STABILIZATION BY SPARSE CONTROLS FOR A CLASS OF SEMILINEAR PARABOLIC EQUATIONS *

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Abstract. Stabilization problems for parabolic equations with polynomial nonlinearities are investigated in the context of an optimal control formulation with a sparsity enhancing cost functional. This formulation allows that the optimal control completely shuts down once the trajectory is sufficiently close to a stable steady state. Such a property is not present for commonly chosen control mechanisms. To establish these results it is necessary to develop a function space framework for a class of optimal control problems posed on infinite time horizons, which is otherwise not available.

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1. Introduction. In recent years there has been significant interest in the topic of sparse optimal controls. These controls contribute towards the control objective and simultaneously shut down to zero as much as possible. Up to now optimal control problems with sparsity constraints have typically been investigated for tracking problems on finite time horizons. Sparsity with more general cost functionals was investigated in [6]. In this case the objective is to steer the trajectory of the controlled dynamical system to the desired state while simultaneously minimizing the support of the control. By an adapted choice of the cost functional, the sparsity structure can be influenced. The focus in previous work was set on controlling the sparsity structure in spatial directions. This led to choose controls in the spaces like $L^2(0,T;L^1(\Omega))$ or $L^1(\Omega;L^2(0,T))$ where for technical reasons the L^1-spaces have to be replaced by spaces of measures [5], [10], [7], unless other precautions as for instance constraints on the controls are taken [6], [9].

In the present paper we focus on optimal controls which exhibit temporal sparsity. This can be achieved by choosing a cost-functional for the control variable which is non-smooth in time. We shall concentrate on optimal control formulations for stabilization problems. It will be demonstrated that these problems are particularity well-suited to benefit from the sparsity structure. Specifically, if y_e is a stable equilibrium of a dynamical system, optimal control strategy typically provide controls which asymptotically steer the system to y_e with the control not shutting down to zero even if the controlled trajectory is already in the close vicinity of y_e. Such strategies can be based, for instance, on applying Riccati or Lyapunov techniques to the linearized system or on feedback mechanisms which respect the nature of the differential equation and the control objective, as for example feeding back some weighted difference between the state and y_e; see, for instance, [1], [2], [16]. With temporally sparse controls, on the other hand, it can be guaranteed that the optimal control will automatically shut down to zero in the vicinity of y_e. Of course such a property can

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not be expected unless y_e is stable. In the present paper we develop the necessary concepts for a class of semilinear parabolic equations. This will include in particular to propose a function space framework for open loop infinite time horizon nonlinear optimal control problems. This topic has received very little attention in the literature even in case of smooth cost functionals.

To demonstrate the sparsity nature of optimal controls for stabilization problems we consider

$$\frac{\partial y}{\partial t} - \Delta y + f(y) = u\chi_\omega$$

together with an ininital condition and homogenous Neumann boundary conditions. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function of degree $2m + 1$ with $m \in \mathbb{N}$ arbitrary if $n \leq 2$ and $m = 1$ if $n = 3$, and further specifications to be given below. Due to the choice of Neumann boundary conditions every root of f is an equilibrium of the uncontrolled state equation. Successive distinct roots of f alternate between stable and unstable behavior. As described above here we are interested in the behavior of sparse controls in the neighborhood of a stable equilibrium, which after a possible change of variables is assumed to be the origin. Consequently, we make the following assumption on f:

$$f(0) = 0, f'(0) > 0 \text{ and the leading coefficient of } f \text{ is positive.} \tag{1.1}$$

It will be shown that this condition guarantees that 0 is a stable equilibrium for our dynamical system, see Theorems 2.5 and 2.6.

Let us point out that reaction diffusion systems of polynomial type arise in many interesting applications including models in physiology, for instance in the context of FitzHughNagumo models, which describe the prototype of excitable systems, e.g. see [12], or Schlögl's model which is a canonical example of a chemical reaction system [14]. See [3], [8], [13] for the optimal control of these systems.

