^1(I, \mathfrak{V}, \mathfrak{V}')} \leq D \left|z_0\right|_{\mathfrak{H}}$, with D independent of (N, z_0),
3. we have $Q^N \mathfrak{L} z^N \rightharpoonup \mathfrak{L} z$ in $\mathfrak{V}'$, provided $z^N \rightharpoonup z$ in $\mathfrak{V}$.

Lemma A.3. *Let $(\mathfrak{L}, \mathfrak{V}, \mathfrak{H})$ have the Galerkin approximation property. If there is a constant $B_{\mathfrak{L}} \geq 0$ so that*

$$|\langle \mathfrak{L}h, h\rangle_{\mathfrak{V}',\mathfrak{V}}|_{\mathbb{R}} \leq B_{\mathfrak{L}} |h|^2_{\mathfrak{H}}, \quad \textit{for all} \quad h \in \mathfrak{V},$$

then there exists one, and only one, $(\mathfrak{V}, \mathfrak{H})$-weak solution for system (A.1).

Proof. A solution z is given by a weak-limit, in $W^1(I, \mathfrak{V}, \mathfrak{V}')$, of the Galerkin approximations solving (A.2), $z^N \rightharpoonup z$. The uniqueness follows from $\frac{\mathrm{d}}{\mathrm{d}t} |d|^2_{\mathfrak{H}} = -2\langle \mathfrak{L}d, d\rangle_{\mathfrak{V}',\mathfrak{V}} \leq 2B_{\mathfrak{L}} |d|^2_{\mathfrak{H}}$, where d is the difference between two solutions, which implies $|d(t)|^2_{\mathfrak{H}} \leq \mathrm{e}^{2B_{\mathfrak{L}}(t-s_0)} |d(s_0)|^2_{\mathfrak{H}} = 0$, for all $t \in I$. □

The notion of weak solutions is classical. We refer to [13, Chapter 1, Section 6], [19, Chapter 1, Section 3], and [20, Chapter 3, Sections 1.3, 1.4, and 3.2].

Proof of Proposition 1. We suppose that Assumptions 4–8, and 10 hold true. Recall that $H = L^2(\Omega)$, $V = H^1(\Omega)$ and $\mathrm{D}(A) = \{f \in H^2(\Omega) \mid \frac{\partial f}{\partial \mathbf{n}} = 0\}$. Let us take the spaces

$$\mathfrak{H} = (E_M^\perp \bigcap H) \times H, \quad \mathfrak{V} = (E_M^\perp \bigcap V) \times H, \quad \text{and} \quad \mathfrak{D}(\mathfrak{L}) = (E_M^\perp \bigcap \mathrm{D}(\Delta)) \times H,$$

and the linear operator, in matrix form,

$$\mathfrak{L} \in \mathcal{L}(\mathfrak{V}, \mathfrak{V}'), \qquad \begin{bmatrix} v \\ w \end{bmatrix} \mapsto \begin{bmatrix} P_{E_M^\perp}^{U_M} A + P_{E_M^\perp}^{U_M} A_{\mathrm{r}} & P_{E_M^\perp}^{U_M} \widetilde{\mathcal{S}} w \\ \mathcal{R} & \mathcal{D} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}. \tag{A.3}$$

Next we prove that system (A.1) has the Galerkin property. We set the sequence of eigenfunctions e_i, $i \geq 1$, of $A = -\Delta + 1$ under Neumann boundary conditions. Then we set the basis $\{(e_i, e_j) \mid i > M, \ j \geq 1\} \subset \mathfrak{V}$ for the space $\mathfrak{H}$. It is clear that (1) in Definition A.2 is satisfied. Next we consider the system

$$\dot{z}^N = -Q^N \mathfrak{L} z^N = \begin{bmatrix} -A - Q_1^N P_{E_M^\perp}^{U_M} A_{\mathrm{r}} & -Q_1^N P_{E_M^\perp}^{U_M} \widetilde{\mathcal{S}} \\ -Q_2^N \mathcal{R} & -\delta \mathsf{Id} \end{bmatrix}, \qquad z^N(s_0) = z_0^N \in \mathfrak{H}, \tag{A.4}$$

