

BOUNDARY FEEDBACK STABILIZATION OF THE MONODOMAIN EQUATIONS

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ABSTRACT. Boundary feedback control for a coupled nonlinear PDE-ODE system (in the two and three dimensional cases) is studied. Particular focus is put on the monodomain equations arising in the context of cardiac electrophysiology. Neumann as well as Dirichlet based boundary control laws are obtained by an algebraic operator Riccati equation associated with the linearized system. Local exponential stability of the nonlinear closed loop system is shown by a fixed-point argument. Numerical examples are given for a finite element discretization of the two dimensional monodomain equations.

1. Introduction. This paper is concerned with the problem of boundary feedback stabilization for a coupled nonlinear reaction diffusion system. Both Neumann and Dirichlet boundary control is studied. The dynamics of interest are described by the so-called monodomain equations, a reasonably accurate simplification of the bidomain equations (see, e.g., [23]) which are frequently used to model the electrophysiological activity of the human heart. It is well-known that anomalous behavior such as fibrillation processes can be recognized within the mathematical model in the form of spiral or reentry waves. In this context, the clinical goal consists of a successful termination of these waves by external stimuli, usually called *defibrillation*. In [9], we have shown that the monodomain equations can locally be stabilized around a stationary solution by means of a state feedback law obtained from the linearized system. In contrast to the setup considered therein, a more realistic scenario assumes that the stimulating electrodes are situated at the boundary of the domain. With this in mind, our goal is to extend the concepts from [9] to the case of a boundary control. We further employ slightly different Lipschitz continuity

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estimates than the ones used in [9]. Proceeding this way, we are able to generalize the local stabilization results for the nonlinear system to a broader class of initial perturbations. Mathematically the monodomain equations are an evolution system consisting of a diffusion equation with a polynomial, more specifically, cubic nonlinearity coupled with an ordinary differential equation. Such systems play an important role far beyond their use for the description of the electrical activity of the heart. In fact, many models of this kind arise in cellular biology as described in [19], for example. One such application area is the modeling of wave propagation in excitable systems, as for instance nerve axons. They also arise in modeling chemical reaction dynamics and are known there under the name of ‘‘Schloggl’’ model, see e.g. [12, 28].

Thus, let us consider the following nonlinear reaction diffusion system

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta v - I_{\text{ion}}(v, w) + f_v \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial t} &= \iota v - \kappa w + f_w \quad \text{in } \Omega \times (0, \infty), \\ \theta \frac{\partial v}{\partial \nu} &= (\theta - 1)v + m\tilde{u} + g \quad \text{on } \Gamma \times (0, \infty), \\ v(x, 0) &= \bar{v} + y_0 \text{ and } w(x, 0) = \bar{w} + z_0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ denotes a bounded open set with smooth boundary $\Gamma = \partial\Omega$ and constants $\iota, \kappa > 0$. The factor $\theta \in \{0, 1\}$ is used to model both the Neumann and the Dirichlet boundary control setup, respectively. The nonlinearity $I_{\text{ion}}(v, w)$ is assumed to be of FitzHugh-Nagumo type, i.e.

$$I_{\text{ion}}(v, w) = av^3 - bv^2 + cv + dw, \tag{2}$$

with $a, b, c, d \in \mathbb{R}_+$. Here, f_v, f_w and g are external forcing functions and $(\bar{v}, \bar{w}) \in H^3(\Omega) \times L^\infty(\Omega)$, $s > 0$ denotes a stationary solution to

$$\begin{aligned} 0 &= \Delta \bar{v} - I_{\text{ion}}(\bar{v}, \bar{w}) + f_v \quad \text{in } \Omega \\ 0 &= \iota \bar{v} - \kappa \bar{w} + f_w \quad \text{in } \Omega \\ \theta \frac{\partial \bar{v}}{\partial \nu} &= (\theta - 1)\bar{v} + g \quad \text{on } \Gamma. \end{aligned} \tag{3}$$

The function m is assumed to localize the control in a part of the boundary Γ . For the precise assumptions on m , we follow the setup from [29]. In particular, if Γ is of class C^3 , we assume that $m \in C^2(\Gamma)$, $m \geq 0$ and $m(x) = 1$, $x \in \Gamma_c$, where Γ_c denotes an open control subset in Γ .

Similar to [9], we want to find a control input function $u \in L^2(0, \infty; L^2(\Gamma))$ such that the solution (v, w) to (1) locally decays exponentially, i.e., provided the initial perturbation $\mathbf{y}_0 = (y_0, z_0)$ is small in an appropriate sense. For $0 \leq \sigma < \kappa$, let us introduce $\mathbf{y} = (y, z) := (e^{\sigma t}(v - \bar{v}), e^{\sigma t}(w - \bar{w}))$, $u = e^{\sigma t}\tilde{u}$ and focus on

$$\begin{aligned} \frac{\partial y}{\partial t} &= \Delta y + \alpha(x)y - dz - ae^{-2\sigma t}y^3 + (b - 3a\bar{v})e^{-\sigma t}y^2 \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial z}{\partial t} &= \iota y - (\kappa - \sigma)z \quad \text{in } \Omega \times (0, \infty), \\ \theta \frac{\partial y}{\partial \nu} &= (\theta - 1)y + mu \quad \text{on } \Gamma \times (0, \infty), \\ y(x, 0) &= y_0 \text{ and } z(x, 0) = z_0 \quad \text{in } \Omega, \end{aligned} \tag{4}$$

where $\alpha(x) = -3a\bar{v}^2 + 2b\bar{v} - c + \sigma$. Instead of the stabilization of (1) around a solution of (3), we rather consider the stabilization of (4) around zero. The strategy discussed in the following is based on a linear quadratic infinite horizon optimal control problem for the linearized system and has been successfully employed for many different systems, see, e.g., [3, 4, 6, 25, 29]. In particular, for the linearized system, we closely follow the abstract Riccati theory for boundary control systems as presented in [2, 21].

The structure of the paper is as follows. In Section 2 we study the case of a Neumann boundary control together with a partial Dirichlet boundary observation. Based on the general results from [21, Chapter 2] and [2], we analyze the effect of a Riccati-based feedback approach obtained for the linearized system when used in the full nonlinear setting. In particular, we slightly extend the Lipschitz continuity estimates from [9]. This will lead to a local stabilization result (in the two and three dimensional cases) for initial perturbations $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$, $\varepsilon \in (\frac{1}{2}, 1]$. In Section 3 we investigate the possibility of using a Dirichlet boundary control. Similar to [25, 29], for $\varepsilon \in (0, \frac{1}{2})$ we first have to show that the involved Riccati operator $\mathbf{\Pi}$ satisfies $\mathbf{\Pi} \in \mathcal{L}(H^\varepsilon(\Omega) \times L^2(\Omega), H^{\varepsilon+2}(\Omega) \times L^2(\Omega))$. As a result, in the two dimensional case, for $\varepsilon \in (0, \frac{1}{2})$, we obtain a local stabilization result for the nonlinear system provided the initial perturbation $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ is small enough. In Section 4 we illustrate some of the theoretical results in a numerical setup. We conclude with a short summary of our contributions in Section 5.

Notation. For $p \geq 1$ and $s \geq 0$, we denote by $L^p(\Omega)$ and $H^s(\Omega)$ the usual Lebesgue and Sobolev spaces. We define $H_\nu^s(\Omega) := \left\{ y \in C^\infty(\Omega) \mid \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma \right\}$, where $C^\infty(\Omega)$ denotes the space of infinitely differentiable functions in Ω and where the closure is taken with respect to the $\|y\|_{H^s(\Omega)}$, $s \geq 0$, norm. For $s > \frac{3}{2}$, we have $H_\nu^s(\Omega) = \left\{ y \in H^s(\Omega) \mid \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma \right\}$ and for $s \in [0, \frac{3}{2})$, we have $H_\nu^s(\Omega) = H^s(\Omega)$, see, e.g., [8, Section II-1]. Furthermore, we set $H_0^s(\Omega) = \overline{\{y \in C_0^\infty(\Omega)\}}$, where the closure is taken with respect to the $\|y\|_{H^s(\Omega)}$, $s \geq 0$, norm. For $s > \frac{1}{2}$, we have that $H_0^s(\Omega) = \{y \in H^s(\Omega) \mid y = 0 \text{ on } \Gamma\}$. Given $p \geq 1$, an interval $I \subset \mathbb{R}$ and a Hilbert space X , we denote with $L^p(I; X)$ (Bochner) p -integrable functions on I with values in X . We define the space

$$W(0, T; \mathbf{X}, \mathbf{Y}) := \left\{ \mathbf{y} \in L^2(0, T; \mathbf{X}) \mid \frac{d\mathbf{y}}{dt} \in L^2(0, T; \mathbf{Y}) \right\}.$$

For $Q_\infty = \Omega \times (0, \infty)$ and $r \geq 0, s \geq 0$ we define the anisotropic Sobolev spaces $H^{r,s}(Q_\infty)$ by

$$H^{r,s}(Q_\infty) = L^2(0, \infty; H^r(\Omega)) \cap H^s(0, \infty; L^2(\Omega)).$$

Similar, we use Σ_∞ . We further use $\mathbf{L}^2(\Omega)$ instead of $L^2(\Omega) \times L^2(\Omega)$ and define

$$\mathbf{H}^{r,s}(Q_\infty) := (L^2(0, \infty; H^r(\Omega)) \cap H^s(0, \infty; L^2(\Omega)))^2.$$

In more general, boldface letters are always associated with the PDE-ODE system while italic letters refer to either the PDE or the ODE part. Let \mathbf{X} and \mathbf{Y} denote Hilbert spaces. For a closed densely defined linear operator $\mathbf{A}: \mathcal{D}(\mathbf{A}) \subset \mathbf{Y} \rightarrow \mathbf{Y}$, the resolvent set of \mathbf{A} is denoted by $\rho(\mathbf{A})$. Given its infinitesimal generator \mathbf{A} with $e^{\mathbf{A}t}$ we denote the associated semigroup. For the definition and calculus involving interpolation spaces $[\mathbf{X}, \mathbf{Y}]_\theta$, we refer to e.g. [8, 22, 30]. By C, C_1 and C_2 we denote generic constants that may vary throughout consecutive calculations.

2. Neumann boundary control. We start with the linearization of (4) for $\theta = 1$. Hence, let us consider

$$\begin{aligned} \frac{\partial y}{\partial t} &= \Delta y + \alpha(x)y - dz \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial z}{\partial t} &= \iota y - (\kappa - \sigma)z \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial y}{\partial \nu} &= mu \quad \text{on } \Gamma \times (0, \infty), \\ y(x, 0) &= y_0 \text{ and } z(x, 0) = z_0 \quad \text{in } \Omega, \end{aligned} \tag{5}$$

as a boundary control system as described in, e.g., [8, 21, 31]. For this purpose, we introduce the operators $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ and $(\mathbf{A}^*, \mathcal{D}(\mathbf{A}^*))$ with

$$\begin{aligned} \mathbf{A}\mathbf{y} &= \begin{pmatrix} \Delta y + \alpha(x)y - dz \\ \iota y - (\kappa - \sigma)z \end{pmatrix}, \quad \mathcal{D}(\mathbf{A}) = H_\nu^2(\Omega) \times L^2(\Omega), \\ \mathbf{A}^*\mathbf{y} &= \begin{pmatrix} \Delta y + \alpha(x)y + \iota z \\ -dy - (\kappa - \sigma)z \end{pmatrix}, \quad \mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\mathbf{A}). \end{aligned} \tag{6}$$

For the rest of this section, let $\lambda_0 \in \rho(\mathbf{A})$ be such that $\widehat{\mathbf{A}} := \lambda_0 I - \mathbf{A}$ with $\mathcal{D}(\widehat{\mathbf{A}}) = \mathcal{D}(\mathbf{A})$ satisfies

$$\begin{aligned} (\widehat{\mathbf{A}}\mathbf{y}, \mathbf{y})_{\mathbf{L}^2(\Omega)} &\geq \|\mathbf{y}\|_{H^1(\Omega) \times L^2(\Omega)}^2 \quad \text{for all } \mathbf{y} \in \mathcal{D}(\mathbf{A}), \\ (\widehat{\mathbf{A}}^*\mathbf{y}, \mathbf{y})_{\mathbf{L}^2(\Omega)} &\geq \|\mathbf{y}\|_{H^1(\Omega) \times L^2(\Omega)}^2 \quad \text{for all } \mathbf{y} \in \mathcal{D}(\mathbf{A}^*). \end{aligned}$$