We are now prepared to formulate the optimal control problem which will be analyzed in this work:

$$\text{(P)} \qquad \min_{u\in U} J(u) = \frac{1}{2}\int_0^\infty \int_\Omega y_u^2 \, dx\, dt + \frac{\nu}{2}\int_0^\infty \int_\omega u^2 \, dx\, dt + \alpha \int_0^\infty \Big(\int_\omega u^2 \, dx \Big)^{1/2} dt,$$

where

$$U = L^2(0,\infty; L^2(\omega)) \cap L^1(0,\infty; L^2(\omega)), \quad \nu > 0, \quad \alpha > 0.$$

Here y_u denotes the solution of the following parabolic equation

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) & = & u\chi_\omega & \text{in } Q = \Omega \times (0,\infty), \\ \partial_n y & = & 0 & \text{on } \Sigma = \Gamma \times (0,\infty), \\ y(0) & = & y_0 & \text{in } \Omega, \end{array} \right. \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $1 \leq n \leq 3$, with a Lipschitz boundary Γ, ω is a subdomain of Ω, χ_ω denotes the characteristic function of ω, $a \in L^\infty(\Omega)$, $0 \leq a \not\equiv 0$, and $y_0 \in H^1(\Omega)$.

Remark 1.1. *The assumption $a \not\equiv 0$ has been introduced just for simplicity of the presentation, but it is not necessary. All the results of this paper remain valid if we take $a \equiv 0$. Indeed, from the assumptions on f we know that $f(s) = a_1 s + \cdots a_{2m+1} s^{2m+1}$*

with $a_1 > 0$. Therefore we can take $0 < \tilde{a}_1 < a_1$ so that the polynomial $\tilde{f}(s) = \tilde{a}_1 s + a_2 s^2 + \cdots a_{2m+1} s^{2m+1}$ has the same properties as f. If $a \equiv 0$ we set $a = a_1 - \tilde{a}_1$ and all the above assumptions are fulfilled.

Concerning the three terms in the cost functional J, the first one reflects the objective of stabilization to 0. The second one is required for the well-posedness of the control problem, and the last one promotes the temporal sparsity of the optimal controls. We observe that cost J can take the value ∞ for some controls $u \in U$, because $y_u \notin L^2(Q)$. We will say that $u \in U$ is a feasible control if $J(u) < \infty$.

We shall frequently use the following property of f which is a consequence of assumption (1.1) and the fact that the polynomial is of odd degree

$$\exists \Lambda > 0 \text{ such that } f'(t) \geq -\Lambda \quad \forall t \in \mathbb{R}. \tag{1.3}$$

The plan of this paper is as follows. Section 2 contains the analysis of the controlled state equation in spaces with infinite time horizons. Moreover two theorems are presented which provide sufficient conditions for exponential stability of the uncontrolled trajectories provided that the initial condition is ether sufficiently small or it is appropriately located with respect to the roots of the polynomial. In Section 3 existence of optimal controls is proved and the sensitivity and adjoint equations are analyzed. Finally Section 4 contains the optimality system. It allows to deduce the sparsity properties of the optimal controls.

2. Analysis of the State Equation. We shall denote by $L^2_{loc}(0,\infty;H^1(\Omega))$ the space of functions y belonging to $L^2(0,T;H^1(\Omega))$ for every $0 < T < \infty$. Analogously we define $L^2_{loc}(0,\infty;L^2(\Omega))$ and $L^\infty_{loc}(0,\infty;L^2(\Omega))$.

DEFINITION 2.1. *We call y a solution to (1.2) if $y \in L^2_{loc}(0,\infty;H^1(\Omega)) \cap C([0,\infty);L^2(\Omega))$, $f(y) \in L^2_{loc}(0,\infty;L^2(\Omega))$ and for every $T > 0$ the restriction of y to $Q_T = \Omega \times (0,T)$ satisfies in the usual variational sense the equation*

$$\left\{\begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(y) &=& u\chi_\omega & \text{in } Q_T, \\ \partial_n y &=& 0 & \text{on } \Sigma_T, \\ y(0) &=& y_0 & \text{in } \Omega, \end{array}\right. \tag{2.1}$$

We have the following existence and uniqueness result.

