EXPLICIT EXPONENTIAL STABILIZATION OF NONAUTONOMOUS LINEAR PARABOLIC-LIKE SYSTEMS BY A FINITE NUMBER OF LOCALLY SUPPORTED ACTUATORS

Karl Kunisch[1] and Sérgio S. Rodrigues[2]

Abstract. An explicit feedback controller is proposed for stabilization of linear parabolic equations, with a time-dependent reaction-convection operator. The range of the feedback controller is finite-dimensional, and its dimension depends polynomially on a suitable norm of the reaction-convection operator. A sufficient condition for stabilizability is given, which involves the asymptotic behavior of the eigenvalues of the (time-independent) diffusion operator, the norm of the reaction-convection operator, and the norm of the nonorthogonal projection onto the controller's range along a suitable infinite dimensional (higher-modes) eigenspace. To construct the explicit feedback, the essential step consists in computing the nonorthogonal projection. Numerical simulations are presented, in 1D and 2D, showing the practicability of the controller and its response to measurement errors.

2010 Mathematics Subject Classification. 93D15, 93B52, 93C05, 93C20.

.

1. Introduction

We consider a nonautonomous evolutionary system in the abstract form

$$\dot{y}(t) + Ay(t) + A_{\mathrm{rc}}(t)y(t) - \sum_{i=1}^{M} u_i(t)\Psi_i = 0, \quad y(0) = y_0, \tag{1.1}$$

where y is the unknown state, y_0 and Ψ_i, $i \in \{1, 2, \ldots, M\}$, are given in a Hilbert space H, and $u(t) = (u_1, \ldots, u_M)(t)$ is a control function at our disposal, taking values in $\mathbb{R}^M$.

We want to find general conditions on the linear operators A, and A_{rc}, and on the family of actuators $\{\Psi_i \mid i \in \{1, 2, \ldots, M\}\}$, which will allow us to guarantee the existence of a stabilizing feedback control. Roughly speaking, the operators A and A_{rc} will play the roles of a diffusion-like and a reaction-convection-like operator, respectively. For example, we may think of the parabolic equation (e.g., under homogeneous Dirichlet boundary

Keywords and phrases: Exponential stabilization, nonautonomous parabolic systems, finite-dimensional controller, explicit feedback

[1] Institute for Mathematics and Scientic Computing, Karl-Franzens-Universität, Heinrichstr. 36, 8010 Graz, Austria.
e-mail: `karl.kunisch@uni-graz.at`

[2] Johann Radon Institute for Computational and Applied Mathematics, ÖAW,Altenbergerstraße 69, 4040 Linz, Austria.
e-mail: `sergio.rodrigues@ricam.oeaw.ac.at`

conditions)

$$\dot{y}(x,t) - \nu\Delta y(x,t) + a(x,t)y(x,t) + \nabla\cdot\big(b(x,t)y(x,t)\big) - \sum_{i=1}^{M} u_i(t)\Psi_i(x) = 0, \quad y(0) = y_0, \tag{1.2}$$

where $\nu > 0$ and the given functions a and b, defined for $(x,t) \in \Omega \times [0,+\infty)$, take values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. In this case $Ay = -\nu\Delta y$, and $A_{\mathrm{rc}}y = ay + \nabla\cdot(by)$.

The motivation for considering such a system (1.1) comes mainly from recent works on stabilization to time-dependent trajectories [1, 7, 11, 15, 16, 21, 25], where a system in the form (1.1) arises from the linearization around the targeted trajectory. *Local* stabilization to time-dependent trajectories of a given *nonlinear* system will (under some conditions on the nonlinearity) follow from the stabilization to zero of system (1.1), together with a suitable fixed point argument.

In [7] the controllers Ψ_i are localized in a small subset of $\omega \subset \Omega$ and are constructed from suitably truncated eigenfunctions of the Dirichlet Laplacian in $L^2(\Omega)^3$ operator and it is proven that if we use a big enough number of eigenfunctions then we can stabilize the system. In [16] the controllers are constructed in a similar way (though the eigenfunctions are those of the Dirichlet Laplacian in $L^2(\mathcal{O})$ for an interval $\mathcal{O} \subseteq \Omega$ containing the support of the control) and some estimates on the number of actuators are given, which depend exponentially on a suitable norm $|A_{\mathrm{rc}}|_X$ of the operator A_{rc}, where X is a suitable Banach space. Again in [16], the authors perform numerical simulations which suggest that a better estimate might exist. They also show that, taking eigenfunctions of the Laplacian in $L^2(\Omega)$ as actuators, leads to an estimate which depends polynomially on $|A_{\mathrm{rc}}|_X$. Notice that, in this case the support $\overline{\omega}$ of the controller is $\overline{\Omega}$.

In [11, 15, 21], the actuators are not anymore necessarily constructed explicitly from the truncated eigenfunctions of the Laplacian operator. Moreover, a sufficient condition for stabilizability is given depending on a suitable norm $|1_\omega(1-P_{\mathcal{U}})1_\omega|_X$ where $P_{\mathcal{U}}$ is the orthogonal projection onto $\mathcal{U} := \mathrm{span}\{\Psi_i \mid i \in \{1,2,\dots,M\}\}$. In this case similar estimates on the number M of actuators can be given, for suitably chosen piecewise constant actuators, but again depending *exponentially* on a suitable norm $|A_{\mathrm{rc}}|_X$. The numerical simulations we find in these works also suggest that a better estimate for the number of actuators could exist. The reason for the exponential dependence of the estimates on M, in the above works, is due to the fact that the null controllability of the system (for controls in $L^2(\omega)$) is used, for which the norm of the associated control depends exponentially on $|A_{\mathrm{rc}}|_X$. In this manuscript we follow a different procedure where we do not use/need the null controllability of the system and in this way we can remove the exponential dependence of the numbers of controls on $|A_{\mathrm{rc}}|_X$.

Of course, the number of required actuators will depend on the type of actuators we have at our disposal. We will keep this in mind hereafter (see, in particular, the condition for stabilizability (3.2:**CS**a)). In fact, for $A_{\mathrm{rc}}y = \rho y$ with $\rho < 0$ we cannot stabilize (1.1) if all of our actuators are orthogonal to a given eigenfunction ϕ of A whose associated eigenvalue α satisfies $\rho + \alpha < 0$.

We are going to present a setting, for the parabolic equation (1.2), where the number of actuators, which are needed for stabilizability, depends *polynomially* on the norm of A_{rc}. Still, the support of the control can be a small subset, and the actuators can be piecewise constant.

Furthermore, while in [11, 15, 16, 21] the stabilizing feedback control operator was taken as the solution of an appropriate differential Riccati equation, here we give an explicit simple form of the feedback operator. In particular, for the simulations we do not need to solve any matrix Riccati equation. We recall that solving a matrix Riccati equation is a difficult numerical task for large matrices, which arise from high resolution approximations.

The feedback control. Let $P_F\colon H \to F$ stand for the orthogonal projection in H onto a given finite-dimensional space $F \subset H$. Further, let E_M be the span of the eigenfunctions of the operator A associated with the (repeated) first M eigenvalues. We denote the linear span of our actuators by $\mathcal{U} = \mathrm{span}\{\Psi_i \mid i \in \{1, 2, \dots, M\}\}$. We will assume that our actuators satisfy $H = \mathcal{U} \oplus E_M^{\perp}$. We can see that $\mathcal{U} = E_M$ satisfies the last condition, but we do not assume that our actuators are eigenfunctions. Actually, we are particularly interested in the case where the actuators Ψ_i are locally supported indicator functions, which are more appropriate for real-world applications.

To describe a consequence of the central result of this paper, let $P_{\mathcal{U}}^{E_M^\perp} : H \to \mathcal{U}$ stand for the (nonorthogonal) projection onto $\mathcal{U}$ along $E_M^\perp$. If

$$\overline{\mu}_M \coloneqq \alpha_{M+1} - \left(6 + 4\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right)|A_{\mathrm{rc}}|_X^2 > 0, \tag{1.3}$$

for a suitable Banach space X, then a feedback stabilizing control is given by

$$y \to \mathcal{K}(t)y \coloneqq P_{\mathcal{U}}^{E_M^\perp}\left(Ay + A_{\mathrm{rc}}(t)y - \lambda y\right),$$

for any given constant $\lambda > 0$. More precisely, the system

$$\dot{y}(t) + Ay(t) + A_{\mathrm{rc}}(t)y(t) - \mathcal{K}(t)y(t) = 0, \quad y(0) = y_0,$$

is exponentially stable: there exist suitable constants $\mu > 0$ and $D \ge 1$ such that

$$|y(t)|_H^2 \le D\mathrm{e}^{-\mu(t-s)}\,|y(s)|_H^2\,, \quad \text{for all} \quad t \ge s \ge 0.$$

In the case of the parabolic equation (1.2), we know that $\alpha_{M+1} \to +\infty$ as $M \to +\infty$. So (1.3) will hold for a big enough M providing we can construct $\mathcal{U} = \mathcal{U}(M)$ so that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ increases slower than α_{M+1}, that is, so that $\lim\limits_{M\to+\infty} \alpha_{M+1}\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^{-2} \to +\infty$. Notice that in the case $\mathcal{U} = E_M$, then $P_{E_M}^{E_M^\perp}$ is an orthogonal projection, $\left|P_{E_M}^{E_M^\perp}\right|_{\mathcal{L}(H)} = 1$, and the latter condition reduces to $\lim\limits_{M\to+\infty} \alpha_{M+1} = +\infty$, which is known to hold. Moreover (1.3) reads $\alpha_{M+1} > 10\,|A_{\mathrm{rc}}|_X^2$, which will clearly be satisfied for big enough M.

In the case $\mathcal{U} \ne E_M$, the projection $P_{\mathcal{U}}^{E_M^\perp}$ is nonorthogonal and $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)} > 1$. In this case condition (1.3) is not trivial and will be checked numerically. The simulations, for system (1.2), show that for piecewise constant actuators and by fixing the total volume of the support of the actuators, we can construct $\mathcal{U}(M)$ so that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ remains bounded. Consequently, in case we know the asymptotic behavior of the eigenvalues α_M, we can also find an estimate for the number M of actuators. Suppose we can indeed construct $\mathcal{U}(M)$ so that $\left|P_{\mathcal{U}(M)}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2 \le C_1$ for all $M \ge 1$. For a parabolic equation as (1.2) we know that the eigenvalues of the Laplacian satisfy $\alpha_M \ge C_2 M^{\frac{2}{d}}$, see [18]. Thus, (1.3) will follow from

$$(M+1) > C_2^{-\frac{d}{2}}\,(6+4C_1)^{\frac{d}{2}}\,|A_{\mathrm{rc}}|_X^d\,. \tag{1.4}$$

In particular M depends polynomially on $|A_{\mathrm{rc}}|_X$, which improves the exponential dependence derived in [11, 15, 16, 21]. Differently from those previous works, here the support $\overline{\omega} = \bigcup\limits_{i=1}^{M} \operatorname{supp}\Psi_i$ of the controller is not fixed a priori (still, its volume $|\omega|$ can be small).

While we are particularly interested in the nonautonomous case, the results are, of course, valid also in the autonomous case. However, we recall that in the autonomous case other tools, like the spectral properties of the system operator $A + A_{\mathrm{rc}}$ can be used to construct a stabilizing controller and to give estimates on the dimension of its range (in both cases of internal and boundary controls). We refer to the works [2, 4–6, 8, 24] and references therein. Unfortunately, the spectral properties of $A+A_{\rm rc}(t)$ seem to be (at least, by themselves) not appropriate for studying the stability of the corresponding nonautonomous system, see [28]. In [4, 5] the feedbacks are constructed explicitly, while in [2, 6, 8, 24] they are Riccati based feedbacks.

The results in this work will be applied to internal controls for parabolic equations. The extension of our procedure to the case of boundary controls is not clear yet, and will be addressed in a future work.

The rest of the paper is organized as follows. In section 2 we recall some results concerning weak solutions for parabolic-like systems and the properties of nonorthogonal projections. In section 3 we prove our main result. In section 4 we discuss the sufficient condition (1.3). In section 5 we present the results of some numerical simulations showing the performance of our feedback and its robustness against estimation errors. Finally, in section 6 we present additional remarks concerning our results.

Notation. We follow [11, 15, 16, 21]. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_r := (r,\,+\infty)$, for $r\in\mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N}\setminus\{0\}$. We denote by $\Omega\subset\mathbb{R}^d$ a bounded open connected subset, with $d\in\mathbb{N}_0$.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm, by X' its dual, and by $\langle\cdot,\cdot\rangle_{X',X}$ the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} := \sup\{\langle f,x\rangle_{X',X} \mid x\in X \text{ and } |x|_X = 1\}$. In case X is a Hilbert space we denote the inner product by $(\cdot,\cdot)_X$.

Given an open interval $I\subseteq\mathbb{R}$ and two Banach spaces $X,\,Y$, we write $W(I,\,X,\,Y) := \{f\in L^2(I,\,X)\mid \partial_t f\in L^2(I,\,Y)\}$, where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I,\,X,\,Y)} := \left(|f|^2_{L^2(I,\,X)} + |\partial_t f|^2_{L^2(I,\,Y)}\right)^{1/2}$. In the case $X=Y$ we write $H^1(I,\,X) := W(I,\,X,\,X)$.

If the inclusions $X\subseteq Z$ and $Y\subseteq Z$ are continuous, where Z is a Hausdorff topological space, then we can define the Banach spaces $X\times Y$, $X\cap Y$, and $X+Y$, endowed with the norms $|(a,\,b)|_{X\times Y} := \left(|a|_X^2 + |b|_Y^2\right)^{\frac12}$; $|a|_{X\cap Y} := |(a,\,a)|_{X\times Y}$; and $|a|_{X+Y} := \inf_{(a^X,\,a^Y)\in X\times Y}\left\{|(a^X,\,a^Y)|_{X\times Y}\mid a = a^X + a^Y\right\}$, respectively. We can show that, if X and Y are endowed with a scalar product, then also $X\times Y$, $X\cap Y$, and $X+Y$ are. In case we know that $X\cap Y=\{0\}$, we say that $X+Y$ is a direct sum and we write $X\oplus Y$ instead.

Again, if X and Y are endowed with a scalar product, then also $W(I,\,X,\,Y)$ is. The space of continuous linear mappings from X into Y will be denoted by $\mathcal{L}(X\to Y)$. When $X=Y$ we simply denote $\mathcal{L}(X) := \mathcal{L}(X\to X)$.

If the inclusion $X\subseteq Y$ is continuous, we write $X\hookrightarrow Y$; we write $X\overset{\rm d}{\hookrightarrow} Y$, respectively $X\overset{\rm c}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact.

The kernel and range of a linear mapping $A\colon Z\to W$, between vector spaces Z and W, will be denoted $\operatorname{Ker}(A) := \{x\in Z\mid Ax=0\}$ and $\operatorname{Ran}(A) := \{Ax\mid x\in Z\}$, respectively.

$\overline{C}_{[a_1,\dots,a_k]}$ denotes a nonnegative function of nonnegative variables a_j that increases in each of its arguments.

Finally, C, C_i, $i=0,\,1,\,\dots$, stand for unessential positive constants.

2. Preliminaries

Here we introduce the general properties we ask for the operators A, and $A_{\rm rc}$, and derive some results on the regularity of the solutions for system (1.1).

2.1. Assumptions on the state operators

Let us be given a Hilbert space H that we will consider as pivot space, that is, $H'=H$. Let V be another Hilbert space with $V\subseteq H$.

Assumption 2.1. *$A\in\mathcal{L}(V\to V')$, and $(y,\,z)\mapsto\langle Ay,\,z\rangle_{V',\,V}$ is a complete scalar product in V.*

From now we will suppose that V is endowed with the scalar product $(y,\,z)_V := \langle Ay,\,z\rangle_{V',\,V}$, which still makes V a Hilbert space. Necessarily, A is symmetric and $A\colon V\to V'$ is an isometry.

Assumption 2.2. *The inclusion $V\subseteq H$ is dense, continuous, and compact. That is, $V\xhookrightarrow{\rm d,\,c} H$.*

Necessarily, we have that

$$\langle y,\, z\rangle_{V',\, V} = (y,\, z)_H, \quad \text{for all } (y,\, z) \in H \times V,$$

and also that the operator A is densely defined in H, with domain $\mathrm{D}(A) \coloneqq \{u \in V \mid Au \in H\}$ endowed with the scalar product $(y,\, z)_{\mathrm{D}(A)} \coloneqq (Ay,\, Az)_H$, and the inclusions

$$\mathrm{D}(A) \xhookrightarrow{\text{d, c}} V \xhookrightarrow{\text{d, c}} H \xhookrightarrow{\text{d, c}} V' \xhookrightarrow{\text{d, c}} \mathrm{D}(A)'.$$

Further, A has a compact inverse $A^{-1} \colon H \to \mathrm{D}(A)$, and we can find a nondecreasing system of (repeated) eigenvalues $(\alpha_n)_{n\in\mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_n)_{n\in\mathbb{N}_0}$:

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \alpha_{n+1} \to +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n.$$

We can define, for every $\beta \in \mathbb{R}$, the fractional powers A^β, of A, by

$$A^\beta \sum_{n=1}^{+\infty} y_n e_n \coloneqq \sum_{n=1}^{+\infty} \alpha_n^\beta y_n e_n,$$

and the corresponding domains $\mathrm{D}(A^{|\beta|}) \coloneqq \{y \in H \mid A^{|\beta|} y \in H\}$, and $\mathrm{D}(A^{-|\beta|}) \coloneqq \mathrm{D}(A^{|\beta|})'$. We have that $\mathrm{D}(A^\beta) \xhookrightarrow{\text{d, c}} \mathrm{D}(A^{\beta_1})$, for all $\beta > \beta_1$, and we can see that $\mathrm{D}(A^0) = H$, $\mathrm{D}(A^1) = \mathrm{D}(A)$, $\mathrm{D}(A^{\frac{1}{2}}) = V$.