As a consequence, $-\widehat{\mathbf{A}}$ generates an analytic exponentially stable semigroup on $\mathbf{Y} = \mathbf{L}^2(\Omega)$ and, additionally, its fractional powers $\widehat{\mathbf{A}}^\gamma$ (see [24]) are well-defined. Further, we introduce the Neumann map $\mathbf{N}_{\widehat{\mathbf{A}}}$ by $\mathbf{N}_{\widehat{\mathbf{A}}}u = \mathbf{y}$ iff

$$\begin{aligned} \lambda_0 y - \Delta y - \alpha(x)y + dz &= 0 \quad \text{in } \Omega, & \frac{\partial y}{\partial \nu} &= u \quad \text{on } \Gamma, \\ \lambda_0 z - \iota y + (\kappa - \sigma)z &= 0 \quad \text{in } \Omega. \end{aligned} \tag{7}$$

By explicitly solving the second equation in (7) for z , we equivalently obtain

$$\lambda_0 y - \Delta y - \alpha(x)y + \frac{dt}{\lambda_0 + \kappa - \sigma}y = 0 \quad \text{in } \Omega, \quad \frac{\partial y}{\partial \nu} = u \quad \text{on } \Gamma.$$

Recall that by assumption $\bar{v} \in H^3(\Omega)$ which for $n = 2, 3$ implies that $\bar{v}^2 \in H^3(\Omega)$, see, e.g., [18, Proposition B.1]. Hence, $\alpha \in H^3(\Omega)$ and with [1, Theorem 4.12], we obtain that $\alpha \in C_b^1(\Omega)$. From [17, Theorem 2.4.2.7/Remark 2.5.1.2], it thus follows that

$$\mathbf{N}_{\widehat{\mathbf{A}}}: \text{continuous } H^s(\Gamma) \rightarrow \mathbf{H}^{s+\frac{3}{2}}(\Omega), \quad 0 \leq s \leq \frac{3}{2}. \tag{8}$$

In particular, for $s = 0$ let us emphasize that

$$\mathbf{N}_{\widehat{\mathbf{A}}}: \text{continuous } L^2(\Gamma) \rightarrow \mathbf{H}^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\varepsilon}(\Omega) \times L^2(\Omega) = \mathcal{D}(\widehat{\mathbf{A}}^{\frac{3}{4}-\varepsilon}), \quad \varepsilon > 0. \tag{9}$$

We now may rewrite the linearized system as a boundary control system of the form

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}Mu(t), \quad \mathbf{y}_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \tag{10}$$

where $\mathbf{B} = \widehat{\mathbf{A}}\mathbf{N}_{\widehat{\mathbf{A}}} \in \mathcal{L}(L^2(\Gamma), [\mathcal{D}(\mathbf{A}^*)]')$ and M is the multiplication operator associated with m .

Lemma 2.1. For all $\mathbf{p} = (p, q) \in \mathcal{D}(\mathbf{A}^*)$, it holds that

$$\mathbf{B}^* \mathbf{p} = \mathbf{N}_{\widehat{\mathbf{A}}}^* \widehat{\mathbf{A}}^* \mathbf{p} = p|_{\Gamma}. \quad (11)$$

For $\mathbf{p} \in H^{\frac{1}{2}+s}(\Omega) \times L^2(\Omega)$, $s > 0$, it further holds

$$\|\mathbf{B}^* \mathbf{p}\|_{H^s(\Gamma)} \leq C \|\mathbf{p}\|_{H^{\frac{1}{2}+s}(\Omega) \times L^2(\Omega)}.$$

Proof. Assume that $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*)$ and $u \in L^2(\Gamma)$. With $\mathbf{y} = (y, z)$ let us denote $\mathbf{N}_{\widehat{\mathbf{A}}} u = \mathbf{y}$. We then have

$$\begin{aligned} (\mathbf{B}u, \mathbf{p})_{[\mathcal{D}(\mathbf{A}^*)]', \mathcal{D}(\mathbf{A}^*)} &= \left(\mathbf{N}_{\widehat{\mathbf{A}}} u, \widehat{\mathbf{A}}^* \mathbf{p} \right)_{L^2(\Omega)} = \left(\mathbf{y}, \widehat{\mathbf{A}}^* \mathbf{p} \right)_{L^2(\Omega)} \\ &= (y, \lambda_0 p - \Delta p - \alpha(x)p - \iota q)_{L^2(\Omega)} + (z, \lambda_0 q + (\kappa - \sigma)q + dp)_{L^2(\Omega)} \\ &= (\lambda_0 y - \Delta y - \alpha(x)y + dz, p)_{L^2(\Omega)} + (\lambda_0 z - \iota y + (\kappa - \sigma)z, q)_{L^2(\Omega)} \\ &\quad - \left(y, \frac{\partial p}{\partial \nu} \right)_{L^2(\Gamma)} + \left(\frac{\partial y}{\partial \nu}, p \right)_{L^2(\Gamma)}. \end{aligned}$$

By (7) and the fact that $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*)$, we thus obtain

$$(\mathbf{B}u, \mathbf{p})_{[\mathcal{D}(\mathbf{A}^*)]', \mathcal{D}(\mathbf{A}^*)} = (u, p)_{L^2(\Gamma)} = (u, \mathbf{B}^* \mathbf{p})_{L^2(\Gamma)}.$$

□

Finally, note that (9) implies that

$$\widehat{\mathbf{A}}^{-\gamma} \mathbf{B} = \widehat{\mathbf{A}}^{1-\gamma} \mathbf{N}_{\widehat{\mathbf{A}}} \in \mathcal{L}(L^2(\Gamma), \mathbf{Y}), \quad \gamma = \frac{1}{4} + \varepsilon, \quad \varepsilon > 0. \quad (12)$$

2.1. Existing Riccati theory for the linearized system. With system (5), we associate the following cost functional

$$\begin{aligned} \mathcal{J}(u, \mathbf{y}) &= \int_0^\infty \left(\left\| \widetilde{M} \mathbf{C} \mathbf{y}(t) \right\|_{L^2(\Gamma)}^2 + \|u(t)\|_{L^2(\Gamma)}^2 \right) dt \\ &= \int_0^\infty \left(\left\| \widetilde{M} C y(t) \right\|_{L^2(\Gamma)}^2 + \|u(t)\|_{L^2(\Gamma)}^2 \right) dt, \end{aligned} \quad (13)$$

where the multiplication operator \widetilde{M} denotes the observable part of y on the boundary Γ and $\mathbf{C} \mathbf{y} = C y$ where $C: H^{\frac{1}{2}+s}(\Omega) \rightarrow L^2(\Gamma)$, $s > 0$, is the Dirichlet trace operator $C y = y|_{\Gamma}$. Based on the results in [2, 21], our goal now is to find u in feedback form such that the cost functional (13) is minimized. We therefore make the following hypotheses which we also comment on subsequently:

- (H1) \mathbf{A} is the infinitesimal generator of a strongly continuous analytic semigroup on $\mathbf{Y} := L^2(\Omega)$.
- (H2) $\widehat{\mathbf{A}}^{-\gamma} \mathbf{B} \mathbf{M} \in \mathcal{L}(U, \mathbf{Y})$, $U := L^2(\Gamma)$, for some fixed constant γ , $0 \leq \gamma < 1$.
- (H3) $\widetilde{M} \mathbf{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), Z)$, $Z := L^2(\Gamma)$, for some fixed $0 < \delta < \min\{1 - \gamma, \frac{1}{2}\}$.
- (H4) For each $\mathbf{y}_0 \in \mathbf{Y}$, there exists $u \in L^2(0, \infty; U)$ such that the corresponding solution \mathbf{y} satisfies $\widetilde{M} \mathbf{C} \mathbf{y} \in L^2(0, \infty; Z)$ so that $\mathcal{J}(u, \mathbf{y}) < \infty$.
- (H5) There exists an operator $\mathbf{K} \in \mathcal{L}(Z, \mathcal{D}(\widehat{\mathbf{A}}^\delta))$, such that the strongly continuous analytic semigroup $e^{(\mathbf{A} + \mathbf{K} \widetilde{M} \mathbf{C})t}$ on $\mathcal{D}(\widehat{\mathbf{A}}^\delta)$, generated by $\mathbf{A} + \mathbf{K} \widetilde{M} \mathbf{C}$ is exponentially stable.

As we have shown in [9, Lemma 3.1], \mathbf{A} indeed is the infinitesimal generator of a strongly continuous analytic semigroup on \mathbf{Y} . Hence, (H1) is satisfied. From (12), it follows that (H2) holds for $U = L^2(\Gamma)$ and

$$\frac{1}{4} < \gamma < 1. \quad (14)$$

Lemma 2.1 implies that $\mathbf{C} = \mathbf{B}^*$ and $\widetilde{M}\mathbf{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), Z)$ for

$$\frac{1}{4} < \delta < \min \left\{ 1 - \gamma, \frac{1}{2} \right\} \quad (15)$$

such that (H3) is satisfied. From now on, let us fix $\frac{1}{4} < \delta < \frac{3}{8}$. It remains to show the validity of (H4) and (H5). For this, we may follow the strategy used in [9] and first consider the decoupled problem

$$\begin{aligned} \frac{\partial \tilde{y}}{\partial t} &= \Delta \tilde{y} + \alpha(x)\tilde{y} + \omega \tilde{y} \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial \tilde{y}}{\partial \nu} &= m u \quad \text{on } \Gamma \times (0, \infty), \\ \tilde{y}(x, 0) &= y_0, \end{aligned} \quad (16)$$

where $\omega > 0$, together with the cost functional

$$J(u, \tilde{y}) = \int_0^\infty \left(\|\tilde{y}(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{L^2(\Gamma)}^2 \right) dt, \quad (17)$$

which is finite for $\tilde{y} \in L^2(Q_\infty)$ and $u \in L^2(\Sigma_\infty)$. Using available null controllability results from [15, Theorem 2] and a well-known extension argument (see, e.g., [16, Theorem 2.3]) we particularly conclude that for each $y_0 \in L^2(\Omega)$ there exists $u \in L^2(\Sigma_\infty)$ such that the corresponding solution \tilde{y} satisfies $\tilde{y} \in L^2(Q_\infty)$ so that $J(u, \tilde{y}) < \infty$. In other words, the decoupled system (16) is stabilizable. As a consequence, there exists a linear feedback operator $K \in \mathcal{L}(L^2(\Omega), L^2(\Gamma))$ such that the decoupled closed-loop operator $A := \Delta + \alpha + \omega + BK$ generates an exponentially stable semigroup on $L^2(\Omega)$. According to [9, Lemma 3.2], if we choose $\omega = \varepsilon + \frac{\iota d}{(\kappa - \sigma) - \varepsilon}$, $0 < \varepsilon < \kappa - \sigma$, by means of a Schur-complement type argument, one can show that all eigenvalues of the operator

$$\begin{pmatrix} \Delta + \alpha + \varepsilon + BK & -d \\ \iota & -(\kappa - \sigma - \varepsilon) \end{pmatrix} = \begin{pmatrix} \Delta + \alpha + BK & -d \\ \iota & -(\kappa - \sigma) \end{pmatrix} + \varepsilon I$$

are located in the open left complex half plane. This implies that the coupled system (5) is in fact ε -stabilizable and assumption (H4) is fulfilled. Finally, the detectability condition (H5) of the pair $(\mathbf{A}, \widetilde{M}\mathbf{C})$ is equivalent to the stabilizability of the pair $(\mathbf{A}^*, \mathbf{C}^*\widetilde{M})$. Since we already know that $\mathbf{B} = \mathbf{C}^*$, the arguments provided for (H4) apply here as well and imply (H5). In summary, all of the assumptions (H1)-(H5) are indeed fulfilled for the monodomain equations and we can use the results from [21, Section 2.5] that guarantee the existence of a unique nonnegative self-adjoint solution $\mathbf{\Pi} = \mathbf{\Pi}^* \in \mathcal{L}(\mathbf{Y})$ to the algebraic Riccati equation for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{D}(\widehat{\mathbf{A}}^{\delta+s})$, $s > 0$:

$$(\mathbf{A}^*\mathbf{\Pi}\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{Y}} + (\mathbf{\Pi}\mathbf{A}\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{Y}} + \left(\widetilde{M}\mathbf{C}\mathbf{z}_1, \widetilde{M}\mathbf{C}\mathbf{z}_2 \right)_Z = (M\mathbf{B}^*\mathbf{\Pi}\mathbf{z}_1, M\mathbf{B}^*\mathbf{\Pi}\mathbf{z}_2)_U.$$