THEOREM 2.2. *For every $u \in L^2(0,\infty;L^2(\omega))$ equation (1.2) has a unique solution. Moreover $y \in H^1_{loc}(0,\infty;L^2(\Omega))$ holds.*

Proof. This proof can not rely on the usual techniques for semilinear monotone equations because the right hand side u is not in any $L^p(0,T;L^q(\Omega))$ space with p and q large enough so that the corresponding state belongs to $L^\infty(Q_T)$. First we make the change of variables according to $z(x,t) = \mathrm{e}^{-\Lambda t} y(x,t)$, where Λ is introduced in (1.3). Then the resulting equation for z is given by

$$\left\{\begin{array}{rcll} \dfrac{\partial z}{\partial t} - \Delta z + az + \tilde{f}(t,z) &=& \mathrm{e}^{-\Lambda t} u\chi_\omega & \text{in } Q_T, \\ \partial_n z &=& 0 & \text{on } \Sigma_T, \\ z(0) &=& y_0 & \text{in } \Omega, \end{array}\right. \tag{2.2}$$

where

$$\tilde{f}(t,s) = \mathrm{e}^{-\Lambda t} f(\mathrm{e}^{\Lambda t} s) + \Lambda s \quad \forall (t,s) \in \mathbb{R}^2.$$

For any positive integer k we denote by $\mathrm{Proj}_{[-k,+k]}(s)$ the projection of a real number s on the interval $[-k,+k]$ and we set $\tilde f_k(t,s)=\tilde f(t,\mathrm{Proj}_{[-k,+k]}(s))$. As a consequence of (1.1) and (1.3), we have that $\tilde f_k(t,0)=0$ and $\partial_s\tilde f_k(t,s)=f'(\mathrm{e}^{\Lambda t}s)+\Lambda\ge 0$ if $|s|<k$ and $\partial_s\tilde f_k(t,s)=0$ if $|s|>k$. Moreover, $\partial_s\tilde f_k(t,0)=f'(0)+\Lambda>0$ holds. By an application of Schauder's fixed point theorem we can obtain the existence of a solution $z_k\in L^2(0,T;H^1(\Omega))\cap C([0,T],L^2(\Omega))$ of the following equation

$$\left\{\begin{array}{rcll}\dfrac{\partial z_k}{\partial t}-\Delta z_k+az_k+\tilde f_k(t,z_k) &=& \mathrm{e}^{-\Lambda t}u\chi_\omega & \text{in } Q_T,\\ \partial_n z_k &=& 0 & \text{on } \Sigma_T,\\ z_k(0) &=& y_0 & \text{in } \Omega.\end{array}\right. \tag{2.3}$$

The uniqueness of z_k is a consequence of the monotonicity of $\tilde f_k$. Because of the regularity of $y_0\in H^1(\Omega)$ and the fact that $u\in L^2(0,T;L^2(\omega))$, we know that $z_k\in H^1(0,T;L^2(\Omega))$ as well; see, for instance, [15, Proposition III-2.5]. Multiplying (2.3) by z_k, integrating by parts in Q_T and using that $\tilde f_k(t,z_k)z_k\ge 0$, we get the existence of a constant C_a such that for every $t\in[0,T]$

$$\begin{aligned}&\frac12\|z_k(t)\|^2_{L^2(\Omega)}+C_a\|z_k\|^2_{L^2(0,t;H^1(\Omega)}\le\frac12\|z_k(t)\|^2_{L^2(\Omega)}+\int_{Q_T}(|\nabla z_k|^2+az_k^2)\,dx\,dt\\ &\le\int_0^T\int_\omega \mathrm{e}^{-\Lambda t}uz_k\,dx\,dt+\frac12\|y_0\|^2_{L^2(\Omega)}\le\|u\|_{L^2(0,T;L^2(\omega))}\|z_k\|_{L^2(0,T;H^1(\Omega))}\\ &+\frac12\|y_0\|^2_{L^2(\Omega)}\le\frac{1}{2C_a}\|u\|^2_{L^2(0,T;L^2(\omega))}+\frac{C_a}{2}\|z_k\|^2_{L^2(0,T;H^1(\Omega))}+\frac12\|y_0\|^2_{L^2(\Omega)}.\end{aligned}$$

From here we deduce

$$\|z_k\|_{L^\infty(0,T;L^2(\Omega))}+\|z_k\|_{L^2(0,T;H^1(\Omega))}\le\sqrt{\frac{2}{C_a}}\|u\|_{L^2(0,T;L^2(\omega))}+\sqrt{2}\|y_0\|_{L^2(\Omega)}. \tag{2.4}$$