For the time-dependent operators we assume the following:

Assumption 2.3. *For almost every $t > 0$ we have $A_{\mathrm{rc}}(t) \in \mathcal{L}(H \to V')$, and there is a nonnegative constant C_{rc} such that, $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H\to V'))} \le C_{\mathrm{rc}}$.*

2.2. **Weak solutions**

We recall a regularity result for system

$$\dot y(t) + Ay(t) + A_{\mathrm{rc}}(t)y(t) + f(t) = 0, \quad y(s_0) = y_0. \tag{2.1}$$

That is, for (1.1) with a general external force in place of the control, and with the initial time shifted to $t = s_0 \ge 0$. In what follows s_0 and s_1 stand for two nonnegative real numbers and I for a finite interval as follows.

$$I \coloneqq (s_0,\, s_1), \quad 0 \le s_0 < s_1. \quad \text{Further } |I| \coloneqq s_1 - s_0. \tag{2.2}$$

Throughout the paper we assume that $y_0 \in H$.

Lemma 2.4. *Given $f \in L^2(I,\, V')$, there is a weak solution $y \in W(I,\, V,\, V')$ for (2.1). Moreover y is unique and depends continuously on the data:*

$$|y|^2_{W(I,\, V,\, V')} \le \overline{C}_{[|I|,\, C_{\mathrm{rc}}]} \left(|y(s_0)|^2_H + |f|^2_{L^2(I,\, V')} \right).$$

The proof is omitted since it follows by well known arguments.

Definition 2.5. For $f \in L^2_{\mathrm{loc}}(\mathbb{R}_{s_0},\, V')$, the function y defined in $\mathbb{R}_{s_0}$ by the property that $y|_{(s_0,\, s)}$ coincides with the weak solution of (2.1) in $(s_0,\, s)$ for all $s > s_0$, is well defined. It is called the global weak solution of (2.1) in the half-line $\mathbb{R}_{s_0} = (s_0,\, +\infty)$.

2.3. **Nonorthogonal projections**

We are going to use some nonorthogonal projection operators associated with a suitable direct sum splitting the Hilbert space H.

Definition 2.6. Two closed subspaces $\mathcal{F} \subset H$ and $\mathcal{E} \subset H$ are said complementary in the Hilbert space H if we have the direct sum $H = \mathcal{F} \oplus \mathcal{E}$. The projection onto $\mathcal{F}$ along $\mathcal{E}$ will be denoted

$$P_{\mathcal{F}}^{\mathcal{E}} \colon H \to \mathcal{F}, \quad x \mapsto x_{\mathcal{F}}$$

where $x_{\mathcal{F}}$ is defined by

$$x = x_{\mathcal{F}} + x_{\mathcal{E}} \quad \text{and} \quad (x_{\mathcal{F}}, x_{\mathcal{E}}) \in \mathcal{F} \times \mathcal{E}.$$

Remark 2.7. The continuity of the projection $P_{\mathcal{F}}^{\mathcal{E}}$ is well known, see [12, section 2.4, Theorem 2.10]. Notice that $P_{\mathcal{E}}^{\mathcal{F}} = 1 - P_{\mathcal{F}}^{\mathcal{E}}$. Here "1" is understood to be the identity operator on H.

The projection $P_{\mathcal{F}}^{\mathcal{E}}$ is orthogonal if $\mathcal{E} = \mathcal{F}^{\perp}$. We shall denote orthogonal projections simply as

$$P_{\mathcal{F}} \coloneqq P_{\mathcal{F}}^{\mathcal{F}^{\perp}}.$$

Henceforth, let us fix two sets $\{f_1, f_2, \dots, f_M\} \subset H$ and $\{g_1, g_2, \dots, g_M\} \subset H$ in the Hilbert space H. We assume that the vectors of each set are linearly independent and consider the M-dimensional subspaces $\mathcal{F} \coloneqq \operatorname{span}\{f_1, f_2, \dots, f_M\}$ and $\mathcal{G} \coloneqq \operatorname{span}\{g_1, g_2, \dots, g_M\}$.

We denote the "coordinates to span" mapping as follows

$$[\mathcal{F}] \colon \mathcal{M}_{M\times 1} \to \mathcal{F}, \qquad [\mathcal{F}]v = [\mathcal{F}] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix} \coloneqq \sum_{i=1}^{M} v_i f_i,$$

where $\mathcal{M}_{m\times n}$ denotes the space of $(m \times n)$-matrices with real entries. Notice that $[\mathcal{F}]$ depends on the (ordered) basis $\{f_1, \dots, f_M\}$. We define $[\mathcal{G}] \colon \mathcal{M}_{M\times 1} \to \mathcal{G}$ analogously. We will also denote the matrix

$$[(\mathcal{G}, \mathcal{F})_H] \coloneqq [(g_i, f_j)_H] \in \mathcal{M}_{M\times M}$$

whose entry in the i-th row and j-th column is $(g_i, f_j)_H$. Further for a given vector $y \in H$, we introduce the vectors

$$[(\mathcal{G}, y)_H] \coloneqq \begin{bmatrix} (g_1, y)_H \\ (g_2, y)_H \\ \vdots \\ (g_M, y)_H \end{bmatrix} \quad \text{and} \quad [(y, \mathcal{F})_H] \coloneqq \begin{bmatrix} (y, f_1)_H & (y, f_2)_H & \dots & (y, f_M)_H \end{bmatrix}.$$

Lemma 2.8. *The following conditions are equivalent*

(a): $H = \mathcal{F} \oplus \mathcal{G}^{\perp}$,

(b): $[(\mathcal{G}, \mathcal{F})_H]$ is invertible,

(c): $P_{\mathcal{G}}\mathcal{F} = \mathcal{G}$.

Proof. We will show the implications (a)⇒(c)⇒(b)⇒(a).
(a)⇒(c): $\mathcal{G} = P_{\mathcal{G}}H = P_{\mathcal{G}}(\mathcal{F} \oplus \mathcal{G}^{\perp}) = P_{\mathcal{G}}\mathcal{F}$.
(c)⇒(b): Given $v \in \operatorname{Ker}[(\mathcal{G}, \mathcal{F})_H]$, we have $y = [\mathcal{F}]v \in \mathcal{F} \bigcap \mathcal{G}^{\perp}$, which implies $P_{\mathcal{G}}y = 0$. Since $(P_{\mathcal{G}}|_{\mathcal{F}}) \colon \mathcal{F} \to \mathcal{G}$ is necessarily an isomorphism, and $0 = P_{\mathcal{G}}y = (P_{\mathcal{G}}|_{\mathcal{F}})y$, it follows that $y = 0$ and $v = [\mathcal{F}]^{-1}y = 0$.

(b)⇒(a): Let $h \in H$. We can write

$$h = z + w, \quad \text{with} \quad (z, w) \in (\mathcal{F} + \mathcal{G}^\perp) \times (\mathcal{F} + \mathcal{G}^\perp)^\perp.$$

Therefore we have that $w \in \mathcal{F}^\perp \bigcap \mathcal{G}$, which implies that $0 = [(w, \mathcal{F})_H] = ([\mathcal{G}]^{-1} w)^\top [(\mathcal{G}, \mathcal{F})_H]$, where $B^\top$ denotes the transpose of the matrix (vector) B. Necessarily $[\mathcal{G}]^{-1} w = 0$ and so $w = 0$, which gives us $h = z \in \mathcal{F} + \mathcal{G}^\perp$. We can concude that $H = \mathcal{F} + \mathcal{G}^\perp$. Finally, if $v \in \mathcal{F} \bigcap \mathcal{G}^\perp$ we find that $0 = [(\mathcal{G}, v)_H] = [(\mathcal{G}, \mathcal{F})_H]([\mathcal{F}]^{-1} v)$, which implies that $[\mathcal{F}]^{-1} v = 0$, hence $v = 0$. □

Lemma 2.9. *If $H = \mathcal{F} \oplus \mathcal{G}^\perp$, then the associated propjection onto $\mathcal{F}$ along $\mathcal{G}^\perp$ is given by*

$$P_{\mathcal{F}}^{\mathcal{G}^\perp} y = \left([\mathcal{F}] \circ [(\mathcal{G}, \mathcal{F})_H]^{-1}\right) [(\mathcal{G}, y)_H], \quad \textit{for all} \quad y \in H. \tag{2.3a}$$

If in addition each of the sets $\{f_1, f_2, \dots, f_M\}$ and $\{g_1, g_2, \dots, g_M\}$ is orthonormal, then

$$P_{\mathcal{F}}^{\mathcal{G}^\perp} = P_{\mathcal{F}} \circ [\mathcal{G}] \circ ([(\mathcal{G}, \mathcal{F})_H][(\mathcal{F}, \mathcal{G})_H])^{-1} \circ [\mathcal{G}]^{-1} \circ P_{\mathcal{G}}. \tag{2.3b}$$

Proof. Let us denote the mapping $y \mapsto \mathcal{P} y := \left([\mathcal{F}] \circ [(\mathcal{G}, \mathcal{F})_H]^{-1}\right) [(\mathcal{G}, y)_H]$. It follows that for any pair $(f, h) \in \mathcal{F} \times \mathcal{G}^\perp$,

$$\mathcal{P}(f + h) = \mathcal{P} f.$$

Next notice that $[(\mathcal{G}, \mathcal{F})_H] v = [(\mathcal{G}, [\mathcal{F}] v)_H]$, which implies $v = [(\mathcal{G}, \mathcal{F})_H]^{-1} [(\mathcal{G}, [\mathcal{F}] v)_H]$, for any given vector $v \in \mathcal{M}_{M \times 1}$. Therefore, since $f = [\mathcal{F}][\mathcal{F}]^{-1} f$

$$\mathcal{P}(f + h) = \left([\mathcal{F}] \circ [(\mathcal{G}, \mathcal{F})_H]^{-1}\right) [(\mathcal{G}, [\mathcal{F}][\mathcal{F}]^{-1} f)_H] = [\mathcal{F}][\mathcal{F}]^{-1} f = f,$$

that is, necessarily $P_{\mathcal{F}}^{\mathcal{G}^\perp} = \mathcal{P}$.

If the sets $\{f_1, f_2, \dots, f_M\}$ and $\{g_1, g_2, \dots, g_M\}$ are orthonormal, then it is clear that for all $(y, v) \in H \times \mathcal{M}_{M \times 1}$,

$$[(\mathcal{G}, y)_H] = ([\mathcal{G}]^{-1} \circ P_{\mathcal{G}}) y \quad \text{and} \quad P_{\mathcal{F}}([\mathcal{G}] v) = [\mathcal{F}]([(\mathcal{F}, \mathcal{G})_H] v)$$

which, together with (2.3a), leads us to

$$P_{\mathcal{F}}^{\mathcal{G}^\perp} y = \left([\mathcal{F}] \circ [(\mathcal{G}, \mathcal{F})_H]^{-1}\right) [(\mathcal{G}, y)_H] = (P_{\mathcal{F}} \circ [\mathcal{G}] \circ [(\mathcal{F}, \mathcal{G})_H]^{-1} \circ [(\mathcal{G}, \mathcal{F})_H]^{-1} \circ [\mathcal{G}]^{-1} \circ P_{\mathcal{G}}) y,$$

which is equivalent to (2.3b). □

Now we present a corollary on the computation of the norm $\left| P_{\mathcal{F}}^{\mathcal{G}^\perp} \right|_{\mathcal{L}(H)}$, which we will use in the numerical simulations.

Corollary 2.10. *If each of the sets $\{f_1, f_2, \dots, f_M\}$ and $\{g_1, g_2, \dots, g_M\}$ is orthonormal and if $H = \mathcal{F} \oplus \mathcal{G}^\perp$, then the projection onto $\mathcal{F}$ satisfies*

$$\left| P_{\mathcal{F}}^{\mathcal{G}^\perp} \right|_{\mathcal{L}(H)}^2 = \left(\min_{\theta} \{\theta \textit{ is an eigenvalue of } [(\mathcal{G}, \mathcal{F})_H][(\mathcal{F}, \mathcal{G})_H]\} \right)^{-1}.$$

Proof. First of all, note that it follows, from Lemma 2.8, that $[(\mathcal{G}, \mathcal{F})_H][(\mathcal{F}, \mathcal{G})_H]$ is symmetric and positive definite. It is well known that the orthogonal projections $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ have norm 1. It is also true that the mappings $[\mathcal{G}] \in \mathcal{L}(\mathcal{M}_{M \times 1}, \mathcal{G})$ and $[\mathcal{F}] \in \mathcal{L}(\mathcal{M}_{M \times 1}, \mathcal{F})$ are bijective isometries (we suppose $\mathcal{M}_{M \times 1} \sim \mathbb{R}^M$ endowed with the usual Euclidean scalar product). Now we observe that the mapping $[(\mathcal{G}, \mathcal{F})_H]^{-1} \in \mathcal{L}(\mathcal{M}_{M \times 1})$

is exactly the mapping sending the coordinates k of a vector field $g = [\mathcal{G}]k \in \mathcal{G}$ to the coordinates of the projection $P_{\mathcal{F}}^{\mathcal{G}^\perp} g \in \mathcal{F}$. Therefore we have that

$$\left|P_{\mathcal{F}}^{\mathcal{G}^\perp}\right|_{\mathcal{L}(H)}^2 = \sup_{k\in\mathcal{M}_{M\times1}\setminus\{0\}} \frac{\left|[(\mathcal{G},\mathcal{F})_H]^{-1}k\right|_{\mathcal{M}_{M\times1}}^2}{|k|_{\mathcal{M}_{M\times1}}^2} = \sup_{k\in\mathcal{M}_{M\times1}\setminus\{0\}} \frac{([(\mathcal{G},\mathcal{F})_H]^{-1}k)^\top[(\mathcal{G},\mathcal{F})_H]^{-1}k}{k^\top k}$$
$$= \sup_{k\in\mathcal{M}_{M\times1}\setminus\{0\}} \frac{k^\top([(\mathcal{G},\mathcal{F})_H][(\mathcal{F},\mathcal{G})_H])^{-1}k}{k^\top k} = \max_\beta\{\beta \text{ is an eigenvalue of } ([(\mathcal{G},\mathcal{F})_H][(\mathcal{F},\mathcal{G})_H])^{-1}\}.$$

The proof is finished. □

3. The stabilizing feedback control

Here we present the stabilizing control for system (1.1), provided a general condition is satisfied by the set of actuators.

Given $M \in \mathbb{N}_0$, let E_M be the space spanned by the eigenfunctions associated with the first M eigenvalues of A:

$$E_M \coloneqq \operatorname{span}\{e_n \mid Ae_n = \alpha_n e_n \text{ and } n \in \{1,2,\dots,M\}\}, \quad M \geq 1. \tag{3.1}$$

Remark 3.1. The spaces $E_M = P_{E_M}H$ are well defined as soon as the complete basis of eigenfunctions $\{e_n \mid Ae_n = \alpha_n e_n,\ n \in \mathbb{N}_0\}$ has been fixed (and further ordered for eigenfunctions corresponding to the same eigenvalue).

3.1. The condition for stabilizability

We will show that a sufficient condition for the existence of a stabilizing control taking values in a subspace $\mathcal{U} = \operatorname{span}\{\Psi_1, \Psi_2, \dots, \Psi_M\} \subset H$ is given by

$$H = \mathcal{U} \oplus E_M^\perp, \tag{3.2:$\mathbf{CS}$a}$$
$$\alpha_{M+1} > \inf_{\substack{\gamma\in\mathbb{R}_0^2,\\ (2-\gamma_1-\gamma_2)>0}} \frac{1}{(2-\gamma_1-\gamma_2)}\left(\gamma_1^{-1}\Xi_1 + \gamma_2^{-1}\left(2+2\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right)\Xi_2\right), \tag{3.2:$\mathbf{CS}$b}$$

where

$$\Xi_1 \coloneqq \sup_{(t,Y)\in\mathbb{R}_0\times(E_M^\perp\cap V)} \frac{|\langle A_{\mathrm{rc}}(t)Y,Y\rangle_{V',V}|_{\mathbb{R}}^2}{|Y|_H^2\,|Y|_V^2} \leq \left|P_{E_M^\perp}A_{\mathrm{rc}}P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2, \tag{3.3a}$$
$$\Xi_2 \coloneqq \left|P_{E_M}A_{\mathrm{rc}}P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2. \tag{3.3b}$$

In particular, we note that when $\Xi_2 = 0$, as in the case $A_{\mathrm{rc}}y = \rho y$ for some constant $\rho \in \mathbb{R}$, then the norm $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ of the projection plays no role in (3.2:**CS**b).

Observe that from (3.2:**CS**a) it necessarily follows that $\mathcal{U}$ is a M-dimensional space, because (3.2:**CS**a) implies $E_M = P_{E_M}H = P_{E_M}(\mathcal{U} \oplus E_M^\perp) = P_{E_M}\mathcal{U}$, then the vectors in $\{\Psi_1, \Psi_2, \dots, \Psi_M\}$ are necessarily linearly independent.