Additionally, by [21, Theorem 2.5.2], it holds that

$$(\widehat{\mathbf{A}}^*)^\gamma \mathbf{\Pi} \widehat{\mathbf{A}}^{-\delta} \in \mathcal{L}(\mathbf{Y}), \quad \mathbf{B}^*\mathbf{\Pi} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), U) \quad (18)$$

and $(\mathbf{A}_\Pi, \mathcal{D}(\mathbf{A}_\Pi))$, where $\mathbf{A}_\Pi = \mathbf{A} - \mathbf{B}M^2\mathbf{B}^*\Pi$, is the infinitesimal generator of a strongly continuous semigroup exponentially stable on $\mathcal{D}(\widehat{\mathbf{A}}^\delta)$. In [21, Section 2.5], these results are actually derived by a change of variables

$$\begin{aligned}\bar{\mathbf{B}}M &\equiv \widehat{\mathbf{A}}^\delta \mathbf{B}M: \text{continuous } U \rightarrow (\mathcal{D}(\widehat{\mathbf{A}}^*)^{1+\delta})', \\ \widehat{\mathbf{A}}^{-\bar{\gamma}}\bar{\mathbf{B}}M &= \widehat{\mathbf{A}}^{-(\bar{\gamma}-\delta)}\mathbf{B}M = \widehat{\mathbf{A}}^{-\gamma}\mathbf{B}M \in \mathcal{L}(U, \mathbf{Y}), \quad \bar{\gamma} = \gamma + \delta < 1, \\ \widetilde{M}\bar{\mathbf{C}} &\equiv \widetilde{M}\mathbf{C}\widehat{\mathbf{A}}^{-\delta} \in \mathcal{L}(\mathbf{Y}, Z)\end{aligned}\quad (19)$$

for the transformed system

$$\frac{d}{dt}\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{y}} + \bar{\mathbf{B}}Mu, \quad \bar{\mathbf{y}}(0) = \bar{\mathbf{y}}_0, \quad (20)$$

where $\bar{\mathbf{y}} = \widehat{\mathbf{A}}^\delta \mathbf{y}$ and the modified cost functional

$$\bar{J}(u, \bar{\mathbf{y}}_0) = \int_0^\infty \left(\left\| \widetilde{M}\bar{\mathbf{C}}\bar{\mathbf{y}}(t) \right\|_Z^2 + \|u(t)\|_U^2 \right) dt.$$

In particular, for the corresponding Riccati operator $\bar{\Pi}$ it is shown that

$$M\bar{\mathbf{B}}^*\bar{\Pi} \in \mathcal{L}(\mathbf{Y}, U), \quad (\widehat{\mathbf{A}}^*)^\theta \bar{\Pi} \in \mathcal{L}(\mathbf{Y}), \quad 0 \leq \theta < 1 \quad (21)$$

and that the semigroup generated by $\mathbf{A}_{\bar{\Pi}} = \mathbf{A} - \bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*\bar{\Pi}$ is exponentially stable on \mathbf{Y} , i.e.,

$$\|e^{\mathbf{A}_{\bar{\Pi}}t}\|_{\mathcal{L}(\mathbf{Y})} \leq C_{\mathbf{A}_{\bar{\Pi}}} e^{-\omega_{\mathbf{A}_{\bar{\Pi}}}t},$$

with constants $C_{\mathbf{A}_{\bar{\Pi}}} \geq 1$ and $\omega_{\mathbf{A}_{\bar{\Pi}}} > 0$, see [21, Theorem 2.5.1/2.5.2]. Moreover, we have the relation

$$\mathbf{A}_{\bar{\Pi}} = \mathbf{A} - \mathbf{B}M^2\mathbf{B}^*\Pi = \widehat{\mathbf{A}}^{-\delta}\mathbf{A}_{\bar{\Pi}}\widehat{\mathbf{A}}^\delta. \quad (22)$$

2.2. The nonhomogeneous linear system. Before we turn to the nonlinear setting, we first consider the nonhomogeneous equation

$$\frac{d}{dt}\mathbf{y} = \mathbf{A}_{\bar{\Pi}}\mathbf{y} + \mathbf{f} = (\widehat{\mathbf{A}}^{-\delta}\mathbf{A}_{\bar{\Pi}}\widehat{\mathbf{A}}^\delta)\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(0) = \mathbf{y}_0, \quad (23)$$

with $\mathbf{f} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$. For what follows, we cite a general result concerning the maximal regularity of the nonhomogeneous system (23).

Theorem 2.2. ([8, Chapter 3, Theorem 2.2]) *Let \mathbf{Y} be a Hilbert space and suppose that \mathbf{A} is the infinitesimal generator of an analytic semigroup of negative type on \mathbf{Y} . Then, for all $0 \leq \alpha \leq 1$, the mapping*

$$\begin{aligned}h: \mathbf{y} &\mapsto \left(\frac{d}{dt}\mathbf{y} - \mathbf{A}\mathbf{y}, \mathbf{y}(0) \right) \\ &W(0, \infty; [\mathcal{D}(\mathbf{A}), \mathbf{Y}]_\alpha, [\mathcal{D}(\mathbf{A}^*), \mathbf{Y}]'_{1-\alpha}) \\ &\rightarrow L^2(0, \infty; [\mathcal{D}(\mathbf{A}^*), \mathbf{Y}]'_{1-\alpha}) \times [[\mathcal{D}(\mathbf{A}), \mathbf{Y}]_\alpha, [\mathcal{D}(\mathbf{A}^*), \mathbf{Y}]'_{1-\alpha}]_{\frac{1}{2}}\end{aligned}$$

is an isomorphism.

The idea now is to apply this isomorphism to the (transformed) system (23) associated with $\mathbf{A}_{\bar{\Pi}}$. In particular, for $\mathbf{A}_{\bar{\Pi}}$, we have to characterize the interpolation spaces $[\mathcal{D}(\mathbf{A}_{\bar{\Pi}}), \mathbf{Y}]_\alpha$ and $[\mathcal{D}(\mathbf{A}_{\bar{\Pi}}^*), \mathbf{Y}]'_{1-\alpha}$. As pointed out in [2], the fact that $\widehat{\mathbf{A}}$ has bounded imaginary powers together with the perturbation results from [14, Proposition 2.7] allows us to conclude that $\mathbf{A}_{\bar{\Pi}}$ has bounded imaginary powers.

According to [30, Theorem 1.15.3] we thus have that $[\mathcal{D}(\mathbf{A}_{\bar{\Pi}}), \mathbf{Y}]_{\alpha} = \mathcal{D}(\mathbf{A}_{\bar{\Pi}}^{1-\alpha})$. Furthermore, since $\bar{\Pi}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*$ is \mathbf{A}^* -bounded, we obtain that $\mathcal{D}(\mathbf{A}_{\bar{\Pi}}^*) = \mathcal{D}(\widehat{\mathbf{A}}^*)$. Hence, it follows that $[\mathcal{D}(\mathbf{A}_{\bar{\Pi}}^*), \mathbf{Y}]'_{1-\alpha} = [\mathcal{D}((\widehat{\mathbf{A}}^*)^{\alpha})]'$. Following [2], let us then introduce $H_{\bar{\Pi}}^s := \mathcal{D}(\mathbf{A}_{\bar{\Pi}}^s)$ and recall from [2] the characterization of $H_{\bar{\Pi}}^s$:

$$H_{\bar{\Pi}}^s = \begin{cases} \left\{ \mathbf{y} \in \mathbf{L}^2(\Omega) \mid \mathbf{y} + \widehat{\mathbf{A}}^{-1}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*\bar{\Pi}\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s) \right\} & \text{if } s \in [0, 1], \\ [\mathcal{D}((\widehat{\mathbf{A}}^*)^{-s})]' & \text{if } s \in [-1, 0]. \end{cases} \quad (24)$$

In summary, Theorem 2.2 implies that

$$h: W(0, \infty; H_{\bar{\Pi}}^{r+\frac{1}{2}}, H_{\bar{\Pi}}^{r-\frac{1}{2}}) \rightarrow L^2(0, \infty; H_{\bar{\Pi}}^{r-\frac{1}{2}}) \times H_{\bar{\Pi}}^r \quad (25)$$

is an isomorphism. Note that for a result like (25), in [2, Corollary 2] the author assumes that $\bar{\mathbf{C}}$ satisfies $\bar{\mathbf{C}}^*\bar{\mathbf{C}} \in \mathcal{L}(\mathcal{D}((\widehat{\mathbf{A}}^*)^{\frac{1}{2}}), \mathcal{D}(\widehat{\mathbf{A}}^{\frac{1}{2}}))$ which does not hold in our case. Before we proceed with system (23), we give a more precise characterization of $H_{\bar{\Pi}}^s$, see also [2, Section 3/4] for the corresponding results in the context of the Navier-Stokes equations.

Proposition 1. *For $s \in [0, \frac{3}{4}]$ it holds that $H_{\bar{\Pi}}^s = H^{2s}(\Omega) \times L^2(\Omega)$. Moreover, for $\varepsilon \in (\frac{1}{2}, 1]$ it holds that*

$$\widehat{\mathbf{A}}^{\delta}\mathbf{y} \in H_{\bar{\Pi}}^{\frac{1+\varepsilon}{2}-\delta} \Leftrightarrow \mathbf{y} \in H^{1+\varepsilon}(\Omega) \times L^2(\Omega), \quad \frac{\partial \mathbf{y}}{\partial \nu} + M^2\mathbf{B}^*\bar{\Pi}\mathbf{y} = 0 \text{ on } \Gamma.$$

Proof. Let us first show that $\mathbf{y} \in \mathbf{L}^2(\Omega)$ already implies that $\widehat{\mathbf{A}}^{-1}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*\bar{\Pi}\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s)$, $s \in [0, \frac{3}{4}]$. Let us define $\theta := 2\delta + \frac{1}{4}$. Since $\delta < \frac{3}{8}$ we have that $\theta < 1$ and, by (21) it follows that

$$(\widehat{\mathbf{A}}^*)^{\delta}\bar{\Pi}\mathbf{y} \in \mathcal{D}((\widehat{\mathbf{A}}^*)^{\theta-\delta}) = H^{2(\theta-\delta)}(\Omega) \times L^2(\Omega) = H^{2\delta+\frac{1}{2}}(\Omega) \times L^2(\Omega).$$

Lemma 2.1 then yields $\bar{\mathbf{B}}^*\bar{\Pi}\mathbf{y} = \mathbf{B}^*(\widehat{\mathbf{A}}^*)^{\delta}\bar{\Pi}\mathbf{y} \in H^{2\delta}(\Gamma)$ and, from (8), we obtain that $\mathbf{N}_{\widehat{\mathbf{A}}}M^2\bar{\mathbf{B}}^*\bar{\Pi}\mathbf{y} \in \mathbf{H}^{2\delta+\frac{3}{2}}(\Omega)$. Hence, we can conclude that

$$\widehat{\mathbf{A}}^{-1}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*\bar{\Pi}\mathbf{y} = \widehat{\mathbf{A}}^{\delta}\mathbf{N}_{\widehat{\mathbf{A}}}\bar{\mathbf{B}}^*\bar{\Pi}\mathbf{y} \in H^{\frac{3}{2}}(\Omega) \times L^2(\Omega) \subset \mathcal{D}(\widehat{\mathbf{A}}^s), \quad s \in [0, \frac{3}{4}].$$

The first assertion now easily follows since

$$\mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega) = \mathcal{D}(\widehat{\mathbf{A}}^s) \text{ iff } \mathbf{y} + \widehat{\mathbf{A}}^{-1}\bar{\mathbf{B}}\bar{\mathbf{B}}^*M^2\bar{\Pi}\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s).$$