with $Q^N \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} Q_1^N f \\ Q_2^N g \end{bmatrix} \coloneqq \begin{bmatrix} \sum_{i=1}^N (f, e_{M+i})_H e_{M+i} \\ \sum_{i=1}^N (g, e_i)_H e_i. \end{bmatrix}$. Since Q^N is an orthogonal projection in $\mathfrak{H}$, with $z^N = \begin{bmatrix} z_1^N \\ z_2^N \end{bmatrix}$ we have $\langle Q_1^N P_{E_M^\perp}^{U_M} A z_1^N, z_1^N\rangle_{V',V} = (P_{E_M^\perp}^{U_M} A z_1^N, z_1^N)_H = |z_1^N|^2_V = \langle A z_1^N, z_1^N\rangle_{V',V}$, and $\langle Q_2^N \mathcal{D} z_2^N, z_2^N\rangle_{V',V} = \delta |z_1^N|^2_H = \langle \mathcal{D} z_2^N, z_2^N\rangle_{V',V}$.

Notice that the estimates in the proof of Lemma 2.2, by taking $\mathcal{A} = A + Q_1^N P_{E_M^\perp}^{U_M} A_{\mathrm{r}}$, $\mathcal{D} = \delta \mathsf{Id}$, $\mathcal{R} = Q_2^N \mathcal{R}$, and $\mathcal{S} = Q_1^N P_{E_M^\perp}^{U_M} \widetilde{\mathcal{S}}$, also hold true for system (A.4). From Remark 4 we know that the system $\dot{z}^N = -\mathcal{A} z^N$ is stable if $\widetilde{\mathcal{S}} = 0$. Therefore, we can conclude that the assumptions in Lemma 2.2 are satisfied. Further, from Assumption 10, it follows that the assumption in Corollary 1 is also satisfied. Thus, from Corollary 1, the Galerkin approximations satisfy

$$|z^N|_{L^\infty(\mathbb{R}_{s_0}, \mathfrak{H})} \leq D_{\mathrm{c}} |z_0^N|_{\mathfrak{H}}.$$

This inequality, together with

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\left|z^N\right|_{\mathfrak{H}}^2 &\le -2\left|z_1^N\right|_V^2 - 2\delta\left|z_2^N\right|_H^2 + 2\left|Q_1^N P_{E_M^\perp}^{U_M} A_{\mathrm{r}}\right|_{\mathcal{L}(H,V')}\left|z_1^N\right|_H\left|z_1^N\right|_V\\ &\quad + 2\left|Q_1^N P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}\right|_{\mathcal{L}(H,H)}\left|z_2^N\right|_H\left|z_1^N\right|_H + 2\left|Q_2^N\mathcal{R}\right|_{\mathcal{L}(H,H)}\left|z_1^N\right|_H\left|z_2^N\right|_H\\ &\le -\left|z_1^N\right|_V^2 - \delta\left|z_2^N\right|_H^2 + C_1\left|P_{E_M^\perp}^{U_M} A_{\mathrm{r}}\right|_{\mathcal{L}(H,V')}^2\left|z_1^N\right|_H^2\\ &\quad + \tfrac{C_1}{\delta}\left|P_{E_M^\perp}^{U_M}\widetilde{\mathcal{S}}\right|_{\mathcal{L}(H,H)}^2\left|z_1^N\right|_H^2 + \tfrac{C_1}{\delta}\left|\mathcal{R}\right|_{\mathcal{L}(H,H)}^2\left|z_1^N\right|_H^2,\end{aligned}$$

which we can obtain after multiplying (A.4) by z^N, imply that

$$\left|z^N\right|_{L^2((s_0,s_0+t),\mathfrak{V})}^2 \le (1+D_2t)\left|z_0^N\right|_{\mathfrak{H}}^2, \qquad t\in(0,s_1-s_0),$$

with D_2 independent of N and z_0. Hence, by (A.4), we obtain

$$\begin{aligned}&\left|\dot{z}^N\right|_{L^2((s_0,s_0+t),\mathfrak{V}')}^2 = \left|Q^N\mathfrak{L}z^N\right|_{L^2((s_0,s_0+t),\mathfrak{V}')}^2\\ &\le \left|z^N\right|_{L^2((s_0,s_0+t),\mathfrak{V})}^2\left|\mathfrak{L}\right|_{\mathcal{L}(\mathfrak{V},\mathfrak{V}')}^2\left|Q^N\right|_{\mathcal{L}(\mathfrak{V}')}^2 \le D_3\left|z^N\right|_{L^2((s_0,s_0+t),\mathfrak{V})}^2,\end{aligned}$$