Lemma 3.2. *Assume that $\mathcal{U}$ satisfies* (3.2:**CS**a)*, then we also have $V' = \mathcal{U} \oplus E_M^{\perp,V'}$. Here the space $E_M^{\perp,V'}$ stands for the orthogonal complement of E_M in V'.*

Proof. Let $x \in V'$. Then $x \in E_M \oplus E_M^{\perp,V'} \subseteq \mathcal{U} + E_M^{\perp} + E_M^{\perp,V'} \subseteq \mathcal{U} + E_M^{\perp,V'}$. We can conclude that $V' = \mathcal{U} + E_M^{\perp,V'}$. Now if $y \in \mathcal{U} \bigcap E_M^{\perp,V'}$, then for all $w \in E_M$

$$(y, w)_H = (A^{-\frac{1}{2}} y, A^{-\frac{1}{2}} Aw)_H = (y, Aw)_{V'} = 0,$$

since A maps E_M onto itself. That is, $y \in \mathcal{U} \bigcap E_M^{\perp} = \{0\}$. $\square$

From Lemma 3.2 the projections $P_{\mathcal{U}}^{E_M^{\perp}} \in \mathcal{L}(H)$ and $P_{E_M^{\perp}}^{\mathcal{U}} \in \mathcal{L}(H)$ can be extended to projections $P_{\mathcal{U}}^{E_M^{\perp,V'}} \in \mathcal{L}(V')$ and $P_{E_M^{\perp,V'}}^{\mathcal{U}} \in \mathcal{L}(V')$. Hereafter, for simplicity, we will still denote the latter by $P_{\mathcal{U}}^{E_M^{\perp}} \in \mathcal{L}(V')$ and $P_{E_M^{\perp}}^{\mathcal{U}} \in \mathcal{L}(V')$ instead.

Lemma 3.3. *Assume that $\mathcal{U}$ satisfies* (3.2:**CS**a)*, then we have the following properties.*

$$P_{E_M} = P_{E_M} P_{\mathcal{U}}^{E_M^{\perp}}, \qquad P_{\mathcal{U}}^{E_M^{\perp}} = P_{\mathcal{U}}^{E_M^{\perp}} P_{E_M}, \qquad \text{and} \qquad P_{E_M^{\perp}}^{\mathcal{U}} = P_{E_M^{\perp}} + P_{E_M^{\perp}}^{\mathcal{U}} P_{E_M}.$$

Proof. The proof is straightforward. $\square$

We recall the interval $I = (s_0, s_1)$ in (2.2).

Lemma 3.4. *Let $\mathcal{U} \subset H$ satisfy* (3.2:**CS**a)*, $w_0 \in E_M^{\perp}$, and $q \in H^1(I, E_M)$. Then there exists a weak solution, taking its values in $E_M^{\perp} \subset H$, for the system*

$$\dot{Q} + P_{E_M^{\perp}}^{\mathcal{U}} \Big((A + A_{\mathrm{rc}}) (Q + q) \Big) + P_{E_M^{\perp}}^{\mathcal{U}} \dot{q} = 0, \qquad Q(s_0) = w_0. \tag{3.4}$$

Moreover the solution is unique and depends continuously on the data as follows:

$$|Q|_{W(I,V,V')}^2 \le \overline{C}_{\left[|I|, M, C_{\mathrm{rc}}, \left|P_{E_M^{\perp}}^{\mathcal{U}}\right|_{\mathcal{L}(V')}, \left|P_{\mathcal{U}}^{E_M^{\perp}}\right|_{\mathcal{L}(H)}\right]} \Big(|w_0|_H^2 + |q|_{H^1(I,H)}^2 \Big). \tag{3.5}$$

Proof. Existence of a solution satisfying (3.4) follows by standard arguments. We restrict ourselves to the following estimates, which will in particular show the structure of the constant $\overline{C}$ in (3.5). We look for Q taking its values in $V \bigcap E_M^{\perp} = V \bigcap P_{E_M^{\perp}}^{\mathcal{U}} H$. Observe that A maps $V \bigcap E_M^{\perp}$ into $E_M^{\perp,V'} \subset V'$. For $Q \in V \bigcap E_M^{\perp}$ we have that $AQ \in E_M^{\perp,V'}$, $P_{E_M^{\perp}}^{\mathcal{U}} AQ = AQ$, and in particular $\langle P_{E_M^{\perp}}^{\mathcal{U}} AQ, Q \rangle_{V',V} = |Q|_V^2$. Taking the duality product with $2Q$ in (3.4), and using Lemma 3.3, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} |Q|_H^2 &= -2 |Q|_V^2 - 2\langle P_{E_M^{\perp}}^{\mathcal{U}} Aq, Q \rangle_{V',V} - 2\langle P_{E_M^{\perp}}^{\mathcal{U}} A_{\mathrm{rc}}(Q + q), Q \rangle_{V',V} - 2\langle P_{E_M^{\perp}}^{\mathcal{U}} \dot{q}, Q \rangle_{V',V} \\
&= -2 |Q|_V^2 - 2\langle P_{E_M^{\perp}} A_{\mathrm{rc}} Q, Q \rangle_{V',V} - 2\langle P_{E_M^{\perp}}^{\mathcal{U}} P_{E_M} A_{\mathrm{rc}} Q, Q \rangle_{V',V} \\
&\quad - 2\langle P_{E_M^{\perp}}^{\mathcal{U}} Aq, Q \rangle_{V',V} - 2\langle P_{E_M^{\perp}}^{\mathcal{U}} A_{\mathrm{rc}} q, Q \rangle_{V',V} - 2\langle P_{E_M^{\perp}}^{\mathcal{U}} \dot{q}, Q \rangle_{V',V}
\end{aligned}$$

and for any given positive constants γ_1, γ_2, and γ_3,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} |Q|_H^2 \le{}& -(2 - \gamma_1 - \gamma_2 - 3\gamma_3) |Q|_V^2 + \gamma_1^{-1} \Xi_1 |Q|_H^2 + \gamma_2^{-1} \left| P_{E_M^{\perp}}^{\mathcal{U}} P_{E_M} \right|_{\mathcal{L}(V')}^2 \Xi_2 |Q|_H^2 \\
&+ \gamma_3^{-1} \left| P_{E_M^{\perp}}^{\mathcal{U}} P_{E_M} \right|_{\mathcal{L}(V')}^2 \Big(|q|_V^2 + |\dot{q}|_{V'}^2 \Big) + \gamma_3^{-1} \left| P_{E_M^{\perp}}^{\mathcal{U}} \right|_{\mathcal{L}(V')}^2 \Xi_3 |q|_H^2
\end{aligned}$$

with Ξ_1 and Ξ_2 as in (3.3a)-(3.3b) and with

$$\Xi_3 := |P_{\mathcal{U}^\perp} A_{\mathrm{rc}} P_{E_M}|^2_{\mathcal{L}(V')}.$$

For $X \hookrightarrow Y$, let us denote by $|\iota|_{\mathcal{L}(X\to Y)}$ the norm of the inclusion mapping $\iota(x) = x$ for all $x \in X$. If $X_0 \subseteq X$ is a subspace of X we also denote by $\left|\iota|_{X_0}\right|_{\mathcal{L}(X\to Y)}$ the norm of the inclusion $X_0 \hookrightarrow Y$. Recall that for $p \in E_M$, we have $|p|^2_H \le \alpha_M |p|^2_{V'}$. Also, for $z \in E_M^\perp$ we have $|z|^2_{V'} \le \alpha_{M+1}^{-1} |p|^2_H$. Therefore, using Lemma 3.3 and the identity $1 = P_{\mathcal{U}}^{E_M^\perp} + P_{E_M^\perp}^{\mathcal{U}}$, we find

$$\begin{aligned}\left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|^2_{\mathcal{L}(V')} &\le \left|\iota|_{E_M^\perp}\right|^2_{\mathcal{L}(H,V')} \left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|^2_{\mathcal{L}(H,H)} \left|\iota|_{E_M}\right|^2_{\mathcal{L}(V',H)} |P_{E_M}|^2_{\mathcal{L}(V',V')}\\ &\le \alpha_{M+1}^{-1} \left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|^2_{\mathcal{L}(H)} \alpha_M \le \left|P_{E_M} - P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(H)} \le 2 + 2\left|P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(H)},\end{aligned}$$

which, since $|\iota|^2_{\mathcal{L}(H,V')} \le \alpha_1^{-1}$, leads us to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} |Q|^2_H \le{}& -(2-\gamma_1-\gamma_2-3\gamma_3)|Q|^2_V + \left(\gamma_1^{-1}\Xi_1 + \gamma_2^{-1}\left(2 + 2\left|P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(H)}\right)\Xi_2\right)|Q|^2_H\\ &+ \gamma_3^{-1}\left(\left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|^2_{\mathcal{L}(V')} + \left|P_{E_M^\perp}^{\mathcal{U}}\right|^2_{\mathcal{L}(V')}\Xi_3\right)\left(\alpha_M |q|^2_H + |q|^2_H + \alpha_1^{-1}|\dot q|^2_H\right).\end{aligned} \tag{3.6}$$

For any time $t \in I$ and any triple $(\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}_0^3$ such that that $2-\gamma_1-\gamma_2-3\gamma_3 > 0$, by Gronwall's inequality we obtain

$$|Q(t)|^2_H \le \mathrm{e}^{D_1(t-s_0)}\left(|w_0|^2_H + D_2 |q|^2_{H^1(I,H)}\right) \tag{3.7}$$

with

$$\begin{aligned}D_1 &:= C_{\mathrm{rc}}^2\left(\gamma_1^{-1} + 2\gamma_2^{-1} + 2\gamma_2^{-1}\left|P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(H)}\right),\\ D_2 &:= \gamma_3^{-1}\left(\left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|^2_{\mathcal{L}(V')} + \left|P_{E_M^\perp}^{\mathcal{U}}\right|^2_{\mathcal{L}(V')}\Xi_3\right)\left(1 + \alpha_M + \alpha_1^{-1}\right),\end{aligned}$$

where C_{rc}^2 was defined in Assumption 2.3. Some more standard estimates lead us to (3.5).

Finally, the uniqueness of Q follows from the fact that the difference between two solutions will solve (3.4) with $q = 0$ and $w_0 = 0$, and in that case the right hand side of (3.7) vanishes. □

3.2. The explicit closed loop system

Hereafter we suppose that Assumptions 2.1, 2.2, and 2.3 are satisfied. The following theorem shows that condition (3.2:**CS**) guarantees the existence of a finite dimensional explicit stabilizing feedback operator.

Theorem 3.5. *Let $\lambda > 0$. If $\mathcal{U}$ satisfies (3.2:**CS**), then the system*

$$\dot y(t) + Ay(t) + A_{\mathrm{rc}}(t)y(t) - P_{\mathcal{U}}^{E_M^\perp}\left(Ay(t) + A_{\mathrm{rc}}(t)y(t) - \lambda y(t)\right) = 0, \tag{3.8a}$$

$$y(0) = y_0, \tag{3.8b}$$

has a unique weak solution. Moreover there is a pair of constants $D \ge 1$ and $\mu > 0$ such that

$$|y(t)|^2_H \le D\mathrm{e}^{-\mu(t-s_0)} |y(s_0)|^2_H, \quad \textit{for all} \quad t \ge s_0 \ge 0. \tag{3.9}$$

Proof. Let us take $q = \mathrm{e}^{-\lambda t} P_{E_M} y_0$ and $w_0 = P_{E_M^\perp} y_0$, and let Q be the corresponding solution to system (3.4). Then, we observe that $y = Q + q$ solves (3.8). In fact

$$\begin{aligned}
\dot y &= -P_{E_M^\perp}^{\mathcal{U}} \Big((A + A_{\mathrm{rc}})\,(Q+q) \Big) - P_{E_M^\perp}^{\mathcal{U}} \dot q + \dot q = -P_{E_M^\perp}^{\mathcal{U}} \Big((A + A_{\mathrm{rc}})\,y \Big) + P_{\mathcal{U}}^{E_M^\perp} \dot q\\
&= -\Big(1 - P_{\mathcal{U}}^{E_M^\perp}\Big)\Big((A + A_{\mathrm{rc}})\,y\Big) - \lambda P_{\mathcal{U}}^{E_M^\perp} q,
\end{aligned}$$

where we have used $1 = P_{\mathcal{U}}^{E_M^\perp} + P_{E_M^\perp}^{\mathcal{U}}$ (cf. Remark 2.7). Therefore (3.8a) follows. Clearly we also have $y(0) = Q(0) + q(0) = w_0 + P_{E_M} y_0 = y_0$.

From (3.2:**CS**b) we can choose a pair $(\widetilde\gamma_1, \widetilde\gamma_2)$ such that $2 - \widetilde\gamma_1 - \widetilde\gamma_2 > 0$ and

$$\mu := (2 - \widetilde\gamma_1 - \widetilde\gamma_2)\alpha_{M+1} - \left(\widetilde\gamma_1^{-1}\Xi_1 + \widetilde\gamma_2^{-1} \left(2 + 2 \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^2 \right) \Xi_2 \right) > 0.$$

Then we can also choose a small enough $\widetilde\gamma_3 > 0$ such that

$$\mu_{\widetilde\gamma} := (2 - \widetilde\gamma_1 - \widetilde\gamma_2 - 3\widetilde\gamma_3)\alpha_{M+1} - \left(\widetilde\gamma_1^{-1}\Xi_1 + \widetilde\gamma_2^{-1} \left(2 + 2 \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^2 \right) \Xi_2 \right) > 0.$$

Therefore, from (3.6) and the Gronwall's inequality, it follows that, for any $t \ge s_0 \ge 0$,

$$\begin{aligned}
|Q(t)|_H^2 &\le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + \widetilde D_2 \int_{s_0}^t \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s)} (|q(s)|_H^2 + |\dot q(s)|_H^2 \,\mathrm{d}s\\
&\le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + \widetilde D_2 (1+\lambda^2) \int_{s_0}^t \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s)} \mathrm{e}^{-2\lambda(s-s_0)} |q(s_0)|_H^2 \,\mathrm{d}s
\end{aligned} \tag{3.10}$$

with $\widetilde D_2 := \widetilde\gamma_3^{-1} \left(\left| P_{E_M^\perp}^{\mathcal{U}} P_{E_M} \right|_{\mathcal{L}(V')}^2 + \left| P_{E_M^\perp}^{\mathcal{U}} \right|_{\mathcal{L}(V')}^2 \Xi_3 \right) \left(1 + \alpha_M + \alpha_1^{-1} \right)$.

Now, if $2\lambda \ne \mu_{\widetilde\gamma}$, we obtain

$$\begin{aligned}
|Q(t)|_H^2 &\le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + \widetilde D_2 (1+\lambda^2) |q(s_0)|_H^2 \mathrm{e}^{-\mu_{\widetilde\gamma} t + 2\lambda s_0} \frac{\mathrm{e}^{(\mu_{\widetilde\gamma} - 2\lambda)t} - \mathrm{e}^{(\mu_{\widetilde\gamma} - 2\lambda)s_0}}{\mu_{\widetilde\gamma} - 2\lambda}\\
&\le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + \widetilde D_2 (1+\lambda^2) |q(s_0)|_H^2 |\mu_{\widetilde\gamma} - 2\lambda|^{-1} \mathrm{e}^{-\mu(t-s_0)}
\end{aligned}$$

with $\mu := \min\{\mu_{\widetilde\gamma}, 2\lambda\} > 0$.

On the other hand, if $2\lambda = \mu_{\widetilde\gamma}$ we obtain

$$\begin{aligned}
|Q(t)|_H^2 &\le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + \widetilde D_2 (1+\lambda^2) |q(s_0)|_H^2 \int_{s_0}^t \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} \,\mathrm{d}s\\
&\le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + \widetilde D_2 (1+\lambda^2) |q(s_0)|_H^2 \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} (t - s_0)\\
&\le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + \widetilde D_2 (1+\lambda^2) |q(s_0)|_H^2 \mathrm{e}^{-\mu(t-s_0)} e^{-1} (\mu_{\widetilde\gamma} - \mu)^{-1}
\end{aligned}$$

for any $\mu < \mu_{\widetilde\gamma}$. Notice that $e^{-1}(\mu_{\widetilde\gamma} - \mu)^{-1} = \max_{s>0} \mathrm{e}^{-(\mu_{\widetilde\gamma} - \mu)s} s$.

Thus, in either case, there exists a constant D_3 such that

$$|Q(t)|_H^2 \le \mathrm{e}^{-\mu_{\widetilde\gamma}(t-s_0)} |Q(s_0)|_H^2 + D_3 \mathrm{e}^{-\mu(t-s_0)} |q(s_0)|_H^2,$$

with $\mu < \min\{\mu_{\widetilde{\gamma}}, 2\lambda\}$. This implies that

$$\begin{aligned} |y(t)|_H^2 &= |Q(t)|_H^2 + |q(t)|_H^2 \le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)} |Q(s_0)|_H^2 + (1+D_3)\mathrm{e}^{-\mu(t-s_0)} |q(s_0)|_H^2 \\ &\le D\mathrm{e}^{-\mu(t-s_0)} |y(s_0)|_H^2 \end{aligned} \tag{3.11}$$

with $\mu < \min\{\mu_{\widetilde{\gamma}}, 2\lambda\} > 0$ and $D \coloneqq 1 + D_3 \ge 1$. □

Next we prove that our feedback operator in Theorem 3.5 is bounded.