For the second assertion, let us assume that $\mathbf{y} \in H^{1+\varepsilon}(\Omega) \times L^2(\Omega)$ with $\frac{\partial \mathbf{y}}{\partial \nu} + M^2(\mathbf{B}^*\bar{\Pi}\mathbf{y}) = 0$ on Γ . We then obtain that $\widehat{\mathbf{A}}^{\delta}\mathbf{y} \in H^{1+\varepsilon-2\delta}(\Omega) \times L^2(\Omega)$, which, by the first part, equals $H_{\bar{\Pi}}^{\frac{1+\varepsilon}{2}-\delta}$. On the other hand, if $\widehat{\mathbf{A}}^{\delta}\mathbf{y} \in H_{\bar{\Pi}}^{\frac{1+\varepsilon}{2}-\delta}$ from (24) we conclude that $\widehat{\mathbf{A}}^{\delta}\mathbf{y} \in \mathbf{L}^2(\Omega)$ as well as

$$\widehat{\mathbf{A}}^{\delta}\mathbf{y} + \widehat{\mathbf{A}}^{-1}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*\bar{\Pi}\widehat{\mathbf{A}}^{\delta}\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^{\frac{1+\varepsilon}{2}-\delta}).$$

By definition of $\bar{\mathbf{B}}$ this also implies that

$$\widehat{\mathbf{A}}^{\frac{1+\varepsilon}{2}} \left(\mathbf{y} + \widehat{\mathbf{A}}^{-1}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*(\widehat{\mathbf{A}}^*)^{\delta}\bar{\Pi}\widehat{\mathbf{A}}^{\delta}\mathbf{y} \right) \in \mathbf{L}^2(\Omega).$$

Using the relation between $\bar{\Pi}$ and $\bar{\bar{\Pi}}$ we get

$$\widehat{\mathbf{A}}^{\frac{1+\varepsilon}{2}} (\mathbf{y} + \mathbf{N}_{\widehat{\mathbf{A}}}M^2\mathbf{B}^*\bar{\Pi}\mathbf{y}) \in \mathbf{L}^2(\Omega).$$

For $\gamma = \frac{1}{2}$, from (18) we obtain that $\bar{\Pi}\widehat{\mathbf{A}}^{-\delta}\widehat{\mathbf{A}}^{\delta}\mathbf{y} \in H^1(\Omega) \times L^2(\Omega)$ and, hence, that $\mathbf{B}^*\bar{\Pi}\mathbf{y} \in H^{\frac{1}{2}}(\Gamma)$. With (8) we thus have that $\mathbf{N}_{\widehat{\mathbf{A}}}M^2\mathbf{B}^*\bar{\Pi}\mathbf{y} \in \mathbf{H}^2(\Omega)$. This

particularly yields that $\mathbf{y} \in H^{1+\varepsilon}(\Omega) \times L^2(\Omega)$. The boundary conditions of \mathbf{y} follow immediately from the definition of $\mathbf{N}_{\widehat{\mathbf{A}}}$. \square

As a consequence of the previous results, for (23), we finally obtain.

Theorem 2.3. *Let $\varepsilon \in (\frac{1}{2}, 1]$. If $f_1 \in L^2(0, \infty; H^{\varepsilon-1}(\Omega))$, $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$, then (23) has a unique solution*

$$\mathbf{y} \in \left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \cap C_b([0, \infty); H^\varepsilon(\Omega)) \right) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))$$

satisfying

$$\|\mathbf{y}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq C_1 (\|\mathbf{y}_0\|_{H^\varepsilon(\Omega) \times L^2(\Omega)} + \|f_1\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))}).$$

Proof. Due to the relation between $\mathbf{\Pi}$ and $\widehat{\mathbf{\Pi}}$, it holds that

$$\frac{d}{dt}\mathbf{y} = \mathbf{A}_{\mathbf{\Pi}}\mathbf{y} + \mathbf{f} = (\widehat{\mathbf{A}}^{-\delta}\mathbf{A}_{\widehat{\mathbf{\Pi}}}\widehat{\mathbf{A}}^\delta)\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(0) = \mathbf{y}_0.$$

Hence, instead of (23), let us consider the transformed system

$$\frac{d}{dt}\mathbf{z} = \mathbf{A}_{\widehat{\mathbf{\Pi}}}\mathbf{z} + \widehat{\mathbf{A}}^\delta\mathbf{f}, \quad \mathbf{z}(0) = \mathbf{z}_0 = \widehat{\mathbf{A}}^\delta\mathbf{y}_0,$$

where $\mathbf{z} = \widehat{\mathbf{A}}^\delta\mathbf{y}$. Since $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega) \subset \mathcal{D}(\widehat{\mathbf{A}}^{\frac{\varepsilon}{2}})$, for $\varepsilon \geq 2\delta$ we obtain that $\mathbf{z}_0 \in \mathcal{D}(\widehat{\mathbf{A}}^{\frac{\varepsilon}{2}-\delta})$. By Proposition 1 this is equivalent to $\mathbf{z}_0 \in H_{\widehat{\mathbf{\Pi}}}^{\frac{\varepsilon}{2}-\delta}$. For $\varepsilon < 2\delta$, it follows that $\mathbf{z}_0 \in [\mathcal{D}((\widehat{\mathbf{A}}^*)^{\delta-\frac{\varepsilon}{2}})]'$ which, due to (24), again means that $\mathbf{z}_0 \in H_{\widehat{\mathbf{\Pi}}}^{\frac{\varepsilon}{2}-\delta}$. Similarly, since $H^{\varepsilon-1}(\Omega) = [\mathcal{D}((\widehat{\mathbf{A}}^*)^{\frac{1-\varepsilon}{2}})]'$ from $\mathbf{f} \in L^2(0, \infty; H^{\varepsilon-1}(\Omega))$ we obtain that $\widehat{\mathbf{A}}^\delta\mathbf{f} \in L^2(0, \infty; [\mathcal{D}((\widehat{\mathbf{A}}^*)^{\frac{1-\varepsilon}{2}+\delta})])'$ which is equivalent to $\widehat{\mathbf{A}}^\delta\mathbf{f} \in H_{\widehat{\mathbf{\Pi}}}^{\frac{\varepsilon-1}{2}-\delta}$. Equation (25) then yields that $\mathbf{z} \in W(0, \infty; H_{\widehat{\mathbf{\Pi}}}^{\frac{\varepsilon+1}{2}-\delta}, H_{\widehat{\mathbf{\Pi}}}^{\frac{\varepsilon-1}{2}-\delta})$. Once more, with Proposition 1, we find that

$$\mathbf{z} \in W(0, \infty; H^{1+\varepsilon-2\delta}(\Omega) \times L^2(\Omega), [H^{1-\varepsilon-2\delta}(\Omega) \times L^2(\Omega)]').$$

Since $\mathbf{z} = \widehat{\mathbf{A}}^\delta\mathbf{y}$, from Proposition 1 we find that

$$\mathbf{y} \in W(0, \infty; H^{1+\varepsilon}(\Omega) \times L^2(\Omega), [H^{1-\varepsilon}(\Omega) \times L^2(\Omega)]').$$

By the intermediate derivatives theorem ([22, Theorem 4.1]) and [22, Theorem 4.2] it follows that

$$\begin{aligned} \mathbf{y} &\in H^{\frac{1+\varepsilon}{2}}(0, \infty; [H^{1+\varepsilon}(\Omega) \times L^2(\Omega), [H^{1-\varepsilon}(\Omega) \times L^2(\Omega)]']^{\frac{1+\varepsilon}{2}}), \\ \mathbf{y} &\in C_b([0, \infty); [H^{1+\varepsilon}(\Omega) \times L^2(\Omega), [H^{1-\varepsilon}(\Omega) \times L^2(\Omega)]']^{\frac{1}{2}}). \end{aligned}$$

Using [22, Theorem 12.5], this yields $\mathbf{y} \in H^{\frac{1+\varepsilon}{2}}(0, \infty; L^2(\Omega))$ as well as $\mathbf{y} \in C_b([0, \infty); H^\varepsilon(\Omega) \times L^2(\Omega))$. The assertion for the ODE variable immediately follows from

$$\frac{d}{dt}z = \nu y - (\kappa - \sigma)z \in H^{\frac{1+\varepsilon}{2}}([0, \infty); L^2(\Omega)).$$

\square

2.3. Results for the nonlinear system. In [9] we have shown some Lipschitz estimates for nonlinearities of the form (2). In what follows, we extend these results to cover a broader range of parameters ε .

Proposition 2. *Let $\Omega \subset \mathbb{R}^n, n \in \{2, 3\}$. Assume that $f, g, h \in H^{1+\varepsilon}(\Omega), \varepsilon \in (0, 1]$ ($\varepsilon \in [\frac{1}{2}, 1]$ for $n = 3$). Then*

$$\begin{aligned} \|fg\|_{H^{\varepsilon-1}(\Omega)} &\leq C \|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)} \|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}, \\ \|fgh\|_{H^{\varepsilon-1}(\Omega)} &\leq C \|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)} \|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)} \|h\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}. \end{aligned}$$

Proof. Let us first assume that $\varepsilon \in (0, \frac{1}{2})$ and $n = 2$. Choosing $\lambda = \varepsilon - 1$ and $\mu = \omega = \frac{4}{3}$ in [18, Proposition B.1] we have that $\mu + \omega + \lambda = \frac{5}{3} + \varepsilon > 1$ and $2\lambda + \mu + \omega = \frac{2}{3} + 2\varepsilon > 0$. This shows the first inequality. For the second one, we note that with $\lambda = \varepsilon - 1, \mu = \frac{2}{3} - \varepsilon$ and $\omega = \frac{4}{3}$ it holds that $\lambda + \mu + \omega = 1$ as well as $2\lambda + \mu + \omega = \varepsilon > 0$. This yields

$$\|fgh\|_{H^{\varepsilon-1}(\Omega)} \leq C \|fg\|_{H^{-\frac{1}{3}}(\Omega)} \|h\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

Moreover, with $\lambda = -\frac{1}{3}$ and $\mu = \omega = \frac{2}{3} + \varepsilon$ in [18, Proposition B.1] we find that

$$\|fg\|_{H^{-\frac{1}{3}}(\Omega)} \leq C \|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)} \|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

Next assume that $\varepsilon \in [\frac{1}{2}, 1]$ and $n \in \{2, 3\}$. The first inequality follows exactly as before since it still holds that $\frac{5}{3} + \varepsilon > \frac{n}{2}$. For the second one, we choose $\lambda = \varepsilon - 1, \mu = \frac{7}{6} - \varepsilon$ and $\omega = \frac{4}{3}$ in [18, Proposition B.1]. This implies $\lambda + \mu + \omega = \frac{3}{2}$ and $2\lambda + \mu + \omega > 0$. Hence, we find that

$$\|fgh\|_{H^{\varepsilon-1}(\Omega)} \leq C \|fg\|_{H^{\frac{1}{6}}(\Omega)} \|h\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

Moreover, with $\lambda = \frac{1}{6}$ and $\mu = \omega = \frac{1}{6} + \varepsilon$ in [18, Proposition B.1] this leads to

$$\|fg\|_{H^{\frac{1}{6}}(\Omega)} \leq C \|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)} \|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

This shows the second inequality. \square

Proposition 2 allows to show the following results.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n, n \in \{2, 3\}$. Assume that $\sigma \geq 0$ and $\varepsilon \in (0, 1]$ ($\varepsilon \in [\frac{1}{2}, 1]$ for $n = 3$). Then for $y_1, y_2 \in H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)$, it holds that*

$$\begin{aligned} \|e^{-\sigma t}(y_1^2 - y_2^2)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))} &\leq C \|y_1 - y_2\|_{1+\varepsilon, \frac{1+\varepsilon}{2}} \left(\|y_1\|_{1+\varepsilon, \frac{1+\varepsilon}{2}} + \|y_2\|_{1+\varepsilon, \frac{1+\varepsilon}{2}} \right), \\ \|e^{-2\sigma t}(y_1^3 - y_2^3)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))} &\leq C \|y_1 - y_2\|_{1+\varepsilon, \frac{1+\varepsilon}{2}} \left(\|y_1\|_{1+\varepsilon, \frac{1+\varepsilon}{2}}^2 + \|y_2\|_{1+\varepsilon, \frac{1+\varepsilon}{2}}^2 \right). \end{aligned}$$