Next we prove that $\{\tilde f_k(\cdot,z_k)\}_{k=1}^\infty$ is a bounded sequence in $L^2(Q_T)$. Since $\tilde f(t,s)$ is a polynomial in the variable s of degree $2m+1$ with coefficients depending on t, but uniformly bounded in $[0,T]$, and leading positive coefficient, elementary calculus leads to the existence of two constants $C_1>0$ and $C_2\ge0$ such that

$$\tilde f(t,s)^2\le C_1\tilde f(t,s)s^{2m+1}+C_2\quad\forall(t,s)\in[0,T]\times\mathbb{R}.$$

Now, setting $\hat z_k=\mathrm{Proj}_{[-k,+k]}(z_k)$ we have that $\tilde f_k(t,z_k)=\tilde f(t,\hat z_k)$ and

$$\tilde f_k(t,z_k)^2\le C_1\tilde f(t,\hat z_k)\hat z_k^{2m+1}+C_2=C_1\tilde f_k(t,z_k)\hat z_k^{2m+1}+C_2\quad\forall t\in[0,T].$$

Since the sign of the functions $\hat z_k$ coincide with the sign of z_k we get $\tilde f_k(t,z_k)\hat z_k^{2m+1}\ge0$. Hence the boundedness of $\{\tilde f_k(\cdot,z_k)\}_{k=1}^\infty$ in $L^2(Q_T)$ will follow from the boundedness of $\{\tilde f_k(\cdot,z_k)\hat z_k^{2m+1}\}_{k=1}^\infty$ in $L^1(Q_T)$. To prove this boundedness first we observe that $\hat z_k^{2m+1}\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ because z_k belongs to this space and $\hat z_k\in L^\infty(Q_T)$. Then we can multiply the state equation (2.3) by $\hat z_k^{2m+1}$ and integrate

by parts. To this end we observe that

$$\int_0^T\int_\Omega \frac{\partial z_k}{\partial t}\hat z_k^{2m+1}\,dx\,dt = \int_\Omega z_k(T)\hat z_k^{2m+1}(T)\,dx - \int_\Omega y_0 y_{0k}^{2m+1}\,dx$$
$$-(2m+1)\int_0^T\int_\Omega z_k\frac{\partial \hat z_k}{\partial t}\hat z_k^{2m}\,dx\,dt \geq \int_\Omega \hat z_k^{2m+2}(T)\,dx - \int_\Omega y_0^{2m+2}\,dx$$
$$-(2m+1)\int_0^T\int_\Omega \frac{\partial \hat z_k}{\partial t}\hat z_k^{2m+1}\,dx\,dt, \tag{2.5}$$

where $y_{0k} = \mathrm{Proj}_{[-k,+k]}(y_0)$. Recall that $y_0 \in H^1(\Omega) \subset L^{2m+2}(\Omega)$ for any positive integer m if $n \leq 2$ and for $m \leq 2$ if $n = 3$ and so the above integral involving y_0 is finite. The last integral can be computed as follows

$$\int_0^T\int_\Omega \frac{\partial \hat z_k}{\partial t}\hat z_k^{2m+1}\,dx\,dt = \frac{1}{2m+2}\int_0^T\frac{d}{dt}\int_\Omega \hat z_k^{2m+2}\,dx\,dt$$
$$= \frac{1}{2m+2}\Big(\int_\Omega \hat z_k^{2m+2}(T)\,dx - \int_\Omega y_{0k}^{2m+2}\,dx\Big).$$

Inserting this identity in (2.5) we get

$$\int_0^T\int_\Omega \frac{\partial z_k}{\partial t}\hat z_k^{2m+1}\,dx\,dt \geq \frac{1}{2m+2}\int_\Omega \hat z_k^{2m+2}(T)\,dx - \int_\Omega y_0^{2m+2}\,dx.$$

Moreover we have that $\nabla z_k \nabla \hat z_k^{2m+1} = (2m+1)\hat z_k^{2m}\nabla z_k\nabla \hat z_k = (2m+1)\hat z_k^{2m}|\nabla \hat z_k|^2$. Hence combining these facts, we deduce from equation (2.3)

$$\frac{1}{2m+2}\int_\Omega \hat z_k^{2m+2}(T)\,dx + \int_{Q_T}\hat z_k^{2m}[(2m+1)|\nabla \hat z_k|^2 + a\hat z_k^2]\,dx\,dt$$
$$+\int_{Q_T}\tilde f_k(t,z_k)\hat z_k^{2m+1}\,dx\,dt \leq \int_0^T\int_\omega \mathrm{e}^{-\Lambda t}u\hat z_k^{2m+1}\,dx\,dt + \int_\Omega y_0^{2m+2}\,dx$$
$$\leq \|u\|_{L^2(0,T;L^2(\omega))}\Big(\int_{Q_T}\hat z_k^{4m+2}\,dx\,dt\Big)^{1/2} + C\|y_0\|_{H^1(\Omega)}^{2m+2}.$$