Theorem 3.6. *The feedback operator in Theorem 3.5*

$$y \to \mathcal{K}(t)y \coloneqq P_{\mathcal{U}}^{E_M^\perp}(Ay + A_{\mathrm{rc}}(t)y - \lambda y)$$

and its associated control $\eta(t) \coloneqq \mathcal{K}(t)y(t)$ are bounded:

$$|\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \le \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(V',H)} \left(\alpha_M^{\frac{1}{2}} + |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} + \lambda\alpha_1^{-\frac{1}{2}}\right), \tag{3.12a}$$

$$\left|\mathrm{e}^{\frac{\varepsilon}{2}(t-s_0)}\eta\right|_{L^2(\mathbb{R}_{s_0},H)}^2 \le |\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))}^2 \frac{D}{\mu-\varepsilon} |y(s_0)|_H^2, \quad \textit{for any} \quad \varepsilon < \mu, \tag{3.12b}$$

with μ as in (3.9)*.*

Proof. Recalling that $P_{\mathcal{U}}^{E_M^\perp} = P_{\mathcal{U}}^{E_M^\perp} P_{E_M}$ and $P_{E_M}A = AP_{E_M}$, we find

$$|\mathcal{K}(t)|_{\mathcal{L}(H)} \le \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(V',H)} \left(\left|A|_{E_M}\right|_{\mathcal{L}(H,V')} + |A_{\mathrm{rc}}(t)|_{\mathcal{L}(H,V')} + \lambda\alpha_1^{-\frac{1}{2}}\right),$$

which implies that

$$|\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \le \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(V',H)} \left(\alpha_M^{\frac{1}{2}} + |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} + \lambda\alpha_1^{-\frac{1}{2}}\right),$$

Now from (3.9) we find that the associated control $\eta(t)$ satisfies, for any $\varepsilon < \mu$,

$$\begin{aligned} \left|\mathrm{e}^{\frac{\varepsilon}{2}(t-s_0)}\eta\right|_{L^2(\mathbb{R}_{s_0},H)}^2 &\le |\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))}^2 \left|\mathrm{e}^{\frac{\varepsilon}{2}(t-s_0)}y\right|_{L^2(\mathbb{R}_{s_0},H)}^2 \\ &\le |\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))}^2 \frac{D}{\mu-\varepsilon} |y(s_0)|_H^2, \end{aligned}$$

which completes the proof. □

Remark 3.7. We can prove that under the same condition (3.2:**CS**) the feedback given by $y \to \mathcal{K}_*(t)y \coloneqq -P_{\mathcal{U}}^{E_M^\perp}(A_{\mathrm{rc}}(t)y - \lambda y)$ is also stabilizing (which corresponds to take $q_* = P_{E_M}y$ as the solution of the system $\dot{q}_* = -Aq_* - \lambda q_*, \quad q_*(0) = P_{E_M}y(0)$). In this case, in (3.12a) we obtain a better estimate $|\mathcal{K}_*|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \le \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(V',H)} \left(|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} + \lambda\alpha_1^{-\frac{1}{2}}\right)$. However, from simulations we have performed (and will not present here) it is not clear whether the associated bound constant D in (3.9) (and in (3.12b)) will be smaller for $\mathcal{K}_*$ than for $\mathcal{K}$. Thus, we have chosen to consider the feedback $\mathcal{K}$ only, whose associated projection $q = P_{E_M}y$ is more explicit.

Remark 3.8. We observe that the stabilization result in Theorem 3.5 still holds true when E_M is replaced by a space spanned by any set of M linearly independent eigenfunctions, say by $\widetilde{E}_M = \mathrm{span}\{e_{\sigma(i)} \mid i \in \{1, 2, \dots, M\}\}$

with $\sigma\colon \mathbb{N}_0 \to \mathbb{N}_0$ being an increasing function. For that we just need to replace the condition (3.2:**CS**) by the analogous one with

$$(\widetilde{E}_M, \min\{\alpha_j \mid \alpha_j \notin \{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots \alpha_{\sigma(M)}\}\})$$

in the place of (E_M, α_{M+1}). This will be used in section 4.8.

4. Remarks on the proposed sufficient condition for stability

By considering the values $(\gamma_1, \gamma_2) = (\frac{1}{2}, \frac{1}{2})$ we conclude that (3.2:**CS**b) holds if

$$\alpha_{M+1} > \left(6 + 4 \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^2 \right) |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))}^2 . \tag{4.1}$$

which is exactly (1.3), in the Introduction.

Once we know $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))}^2$ and α_{M+1}, we can investigate (4.1), by computing $\left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^2$.

In applications the actuators will be in a suitable class $\mathbb{U}$ (of functions). For example, for parabolic equations (1.2), we could consider actuators which are supported in an apriori given subset, or we could consider piecewise constant actuators.

The question is: given $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))}^2$ can we find M and actuators $\{\Psi_i^M \mid i = 1, 2, \dots, M\} \subset \mathbb{U}$ so that $\mathcal{U} = \mathcal{U}(M) = \operatorname{span}\{\Psi_i^M \mid i = 1, 2, \dots, M\}$ satisfies (4.1)?

For a general $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))}^2$ we can say that (4.1) will hold for big enough M if we know that

$$\lim_{M \to +\infty} \alpha_{M+1} \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^{-2} = +\infty. \tag{4.2}$$

This results in the question: can we find a large enough M and a set of actuators $\{\Psi_i^M \mid i = 1, 2, \dots, M\}$ so that their span, $\mathcal{U} = \mathcal{U}(M) = \operatorname{span}\{\Psi_i^M \mid i = 1, 2, \dots, M\}$, satisfies (4.2)? We recall that the answer is not trivial when $\mathcal{U} \neq E_M$ because $\left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)} > 1$ and, depending on $\mathcal{U}$, may take arbitrary large values.

4.1. The case $M = 0$. Free dynamics

Taking no control in (3.8a) is equivalent to taking $\mathcal{U} = \{0\}$. That is, the free dynamics is followed. This corresponds to take $M = 0$, once we define $E_0 \coloneqq \{0\}$. Indeed, we have $H = \mathcal{U} \oplus H = \mathcal{U} \oplus E_M^\perp$, and $P_{E_M} = 0 = P_{\mathcal{U}}^{E_M^\top}$. In particular (3.2:**CS**a) is satisfied.

Since, for $M = 0$ we have that $\Xi_2 = 0$, the condition for stability (3.2:**CS**b) reads for free dynamics

$$\alpha_1 > \inf_{\substack{\gamma_1 \in \mathbb{R}_0, \\ (2-\gamma_1) > 0}} \frac{1}{(2-\gamma_1)\gamma_1} \Xi_1 = \Xi_1 = \sup_{(t,Y) \in \mathbb{R}_0 \times V} \frac{|\langle A_{\mathrm{rc}}(t) Y, Y \rangle_{V',V}|_{\mathbb{R}}^2}{|Y|_H^2 \, |Y|_V^2}.$$

4.2. Checking the proposed sufficient condition numerically

We assume here that the complete system of eigenfunctions $\{e_i \mid i \in \mathbb{N}_0\}$ of A is orthonormal in the H-scalar product. We can also suppose that $\{\Psi_i^M \mid i = 1, 2, \dots, M\}$ is an orthonormal family in the H-scalar product, otherwise we just orthonormalize it.

From Corollary 2.10, condition (4.2) will follow from

$$\lim_{M \to +\infty} \alpha_{M+1} \min \operatorname{Eig}([\Theta(M)]) = +\infty, \tag{4.3}$$

where for simplicity we have denoted by $\mathrm{Eig}(([\Theta(M)])$ the set of eigenvalues of the symmetric matrix

$$[\Theta(M)] \coloneqq [(E_M,\mathcal{U})_H][(\mathcal{U},E_M)_H]. \tag{4.4}$$

Now we show how we are going to check (4.3) numerically, in the case of a parabolic equation as in (1.2) under homogeneous Dirichlet boundary conditions, with $A = -\nu\Delta\colon V \to V'$, $\nu > 0$.

In this setting we will have the spaces $H = L^2(\Omega)$, $V = H^1_0(\Omega)$, and $V' = H^{-1}(\Omega)$.

We will consider a given triangulation Ω_D of the domain Ω with nodes $\mathbf{x}$. As basis functions we take the usual piecewise linear (hat) functions.

For a given (regular enough) function u, let $u(\mathbf{x})$ denote the column vector of the values of u at the nodes.

Let $\mathbf{M}$ denote the mass matrix associated to our finite-element subspace in H. We take the following discrete approximations:

$$\begin{aligned}
[(\mathcal{U},E_M)_H] &\approx \mathbf{P}(M) \coloneqq \begin{bmatrix}\Psi_1^M(\mathbf{x})^\top\\ \Psi_2^M(\mathbf{x})^\top\\ \vdots\\ \Psi_M^M(\mathbf{x})^\top\end{bmatrix}\mathbf{M}\begin{bmatrix}e_1(\mathbf{x}) & e_2(\mathbf{x}) & \dots & e_M(\mathbf{x})\end{bmatrix},\\
[\Theta(M)] &\approx [\boldsymbol{\Theta}(M)] \coloneqq \mathbf{P}(M)^\top\mathbf{P}(M),
\end{aligned}$$

and

$$\min\mathrm{Eig}([\Theta(M)]) \approx \min\mathrm{Eig}([\boldsymbol{\Theta}(M)]).$$

The matrix $\mathbf{P}(M)^\top\mathbf{P}(M)$ is symmetric and positive semidefinite. By (3.2:**CS**a) and Lemma 2.8 the matrix $[\Theta(M)]$ is invertible. Up to discretization, this translates to the assumption that $[\boldsymbol{\Theta}(M)]$ is positive definite, that is, $\min\mathrm{Eig}([\boldsymbol{\Theta}(M)]) > 0$.

Denoting $\vartheta_M \coloneqq \min\mathrm{Eig}([\boldsymbol{\Theta}(M)])$, condition (4.3) will read

$$\lim_{M\to+\infty} \alpha_{M+1}\vartheta_M = +\infty. \tag{4.5}$$

4.3. Construction of the actuators in one dimension

We will perform some 1D simulations on the interval $\Omega = (0,L)$, for piecewise constant actuators and $A = -\nu\Delta = -\nu\partial_{xx}$. For a suitable chosen $\mathcal{U} = \mathcal{U}(M)$, the simulations suggest that $\vartheta_M \ge \delta > 0$ remains away from zero, with δ independent of M, when the total length of the actuators support is fixed. This confirms (4.5).

In this 1D case the finite elements are chosen with respect to the partition $\mathbf{x} = \begin{bmatrix}0h & 1h & \dots & Nh\end{bmatrix}^\top$ of Ω, where $h = L/N$. We fix the total length of the support of the control, by setting the total volume to be rL, with $r \in (0,1)$.

Piecewise constant actuators $\{\Psi_i^M \mid i = 1,2,\dots,M\}$ are constructed as follows: consider the M-th normalized eigenfunction $e_M(x) = (\frac{2}{L})^{\frac12}\sin(\frac{M\pi x}{L})$ whose extrema are located at

$$q_j = (2j-1)\frac{L}{2M}, \quad j = 1,2,\dots,M.$$

Now we set the actuators as the indicator functions

$$\Psi_i^M(x) \coloneqq 1_{\omega_i}(x) \coloneqq \begin{cases}1, & \text{if } x\in\omega_i;\\ 0, & \text{if } x\in(0,L)\setminus\overline{\omega_i};\end{cases},$$

with

$$\omega_i \coloneqq (q_j - \frac{rL}{2M}, q_j + \frac{rL}{2M}), \quad i = 1, 2, \dots, M.$$

Note that all actuators have the same support length: $\mathrm{length}(\mathrm{supp}(\Psi_i^M)) = \frac{rL}{M}$, and the total length of the support of the control is independent of M: $\mathrm{length}\left(\bigcup_{i=1}^{M} \mathrm{supp}(\Psi_i^M)\right) = \sum_{i=1}^{M} \mathrm{length}(\mathrm{supp}(\Psi_i^M)) = rL$.

4.4. The particular case of a constant reaction

From condition (3.2:**CS**) we see that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}$ will play no role when $\Xi_2 = \left|P_{E_M} A_{\mathrm{rc}} P_{E_M^\perp}\right|^2_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))} = 0$. This is the case for a constant reaction $A_{\mathrm{rc}} y = \rho y$.

With $\rho < 0$, the uncontrolled system

$$\dot{y} + Ay + \rho y = 0$$

is not stable if $\rho < -\alpha_1$. In this case $\left|P_{E_M^\perp} A_{\mathrm{rc}} P_{E_M^\perp}\right|^2_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))} = \alpha_{M+1}^{-1}\rho^2$, and the sufficient condition (3.2:**CS**) reduces to

$$\begin{aligned} H &= \mathcal{U} \oplus E_M^\perp \\ \alpha_{M+1} &> \inf_{\substack{\gamma_1 \in \mathbb{R}_0, \\ (2-\gamma_1)>0}} \frac{1}{(2-\gamma_1)} \gamma_1^{-1} \alpha_{M+1}^{-1} \rho^2 = \alpha_{M+1}^{-1} \rho^2. \end{aligned}$$

In particular, here the norm $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}$ plays no role. That is, it is sufficient to take M actuators such that $\alpha_{M+1} > \rho$ and $H = \mathcal{U} \oplus E_M^\perp$. Recall that the latter identity is equivalent to the invertibility of $[\Theta(M)]$, which can be checked numerically.

Notice, however that $\left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|^2_{\mathcal{L}(V')}$ and $\left|P_{E_M^\perp}^{\mathcal{U}}\right|^2_{\mathcal{L}(V')} \left|P_{\mathcal{U}^\perp} A_{\mathrm{rc}} P_{E_M}\right|^2_{\mathcal{L}(V')}$ still play a role to the constant D in (3.11), and thus also in the norm of the feedback control in (3.12b).

4.5. The particular case of a conservative convection

It is well known that the parabolic system (1.2), under homogeneous Dirichlet boundary conditions is stable with $a = 0$ and with a general $b \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}^d)$ satisfying $\nabla \cdot b = 0$. We can see that condition (3.2:**CS**) reflects this fact. Indeed, in this case we have $\langle A_{\mathrm{rc}} y, y\rangle_{V',V} = \langle \nabla \cdot (by), y\rangle_{V',V} = 0$. Thus, for $M = 0$, we obtain that $\Xi_1 = 0$. From section 4.1 we conclude that the uncontrolled system is stable, since $\alpha_1 > 0$.

4.6. Numerical examples

Here we perform simulations concerning the computation of $\vartheta_M = \left|P_{\mathcal{U}}^{E_M^\perp}\right|^{-2}_{\mathcal{L}(H)}$ appearing in the sufficient condition (4.5). We would like to have $\vartheta_M > 0$ as large as possible, and remaining away from 0 as M increases.

Our actuators $1_{\omega_i} = 1_{\omega_i}(x)$ will be constructed as in section 4.3 (in the figures "$\mathcal{D}_{\mathrm{act}} = \mathrm{mxe}$" underlines that the actuators are distributed following the extrema of the eigenfunction e_M). Figure 1 show those actuators for selected values of M, for $L = 1$.

To compute ϑ_M we have proceeded as follows. We have considered the basis $\{\sin(\frac{i\pi x}{L}) \mid i \in \{1, 2, \dots, M\}\}$ for E_M and the basis $\{1_{\omega_i} \mid i \in \{1, 2, \dots, M\}\}\}$ for $\mathcal{U}$. Then we have orthonormalized these bases, constructed $[\mathbf{\Theta}(M)]$, and computed ϑ_M. The orthonormalization have been done numerically through the Gram–Schmidt procedure (in the "mass" scalar product $(v, w) \mapsto w^\top \mathbf{M} v$).

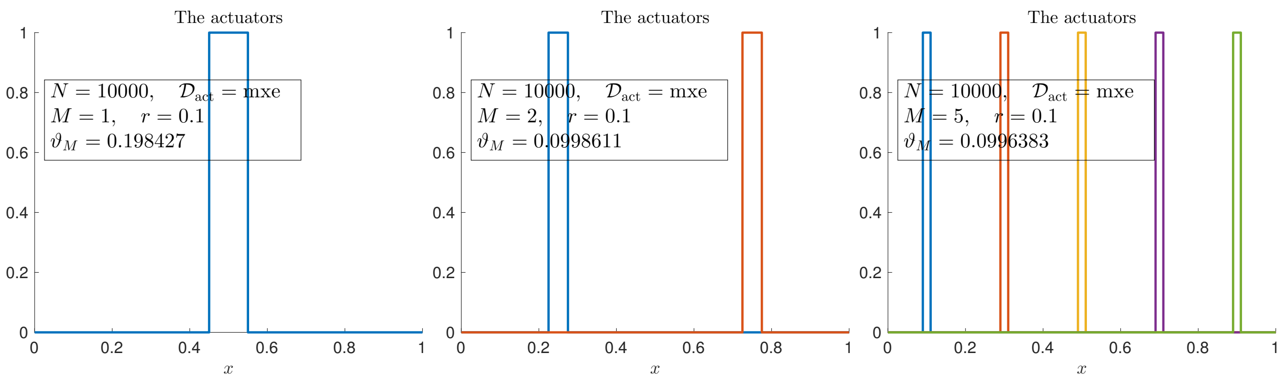
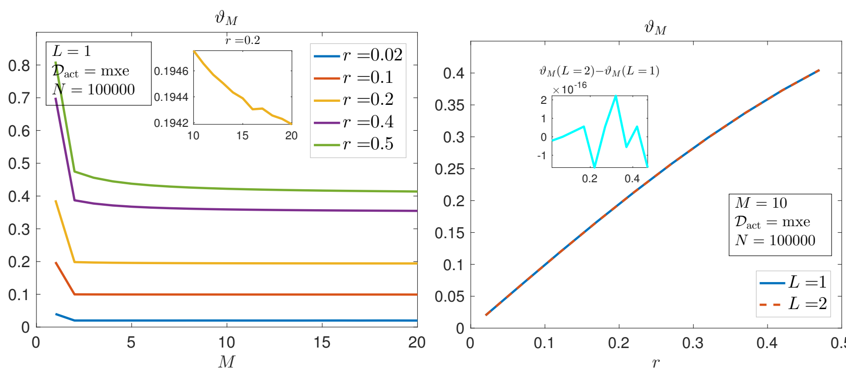


FIGURE 1. The family of actuators.

In Figure 2(a) we see how ϑ_M depends on M, for several rates r of the control support volume rL. It seems that, for fixed r, ϑ_M tends to a positive constant, thus remaining away from zero.

Further ϑ_M increases with $r \in (0, \frac{1}{2})$. That is, as we would intuitively expect, the norm of $P_{\mathcal{U}}^{E_M}$ gets larger as the volume of the control support gets smaller.



(a) Dependence on M. (b) Dependence on r

FIGURE 2. The smallest eigenvalue of $[\Theta(M)]$.

In Figure 2(b) we see how ϑ_M depends on r, for a fixed given pair (M, L).