with D_3 independent of (N,s_0). Here we use that the orthogonal projection $Q^N\in\mathcal{L}(\mathfrak{H})$ is also defined and orthogonal in $\mathfrak{V}'$ (cf. [11, Lemma 3.3]), which implies that $\left|Q^N\right|_{\mathcal{L}(\mathfrak{V}')}^2=1$. Note that for two eigenpairs (α_i,e_i), (α_j,e_j) of $A=-\nu\Delta+\mathrm{Id}$ we have $(e_i,e_j)_{V'} \coloneqq (A^{-\frac12}e_i, A^{-\frac12}e_j)_{L^2} = (A^{-1}e_i,e_j)_{L^2} = \alpha_i^{-1}(e_i,e_j)_{L^2}=0$. Now, it is straightforward to check that points (2) and (3), in Definition A.2, hold true.

Since the assumption in Lemma A.3 is satisfied, the existence and uniqueness of a weak solution for system (A.1), with $\mathfrak{L}$ as in (A.3). This finishes the proof of Proposition 1. □

Proof of Proposition 2. From (42) and $(y(s_0),z(s_0))\in V\times H$, the fact that the solution for system (9) is strong follows by standard arguments. Indeed, recalling (33c), system (9) becomes

$$\dot{y} = -Ay - A_{\mathrm{r}}y - \widetilde{\mathcal{S}}z + \mathcal{K}_{U_M}(y,z), \qquad y(s_0)=y_0\in V, \tag{A.5a}$$

$$\dot{z} = -\delta z - \mathcal{R}y, \qquad z(s_0)=z_0\in H. \tag{A.5b}$$

Multiplying the dynamics by $\mathfrak{L}z$, where $\mathfrak{L}=\begin{bmatrix}A & 0\\ 0 & \delta\end{bmatrix}$, $\mathfrak{H}=H\times H$, $\mathfrak{V}=V\times H$, and $\mathrm{D}(\mathfrak{L})=\mathrm{D}(A)\times H$, we obtain (47) with $g=0$. Using (47) we can now conclude that $(y,z)\in W((s_0,s_0+T),\mathrm{D}(\mathfrak{L}),\mathfrak{H})\subseteq C([s_0,s_0+T],\mathrm{D}(\mathfrak{L}),\mathfrak{H})$. The same argument leads us to

$$|(y,z)|_{L^\infty((s,s+1),\mathfrak{V})}^2 + |(y,z)|_{L^2((s,s+1),\mathrm{D}(\mathfrak{L}))}^2 \le D_4\left|(y(s),z(s))\right|_{\mathfrak{V}}^2, \quad \text{for} \quad s\ge s_0\ge 0, \tag{A.6}$$

with $D_4=D_3(1)$ independent of s.

Furthermore, we have the following smoothing property

$$\left|(y(s+1),z(s+1))\right|_{\mathfrak{V}}^2 \le D_5\left|(y(s),z(s))\right|_{\mathfrak{H}}^2, \quad \text{for all} \quad s\ge s_0\ge 0, \tag{A.7}$$

with D_5 independent of s. This can be derived as follows. We consider, for $t\ge s$, the function $(\breve{y}(t),\breve{z}(t)) \coloneqq (t-s)(y(t),z(t))$, which still solves system (46) with different data:

$$\dot{\breve{y}} = -A\breve{y} - A_{\mathrm{r}}\breve{y} - \widetilde{\mathcal{S}}\breve{z} + \mathcal{K}_{U_M}(\breve{y},\breve{z}) + y, \qquad \breve{y}(s) = 0 \in V,$$
$$\dot{\breve{z}} = -\delta\breve{z} - \mathcal{R}\breve{y} + z, \qquad \breve{z}(s) = 0 \in H,$$

from which (cf. estimate (47) with $g = (y,z)$) we can obtain the estimate

$$\begin{aligned}|(\breve{y},\breve{z})|^2_{L^\infty((s,s+1),\mathfrak{V})} + |(\breve{y},\breve{z})|^2_{L^2((s,s+1),\mathrm{D}(\mathfrak{L}))} &\le D_6\left(|y|^2_{L^2((s,s+1),\mathfrak{H})} + |z|^2_{L^2((s,s+1),\mathfrak{H})}\right)\\ &\le D_6 C\,|(y,z)(s)|^2_{\mathfrak{H}}\,,\end{aligned}$$

where the last inequality follows from Corollary 4 (with s in the role of s_0). In particular (A.7) holds true.