Since $\tilde f(t,s)s^{m+1}$ is a polynomial in s of even degree $4m+2$ and positive leading coefficient, there exist constants $C_3 > 0$ and $C_4 \leq 0$ such that $\tilde f(t,s)s^{m+1} \geq C_3 s^{4m+2} + C_4$. Hence we have that $\tilde f_k(t,z_k)\hat z_k^{2m+1} = \tilde f(t,\hat z_k)\hat z_k^{2m+1} \geq C_3\hat z_k^{4m+2} + C_4$. Using this fact in the above inequality we infer with the Young inequality

$$\int_{Q_T}\tilde f_k(t,z_k)\hat z_k^{2m+1}\,dx\,dt \leq \|u\|_{L^2(0,T;L^2(\omega))}\Big(\int_{Q_T}\hat z_k^{4m+2}\,dx\,dt\Big)^{1/2} + C\|y_0\|_{H^1(\Omega)}^{2m+2}$$
$$\leq \frac{1}{2C_3}\|u\|_{L^2(0,T;L^2(\omega))}^2 + \frac{C_3}{2}\int_{Q_T}\hat z_k^{4m+2}\,dx\,dt + C\|y_0\|_{H^1(\Omega)}^{2m+2}$$
$$\leq \frac{1}{2C_3}\|u\|_{L^2(0,T;L^2(\omega))}^2 + \frac{1}{2}\int_{Q_T}\tilde f_k(t,z_k)\hat z_k^{2m+1}\,dx\,dt - \frac{C_4|Q_T|}{2} + C\|y_0\|_{H^1(\Omega)}^{2m+2},$$

where $|Q_T|$ denotes the Lebesgue measure of Q_T, and hence

$$\int_{Q_T}\tilde f_k(t,z_k)\hat z_k^{2m+1}\,dx\,dt \leq \frac{1}{C_3}\|u\|_{L^2(0,T;L^2(\omega))}^2 - C_4|Q_T| + 2C\|y_0\|_{H^1(\Omega)}^{2m+2}.$$

Thus we have the boundedness of $\{\tilde f_k(\cdot,z_k)\}_{k=1}^\infty$ in $L^2(Q_T)$ as desired. Now using (2.3) we deduce that $\{\frac{\partial z_k}{\partial t}-\Delta z_k\}_{k=1}^\infty$ is bounded in $L^2(Q_T)$. Since $y_0\in H^1(\Omega)$, from the maximal parabolic regularity property of the heat equation it follows that $\{\frac{\partial z_k}{\partial t}\}_{k=1}^\infty$ is bounded in $L^2(Q_T)$ as well. Combining this fact with (2.4), we deduce the existence of a subsequence such that

$$z_k \rightharpoonup z \quad \text{in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \frac{\partial z_k}{\partial t}\rightharpoonup \frac{\partial z}{\partial t} \quad \text{in } L^2(Q_T),$$

which implies the strong convergence $z_k\to z$ in $L^2(Q_T)$. By taking a new subsequence, if necessary, we can assume that $z_k(x,t)\to z(x,t)$ for almost every point $(x,t)\in Q_T$. This implies the pointwise convergence $\tilde f_k(t,z_k(x,t))\to\tilde f(t,z(x,t))$ for almost all $(x,t)\in Q_T$. This together with the boundedness of $\{\tilde f_k(\cdot,z_k)\}_{k=1}^\infty$ in $L^2(Q_T)$ implies that $f(\cdot,z)\in L^2(Q_T)$ and $f_k(\cdot,z_k)\rightharpoonup f(\cdot,z)$ in $L^2(Q_T)$. Moreover, since the space $L^2(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega)$ is continuously embedded in $C([0,T];L^2(\Omega))$, we deduce that z is in this space. According to the convergence properties of $\{z_k\}_{k=1}^\infty$ described above, it is immediate to pass to the limit in (2.3) and deduce that z is a solution of (2.2). Its uniqueness follows from the monotonicity of $\tilde f$. From the equivalence between equations (2.1) and (2.2), we get that $y=\mathrm{e}^{\Lambda t}z$ is the unique solution of (2.1). Since T was an arbitrary positive number we conclude that y satisfies all the requirements of Definition 2.1, and therefore y is the unique solution of (1.2).