The simulations suggest that there exist a constant $C = C(r) > 0$ so that $\left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^2 \le C(r)$. Notice that for $\Omega = (0, L)$, the eigenvalues of $A = -\nu\Delta$ are $\nu \frac{\pi^2}{L^2} M^2$ which allows us to obtain an estimate

$$M > \nu^{-\frac{1}{2}} \frac{L}{\pi} (6 + 4C(r))^{\frac{1}{2}} \left| A_{\mathrm{rc}} \right|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H, V'))}$$

on the number of actuators which allow us to stabilize the system.

In Figure 2(b) we can also see that such dependence does not depend on L, once (r, M) is given. Indeed this follows from the fact that for a given two intervals $(0, L_1)$ and $(0, L_2)$, for normalized eigenfunctions $e_i^{L_n} = (\frac{2}{L_n})^{\frac{1}{2}} \sin(\frac{i\pi x}{L_n}) \in L^2((0, L_n))$, and for normalized actuators $\psi_{a,b}^{L_n}(x) = (\frac{1}{L_n(b-a)})^{\frac{1}{2}} 1_{(L_n a, L_n b)}(x) \in L^2((0, L_n))$, with $1 > b > a > 0$, we have that,

$$(e_i^{L_1}, \psi_{a,b}^{L_1})_{L^2((0,L_1))} = (e_i^{L_1}, \psi_{a,b}^{L_2})_{L^2((0,L_2))}.$$

This means that, for normalized eigenfunctions and actuators, the entries in $[(E_M,\mathcal{U})_H]$ and in $[\Theta(M)]$ do not depend on the lenght L_n provided, when taking actuators $1_{(L_1a,L_1b)}$ in $(0,L_1)$, we take the corresponding actuators $1_{(L_2a,L_2b)}$ on $(0,L_2)$.

4.7. On the placement of the actuators

Intuitively we expect that distributing the actuators over the interval $\Omega=(0,L)$ is better than concentrating them in some region. This is confirmed in Figures 3 and 4. The uniformly distributed actuators 1_{ω_i} ("$\mathcal{D}_{\rm act}=$ uni" in figures) correspond to

$$\omega_i=\left(\tfrac{iL}{M+1}-\tfrac{rL}{2M},\tfrac{iL}{M+1}+\tfrac{rL}{2M}\right),\quad i=1,\dots,M,$$

while the concentrated actuators ("$\mathcal{D}_{\rm act}=$ con" in figures) correspond to a uniform partition of $(\frac{L}{2}-\frac{rL}{2},\frac{L}{2}+\frac{rL}{2})$:

$$\omega_j=\left(\tfrac{L}{2}-\tfrac{rL}{2}+(j-1)\tfrac{rL}{M},\tfrac{L}{2}-\tfrac{rL}{2}+j\tfrac{rL}{M}\right),\quad j=1,\dots,M.$$

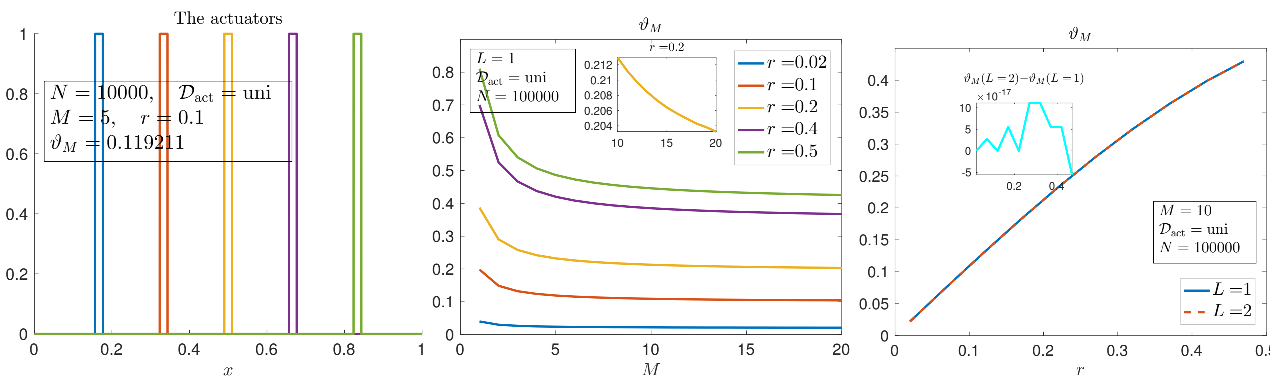


FIGURE 3. Uniformly distributed actuators.

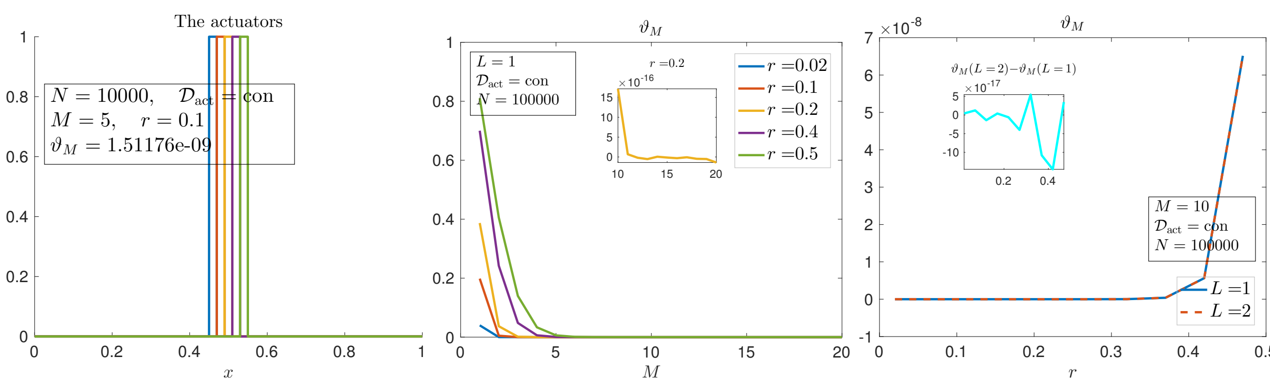


FIGURE 4. Concentrated actuators.

We see that the distributed actuators present an analogous behavior as those constructed as in section 4.3, and ϑ_M remains away from 0. For the concentrated actuators ϑ_M reaches values very close to zero. The simulations were done in Matlab with machine "precision" `eps` $\approx\frac{2.2204}{10^{16}}$.

4.8. Construction of the actuators in higher dimensional domains

For general domains $\Omega \subset \mathbb{R}^d$ the eigenvalues of the Dirichlet Laplacian, see [18, Corollary 1], satisfy $\alpha_M \geq C_d M^{\frac{2}{d}}$, with $C_d = (2\pi) |\mathcal{B}_d|^{-\frac{2}{d}} |\Omega|^{-\frac{2}{d}} \frac{d}{d+2}$, where $|\mathcal{B}_d|$ denotes the volume of the unit ball in $\mathbb{R}^d$ and $|\Omega|$ the volume of Ω. Therefore, condition (4.1) follows from

$$M + 1 > C_d{}^{-\frac{d}{2}} \left(6 + 4 \left| P_{\mathcal{U}}^{E_M^\perp} \right|^2_{\mathcal{L}(H)} \right)^{\frac{d}{2}} |A_{\mathrm{rc}}|^d_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))}, \tag{4.6}$$

which gives us an estimate for M which depends polynomially on $|A_{\mathrm{rc}}|^d_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))}$.

Now the question we may ask is whether we can also construct the actuators $\mathcal{U}(M)$ so that the norm $\left| P_{\mathcal{U}}^{E_M^\perp} \right|^2_{\mathcal{L}(H)}$ remains bounded as M increases.

4.8.1. *The case of a Rectangle*

In the case of a rectangle $\Omega^\times = \bigtimes_{n=1}^{d} (0, L_n)$ we know that the eigenfunctions of Δ are the products of the 1D eigenfunctions. Now we fix $r \in (0,1)$ and define, in each interval $(0, L_n)$, the 1D actuators as in section 4.3 (say covering a total region of volume rL_n), and their linear span $\mathcal{U} = (\mathcal{U})_n$. The notation $(\cdot)_n$ simply means that we are referring to the domain $(0, L_n)$. Next, we consider actuators defined in $\Omega^\times$ which are the products of those 1D actuators. That is, normalized eigenfunctions and actuators read, respectively

$$e^\times_{[j]} = e^1_{j_1}(x_1) e^2_{j_2}(x_2) \ldots e^d_{j_d}(x_d) \quad \text{and} \quad \Psi^\times_{[j]} = \overline{\Psi}^1_{j_1}(x_1) \overline{\Psi}^2_{j_2}(x_2) \ldots \overline{\Psi}^d_{j_d}(x_d), \qquad j \in \mathbb{N}_0^d,$$

where $x = (x_1, x_2, \ldots, x_d) \in \Omega^\times$. For example, we have $e^2_{j_2}(x_2) = (\frac{2}{L_2})^{\frac{1}{2}} \sin(\frac{j_2 \pi x_2}{L_2}) \in (E_M)_2$ and $\overline{\Psi}^3_{j_3}(x_3) = \frac{1}{|\omega_{j_3}|} 1_{\omega_{j_3}}(x_3) \in (\mathcal{U})_3$.

Now we recall Remark 3.8 and consider the spaces

$$\widetilde{E}_{M^d} = E^\times_M := \operatorname{span} \left\{ e^\times_{[j]} \mid j \in \{1, 2, \ldots, M\}^d \right\} \quad \text{and} \quad \mathcal{U}^\times = \operatorname{span} \left\{ \Psi^\times_{[j]} \mid j \in \{1, 2, \ldots, M\}^d \right\}.$$

Let us denote $H_n := L^2((0, L_n))$, for $n \in \{1, 2, \ldots, d\}$, and $H^\times := L^2(\Omega^\times)$. We will also use the notation $(\cdot)_n$ for operators, to underline that the operator $(P)_n$ is understood to be in $\mathcal{L}(H_n)$. Namely, the projections $(P_{\mathcal{U}}^{E_M^\perp})_n$ and $(P_{E_M})_n$ are in $\mathcal{L}(H_n)$.

We "extend" the projections $(P_{\mathcal{U}}^{E_M^\perp})_n$ to operators $(P_{\mathcal{U}}^{E_M^\perp})^\times_n$ in $\mathcal{L}(H^\times)$ defined as

$$(P_{\mathcal{U}}^{E_M^\perp})^\times_n f(x) := \sum_{j_n=1}^{M} \left(\int_0^{L_n} f(x_1, x_2, \ldots, x_n, \ldots, x_d) e^n_{j_n}(x_n) \, \mathrm{d}x_n \right) (P_{\mathcal{U}}^{E_M^\perp})_n e^n_{j_n}(x_n). \tag{4.7}$$

We have the following Lemmas 4.1 and 4.2, whose proofs are given in the appendix.

Lemma 4.1. *We have $H^\times = \mathcal{U}^\times \oplus E^\times_M$, and the projection $P_{\mathcal{U}^\times}^{E_M^{\times\perp}}$ coincides with the composition*

$$\mathcal{P}_\circ := (P_{\mathcal{U}}^{E_M^\perp})^\times_d \circ (P_{\mathcal{U}}^{E_M^\perp})^\times_{d-1} \circ \cdots \circ (P_{\mathcal{U}}^{E_M^\perp})^\times_2 \circ (P_{\mathcal{U}}^{E_M^\perp})^\times_1.$$

Recall that, from section 4.6, we know that $\vartheta_M^{-1} = \left| (P_{\mathcal{U}}^{E_M^\perp})_n \right|^2_{H_n}$ is independent of L_n (cf. Figure 2(b)).

Lemma 4.2. *We have that $|\mathcal{P}_\circ|^2_{\mathcal{L}(H^\times)} = \vartheta_M^{-d}$.*

Now, recalling Remark 3.8, we can see that $\widetilde{\alpha}_M := \frac{(M+1)^2\pi^2}{L_{\overline{n}}^2} + \sum\limits_{n\in\{1,2,\dots,d\}\setminus\overline{n}} \frac{\pi^2}{L_n^2}$ is the smallest eigenvalue of the Dirichlet Laplacian in $\widetilde{E}_{M^d}^{\perp} = E_M^{\times\perp}$, which corresponds to the eigenfunction $e_{[j]}^{\times}$ with $[j]$ defined by $j_n = \begin{cases} 1, & \text{if } n \neq \overline{n}, \\ M+1, & \text{if } n = \overline{n}, \end{cases}$ where $\overline{n} := \min\{n \in \{1,2,\dots,d\} \mid L_n = \max\limits_{m\in\{1,2,\dots,d\}} L_m\}$.

Hence, for the parabolic equation (1.2) in the rectangle $\Omega^{\times}$, recalling (4.1) and Remark 3.8, we have that the stability of the corresponding system (3.8) follows from $\widetilde{\alpha}_M > \left(6 + 4\vartheta_M^{-d}\right) |A_{\rm rc}|^2_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$. Since $\widetilde{\alpha}_M \geq \frac{\pi^2}{L_{\overline{n}}^2}\left((M+1)^2 + d - 1\right)$, the relation $M+1 \geq \frac{L_{\overline{n}}}{\pi}\left(6 + 4\vartheta_M^{-d}\right)^{\frac{1}{2}} |A_{\rm rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$ implies stability of system (3.8). Recalling that there are M^d actuators in $E_M^{\times\perp}$, we can derive the estimate $M_{\rm suff}+1 \geq \left(\frac{L_{\overline{n}}}{\pi}\right)^d \left(6 + 4\vartheta_M^{-d}\right)^{\frac{d}{2}} |A_{\rm rc}|^d_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$ on the number of actuators which allow us to stabilize the system. The latter estimate is analogous to (4.6), differing only by a constant factor. Notice also that the eigenvalues $\widetilde{\alpha}_M$ satisfy $\widetilde{\alpha}_M \geq M^2$, and the subsequence of the (ordered and repeated) eigenvalues α_{M^d} satisfy $\alpha_{M^d} \geq C_d(M^d)^{\frac{2}{d}} = C_d M^2$. So the asymptotic behavior of $\widetilde{\alpha}_M$ and α_{M^d} are analogous.

Notice, however, that in 1D the actuators cover, in each $(0, L_n)$, a total volume $|\omega_{L_n}| = rL_n$, while in dD they cover a volume $|\omega^{\times}| = r^d \bigtimes_{n=1}^d L_n$. That is, the relative volume covered by the actuators is smaller: $\frac{|\omega^{\times}|}{|\Omega^{\times}|} = r^d < r = \frac{|\omega_{L_n}|}{|(0,L_n)|}$.

Summarizing, we have seen that $\left|P_{\mathcal{U}^{\times}}^{\widetilde{E}_{M^d}^{\perp}}\right|^2_{\mathcal{L}(L^2(\Omega^{\times}))} \leq \vartheta_M^{-d}$ remains bounded, provided the 1D projections do: $\left|(P_{\mathcal{U}}^{E_M^{\perp}})_n\right|^2_{\mathcal{L}(L^2((0,L_n)))} = \vartheta_M^{-1} \leq C$ for all $M \in \mathbb{N}$. From the simulations in section 4.6, $\vartheta_M^{-1} \leq C(r)$ remains bounded. For rectangles $\Omega^{\times} \subset \mathbb{R}^d$ we can always find/construct a big enough number $M = M_{\rm suff}$ of actuators depending polynomially on $|A_{\rm rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$ (linearly on $|A_{\rm rc}|^d_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$), so that system (3.8) is stable.

4.8.2. *The case of a general domain*

We conjecture that, for general (regular enough) domains $\Omega \subset \mathbb{R}^d$, the norm $\left|P_{\mathcal{U}}^{E_M^{\perp}}\right|^2_{\mathcal{L}(L^2(\Omega))}$ will remain bounded provided we distribute piecewise constant actuators as uniformly as possible over Ω. And thus, we will be able to find a number M, as (4.6), of actuators, so that system (3.8) is stable.

5. Simulations for the closed-loop system

As described in section 4.2 the spatial discretization is carried out by piecewise linear finite elements. For the full discretization the Crank–Nicolson scheme is used for temporal discretization for the grid defined by $\{t_j = jk : j \in \mathbb{N}\}$, with the time-step $k > 0$.

5.1. **A constant reaction in** 1D

We consider system (1.2) on the interval $\Omega = (0, L)$ with a constant reaction and no convection

$$a(x,t) = \rho := -35\nu(\tfrac{\pi}{L})^2, \qquad b(x,t) = 0, \tag{5.1a}$$

and the initial condition

$$y_0 = 0.05\sin(\tfrac{1\pi x}{L}). \tag{5.1b}$$

That is, $A_{\rm rc} = \rho < 0$. Under our feedback control, as in Theorem 3.5, the system reads

$$\dot y - \nu\Delta y + \rho y - P_{\mathcal U}^{E_M^\perp}(-\nu\Delta y + \rho y - \lambda y) = 0, \tag{5.2}$$

$$y|_{\partial\Omega} = 0, \quad \text{and} \quad y(0) = y_0. \tag{5.3}$$

We will take $L = 1$ and $\nu = 0.1$. For the ordered eigenvalues $\alpha_i = \nu(\frac{\pi}{L})^2 i^2$ of $A = -\nu\Delta$, we can find $-\alpha_1 - \rho > 0$, which implies that the free (uncontrolled) dynamics system $\dot y_{\rm unc} - \nu\Delta y_{\rm unc} + \rho y_{\rm unc} = 0$ is unstable. Indeed, the free dynamics solution corresponding to $y_{\rm unc}(0) = y_0$ is $y_{\rm unc}(t) = \mathrm{e}^{(-\rho-\alpha_1)t} y_0$, $t \ge 0$.