For simplicity let us write

$$\Psi_s := |(y,z)|^2_{L^\infty((s,s+1),\mathfrak{V})} + |(y,z)|^2_{L^2((s,s+1),\mathrm{D}(\mathfrak{L}))}\,.$$

Then from estimates (A.7) and (A.6) together with Corollary 4, it follows, for all $s \ge s_0 \ge 0$:

- if $s \le s_0 + 1$, then

$$\Psi_s \le D_4\,|(y(s),z(s))|^2_{\mathfrak{V}} \le D_4^2\,|(y(s_0),z(s_0))|^2_{\mathfrak{V}} \le D_4^2\mathrm{e}^{2\mu}\mathrm{e}^{-2\mu(s-s_0)}\,|(y(s_0),z(s_0))|^2_{\mathfrak{V}}\,,$$

- if $s \ge s_0 + 1$, then

$$\Psi_s \le D_4 D_5\,|(y(s-1),z(s-1))|^2_{\mathfrak{H}} \le D_4 D_5 C\mathrm{e}^{-2\mu(s-1-s_0)}\,|(y(s_0),z(s_0))|^2_{\mathfrak{H}}\,.$$

We finish the proof by showing the estimate in Proposition 2. It is clear now that we have

$$\sup_{s\ge s_0} \mathrm{e}^{2\mu(s-s_0)}\,|(y(s),z(s))|^2_{\mathfrak{V}} \le \widehat{C}\,|(y(s_0),z(s_0))|^2_{\mathfrak{V}}\,. \tag{A.8}$$

with $\widehat{C} = \mathrm{e}^{2\mu}\max\{D_4^2, D_4D_5C\}$. By direct computations we also obtain

$$\begin{aligned}&\int_s^{s+1} \mathrm{e}^{2\mu(t-s_0)}\,|(y(t),z(t))|^2_{\mathrm{D}(\mathfrak{L})}\,\mathrm{d}t \le \mathrm{e}^{2\mu(s+1-s_0)}\int_s^{s+1}|(y(t),z(t))|^2_{\mathrm{D}(\mathfrak{L})}\,\mathrm{d}t\\ &\le \mathrm{e}^{2\mu(s+1-s_0-s+s_0)}\widehat{C}\,|(y(s_0),z(s_0))|^2_{\mathfrak{V}} = \mathrm{e}^{2\mu}\widehat{C}\,|(y(s_0),z(s_0))|^2_{\mathfrak{V}}\,.\end{aligned} \tag{A.9}$$

Using the dynamics in (A.5), we also obtain for $(\tilde{y},\tilde{z}) := \mathrm{e}^{\mu(t-s_0)}(y(t),z(t))$,

$$\begin{aligned}\left|(\dot{\tilde{y}},\dot{\tilde{z}})\right|_{L^2((s,s+1),\mathfrak{H})} &\le \mu\,|(\tilde{y},\tilde{z})|_{L^2((s,s+1),\mathfrak{H})} + \left|\mathrm{e}^{\mu(t-s_0)}(\dot{y},\dot{z})\right|_{L^2((s,s+1),\mathfrak{H})}\\ &\le \mu\,|(\tilde{y},\tilde{z})|_{L^\infty((s,s+1),\mathfrak{H})} + \left|\mathrm{e}^{\mu(t-s_0)}(y,z)\right|_{L^2((s,s+1),\mathrm{D}(\mathfrak{L}))},\end{aligned} \tag{A.10}$$

Therefore, recalling that $\mathfrak{V} \hookrightarrow \mathfrak{H}$, we conclude from estimates (A.8), (A.9), and (A.10) that

$$\left|\mathrm{e}^{\mu(\cdot-s_0)}(y,z)\right|_{W((s,s+1),\mathrm{D}(\mathfrak{L}),\mathfrak{H})} \le D_8\,|(y(s_0),z(s_0))|_{\mathfrak{H}}\,, \quad \text{for all} \quad s \ge s_0,$$

with D_8 independent of $s \ge s_0$, which is equivalent to the estimate in Proposition 2. □

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