Finally, the fact that $y\in H^1(0,T;L^2(\Omega))$ follows from (2.1) taking into account that $f(y)\in L^2(Q_T)$ and $y_0\in H^1(\Omega)$, [15, Proposition III-2.5]. □

In the next theorem we establish some infinite horizon regularity properties of the solution of (1.2).

Remark 2.3. *We remark that if $y_0\in H^1(\Omega)\cap L^\infty(\Omega)$ and $u\in L^p(0,T;L^q(\omega))$ with $\frac{1}{p}+\frac{n}{2q}<1$, then we have that $y\in L^\infty(Q_T)$ for every $T>0$. Moreover, if $y_0\in C(\bar\Omega)$, then $y\in C(\bar Q_T)$ holds. The proof is standard for the solution of* (2.2) *and from here we deduce the corresponding regularity for $y=e^{\Lambda t}z$.*

THEOREM 2.4. *Let $u\in L^2(0,\infty;L^2(\omega))$ and let y be the solution of* (1.2) *corresponding to u. If $\|y\|_{L^2(Q)}<\infty$, then the following properties hold*

$$f(y), y^{2m+1}\in L^2(Q), \tag{2.6}$$

$$y\in L^2(0,\infty;H^1(\Omega))\cap C([0,\infty);H^1(\Omega)) \quad \textit{and} \quad \frac{\partial y}{\partial t}\in L^2(Q), \tag{2.7}$$

$$\lim_{T\to\infty}\|y(T)\|_{L^2(\Omega)}=0. \tag{2.8}$$

Moreover, there exists a constant C independent of u and y such that

$$\begin{aligned}&\|f(y)\|_{L^2(Q)}+\|y^{2m+1}\|_{L^2(Q)}+\|y\|_{H^1(Q)}+\|y\|_{L^\infty(0,\infty;H^1(\Omega))}\\ &\quad\le C\big(\|y\|_{L^2(Q)}+\|u\|_{L^2(0,\infty;L^2(\omega))}+\|y_0\|_{H^1(\Omega)}^{m+1}\big).\end{aligned} \tag{2.9}$$

Proof. We divide the proof into three parts.

Proof of (2.6). First we demonstrate that $f(y)y^{2m+1}\in L^1(Q)$. Let us write

$$f(s)=\sum_{j=1}^{2m+1}a_js^j \quad \text{and} \quad C_f=\sum_{j=1}^{2m+1}|a_j|. \tag{2.10}$$

Observe that $f(0)=0$ implies that $a_0=0$. From here we infer

$$|f(s)s^{2m+1}|\le C_f s^{4m+2}\ \ \forall |s|\ge 1\ \text{ and }\ |f(s)s^{2m+1}|\le C_f s^2\ \ \forall |s|\le 1. \tag{2.11}$$

Moreover, from the properties assumed for f we deduce the existence of $\mu_0>0$ and $M_0>\mu_0$ such that

$$f'(s)>0 \text{ and } f(s)s\ge 0\ \ \forall |s|\le \mu_0\ \text{ and }\ \forall |s|\ge M_0. \tag{2.12}$$

We will take

$$\mu=\min\{1,\mu_0\}\ \text{ and }\ M>\max\Big\{M_0,\frac{2}{a_{2m+1}}\sum_{j=1}^{2m}|a_j|\Big\}.$$

Let us denote $Q^\delta=\{(x,t)\in Q:|y(x,t)|>\delta\}$ for any real number $\delta>0$, and $Q^{\mu,M}=\{(x,t)\in Q:\mu<|y(x,t)|<M\}$. Since $y\in L^2(Q)$ we have

$$|Q^\delta|\le \frac{1}{\delta^2}\int_{Q^\delta} y^2(x,t)\,dx\,dt\le \frac{1}{\delta^2}\|y\|^2_{L^2(Q)}<\infty\ \ \forall \delta>0, \tag{2.13}$$

hence $|Q^{\mu,M}|\le |Q^\mu|<\infty$ as well. Now we set $C_M=\max_{\mu\le|s|\le M}|f(s)|$. With this notation and using (2.12) and (2.11) we get for every $T>0$

$$\begin{aligned}\Big|\int_{Q_T} f(y)y^{2m+1}\,dx\,dt\Big|&\le C_f\int_{Q_T\setminus Q^\mu} y^2\,dx\,dt+C_M M^{2m+1}|Q_T\cap Q^{\mu,M}|\\&+\int_{Q_T\cap Q^M} f(y)y^{2m+1}\,dx\,dt\le C_f\|y\|^2_{L^2(Q)}+C_M M^{2m+1}|Q^{\mu,M}|\\&+\int_{Q_T\cap Q^M} f(y)y^{2m+1}\,dx\,dt.\end{aligned} \tag{2.14}$$