Since $\alpha_5 < -\rho = |\rho|_{\mathbb R} < \alpha_6$, from the discussion in section 4.4, our closed-loop system (5.2) is stable provided we take $M = 5$ actuators spanning $\mathcal U$ such that $H = \mathcal U \oplus E_M^\perp$. This is satisfied for actuators as in Figures 1, 3, or 4. The stabilizing effect is confirmed in Figures 5, 6, and 7. Note, however, that the location of the actuators has a significant influence on the value of ϑ_M and on the magnitudes reached by the (norm of the) solution and by the control. This confirms the fact, already mentioned in section 4.4, that though in this case the norm of $P_{\mathcal U}^{E_M^\perp}$ plays no role in the condition for stability (3.2:**CS**), it still makes a considerable difference concerning the bound D in (3.11), which provides us a bound for the magnitudes reached by the norm of the solution.

The projection $P_{\mathcal U}^{E_M^\perp}$ has been constructed following (2.3a) from the basis $\{\sin(\frac{i\pi x}{L}) \mid i \in \{1,\dots,M\}\}$ for E_M and from the basis $\{1_{\omega_i} \mid i \in \{1,\dots,M\}\}\}$ for $\mathcal U$. The (numerically) orthonormalized bases have been used only for the computation of ϑ_M.

To see the performance of the proposed feedback, the control will be switched on only during a time interval $FeedOn$. That is, for time $t \notin FeedOn$ we follow the free dynamics. We plot the results on the behavior of the norm of the solution in two cases, $FeedOn = (0.1, 1)$ and $FeedOn = (0.05, 0.75)$. The magnitudes of the control actuators will be plotted only for the latter case. Recall (from sections 4.3 and 4.7) that the actuators 1_{ω_i} are numbered from left to right, that is, $\omega_i = (a_i, b_i)$ with $a_i < b_i \le a_{i+1} < b_{i+1}$ (cf. Figures 1, 3, and 4). In Figures 5, 6, and 7 their corresponding magnitudes are ordered from top to bottom.

Again for the latter case we also plot the control and solution at a selected instant of time $t = 0.1$.

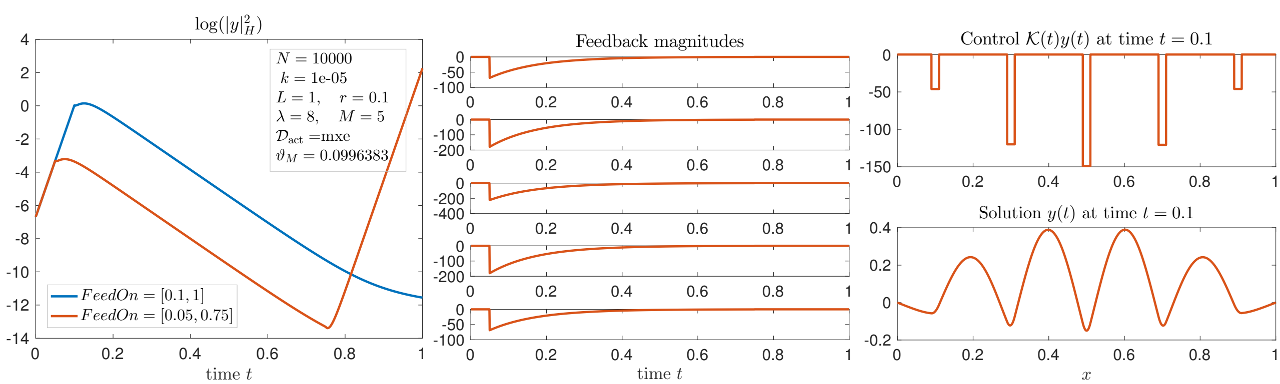


FIGURE 5. Located at extremizers of the M-th eigenfunction.

Finally, in this example, $M = 5$ is sharp, that is, with 4 actuators the proposed feedback is not able to stabilize the system, because the solution of (5.2), with $M = 4$ and issued from $y_0 = e_5$, is given by $y(t) = \mathrm{e}^{(-\alpha_5-\rho)t} y_0$, which coincides with the free dynamics solution and whose norm goes to $+\infty$ with time. Notice that in this case we even cannot stabilize the system if we take the first 4 eigenfunctions as actuators, because the dynamics onto the subspace $E_4^\perp$ will remain free for any given control v (replacing $P_{\mathcal U}^{E_M^\perp}(-\nu\Delta y + \rho y - \lambda y)$ and) taking values in E_4.

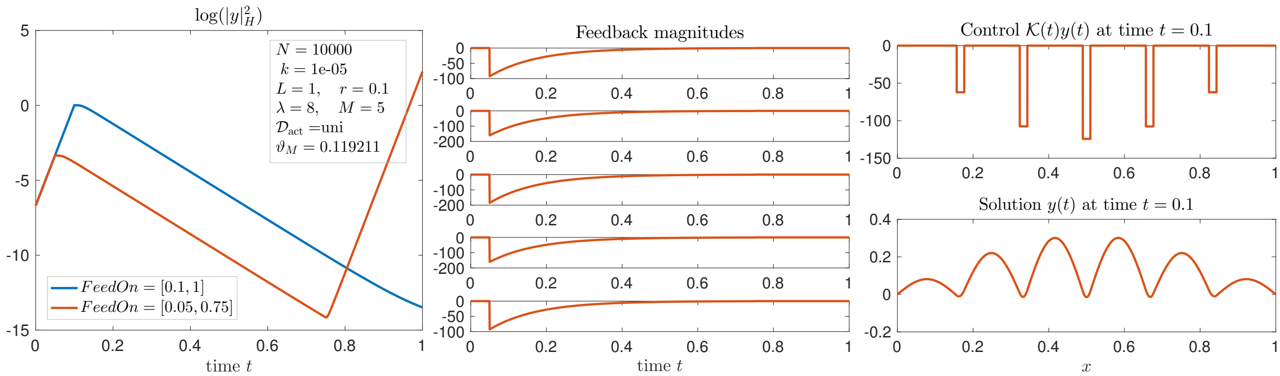


FIGURE 6. Uniformly distributed.

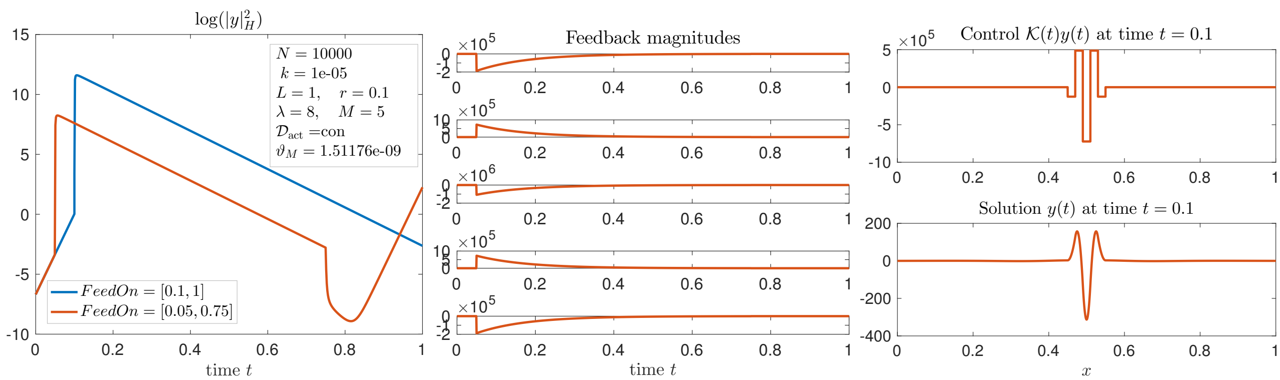


FIGURE 7. Concentrated at the center.

5.2. A general reaction-convection term in 1D

In Figure 8 we see the performance of the proposed feedback for a reaction-convection term depending on both space and time variables. More precisely, we have taken

$$a(x,t) = -35\nu(\tfrac{\pi}{L})^2 - 10\left|\sin(4t)x\cos(xt)\right|_{\mathbb{R}}, \qquad b(x,t) = -4\cos(3\pi t) - 5(\tfrac{L-x}{L})^2 + 2, \tag{5.4a}$$

and the initial condition

$$y_0(x) = \sin(\tfrac{13\pi x}{L}). \tag{5.4b}$$

We see that for small M (in $\{0,1,2\}$) our closed-loop system is not stable, while it is stable for large M. We also see that for $M \ge 7$ the exponential decreasing rate of $|y|_H^2$ is close to $16 = 2\lambda$. This rate cannot be improved (e.g., by adding more actuators) because it coincides with the exact decreasing rate we have imposed for the norm $|P_{E_M} y|_H^2$ of the finite-dimensional projection onto E_M (cf.(3.11), where $\mu < 2\lambda$).

We see that in a neighborhood of $t = 0$ the norm of the (free dynamics) solution is decreasing. This is because our initial condition is in $E_{\widetilde M}^\perp$ with large enough $\widetilde M = 12$. Notice that we have $\frac{\mathrm d}{\mathrm dt}|y|_H^2 \le -2|y|_V^2 + 2|A_{\rm rc}|_{\mathcal L(H,V')}|y|_H|y|_V$, which implies $\frac{\mathrm d}{\mathrm dt}|y|_H^2 \le (-\alpha_{\widetilde M+1} + |A_{\rm rc}|_{\mathcal L(H,V')}^2)|y|_H^2$, at the initial time.

After some time the norm of the uncontrolled solution is increasing. We can conclude that the reaction-convection operator is necessarily transferring energy to the space $E_{\widetilde M}$. In this situation a control is needed to stabilize the system, and we see that our feedback controller is able to do it (for large enough M).

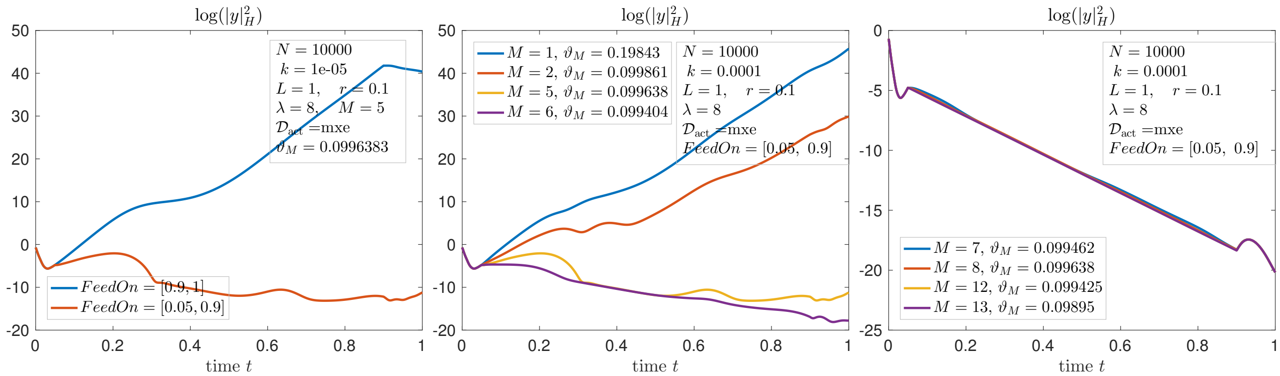


FIGURE 8. Feedback performance. Dependence on the number M of actuators.

5.3. **Response to measurement errors**

To apply a feedback control in applications we will need to know the state $y(t)$ at time t. Then we can compute our control $\mathcal{K}(t)y(t)$ which shall be input to the system. Often it is not possible to know $y(t)$ exactly, but we can obtain suitable estimates for $y(t)$. For example, from measurements of suitable outputs of the system we can sometimes construct a dynamical observer which provides us with an estimate $\widehat{y}(t)$ for $y(t)$.

Once we have an estimate $\widehat{y}(t)$ for $y(t)$, we can compute an estimate of our control as $\mathcal{K}(t)\widehat{y}(t)$. Feedback controls are known to be, in general, able to respond to (small) measurement/estimation errors. In Figure 9 we confirm that our proposed feedback is robust against such errors. As the magnitude of the noise (measurement error), $\eta \coloneqq \widehat{y}(t) - y(t)$, gets smaller the solution goes to a smaller neighborhood of zero.

We present the results corresponding to simulations of the system (3.8) with a perturbed feedback (i.e., with an estimated feedback control $\mathcal{K}(t)\widehat{y}(t)$):

$$\dot{y} - \nu\Delta y + ay + \nabla\cdot(by) - P_{\mathcal{U}}^{E_M^\perp}(-\nu\Delta\widehat{y} + a\widehat{y} + \nabla\cdot(b\widehat{y}) - \lambda\widehat{y}) = 0, \tag{5.5}$$

$$y|_{\partial\Omega} = 0, \quad \text{and} \quad y(0) = y_0. \tag{5.6}$$

where $\widehat{y} \coloneqq y + \eta$.

We will take the noise in the form $\eta = \eta_1 y + \eta_2$ having a component $\eta_1 y$ which is proportional to the state y and a component η_2 which is independent of the state. We will test with three types of hypothetical noise η ("typnoi" in figures):

- $\eta(x,t) = \mathrm{expl}(t,x,\zeta)$
$$\coloneqq \mathrm{e}^{\zeta}\Big(\big(\sin(20\pi t) + 0.1\sin(100\pi(t+x)) - 1\big)y(x,t) + \cos(10\pi t) + .1\cos(200\pi(t+x))\Big),$$
- $\eta(x,t) = \mathrm{rndn}(t,x,\zeta)$
$$\coloneqq \mathrm{e}^{\zeta}\Big((\min\{1, \max\{-1, \mathrm{v}_{\mathrm{ran}_1}(x,t)\}\} - 1)y(x,t) + \min\{1, \max\{-1, \mathrm{v}_{\mathrm{ran}_2}(x,t)\}\}\Big),$$
- $\eta(x,t) = \mathrm{rndm}(t,x,\zeta)$
$$\coloneqq \mathrm{e}^{\zeta}\Big((\mathrm{v}_{\mathrm{ran}3}(x,t) - 1.5)y(x,t) + \mathrm{v}_{\mathrm{ran}4}(x,t) - \mathrm{v}_{\mathrm{ran}5}(x,t)\Big).$$

The functions rndn and rndm are "random" and are to be understood as follows: once we have solved our system up to time $t_m = mk$, say we have just found $y(t_m)$, then we generate random vectors $\mathrm{v}_{\mathrm{ran}_i}(t_m) \in \mathbb{R}^{N+1}$, from which we construct the noise functions rndn and rndm at time $t = t_m$.

For $\mathrm{rndn}(t,x,\zeta)$ the vectors $\mathrm{v}_{\mathrm{ran}_i}(t_m)$, are generated by the Matlab function `randn`, while for $\mathrm{rndm}(t,x,\zeta)$ they are generated by the the Matlab function `rand`.

In Figure 9 we take $M=6$ actuators. The simulations correspond to the data (a,b) as in (5.4a), and the initial condition is taken

$$y_0(x)=10\sin(\tfrac{13\pi x}{L}).$$

We test with several values of ζ ("magnoi" in figures), to see the response of our feedback as the magnitude of the noise decreases. We see that the magnitude of the noise decreases with ζ. When $\zeta=-\infty$ ($-$Inf in the figures) the noise vanishes.

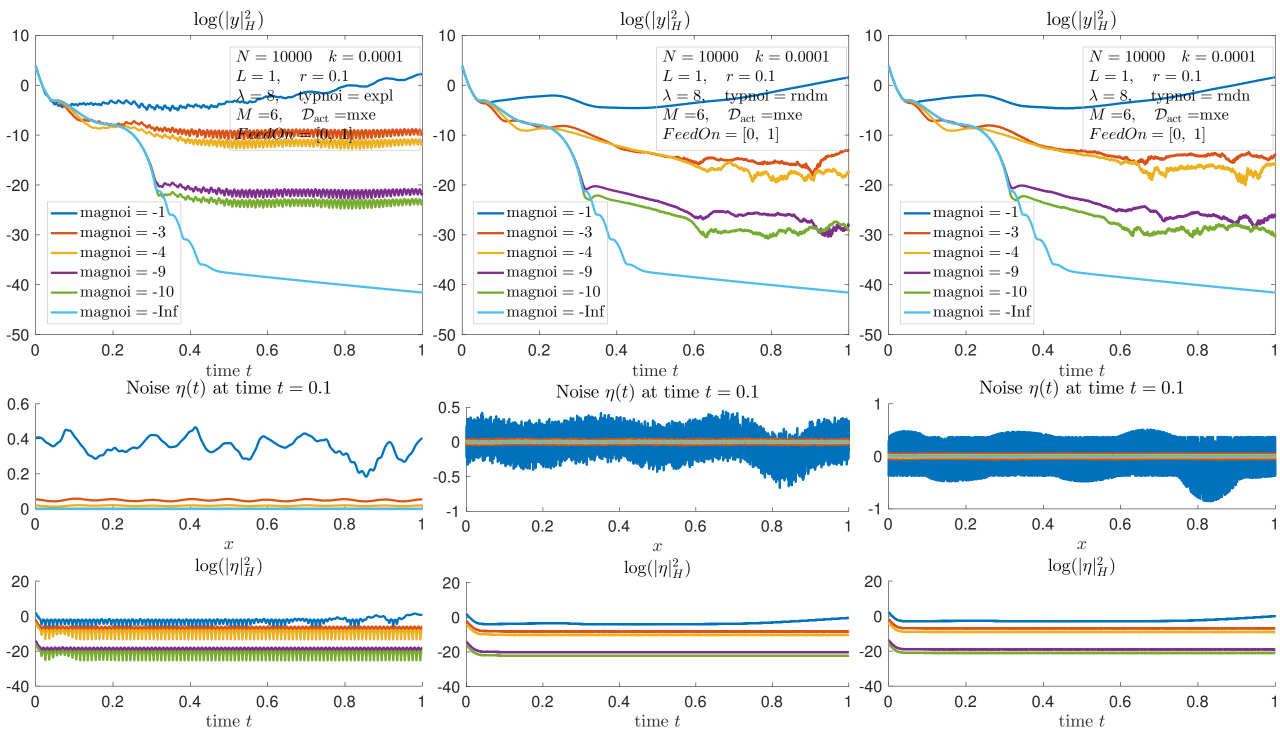


Figure 9. Response to measurement errors.