Proof. Note first that by interpolation ([22, Chapter 1, Theorem 4.1]) we have that

$$y_1, y_2 \in H^{\frac{1}{3}}(0, \infty; [H^{1+\varepsilon}(\Omega), L^2(\Omega)]_{\frac{1}{3}, \frac{1+\varepsilon}{2}}) = H^{\frac{1}{3}}(0, \infty; H^{\frac{1}{3}+\varepsilon}(\Omega))$$

$$y_1, y_2 \in H^{\frac{1}{4}}(0, \infty; [H^{1+\varepsilon}(\Omega), L^2(\Omega)]_{\frac{1}{4}, \frac{1+\varepsilon}{2}}) = H^{\frac{1}{4}}(0, \infty; H^{\frac{1}{2}+\varepsilon}(\Omega)) \subset H^{\frac{1}{4}}(0, \infty; H^{\frac{1}{3}+\varepsilon}(\Omega)).$$

Moreover, by Sobolev embedding (see, e.g., [27, Theorem 7]), it follows that

$$y_1, y_2 \in L^4(0, \infty; H^{\frac{1}{3}+\varepsilon}(\Omega)) \cap L^6(0, \infty; H^{\frac{1}{3}+\varepsilon}(\Omega)).$$

The result now follows from the generalized Hölder inequality and the facts that

$$\begin{aligned} y_1^2 - y_2^2 &= (y_1 - y_2)(y_1 + y_2), \\ y_1^3 - y_2^3 &= (y_1 - y_2)(y_1^2 + 2y_1y_2 + y_2^2) \end{aligned}$$

together with an application of Proposition 2. \square

For the nonlinearity in (4), i.e.,

$$\mathbf{F}(\mathbf{y}) := \begin{pmatrix} (b - 3a\bar{v})e^{-\sigma t}y^2 - ae^{-2\sigma t}y^3 \\ 0 \end{pmatrix},$$

we obtain the following local Lipschitz continuity property.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$. Assume that $\varepsilon \in (0, 1]$ ($\varepsilon \in [\frac{1}{2}, 1]$ for $n = 3$). If*

$$\mathbf{y}_1, \mathbf{y}_2 \in H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega)),$$

then \mathbf{F} is locally Lipschitz continuous from $H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))$ to $L^2(0, \infty; H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)$. In particular, we have that

$$\begin{aligned} & \|\mathbf{F}(\mathbf{y}_1) - \mathbf{F}(\mathbf{y}_2)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)} \\ & \leq C_2 \left(\|y_1\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)} + \|y_2\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)} \right. \\ & \quad \left. + \|y_1\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)}^2 + \|y_2\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)}^2 \right) \|y_1 - y_2\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)}. \end{aligned}$$

Proof. By definition of \mathbf{F} , we have

$$\begin{aligned} & \|\mathbf{F}(\mathbf{y}_1) - \mathbf{F}(\mathbf{y}_2)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)} \\ & = \|(b - 3a\bar{v})e^{-\sigma t}(y_1^2 - y_2^2) - ae^{-2\sigma t}(y_1^3 - y_2^3)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))} \\ & \leq C_2 \left(\|(b - 3a\bar{v})e^{-\sigma t}(y_1^2 - y_2^2)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))} + \|ae^{-2\sigma t}(y_1^3 - y_2^3)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))} \right). \end{aligned}$$

Since $\bar{v} \in H^3(\Omega)$, [18, Proposition B.1], implies that

$$\begin{aligned} & \|\mathbf{F}(\mathbf{y}_1) - \mathbf{F}(\mathbf{y}_2)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)} \\ & \leq C_2 \left(\|e^{-\sigma t}(y_1^2 - y_2^2)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))} + \|e^{-2\sigma t}(y_1^3 - y_2^3)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))} \right). \end{aligned}$$

The claim now follows with Lemma 2.4. \square

Our final results concerns the local stabilization of the full nonlinear system.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ and $\varepsilon \in (\frac{1}{2}, 1]$. Then there exist $\mu_0 > 0$ and a nondecreasing function η from \mathbb{R}^+ into itself such that if $\mu \in (0, \mu_0)$ and $\|\mathbf{y}_0\|_{H^\varepsilon(\Omega) \times L^2(\Omega)} \leq \eta(\mu)$, then*

$$\frac{d}{dt}\mathbf{y} = \mathbf{A}_\Pi \mathbf{y} + \mathbf{F}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

admits a unique solution in the set

$$\begin{aligned} D_\mu = \left\{ \mathbf{y} \in \left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \cap C_b([0, \infty); H^\varepsilon(\Omega)) \right) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega)), \right. \\ \left. \|\mathbf{y}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq \mu \right\}. \end{aligned}$$

Proof. Following the arguments provided for similar results in [9, 26], let us show that the mapping $\mathcal{M}: \mathbf{z} \mapsto \mathbf{y}_z$ defined by

$$\frac{d}{dt}\mathbf{y}_z = \mathbf{A}_\Pi \mathbf{y}_z + \mathbf{F}(\mathbf{z}), \quad \mathbf{y}_z(0) = \mathbf{y}_0, \quad (26)$$

is a contraction in D_μ . As in [9], define

$$\mu_0 = \frac{1}{2} \left(\sqrt{1 + \frac{1}{C_1 C_2}} - 1 \right) \quad \text{and} \quad \eta(\mu) = \frac{3}{4C_1} \mu,$$

with C_1 and C_2 as in Theorem 2.3 and Lemma 2.5, respectively.

Step 1: For $\mathbf{z} \in D_\mu$, from Lemma 2.5 we get

$$\|\mathbf{F}(\mathbf{z})\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)} \leq C_2(\mu + \mu^2)\mu.$$

Utilizing Theorem 2.3 we obtain that

$$\begin{aligned} & \|\mathbf{y}_\mathbf{z}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \\ & \leq C_1 \left(\|\mathbf{y}_0\|_{H^\varepsilon(\Omega) \times L^2(\Omega)} + \|\mathbf{F}(\mathbf{z})\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)} \right) \\ & \leq \frac{3}{4}\mu + C_1 C_2 (\mu + \mu^2)\mu. \end{aligned}$$

Since $\mu \leq \mu_0 = \frac{1}{2} \left(\sqrt{1 + \frac{1}{C_1 C_2}} - 1 \right)$, we conclude that $\mu + \mu^2 \leq \frac{1}{4C_1 C_2}$. This implies

$$\|\mathbf{y}_\mathbf{z}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq \mu.$$

Hence, \mathcal{M} is mapping D_μ to itself.

Step 2: For $\mathbf{z}_1 = (z_1^{(1)}, z_1^{(2)})$, $\mathbf{z}_2 = (z_2^{(1)}, z_2^{(2)}) \in D_\mu$ we further have

$$\frac{d}{dt}(\mathbf{y}_{\mathbf{z}_1} - \mathbf{y}_{\mathbf{z}_2}) = \mathbf{A}_\Pi(\mathbf{y}_{\mathbf{z}_1} - \mathbf{y}_{\mathbf{z}_2}) + \mathbf{F}(\mathbf{z}_1) - \mathbf{F}(\mathbf{z}_2), \quad \mathbf{y}_{\mathbf{z}_1}(0) - \mathbf{y}_{\mathbf{z}_2}(0) = 0.$$

As a consequence, from Theorem 2.3 we know that

$$\|\mathbf{y}_{\mathbf{z}_1} - \mathbf{y}_{\mathbf{z}_2}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq C_1 \|\mathbf{F}(\mathbf{z}_1) - \mathbf{F}(\mathbf{z}_2)\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)}.$$

Using Lemma 2.5, this yields

$$\begin{aligned} \|\mathbf{y}_{\mathbf{z}_1} - \mathbf{y}_{\mathbf{z}_2}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} & \leq 2C_1 C_2 (\mu + \mu^2) \|z_1^{(1)} - z_2^{(1)}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)} \\ & \leq \frac{1}{2} \|z_1^{(1)} - z_2^{(1)}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)}. \end{aligned}$$

The mapping \mathcal{M} is a contraction in D_μ and the proof is complete. \square

As a consequence, for system (1), we obtain the following result.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ and $\varepsilon \in (\frac{1}{2}, 1]$. Let further $0 \leq \sigma < \kappa$ be given. For $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{D}(\mathbf{A})$ let Π be the solution to*

$$(\mathbf{A}^* \Pi \mathbf{z}_1, \mathbf{z}_2)_\mathbf{Y} + (\Pi \mathbf{A} \mathbf{z}_1, \mathbf{z}_2)_\mathbf{Y} + 2\sigma(\mathbf{z}_1, \mathbf{z}_2) + \left(\widetilde{M} \mathbf{C} \mathbf{z}_1, \widetilde{M} \mathbf{C} \mathbf{z}_2 \right)_Z = (M \mathbf{B}^* \Pi \mathbf{z}_1, M \mathbf{B}^* \Pi \mathbf{z}_2)_U.$$

Then there exist $\mu_0 > 0$ and a nondecreasing function η from \mathbb{R}^+ into itself such that if $\mu \in (0, \mu_0)$ and $\|(y_0, z_0)\|_{H^\varepsilon(\Omega) \times L^2(\Omega)} \leq \eta(\mu)$, then

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta v - I_{\text{ion}}(v, w) + f_v \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial t} &= \iota v - \kappa w + f_w \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial \nu} &= g - M^2 \mathbf{B}^* \Pi(v - \bar{v}, w - \bar{w}) \quad \text{on } \Gamma \times (0, \infty), \\ v(x, 0) &= \bar{v} + y_0 \quad \text{and} \quad w(x, 0) = \bar{w} + z_0 \quad \text{in } \Omega, \end{aligned}$$

admits a unique solution (v, w) which satisfies

$$\|e^{\sigma \cdot} (v - \bar{v}, w - \bar{w})\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq \mu.$$

3. Dirichlet boundary control. We now consider the linearization of (4) for $\theta = 0$. Thus, in this section we first focus on

$$\begin{aligned} \frac{\partial y}{\partial t} &= \Delta y + \alpha(x)y - dz \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial z}{\partial t} &= \iota y - (\kappa - \sigma)z \quad \text{in } \Omega \times (0, \infty), \\ y &= mu \quad \text{on } \Gamma \times (0, \infty), \\ y(x, 0) &= y_0 \text{ and } z(x, 0) = z_0 \quad \text{in } \Omega. \end{aligned} \tag{27}$$

Similar as before, let us introduce the operators $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ and $(\mathbf{A}^*, \mathcal{D}(\mathbf{A}^*))$ with

$$\begin{aligned} \mathbf{A}\mathbf{y} &= \begin{pmatrix} \Delta y + \alpha(x)y - dz \\ \iota y - (\kappa - \sigma)z \end{pmatrix}, \quad \mathcal{D}(\mathbf{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega), \\ \mathbf{A}^*\mathbf{y} &= \begin{pmatrix} \Delta y + \alpha(x)y + \iota z \\ -dy - (\kappa - \sigma)z \end{pmatrix}, \quad \mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\mathbf{A}). \end{aligned} \tag{28}$$

With λ_0 and $\widehat{\mathbf{A}}$ as in Section 2 let us define the Dirichlet map $\mathbf{D}_{\widehat{\mathbf{A}}}$ via $\mathbf{D}_{\widehat{\mathbf{A}}}u = \mathbf{y}$ iff

$$\begin{aligned} \lambda_0 y - \Delta y - \alpha(x)y + dz &= 0 \quad \text{in } \Omega, \quad y = u \quad \text{on } \Gamma, \\ \lambda_0 z - \iota y + (\kappa - \sigma)z &= 0 \quad \text{in } \Omega. \end{aligned} \tag{29}$$

As before, by explicitly solving the second equation for z , and using well-known results [17, Theorem 2.5.1.1/Remark 2.5.1.2] we conclude that

$$\mathbf{D}_{\widehat{\mathbf{A}}}: \text{continuous } H^s(\Gamma) \rightarrow \mathbf{H}^{s+\frac{1}{2}}(\Omega), \quad 0 \leq s \leq \frac{5}{2}. \tag{30}$$

Consequently, for the linearized system we now obtain the following boundary control system

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}Mu(t), \quad \mathbf{y}_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \tag{31}$$

where $\mathbf{B} = \widehat{\mathbf{A}}\mathbf{D}_{\widehat{\mathbf{A}}} \in \mathcal{L}(L^2(\Gamma), [\mathcal{D}(\mathbf{A}^*)]')$ and M is the multiplication operator associated with m .