Thus we only need to prove the integrability of $f(y)y^{2m+1}$ in Q^M. To this end, for every $k>M$ we define the projection $y_k=\mathrm{Proj}_{[-k,+k]}(y)$ and we multiply (2.1) by y_k^{2m+1}

$$\begin{aligned}&\int_{Q_T}\frac{\partial y}{\partial t}y_k^{2m+1}\,dx\,dt+\int_Q[\nabla y\nabla y_k^{2m+1}+ayy_k^{2m+1}]\,dx\,dt+\int_{Q_T\cap Q^M}f(y)y_k^{2m+1}\,dx\,dt\\&\le C_f\|y\|^2_{L^2(Q)}+C_M M^{2m+1}|Q^{\mu,M}|+\int_0^T\int_\omega uy_k^{2m+1}\,dx\,dt.\end{aligned} \tag{2.15}$$

Arguing as in the proof of Theorem 2.2 we have

$$\int_0^T\int_\Omega\frac{\partial y}{\partial t}y_k^{2m+1}\,dx\,dt\ge\frac{1}{2m+2}\int_\Omega y_k^{2m+2}(T)\,dx-\int_\Omega y_0^{2m+2}\,dx$$

and $\nabla y\nabla y_k^{2m+1}=(2m+1)y_k^{2m}\nabla y\nabla y_k=(2m+1)y_k^{2m}|\nabla y_k|^2$. Using this in (2.15) and taking into account that (2.12) implies that $f(y(x,t))y_k(x,t)\ge 0$ for every $(x,t)\in$

$Q \setminus Q^{\mu,M}$ we obtain

$$
\begin{aligned}
0 &\le \int_{Q_T\cap Q^M} f(y)y_k^{2m+1}\,dx\,dt \le \int_\Omega y_0^{2m+2}\,dx + C_f\|y\|^2_{L^2(Q)} + C_M M^{2m+1}|Q^{\mu,M}| \\
&+ \|u\|_{L^2(0,+\infty;L^2(\omega))}\Big(\int_{Q_T\cap Q^M} y_k^{4m+2}\,dx\,dt\Big)^{1/2} \\
&\le \|y_0\|^{2m+2}_{H^1(\Omega)} + C_f\|y\|^2_{L^2(Q)} + C_M M^{2m+1}|Q^{\mu,M}| \\
&+ \frac{1}{a_{2m+1}}\|u\|^2_{L^2(0,+\infty;L^2(\omega))} + \frac{a_{2m+1}}{4}\int_{Q_T\cap Q^M} y_k^{4m+2}\,dx\,dt. \qquad (2.16)
\end{aligned}
$$

Due to the choice of M we have for $|s| \ge M$

$$
\begin{aligned}
f(s)s^{2m+1} &\ge s^{4m+2}\Big(a_{2m+1} - \sum_{j=1}^{2m}|a_j|\frac{1}{|s|^{2m+1-j}}\Big) \\
&\ge s^{4m+2}\Big(a_{2m+1} - \frac{1}{M}\sum_{j=1}^{2m}|a_j|\Big) \ge \frac{a_{2m+1}}{2}s^{4m+2}.
\end{aligned}
$$

Since $f'(s) > 0$ for $|s| \ge M$ and $k > M$, we get

$$
f(y(x,t))y_k^{2m+1}(x,t) \ge f(y_k(x,t))y_k^{2m+1}(x,t) \ge \frac{a_{2m+1}}{2}y_k^{4m+2}(x,t), \quad (x,t)\in Q^M.
$$