5.4. Numerical simulations in 2D

Now we present the results of a simulation for a parabolic equation in a domain $\Omega\subset\mathbb{R}^2$. In the previous examples for an 1D interval $(0,L)$ we knew the analytic expression for the Dirichlet Laplacian eigenfunctions, $e_n^{1\mathrm{D}}=(\frac{2}{L})^{\frac{1}{2}}\sin(\frac{n\pi x_1}{L})$. Then for the numerical simulations we can just evaluate this functions at the mesh points and construct the numerical projection $P_{\mathcal{U}}^{E_M^\perp}$ from those vectors. Now we are going to consider a domain $\Omega\subset\mathbb{R}^2$ where we do not know the analytic expression for the eigenfunctions. In this case we have to compute the eigenpairs numerically. In Figure 10 we plot the first 5 computed eigenpairs (e_i,α_i), $i\in\{1,2,3,4,5\}$ of $-\Delta$. We test with 4 actuators 1_{ω_i} where the location of each ω_i is also plotted in Figure 10, together with the triangular mesh we used in our simulations.

We can see that the first 4 eigenvalues are simple. We have chosen to place the actuators close to the extremizers of the 4-th eigenfunction e_4. This was motivated from the results obtained in 1D for the actuators constructed as in section 4.3 and by trying to place them "as uniformly as possible" (as "suggested" at the end of section 4.8). We have the volume ratio $r = \frac{\left|\bigcup_{i=1}^4 \omega_i\right|}{|\Omega|} = \frac{4}{36+2\pi} \approx 0.0946$.

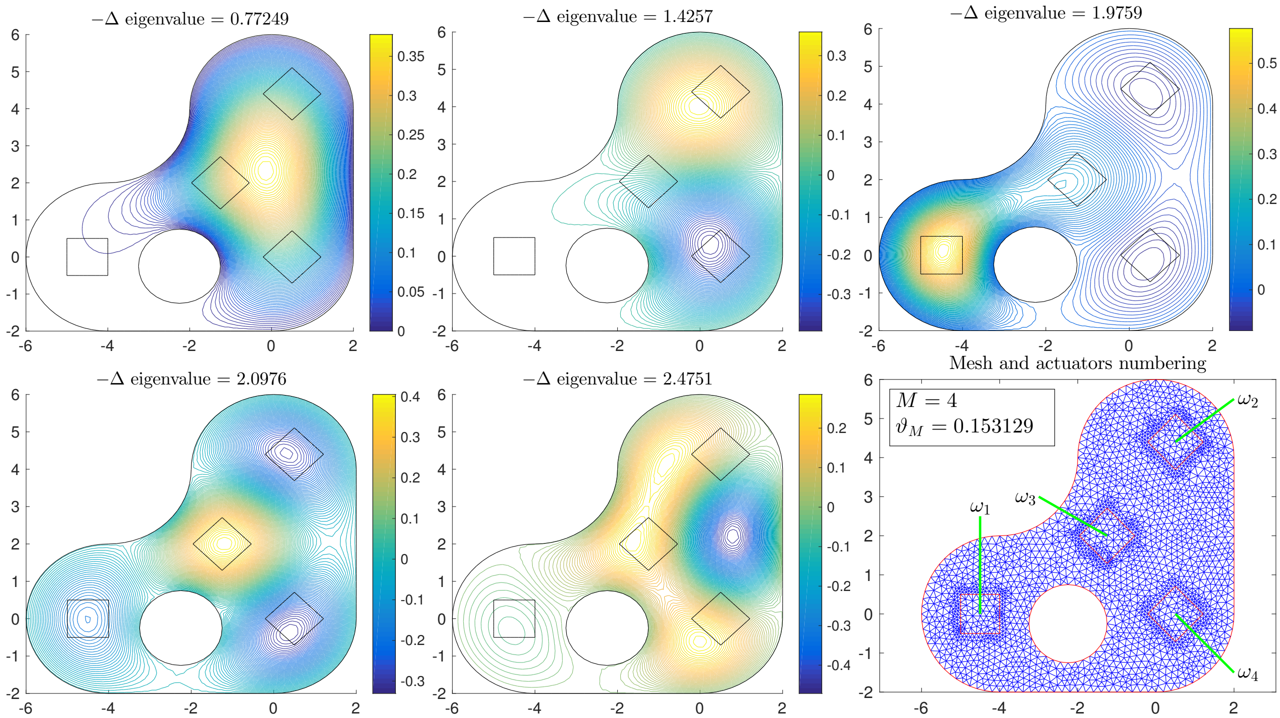


FIGURE 10. Eigenpairs of $-\Delta$ and actuators.

In Figure 11 we see the response of our feedback to measurement errors. Again the robustness of our feedback is confirmed. The norm of solution goes to a smaller neighborhood of 0 as the magnitude of the noise gets smaller. We see also that the numerical noisy solution is more sensitive to the time-step k than the unperturbed solution. However, qualitatively the behavior is similar.

Finally, in Figure 11 in the case of the larger time interval, the norm is plotted only for 1001 equidistant time instants, namely for $t = ik_{\mathrm{sub}}$, $i = 0, 1, \ldots, 1000$.

The results of the simulations correspond to the parabolic equation (1.2) under Dirichlet boundary conditions and

$$a(t, x_1, x_2) = -0.1 - 0.2|\sin(t + x_1)|_{\mathbb{R}}, \quad b(t, x_1, x_2) = \begin{pmatrix} 0.1(x_1 + x_2) \\ 0.1\cos(t)x_1x_2 \end{pmatrix}, \quad \begin{cases} \nu = 0.1, \\ y_0 = 0.01e_1. \end{cases} \tag{5.7}$$

Notice that the first 5 eigenvalues of $-\nu\Delta$ extend from $\alpha_1 \approx 0.077249$ to $\alpha_5 \approx 0.24751$.

5.5. On the parameter λ

We see that our feedback essentially imposes/chooses the projection of the (unperturbed) solution to be $q(t) = P_{E_M}y(t) = \mathrm{e}^{-\lambda t}P_{E_M}y(0)$. The parameter λ is at our disposal and its choice plays a role in the behavior of the closed-loop system. In Figure 12(a), with the data as in section 5.4 and (5.7), we can see that a larger $\lambda > 0$

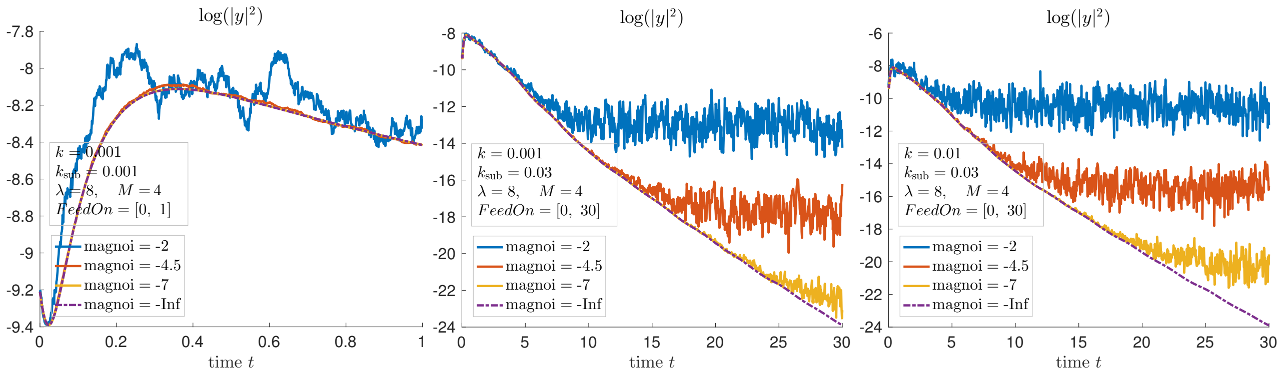
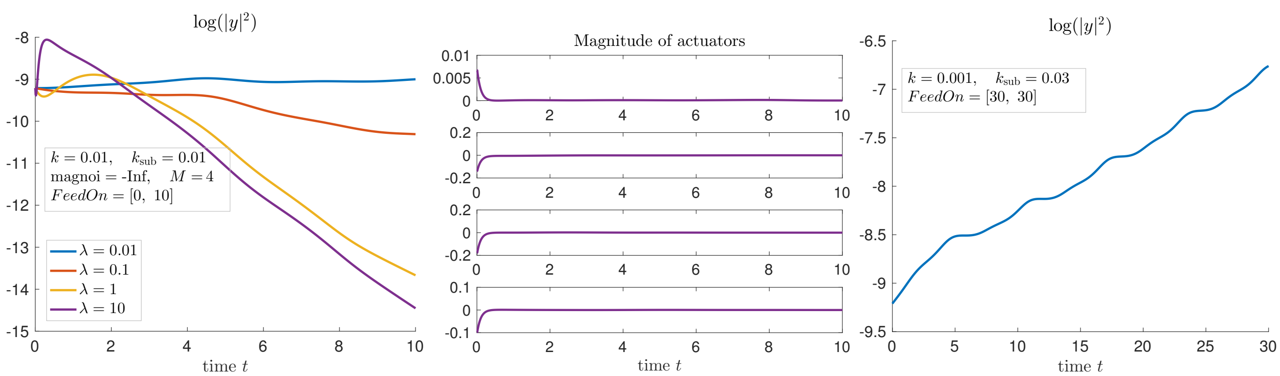


Figure 11. Response to measurement errors.

may provide a faster decreasing, but they also have larger associated transient bounds (the norm of the solution attains larger values for a transient time).

In Figure 12(b) we see the magnitudes (ordered from top to bottom) of the actuators corresponding to the last case plotted in Figure 12(a) (recall the ordering of the actuators in Figure 10). We can also see, in Figure 12(c), that the free dynamics is unstable.

Again for the last case plotted in 12(a) we see the corresponding solution and control, at a selected instant of time, in Figure 13.



(a) Dependence on λ. (b) Magnitude of actuators. (c) Free dynamics.

Figure 12. Dependence on λ and free dynamics.

6. Final remarks

We present here a short discussion on the manuscript results and on related interesting questions to be investigated in future works.

6.1. On the main result

Recall that in Theorem 3.5 we have the Assumptions 2.1, 2.2, and 2.3 and the sufficient condition (3.2:**CS**) for the stability of the closed-loop system (3.8).

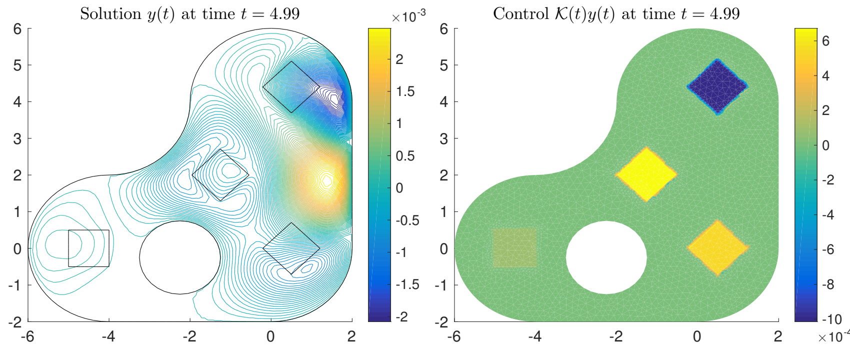


FIGURE 13. Solution and control at a selected time instant

Assumptions 2.1, 2.2, and 2.3 are quite general and are fulfilled for systems other than "pure" parabolic as (1.2). For example, they are satisfied for the Oseen–Stokes system

$$\dot{y} - \nu P_H \Delta y + P_H\big(\langle v \cdot \nabla\rangle y + \langle y \cdot \nabla\rangle v\big) = 0,$$

which can be seen as the linearization of the Navier–Stokes system around a given targeted trajectory v. Here $P_H\colon L^2(\Omega, \mathbb{R}^d) \to H$ is the Leray orthogonal projection onto the space $H = \{z \in L^2(\Omega, \mathbb{R}^d) \mid \operatorname{div} z = 0 \text{ and } z \cdot \mathbf{n} = 0\}$, where $d \in \{2, 3\}$, Ω is an open (smooth) bounded domain, and $\mathbf{n}$ is the unit outward normal to the boundary $\Gamma = \partial\Omega$. For the case of Dirichlet boundary conditions, Assumptions 2.1, 2.2, are satisfied by the Stokes operator $A = -\nu P_H \Delta$ with $V = H_0^1(\Omega, \mathbb{R}^d) \bigcap H$ and $\mathrm{D}(A) = H^2(\Omega, \mathbb{R}^d) \bigcap V$. Assumption 2.3 will be satisfied for a regular enough function v, by using well known estimates for the term $A_{\mathrm{rc}}(t)y = P_H\big(\langle v(t) \cdot \nabla\rangle y + \langle y \cdot \nabla\rangle v(t)\big)$. We refer the reader to [7, 27]. See also [26] and [22, section 6] and references therein for other boundary conditions. In particular, the main stabilization results derived on section 3 are valid for the Oseen–Stokes system. Here we restrict ourselves to the simulations for our closed loop parabolic equations, but it is interesting, in a future work, to perform some simulations of the corresponding closed loop Oseen–Stokes system. And in particular to check condition (3.2:**CS**).

6.2. On the application to nonlinear systems

For future work it is of interest to investigate the response of our proposed feedback when applied to a nonlinear equation. Due to Theorem 3.6, and following a standard argument as in [7, section 4] we can stabilize the nonlinear system (for a suitable class of nonlinearities as in [21], and for strong solutions), provided the initial condition is taken in a small neighborhood of 0. In particular, it can be of interest to compare the size of this neighborhood with the size of the corresponding neighborhood associated with the Riccati based feedback, for example, considered in [11, 15, 16, 21].

6.3. A comparison to Riccati feedback

While in the autonomous case, computing a Riccati based feedback operator Π can be already a difficult numerical task, it becomes even more involved in our nonautonomous setting where (in theory) we would need to solve backwards in time a differential Riccati equation in the time interval $(0, +\infty)$. This is, of course, unfeasible (in practice). In spite of this, in [16] the authors propose an end point condition $\Pi(T)$ for the differential Riccati equation to be solved in a finite interval $(0, T)$, $T > 0$. For works related with the numerical computation/approximation of the Riccati equations we refer the reader to [3, 9, 10, 13, 17].

The construction of the explicit feedback controller $\mathcal{K}$ we propose is in general an easier/faster numerical task. Indeed its construction, with M actuators, boils down to the computation of the first M eigenfunctions of the Laplacian. Furthermore, once those eigenfunctions are available, we can perform the simulations for any other reaction-convection term just by solving the explicit closed-loop system. On the other hand, for Riccati based feedback we need to solve a differential Riccati equation for each reaction-convection term.

6.4. On the location of the actuators

In view of the discussion in sections 4.6 and 4.7, we may ask ourselves the question on the best placement/location of the actuators. For example, it is interesting to know what is the *best location* (i.e., the pairs $\{(\text{center,orientation})_i \mid i \in \{1,2,3,4\}\}$) inside Ω of the 4 actuator regions as in Figure 10, *in order to maximize* ϑ_M. We refer to [19, 20] where the question of optimal actuator location is addressed, with the goal of minimizing a quadratic cost functional. See also to the recent works [14, 23] concerning the problem of optimal actuators *design* (where the goal is again to minimize a suitable cost functional, but where the shape of the control's (or each actuator's) support is not fixed apriori).