Lemma 3.1. *For all $\mathbf{p} = (p, q) \in \mathcal{D}(\mathbf{A}^*)$, it holds that*

$$\mathbf{B}^*\mathbf{p} = \mathbf{D}_{\widehat{\mathbf{A}}}^*\widehat{\mathbf{A}}^*\mathbf{p} = - \left. \frac{\partial p}{\partial \nu} \right|_{\Gamma}. \tag{32}$$

For $\mathbf{p} \in H^{\frac{3}{2}+s}(\Omega) \times L^2(\Omega)$, $s > 0$, it further holds

$$\|\mathbf{B}^*\mathbf{p}\|_{H^s(\Gamma)} \leq C \|\mathbf{p}\|_{H^{\frac{3}{2}+s}(\Omega) \times L^2(\Omega)}.$$

Proof. For $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*)$, $u \in L^2(\Gamma)$ and $\mathbf{D}_{\widehat{\mathbf{A}}}u = \mathbf{y}$ we have

$$\begin{aligned} (\mathbf{B}u, \mathbf{p})_{[\mathcal{D}(\mathbf{A}^*)]', \mathcal{D}(\mathbf{A}^*)} &= \left(\mathbf{D}_{\widehat{\mathbf{A}}}u, \widehat{\mathbf{A}}^*\mathbf{p} \right)_{L^2(\Omega)} = \left(\mathbf{y}, \widehat{\mathbf{A}}^*\mathbf{p} \right)_{L^2(\Omega)} \\ &= (y, \lambda_0 p - \Delta p - \alpha(x)p - \iota q)_{L^2(\Omega)} + (z, \lambda_0 q + (\kappa - \sigma)q + dp)_{L^2(\Omega)} \\ &= (\lambda_0 y - \Delta y - \alpha(x)y + dz, p)_{L^2(\Omega)} + (\lambda_0 z - \iota y + (\kappa - \sigma)z, q)_{L^2(\Omega)} \\ &\quad - \left(y, \frac{\partial p}{\partial \nu} \right)_{L^2(\Gamma)} + \left(\frac{\partial y}{\partial \nu}, p \right)_{L^2(\Gamma)}. \end{aligned}$$

By (29) and the fact that $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*)$, we thus obtain

$$(\mathbf{B}u, \mathbf{p})_{[\mathcal{D}(\mathbf{A}^*)]', \mathcal{D}(\mathbf{A}^*)} = \left(u, -\frac{\partial p}{\partial \nu} \right)_{L^2(\Gamma)} = (u, \mathbf{B}^* \mathbf{p})_{L^2(\Gamma)}.$$

□

Similar as before, let us note that

$$\widehat{\mathbf{A}}^{-\gamma} \mathbf{B} = \widehat{\mathbf{A}}^{1-\gamma} \mathbf{D}_{\widehat{\mathbf{A}}} \in \mathcal{L}(U, \mathbf{Y}) \quad \text{for } \gamma = \frac{3}{4} + \varepsilon, \quad \varepsilon > 0. \quad (33)$$

3.1. Riccati theory for the linearized system. Let us consider the following cost functional corresponding to (27)

$$\begin{aligned} \mathcal{J}(u, \mathbf{y}) &= \int_0^\infty \left(\|\mathbf{C}\mathbf{y}(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{L^2(\Gamma)}^2 \right) dt \\ &= \int_0^\infty \left(\|y(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{L^2(\Gamma)}^2 \right) dt. \end{aligned} \quad (34)$$

Obviously, for the observation operator $\mathbf{C}\mathbf{y}(t) = y(t)$ it holds that $\mathbf{C} \in \mathcal{L}(\mathbf{Y}, Z)$, where $Z = L^2(\Omega)$. As in the Neumann case, conditions (H1)-(H5) hold. From [21, Section 2.1] we now get the existence of a unique nonnegative self-adjoint solution $\mathbf{\Pi} = \mathbf{\Pi}^*$ to an algebraic Riccati equation. We further know ([21, Theorem 2.2.1]) that $(\widehat{\mathbf{A}}^*)^\nu \mathbf{\Pi} \in \mathcal{L}(\mathbf{Y})$ for any $0 \leq \nu < 1$. As emphasized in [21] and investigated in [5, 25], in certain cases it is even possible to take $\nu = 1$ such that $\mathbf{\Pi} \in \mathcal{L}(\mathbf{Y}, \mathcal{D}(\widehat{\mathbf{A}}^*))$. In particular, in [2] it was shown that this always holds true in case that $\mathbf{C}^* \mathbf{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^{\frac{1}{2}}), \mathcal{D}((\widehat{\mathbf{A}}^*)^{\frac{1}{2}}))$. Note that the latter condition is fulfilled for \mathbf{C} as in (34).

Moreover, for the case of the Navier-Stokes equations, in [25], it has been further shown that $\mathbf{\Pi} \in \mathcal{L}(H^\varepsilon(\Omega), H^{2+\varepsilon}(\Omega) \cap H_0^1(\Omega))$, $\varepsilon \in [0, \frac{1}{2}]$. With regard to the desired local stabilization result for the nonlinear system, a property similar to the latter one will be essential.

As before, we consider the space $H_{\mathbf{\Pi}}^s$ and its characterization via (24).

Proposition 3. *For $s \in [0, \frac{3}{4}]$ it holds that*

$$H_{\mathbf{\Pi}}^s = \begin{cases} H^{2s}(\Omega) \times L^2(\Omega) & \text{if } s \in [0, \frac{1}{4}), \\ \{\mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega) \mid y + M^2 \mathbf{B}^* \mathbf{\Pi} \mathbf{y} = 0 \text{ on } \Gamma\} & \text{if } s \in [\frac{1}{4}, \frac{3}{4}). \end{cases}$$

Moreover, for $s \in [0, \frac{1}{4})$, the operator $\mathbf{\Pi}$ satisfies $\mathbf{\Pi} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^s), \mathcal{D}((\widehat{\mathbf{A}}^*)^{1+s}))$.

Proof. Note that for $\mathbf{y} \in \mathbf{L}^2(\Omega)$ it holds that $\mathbf{\Pi} \mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^*)$ [2, Theorem 2]. With Lemma 3.1 it follows that $\mathbf{B}^* \mathbf{\Pi} \mathbf{y} \in H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$. Since $\widehat{\mathbf{A}}^{-1} \mathbf{B} = \mathbf{D}_{\widehat{\mathbf{A}}}$ with (30) we conclude that $\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \mathbf{\Pi} \mathbf{y} \in \mathbf{H}^{\frac{1}{2}}(\Omega) \subset \mathcal{D}(\widehat{\mathbf{A}}^s)$ for $s \in [0, \frac{1}{4})$. Hence, $\mathbf{y} \in H_{\mathbf{\Pi}}^s$ implies that $\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s) = H^{2s}(\Omega) \times L^2(\Omega)$ for $s \in [0, \frac{1}{4})$. Conversely, if $\mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega) = \mathcal{D}(\widehat{\mathbf{A}}^s)$ again it follows that $\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \mathbf{\Pi} \mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s)$. Hence, $\mathbf{y} \in H_{\mathbf{\Pi}}^s$.

Next, let us show that $\mathbf{\Pi} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^s), \mathcal{D}((\widehat{\mathbf{A}}^*)^{1+s}))$ for $s \in [0, \frac{1}{4})$. From [2] we know that for $\mathbf{y} \in \mathbf{L}^2(\Omega)$, the operator $\mathbf{\Pi}$ satisfies

$$\mathbf{A}^* \mathbf{\Pi} \mathbf{y} + \mathbf{\Pi} \mathbf{A} \mathbf{y} - \mathbf{\Pi} \mathbf{B} M^2 \mathbf{B}^* \mathbf{\Pi} \mathbf{y} + \mathbf{y} = 0 \quad \text{in } \mathbf{L}^2(\Omega).$$

We thus also obtain that

$$\widehat{\mathbf{A}}^* \mathbf{\Pi} \mathbf{y} = -\mathbf{\Pi} \widehat{\mathbf{A}} \mathbf{y} + \mathbf{\Pi} \mathbf{B} M^2 \mathbf{B}^* \mathbf{\Pi} \mathbf{y} - (1 + 2\lambda) \mathbf{y}. \quad (35)$$

Since $\Pi \in \mathcal{L}(\mathbf{Y}, \mathcal{D}(\widehat{\mathbf{A}}^*)) \cap \mathcal{L}([\mathcal{D}(\widehat{\mathbf{A}}^*)]', \mathbf{Y})$ by interpolation we have that

$$\Pi \in \mathcal{L}([\mathcal{D}((\widehat{\mathbf{A}}^*)^{1-s})]', \mathcal{D}((\widehat{\mathbf{A}}^*)^s)), \quad s \in [0, \frac{1}{4}).$$

For $\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s)$, $s \in [0, \frac{1}{4})$, this yields

$$\Pi \widehat{\mathbf{A}} \mathbf{y} = (\Pi \widehat{\mathbf{A}}^{1-s})(\widehat{\mathbf{A}}^s \mathbf{y}) \in \mathcal{D}((\widehat{\mathbf{A}}^*)^s).$$

Similarly, using that $\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s)$, $s \in [0, \frac{1}{4})$, we can conclude that

$$\Pi \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} = (\Pi \widehat{\mathbf{A}}^{1-s}) \widehat{\mathbf{A}}^s (\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y}) \in \mathcal{D}((\widehat{\mathbf{A}}^*)^s).$$

By (35) this leads to $\widehat{\mathbf{A}}^* \Pi \mathbf{y} \in \mathcal{D}((\widehat{\mathbf{A}}^*)^s)$. Since $\Pi \in \mathcal{L}(\mathbf{L}^2(\Omega))$ it thus follows that $\Pi: \mathcal{D}((\widehat{\mathbf{A}}^*)^s) \rightarrow \mathcal{D}((\widehat{\mathbf{A}}^*)^{1+s})$ is closed, and hence bounded.

Finally, consider $s \in (\frac{1}{4}, \frac{3}{4})$ and assume that $\mathbf{y} \in H_{\Pi}^s$. As shown above, $\mathbf{y} \in \mathbf{L}^2(\Omega)$ implies that $\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^{\frac{1}{4}-\mu})$, for $\mu > 0$. Hence, from $\mathbf{y} + \widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^s)$ we conclude that $\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^{\frac{1}{4}-\mu})$ and, also, $\Pi \mathbf{y} \in \mathcal{D}((\widehat{\mathbf{A}}^*)^{\frac{5}{4}-\mu})$. Moreover, since $\mathbf{B}^* \Pi \mathbf{y} \in H^{1-2\mu}(\Gamma)$ with (30) we arrive at $\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} \in \mathbf{H}^{\frac{3}{2}-2\mu}(\Omega) \subset H^{2s}(\Omega) \times L^2(\Omega)$. Since this holds for μ arbitrarily small, we conclude that $\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega)$, $s \in (\frac{1}{4}, \frac{3}{4})$. This shows that $\mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega) \in (\frac{1}{4}, \frac{3}{4})$. The boundary conditions for \mathbf{y} follow immediately by the definition of $\mathbf{D}_{\widehat{\mathbf{A}}}$.

Conversely, if $\mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega)$, $s \in (\frac{1}{4}, \frac{3}{4})$ and $y + M^2 \mathbf{B}^* \Pi \mathbf{y} = 0$ on Γ , we immediately have that $\mathbf{y} \in \mathcal{D}(\widehat{\mathbf{A}}^{\frac{1}{4}-\mu})$ for $\mu > 0$. As before, this gives $\widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega)$, $s \in (\frac{1}{4}, \frac{3}{4})$. Together this yields $\mathbf{z} := \mathbf{y} + \widehat{\mathbf{A}}^{-1} \mathbf{B} M^2 \mathbf{B}^* \Pi \mathbf{y} \in H^{2s}(\Omega) \times L^2(\Omega)$. Using the definition of $\mathbf{D}_{\widehat{\mathbf{A}}}$ and the fact that $y + M^2 \mathbf{B}^* \Pi \mathbf{y} = 0$ on Γ , we also obtain that $z = 0$ on Γ . Hence, $\mathbf{z} \in \mathcal{D}(\widehat{\mathbf{A}}^s)$, $s \in (\frac{1}{4}, \frac{3}{4})$. \square

3.2. Results for the nonlinear system. As in the case of Neumann boundary conditions, we first consider a nonhomogeneous equation of the form

$$\frac{d}{dt} \mathbf{y} = \mathbf{A}_{\Pi} \mathbf{y} + \mathbf{f}, \quad \mathbf{y}(0) = \mathbf{y}_0, \quad (36)$$

with $\mathbf{f} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$.

With Theorem 2.2 and Proposition 3, we obtain the following result.

Theorem 3.2. *Let $\varepsilon \in (0, \frac{1}{2})$. If $f_1 \in L^2(0, \infty; H^{\varepsilon-1}(\Omega))$, $\mathbf{y}_0 \in H^{\varepsilon}(\Omega) \times L^2(\Omega)$, then (23) has a unique solution*

$$\mathbf{y} \in \left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_{\infty}) \cap C_b([0, \infty); H^{\varepsilon}(\Omega)) \right) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))$$

satisfying

$$\|\mathbf{y}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_{\infty}) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq C_2 (\|\mathbf{y}_0\|_{H^{\varepsilon}(\Omega) \times L^2(\Omega)} + \|f_1\|_{L^2(0, \infty; H^{\varepsilon-1}(\Omega))}).$$

Moreover, we obtain a local stabilization result for the nonlinear system.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^2$ and $\varepsilon \in (0, \frac{1}{2})$. Then there exist $\mu_0 > 0$ and a nondecreasing function η from \mathbb{R}^+ into itself such that if $\mu \in (0, \mu_0)$ and $\|\mathbf{y}_0\|_{H^{\varepsilon}(\Omega) \times L^2(\Omega)} \leq \eta(\mu)$, then*

$$\frac{d}{dt} \mathbf{y} = \mathbf{A}_{\Pi} \mathbf{y} + \mathbf{F}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

admits a unique solution in the set

$$D_\mu = \left\{ \mathbf{y} \in \left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \cap C_b([0, \infty); H^\varepsilon(\Omega)) \right) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega)), \right. \\ \left. \|\mathbf{y}\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq \mu \right\}.$$

Remark 1. For the results in Theorem 2.6 and Theorem 3.3 we assumed that $U = L^2(\Gamma)$. However, in practical applications it is often more appropriate to utilize *finite dimensional controllers*. In this case, one typically assumes that the control function u is separable, i.e., $u(x, t) = \sum_{i=1}^{\ell} h_i(x)u_i(t)$, where h_i are shape functions and ℓ denotes the dimension of the control space $U = \mathbb{R}^\ell$. An essential tool for showing stabilizability by finite dimensional controllers then is the decomposition of the spectrum into an infinite dimensional stable part and a finite dimensional unstable part, see, e.g., [13, Chapter 5]. In essence, the stabilizability problem then reduces to its finite dimensional counterpart such that, given shape functions $h_i, i = 1, \dots, \ell$, one may utilize the Hautus test for stabilizability. Let us emphasize that due to the structure of the monodomain equations, in [10], we have explicitly specified the spectrum in terms of the spectrum of the PDE part. In particular, as long as the desired stabilization rate is limited by the stability of the ODE, the unstable part of the system is discrete. This way, one may apply the exact same arguments to extend the results in Theorem 2.6 and Theorem 3.3 to the case of finite dimensional controllers.

We end this section with our final result for the local exponential stabilization of system (1) in case that $\theta = 0$.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^2$ and $\varepsilon \in (0, \frac{1}{2})$. Let further $0 \leq \sigma < \kappa$ be given. For $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{D}(\mathbf{A})$ let $\mathbf{\Pi}$ be the solution to*

$$(\mathbf{A}^* \mathbf{\Pi} \mathbf{z}_1, \mathbf{z}_2)_{\mathbf{Y}} + (\mathbf{\Pi} \mathbf{A} \mathbf{z}_1, \mathbf{z}_2)_{\mathbf{Y}} + 2\sigma(\mathbf{z}_1, \mathbf{z}_2) + \left(\widetilde{M} \mathbf{C} \mathbf{z}_1, \widetilde{M} \mathbf{C} \mathbf{z}_2 \right)_Z = (M \mathbf{B}^* \mathbf{\Pi} \mathbf{z}_1, M \mathbf{B}^* \mathbf{\Pi} \mathbf{z}_2)_U.$$

Then there exist $\mu_0 > 0$ and a nondecreasing function η from \mathbb{R}^+ into itself such that if $\mu \in (0, \mu_0)$ and $\|\mathbf{y}_0\|_{H^\varepsilon(\Omega) \times L^2(\Omega)} \leq \eta(\mu)$, then

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta v - I_{\text{ion}}(v, w) + f_v \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial t} &= \nu w - \kappa w + f_w \quad \text{in } \Omega \times (0, \infty), \\ v &= g - M^2 \mathbf{B}^* \mathbf{\Pi}(v - \bar{v}, w - \bar{w}) \quad \text{on } \Gamma \times (0, \infty), \\ v(x, 0) &= \bar{v} + y_0 \quad \text{and } w(x, 0) = \bar{w} + z_0 \quad \text{in } \Omega, \end{aligned}$$

admits a unique solution (v, w) which satisfies

$$\|e^{\sigma \cdot} (v - \bar{v}, w - \bar{w})\|_{H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0, \infty; L^2(\Omega))} \leq \mu.$$

4. Numerical examples. For the numerical validation of the theory presented in this manuscript, we study the following version of the monodomain equations

$$\begin{aligned} \frac{\partial v}{\partial t} &= \alpha \Delta v - av^3 + bv^2 - cv - dw \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial t} &= \iota v - \kappa w \quad \text{in } \Omega \times (0, \infty), \\ \theta \frac{\partial v}{\partial \nu} &= (\theta - 1)v + m\tilde{u} + g \quad \text{on } \Gamma \times (0, \infty), \\ v(x, 0) &= \bar{v} + y_0 \text{ and } w(x, 0) = \bar{w} + z_0 \quad \text{in } \Omega, \end{aligned} \tag{37}$$

where $\Omega = (0, 1) \times (0, 1)$ and all other parameters are to be specified below. A finite element discretization is obtained by the MATLAB[®] PDE toolbox. All results correspond to a 64×64 regular grid with $n = 2 \cdot 4225 = 8450$ degrees of freedom. Given a stationary solution of the form (3), the finite dimensional approximation of (4) with $\sigma = 0$ is given by

$$\mathbf{E}_n \dot{\mathbf{y}}_n = \mathbf{A}_n \mathbf{y}_n + \mathbf{F}_n(\mathbf{y}_n) + \mathbf{B}_n u, \tag{38}$$

where $\mathbf{A}_n, \mathbf{E}_n \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times \ell}$. The nonlinearity $\mathbf{F}_n: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined elementwise according to the nonlinearity in (37). For the discrete control operator, ℓ denotes the dimension of the finite dimensional control space (cf. Remark 1). By $\mathbf{C}_n \in \mathbb{R}^{p \times n}$ we denote the finite dimensional approximation of the output operator appearing in (13) and (34), respectively. In both Neumann and Dirichlet case, the multiplication operator m is taken from [29]. The closed loop system is obtained by solving the following generalized algebraic matrix Riccati equation

$$\mathbf{A}_n^T \mathbf{\Pi}_n \mathbf{E}_n + \mathbf{E}_n^T \mathbf{\Pi}_n \mathbf{A}_n - \mathbf{E}^T \mathbf{\Pi}_n \mathbf{B}_n \mathbf{B}_n^T \mathbf{\Pi}_n \mathbf{E}_n + \mathbf{C}_n^T \mathbf{C}_n = 0.$$

For this, a Kleinman-Newton iteration as described in, e.g., [7, 11, 20], has been used. As a stabilizing initial guess $\mathbf{\Pi}_n^{(0)}$ we solved a k -dimensional algebraic Bernoulli equation corresponding to the unstable subspace (of dimension k) of the matrix pencil $(\mathbf{A}_n, \mathbf{E}_n)$. Each Lyapunov equation arising within the Kleinman-Newton method has been explicitly solved by the MATLAB[®] function `lyap`.

All simulations are generated on an Intel[®]Xeon(R) CPU E31270 @ 3.40 GHz x 8, 16 GB RAM, Ubuntu Linux 14.04, MATLAB[®] Version 8.0.0.783 (R2012b) 64-bit (glnxa64). The solutions of the ODE systems are always obtained by the MATLAB[®] routine `ode23`.

4.1. Neumann boundary control. We start with the Neumann case and thus consider (37) with $\theta = 1$. We further set $\alpha = 0.0015, a = 0.0012, b = 0.1403, c = 1.6140, d = 215.6, \iota = 0.00015, \kappa = 0.015$ and $g = 0$. The control and observation domains are shown in Figure 1a. In particular, we have $\mathbf{B}_n = \mathbf{C}_n^T \in \mathbb{R}^{n \times 12}$. Based on the underlying nonlinear system

$$\begin{aligned} 0 &= -av^3 + bv^2 - cv - dw \\ 0 &= \iota v - \kappa w \end{aligned}$$

we obtain three constant (in space and time) stationary solutions to (37) which can be numerically computed as

$$(\bar{v}_1, \bar{w}_1) = (0, 0), \quad (\bar{v}_2, \bar{w}_2) \approx (44.0222, 0.4402), \quad (\bar{v}_3, \bar{w}_3) \approx (68.9778, 0.6898).$$

In our experiments, we chose (\bar{v}_3, \bar{w}_3) for which the corresponding linearized system exhibited an unstable subspace of dimension 2.

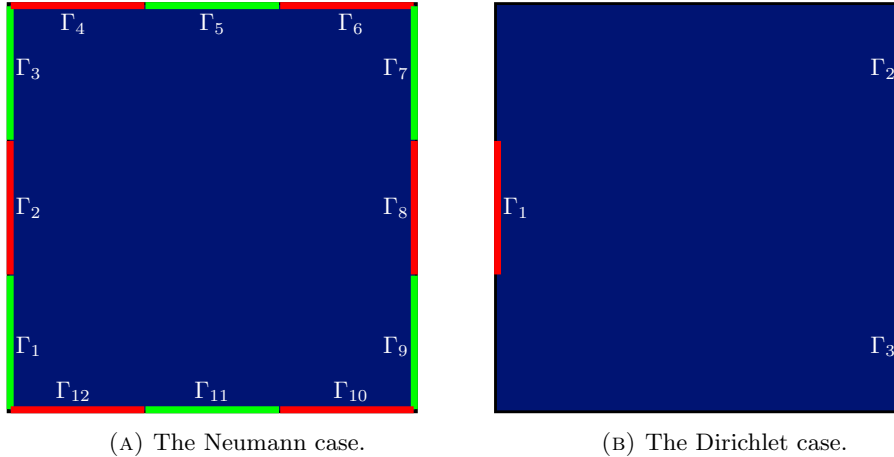


FIGURE 1. Control setup.

Perturbation around constant stationary state. In Figure 2, numerical results for an initial value of the form $(v_0, w_0) = (\bar{v}_3 + \xi_v, \bar{w}_3 + \xi_w)$ with $\xi_v = 0.3 \cdot \text{randn}(\mathbf{n}, 1)$ and $\xi_w = 0.003 \cdot \text{randn}(\mathbf{n}, 1)$ are given. As is seen in Figure 2a the uncontrolled system remains close to the unstable stationary for a certain period of time before it slowly starts to decrease in magnitude. Finally, for $t = 1000$, the system approaches the stable stationary solution (\bar{v}_1, \bar{w}_1) from below. On the other hand, Figure 2b demonstrates the successful local stabilization of the nonlinear system. The different dynamical behavior between uncontrolled and controlled system is even more evident from the results shown in Figure 2d. Here, the temporal evolution of the $L^2(\Omega)$ error between computed and desired state is visualized. Again we emphasize that the uncontrolled system remains close to the desired state first. Figure 2c shows the control law for some of the control domains specified in Figure 1a. In particular, we see that the magnitude of the feedback control is relatively small when compared to the actual state.