— Appendix —

A.1. Proof of Lemma 4.1

We first show that the range of the composition $\mathcal{P}_\circ$ is contained in $\mathcal{U}^\times$. Given a function $f \in H^\times$ we have that $(P_{\mathcal{U}}^{E_M^\perp})_1^\times f = \sum_{j_1=1}^{M} f_{j_1}^1(x_2,\dots,x_d)\overline{\Psi}_{j_1}^1(x_1)$. Similarly, we find the identities $(P_{\mathcal{U}}^{E_M^\perp})_2^\times \circ (P_{\mathcal{U}}^{E_M^\perp})_1^\times f = \sum_{j_1,j_2=1}^{M} f_{(j_1,j_2)}^1(x_3,\dots,x_d)\overline{\Psi}_{j_1}^1(x_1)\overline{\Psi}_{j_2}^2(x_2)$ and $\mathcal{P}_\circ f = \sum_{[j]\in\{1,2,\dots,M\}^d} f_{[j]}^1\Psi_{[j]}^\times$. Thus

$$\mathcal{P}_\circ f \in \mathcal{U}^\times, \quad \text{for all} \quad f \in H^\times. \tag{A.1}$$

Next, we show that $(1-\mathcal{P}_\circ)f \in E_M^{\times\perp}$. From (4.7) and the Fubini's Theorem it follows that, for any $f \in H^\times$ and any eigenfunction $e_{[i]}^\times \in E_M^\times$, we have

$$\left((P_{\mathcal{U}}^{E_M^\perp})_n^\times f, e_{[i]}^\times\right)_{H^\times} = \sum_{j_n=1}^{M}\int_0^{L_1}\mathrm{d}x_1\cdots\int_0^{L_{n-1}}\mathrm{d}x_{n-1}\int_0^{L_{n+1}}\mathrm{d}x_{n+1}\cdots\int_0^{L_d} g_n^{j_n}\,\mathrm{d}x_d\int_0^{L_n} h_n\mathrm{d}x_n$$

with

$$\begin{aligned} g_n^{j_n} &= \left(\int_0^{L_n} f(x_1,x_2,\dots,x_n,\dots,x_d)e_{j_n}^n(x_n)\,\mathrm{d}x_n\right)\bigtimes_{r\in\{1,\dots,d\}\setminus\{n\}} e_{i_r}^r(x_r),\\ h_n &= \left((P_{\mathcal{U}}^{E_M^\perp})_n e_{j_n}^n(x_n)\right)e_{i_n}^n(x_n). \end{aligned}$$

Since $(1-(P_{\mathcal{U}}^{E_M^\perp})_n)e_{j_n}^n(x_n) \in (E_M^\perp)_n$ and $e_{i_n}^n(x_n) \in (E_M)_n$ it follows that

$$\begin{aligned} \int_0^{L_n} h_n\mathrm{d}x_n &= \int_0^{L_n} h_n + \left(1-(P_{\mathcal{U}}^{E_M^\perp})_n\right)e_{j_n}^n(x_n)e_{i_n}^n(x_n)\mathrm{d}x_n = \int_0^{L_n} e_{j_n}^n(x_n)e_{i_n}^n(x_n)\mathrm{d}x_n\\ &= \begin{cases} 1, & \text{if } j_n = i_n, \text{ and}\\ 0, & \text{if } j_n \neq i_n, \end{cases} \end{aligned}$$

from which we obtain that, for all $f \in H^\times$ and all $[i] \in \{1,2,\dots,M\}^d$,

$$\left((P_{\mathcal{U}}^{E_M^\perp})_n^\times f, e_{[i]}^\times\right)_{H^\times} = \int_0^{L_1}\mathrm{d}x_1\cdots\int_0^{L_{n-1}}\mathrm{d}x_{n-1}\int_0^{L_{n+1}}\mathrm{d}x_{n+1}\cdots\int_0^{L_d} g_n^{i_n}\mathrm{d}x_d = \left(f, e_{[i]}^\times\right)_{H^\times}.$$

which implies that for all $g\in H^\times$ and all $[i]\in\{1,2,\dots,M\}^d$,

$$\begin{aligned}(\mathcal{P}_\circ g, e^\times_{[j]})_{H^\times} &= \left((P_{\mathcal{U}}^{E_M^\perp})^\times_d\left((P_{\mathcal{U}}^{E_M^\perp})^\times_{d-1}\circ\cdots\circ(P_{\mathcal{U}}^{E_M^\perp})^\times_1 g\right), e^\times_{[i]}\right)_{H^\times}\\ &= \left((P_{\mathcal{U}}^{E_M^\perp})^\times_{d-1}\circ\cdots\circ(P_{\mathcal{U}}^{E_M^\perp})^\times_1 g, e^\times_{[i]}\right)_{H^\times} = \dots = (g, e^\times_{[i]})_{H^\times}.\end{aligned}$$

That is,

$$g-\mathcal{P}_\circ g\in E_M^{\times\perp},\quad\text{for all}\quad g\in H^\times. \tag{A.2}$$

Finally, we show that $H^\times=\mathcal{U}^\times\oplus E_M^{\times\perp}$. From (A.1) and (A.2) it follows that $H^\times=\mathcal{U}^\times+E_M^{\times\perp}$. Let now $v\in\mathcal{U}^\times\bigcap E_M^{\times\perp}$ which we may write as

$$v=\sum_{[j]\in\{1,2,\dots,M\}^d} v_{[j]}\Psi^\times_{[j]}.$$

Therefore, for any eigenfunction $e^\times_{[i]}\in E_M^\times$ we have $0=(v,e^\times_{[i]})_{H^\times}$, that is,

$$0=\sum_{j\in\{1,2,\dots,M\}^d} v_{[j]}(\Psi^\times_{[j]}, e^\times_{[i]})_{H^\times} \tag{A.3}$$

which gives us

$$0=\sum_{[j]\in\{1,2,\dots,M\}^d} v_{[j]}\bigtimes_{n=1}^{d}(\overline{\Psi}^n_{j_n}, e^n_{i_n})_{H_n}=\sum_{j_d=1}^{M} w_{j_d}(\overline{\Psi}^n_{j_d}, e^n_{i_d})_{H_d}. \tag{A.4}$$

with

$$w_{j_d}\coloneqq\sum_{j\in\{1,2,\dots,M\}^{d-1}} v_{[(j,j_d)]}\left(\bigtimes_{n=1}^{d-1}(\overline{\Psi}^n_{j_n}, e^n_{i_n})_{H_n}\right).$$

We know that $Z\coloneqq[((\mathcal{U})_n,(E_M^\perp)_n)_{H_n}]\in\mathcal{M}_{M\times M}$ is invertible (because so is $[\Theta(M)]=Z^\top Z$, cf. (4.4) and Figure 2).

Now, with $w\coloneqq[w_{j_1},w_{j_2},\dots,w_{j_d}]\in\mathcal{M}_{1\times M}$, from (A.3) we arrive at $0=wZ$, which implies that $w=0$. That is, $0=w_{j_d}$, for all $j_d\in\{1,2,\dots,M\}$.

For a fixed $j_d\in\{1,2,\dots,M\}$ the equation $0=w_{j_d}$ is similar to (A.3). Thus we can repeat the argument to conclude that

$$0=w_{j_{d-1},j_d}\coloneqq\sum_{j\in\{1,2,\dots,M\}^{d-2}} v_{[(j,j_{d-1},j_d)]}\left(\bigtimes_{n=1}^{d-2}(\overline{\Psi}^n_{j_n}, e^n_{i_n})_{H_n}\right),$$

for all $(j_{d-1},j_d)\in\{1,2,\dots,M\}^2$, and

$$0=w_{j_1,\dots,j_d}\coloneqq v_{[j]},\quad\text{for all}\quad j\in\{1,2,\dots,M\}^d.$$

Therefore $v=0$, and

$$H^\times=\mathcal{U}^\times\oplus E_M^{\times\perp}. \tag{A.5}$$

From (A.1), (A.2), and (A.5), we necessarily have $\mathcal{P}_\circ=P_{\mathcal{U}^\times}^{E_M^{\times\perp}}$. □

A.2. **Proof of Lemma 4.2**

Recalling (4.7), we denote, for all $m \in \mathbb{N}_0$ and all $f \in H^\times$,

$$I_m^f = I_m^f(x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_d) \coloneqq \int_0^{L_n} f(x_1, x_2, \dots, x_n, \dots, x_d) e_m^n(x_n)\,\mathrm{d}x_n.$$

Observe that we may write $f = \sum_{m=1}^{+\infty} I_m^f e_m^n(x_n)$,

$$(P_{\mathcal{U}}^{E_M^\perp})_n^\times f = \sum_{j_n=1}^{M} I_{j_n} f (P_{\mathcal{U}}^{E_M^\perp})_n e_{j_n}^n(x_n), \quad \text{and} \quad \int_0^{L_n} f^2\,\mathrm{d}x_n = \sum_{m=1}^{+\infty} (I_m^f)^2,$$

from which we obtain, since I_m^f does not depend on x_n,

$$\begin{aligned}&\left|(P_{\mathcal{U}}^{E_M^\perp})_n^\times f\right|_{H^\times}^2 = \int_{\Omega^\times} \left((P_{\mathcal{U}}^{E_M^\perp})_n^\times f\right)^2 \mathrm{d}\Omega^\times\\
&= \int_0^{L_1} \mathrm{d}x_1 \cdots \int_0^{L_{n-1}} \mathrm{d}x_{n-1} \int_0^{L_{n+1}} \mathrm{d}x_{n+1} \cdots \int_0^{L_d} \mathrm{d}x_d \int_0^{L_n} \left((P_{\mathcal{U}}^{E_M^\perp})_n \sum_{j_n=1}^{M} I_{j_n}^f e_{j_n}^n(x_n) \right)^2 \mathrm{d}x_n.\end{aligned}$$

Then, from

$$\int_0^{L_n} \left((P_{\mathcal{U}}^{E_M^\perp})_n v\right)^2 \mathrm{d}x_n = \left|(P_{\mathcal{U}}^{E_M^\perp})_n v\right|_{H_n}^2 \le \vartheta_M^{-1} |v|_{H_n}^2, \quad \text{for any} \quad v \in H_n,$$

$$\int_0^{L_n} \left(\sum_{j_n=1}^{M} I_{j_n}^f e_{j_n}^n(x_n) \right)^2 \mathrm{d}x_n = \sum_{j_n=1}^{M} (I_{j_n}^f)^2 \le \int_0^{L_n} f^2\,\mathrm{d}x_n,$$

it follows that $\left|(P_{\mathcal{U}}^{E_M^\perp})_n^\times f\right|_{H^\times}^2 \le \vartheta_M^{-1} |f|_{H^\times}^2$, which leads us to $|\mathcal{P}_\circ|_{\mathcal{L}(H^\times)}^2 \le \bigtimes_{n=1}^{d} \left|(P_{\mathcal{U}}^{E_M^\perp})_n^\times\right|_{H^\times}^2 \le \vartheta_M^{-d}$. To finish the proof we now find $\zeta \in H^\times$ such that

$$|\zeta|_{H^\times} = 1 \quad \text{and} \quad |\mathcal{P}_\circ \zeta|_{H^\times} = \vartheta_M^{-\frac{d}{2}}. \tag{A.6}$$

We start by observing that

$$\begin{aligned}\left|(P_{\mathcal{U}}^{E_M^\perp})_n\right|_{\mathcal{L}(H_n)} &= \sup_{\phi \in H_n \setminus \{0\}} \frac{\left|(P_{\mathcal{U}}^{E_M^\perp})_n (P_{E_M})_n \phi\right|_{H_n}}{|\phi|_{H_n}} \le \sup_{\phi \in H_n \setminus \{0\}} \frac{\left|(P_{\mathcal{U}}^{E_M})_n (P_{E_M})_n \phi\right|_{H_n}}{\left|(P_{E_M})_n \phi\right|_{H_n}}\\
&= \sup_{\xi \in (E_M)_n \setminus \{0\}} \frac{\left|(P_{\mathcal{U}}^{E_M})_n \xi\right|_{H_n}}{|\xi|_{H_n}} = \sup_{\substack{\xi \in (E_M)_n\\ |\xi|_{H_n} = 1}} \left|(P_{\mathcal{U}}^{E_M})_n \xi\right|_{H_n}.\end{aligned}$$

Necessarily $\left|(P_{\mathcal{U}}^{E_M^\perp})_n\right|_{\mathcal{L}(H_n)} = \sup_{\substack{\xi \in (E_M)_n\\ |\xi|_{H_n} = 1}} \left|(P_{\mathcal{U}}^{E_M^\perp})_n \xi\right|_{H_n}$ and, since $\{\eta \in (E_M)_n \mid |\eta|_{H_n} = 1\}$ is compact, there exists a maximixer $\overline{\xi}_n \in (E_M)_n$, with $\left|\overline{\xi}_n\right|_{H_n} = 1$ and $\left|(P_{\mathcal{U}}^{E_M^\perp})_n \overline{\xi}_n\right|_{H_n} = \left|(P_{\mathcal{U}}^{E_M^\perp})_n\right|_{\mathcal{L}(H_n)} = \vartheta_M^{-\frac{1}{2}}$. Finally, we see that $\xi^\times(x) \coloneqq \bigtimes_{n=1}^d \overline{\xi}_n(x_n)$ satisfies

$$\left|\xi^\times\right|_{H^\times}^2 = \bigtimes_{n=1}^d \left|\overline{\xi}_n\right|_{H_n}^2 = 1 \quad\text{and}\quad \left|\mathcal{P}_\circ\xi^\times\right|_{H^\times}^2 = \bigtimes_{n=1}^d \left|(P_{\mathcal{U}}^{E_M^\perp})_n\overline{\xi}_n\right|_{H_n}^2 = \vartheta_M^{-d}.$$

That is, $\zeta \coloneqq \xi^\times$ satisfies (A.6). □

Acknowledgements. The work of K. Kunisch was partly supported by the ERC advanced grant 668998 (OCLOC) under the EU's H2020 research program.

References

[1] K. Ammari, T. Duyckaerts, and A. Shirikyan. Local feedback stabilisation to a non-stationary solution for a damped non-linear wave equation. *Math. Control Relat. Fields*, 6(1):1–25, 2016. doi:10.3934/mcrf.2016.6.1.

[2] M. Badra and T. Takahashi. Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: Application to the Navier–Stokes system. *SIAM J. Control Optim.*, 49(2):420–463, 2011. doi:10.1137/090778146.

[3] H. T. Banks and K. Kunisch. The linear regulator problem for parabolic systems. *SIAM J. Control Optim.*, 22(5):684–698, 1984. doi:10.1155/S1024123X00001320.

[4] V. Barbu. Stabilization of Navier–Stokes equations by oblique boundary feedback controllers. *SIAM J. Control Optim.*, 50(4):2288–2307, 2012. doi:10.1137/110837164.

[5] V. Barbu. Boundary stabilization of equilibrium solutions to parabolic equations. *IEEE Trans. Automat. Control*, 58(9):2416–2420, 2013. doi:10.1109/TAC.2013.2254013.

[6] V. Barbu, I. Lasiecka, and R. Triggiani. Abstract settings for tangential boundary stabilization of Navier–Stokes equations by high- and low-gain feedback controllers. *Nonlinear Anal.*, 64:2704–2746, 2006. doi:10.1016/j.na.2005.09.012.

[7] V. Barbu, S. S. Rodrigues, and A. Shirikyan. Internal exponential stabilization to a nonstationary solution for 3D Navier–Stokes equations. *SIAM J. Control Optim.*, 49(4):1454–1478, 2011. doi:10.1137/100785739.

[8] V. Barbu and R. Triggiani. Internal stabilization of Navier–Stokes equations with finite-dimensional controllers. *Indiana Univ. Math. J.*, 53(5):1443–1494, 2004. doi:10.1512/iumj.2004.53.2445.

[9] P. Benner. A MATLAB repository for model reduction based on spectral projection. In *Proceedings of the 2006 IEEE Conference on Computer Aided Control Systems Design*, pages 19–24, October 4-6 2006. doi:10.1109/CACSD-CCA-ISIC.2006.4776618.

[10] P. Benner, A.J. Laub, and V. Mehrmann. Benchmarks for the numerical solution of algebraic Riccati equations. *IEEE Control Systems Magazine*, 17(5):18–28, 1997. doi:10.1109/37.621466.

[11] T. Breiten, K. Kunisch, and S. S. Rodrigues. Feedback stabilization to nonstationary solutions of a class of reaction diffusion equations of FitzHugh–Nagumo type. *SIAM J. Control Optim.*, 55(4):2684–2713, 2017. doi:10.1137/15M1038165.

[12] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, 2011. doi:10.1007/978-0-387-70914-7.

[13] J. A. Burns, E. W. Sachs, and L. Zietsman. Mesh independence of Kleinman–Newton iterations for Riccati equations in Hilbert space. *SIAM J. Control Optim.*, 47(5):2663–2692, 2008. doi:10.1137/060653962.

[14] D. Kalise, K. Kunisch, and K. Sturm. Optimal actuator design based on shape calculus. arXiv:1711.01183, 2017. URL: http://arxiv.org/abs/1711.01183.

[15] A. Kröner and S. S. Rodrigues. Internal exponential stabilization to a nonstationary solution for 1D Burgers equations with piecewise constant controls. In *Proceedings of the 2015 European Control Conference (ECC), Linz, Austria*, pages 2676–2681, July 2015. doi:10.1109/ECC.2015.7330942.

[16] A. Kröner and S. S. Rodrigues. Remarks on the internal exponential stabilization to a nonstationary solution for 1D Burgers equations. *SIAM J. Control Optim.*, 53(2):1020–1055, 2015. doi:10.1137/140958979.

[17] P. Kunkel and V. Mehrmann. Numerical solution of differential algebraic Riccati equations. *Linear Algebra Appl.*, 137/138:39–66, 1990. doi:10.1016/0024-3795(90)90126-W.

[18] P. Li and S.-T. Yau. On the Schrödinger equation and the eigenvalue problem. *Commun. Math. Phys.*, 88(3):309–318, 1983. doi:10.1007/BF01213210.

[19] K. Morris. Linear-quadratic optimal actuator location. *IEEE Trans. Automat. Control*, 56(1):113–124, 2011. doi:10.1109/TAC.2010.2052151.

[20] K. Morris and S. Yang. A study of optimal actuator placement for control of diffusion. In *Proceedings of the 2016 American Control Conference (AAC), Boston, MA, USA*, pages 2566–2571, July 2016. doi:10.1109/ACC.2016.7525303.

[21] D. Phan and S. S. Rodrigues. Stabilization to trajectories for parabolic equations. arXiv:1608.02412v2 [math.OC], 2016. (submitted). URL: http://arxiv.org/abs/1608.02412v2.

[22] D. Phan and S. S. Rodrigues. Gevrey regularity for Navier–Stokes equations under Lions boundary conditions. *J. Funct. Anal.*, 272(7):2865–2898, 2017. doi:10.1016/j.jfa.2017.01.014.

[23] Y. Privat, E. Trélat, and E. Zuazua. Actuator design for parabolic distributed parameter systems with the moment method. *SIAM J. Control Optim.*, 55(2):1128–1152, 2017. doi:10.1137/16M1058418.

[24] J.-P. Raymond and L. Thevenet. Boundary feedback stabilization of the two-dimensional Navier–Stokes equations with finite-dimensional controllers. *Discrete Contin. Dyn. Syst.*, 27(3):1159–1187, 2010. doi:10.3934/dcds.2010.27.1159.

[25] S. S. Rodrigues. Feedback boundary stabilization to trajectories for 3D Navier–Stokes equations. (accepted, conditionally), 2015. arXiv:1508.00829 [math.OC]. URL: http://arxiv.org/abs/1508.00829.

[26] R. Temam. *Navier–Stokes Equations and Nonlinear Functional Analysis*. Number 66 in CBMS-NSF Regional Conf. Ser. Appl. Math. SIAM, Philadelphia, 2nd edition, 1995. doi:10.1137/1.9781611970050.

[27] R. Temam. *Navier–Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, Providence, RI, reprint of the 1984 edition, 2001. URL: http://www.ams.org/bookstore-getitem/item=CHEL-343-H.

[28] M.Y. Wu. A note on stability of linear time-varying systems. *IEEE Trans. Automat. Control*, 19(2):162, 1974. doi:10.1109/TAC.1974.1100529.