Stabilization of a reentry wave. As a second test case, we study the feedback law when the initial value (v_0, w_0) is chosen such that it causes a reentry pattern (as arising in context of fibrillation processes). To be more precise, in Figure 3a we illustrate the wave-like dynamics appearing for the uncontrolled system. Snapshots of the evolution for the closed loop system are provided in Figure 3b. We emphasize that, as we have also seen in the first example, when the control is switched off close to the stationary solution (\bar{v}_3, \bar{w}_3) , the system will converge to the stable resting state (\bar{v}_1, \bar{w}_1) . This way, the feedback law implicitly allows to perform a defibrillation process by first controlling the system to the unstable stationary state. From there the uncontrolled dynamics converges to the origin. Again, a more detailed comparison between controlled and uncontrolled dynamics can be obtained in terms of the $L^2(\Omega)$ error, see Figure 3d. For a precise pattern of the control laws, we refer to 2c.

4.2. Dirichlet boundary control. Next, we consider (37) with $\theta = 0$. For the parameters, we take $\alpha = 0.0015, a = 0.012, b = 0.1395, c = 1.6050, d = 215.6, \iota = 0.00015$ and $\kappa = 0.015$. Again, we use the underlying ODE system to compute three

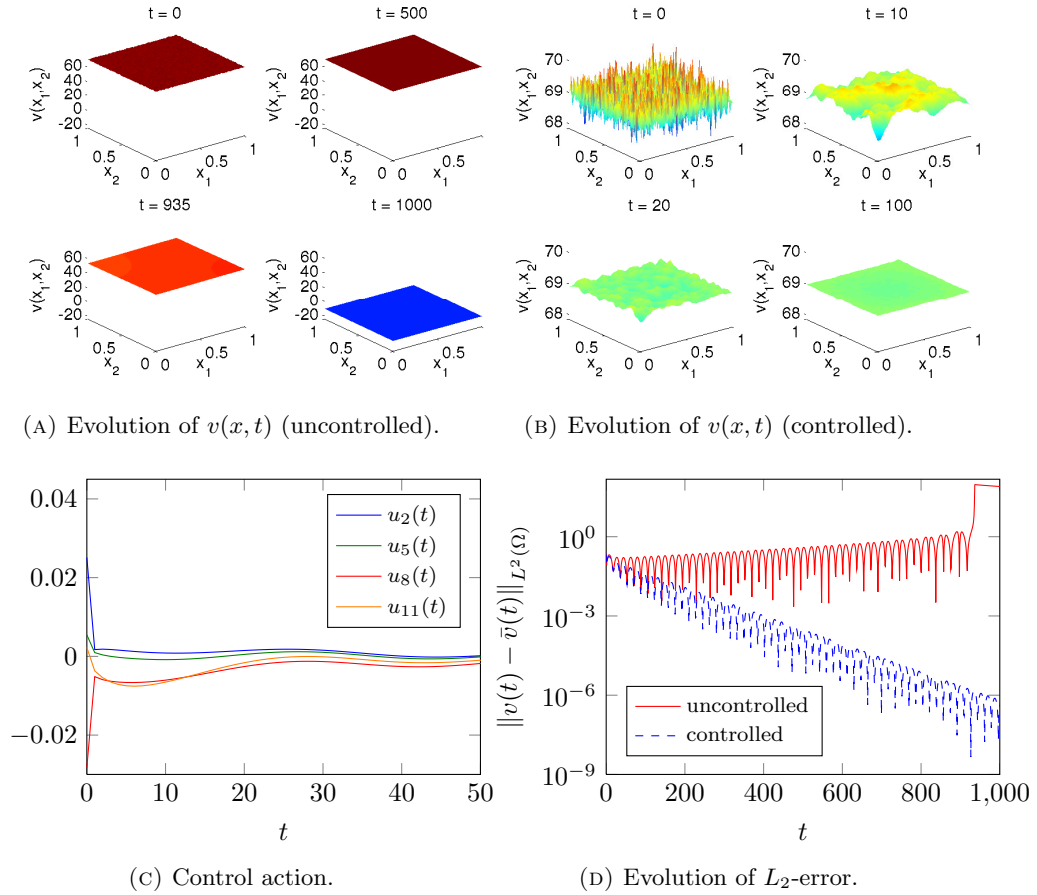


FIGURE 2. Stabilization of perturbed initial state.

constant (in space and time) stationary solutions as

$$(\bar{v}_1, \bar{w}_1) = (0, 0), \quad (\bar{v}_2, \bar{w}_2) \approx (44.4187, 0.4442), \quad (\bar{v}_3, \bar{w}_3) \approx (68.5813, 0.6858).$$

In order to ensure that (\bar{v}_3, \bar{w}_3) is also a stationary solution of the PDE system, in (37) we assume an inhomogeneous Dirichlet boundary condition $g = \bar{v}_3$. The resulting linearized system has 6 eigenvalues with positive real part. In contrast to the Neumann case, we only take three controls that are visualized in Figure 1b.

Perturbation around constant stationary state. The results shown in Figure 4 correspond to a similar setup as in the Neumann case. For both, uncontrolled and controlled system, the initial value was taken as a perturbation ($0.3 \text{ randn}(\mathbf{n}, 1)$) of the unstable stationary solution (\bar{v}_3, \bar{w}_3) . As a consequence, the uncontrolled system shows an oscillatory behavior, see Figure 4a. On the other hand, the feedback law computed from the solution of the algebraic Riccati equation allows to locally stabilize the nonlinear system, see Figure 4b. For a more detailed comparison, we refer to the temporal evolution of the $L^2(\Omega)$ error that is provided in Figure 4d. The performance of the three boundary control patches is visualized in Figure 4c, also underlining the convergence of the closed loop system.

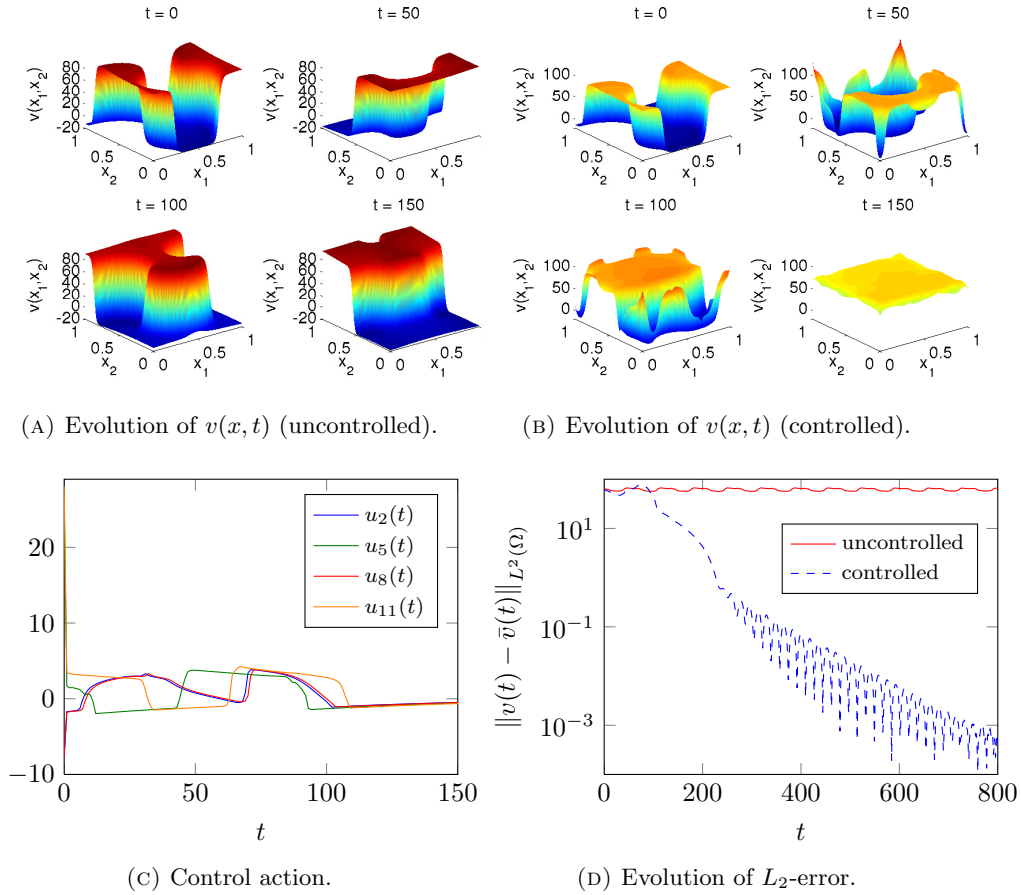


FIGURE 3. Stabilization of a reentry wave.

5. Conclusions. We have studied boundary feedback control problems for a class of nonlinear reaction diffusion systems of PDE-ODE type. A particular emphasis was on the so-called monodomain equations. For the linearized system, we have investigated the use of classical Riccati-based feedback controllers. Based on some new Lipschitz estimates for the cubic non monotone nonlinearity, we could show that the feedback law derived for the linearized system locally stabilizes the nonlinear system as well. While in the case of Neumann boundary conditions, both the two and three dimensional case could be handled, in the Dirichlet case we restricted ourselves to the two dimensional setting. Numerical examples for a FEM discretization of the monodomain equations illustrated the main theoretical findings.

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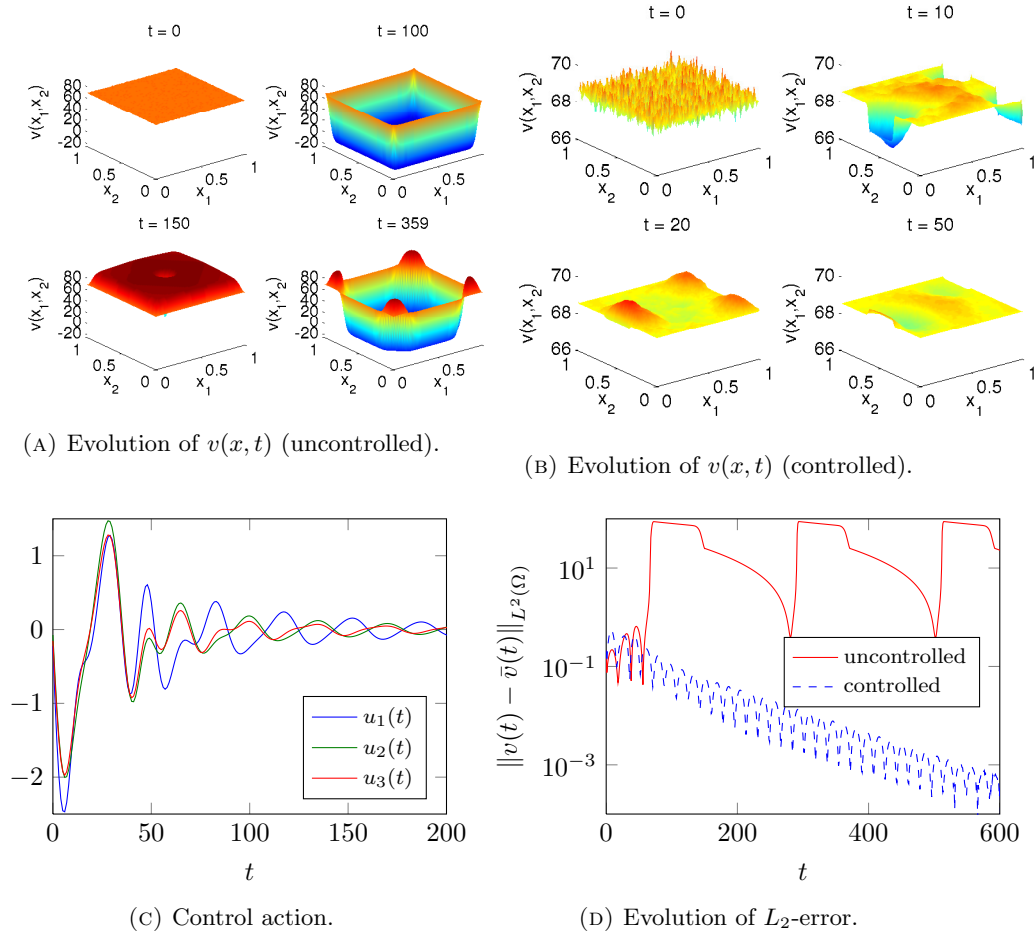


FIGURE 4. Stabilization of perturbed initial state.

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