Inserting this inequality in the right hand side of (2.16) and (2.13) we conclude that

$$
\begin{aligned}
0 &\le \int_{Q_T\cap Q^M} f(y)y_k^{2m+1}\,dx\,dt \\
&\le C\big(\|y_0\|^{2m+2}_{H^1(\Omega)} + \|y\|^2_{L^2(Q)} + \|u\|^2_{L^2(0,\infty;L^2(\omega))}\big) \quad \forall T>0 \text{ and } \forall k,
\end{aligned}
$$

where C only depends on f and M. Since $y_k(x,t) \to y(x,t)$ a.e. in Q, we deduce from the above inequality, (2.13), (2.14), and Fatou's Lemma that

$$
\int_Q |f(y)||y^{2m+1}|\,dx\,dt \le C\big(\|y_0\|^{2m+2}_{H^1(\Omega)} + \|y\|^2_{L^2(Q)} + \|u\|^2_{L^2(0,\infty;L^2(\omega))}\big),
$$

for a new C only depending on f and M. Now we have

$$
\begin{aligned}
\int_Q y^{4m+2}\,dx\,dt &\le \int_{Q\setminus Q^\mu} y^2\,dx\,dt + \int_{Q^{\mu,M}} M^{4m+2}\,dx\,dt + \int_{Q^M} y^{4m+2}\,dx\,dt \\
&\le \|y\|^2_{L^2(Q)} + M^{4m+2}|Q^{\mu,M}| + \frac{2}{a_{2m+1}}\int_{Q^M} f(y)y^{2m+1}\,dx\,dt < \infty,
\end{aligned}
$$

which proves that $y^{2m+1} \in L^2(Q)$. Moreover, since $|f(s)| \le C_f|s|^{2m+1}$ $\forall |s| \ge 1$ and $|f(s)| \le C_f|s|$ $\forall |s| \le 1$, we deduce that

$$
f(s)^2 \le C_f^2(s^2 + s^{4m+2}) \quad \forall s \in \mathbb{R}.
$$

Therefore, the fact that y and $y^{2m+1} \in L^2(Q)$ implies that $f(y) \in L^2(Q)$ and the proof of (2.6) is complete. Additionally, these arguments obviously lead to the estimates for the first two terms of (2.9).

Proof of (2.7). First we observe that $y \in C([0,T];H^1(\Omega))$ for every $T>0$. Indeed, this is a consequence of he fact that $f(y) \in L^2(Q)$ and $y_0 \in H^1(\Omega)$; see [15, Proposition III-2.5]. Hence $y:[0,\infty) \to H^1(\Omega)$ is continuous. To prove that $y \in L^2(0,\infty;H^1(\Omega))$ it is enough to multiply (2.1) by y and integrate in Q_T, $T>0$ arbitrary, to get

$$\begin{aligned}
&\frac{1}{2}\|y(T)\|_{L^2(\Omega)} + \int_{Q_T} (|\nabla y|^2 + ay^2)\,dx\,dt \\
&= \int_0^T \int_\omega uy\,dx\,dt + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)} - \int_{Q_T} f(y)y\,dx\,dt \\
&\le \|u\|_{L^2(0,\infty;L^2(\omega))}\|y\|_{L^2(Q)} + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)} + \|f(y)\|_{L^2(Q)}\|y\|_{L^2(Q)} < \infty,
\end{aligned}$$

Above we have used the assumption $y \in L^2(Q)$. Now it is enough to make $T \to \infty$ to deduce that $y \in L^2(0,\infty;H^1(\Omega))$.

To prove that $y \in L^\infty(0,\infty;H^1(\Omega))$ we take into account that by Theorem 2.2 $y \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for every $T>0$. We can multiply the equation (2.1) by $\frac{\partial y}{\partial t}$ and integrate in Q_T to get

$$\|\frac{\partial y}{\partial t}\|^2_{L^2(Q_T)} + \int_0^T \frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla y|^2 + ay^2)\,dx + \int_{Q_T} f(y)\frac{\partial y}{\partial t}\,dx\,dt = \int_0^T \int_\omega u\frac{\partial y}{\partial t}\,dx\,dt.$$

This implies

$$\begin{aligned}
&\|\frac{\partial y}{\partial t}\|^2_{L^2(Q_T)} + \frac{1}{2}\int_\Omega (|\nabla y(T)|^2 + a_0 y^2(T))\,dx \\
&\le \big(\|u\|_{L^2(0,T;L^2(\omega))} + \|f(y)\|_{L^2(Q)}\big)\|\frac{\partial y}{\partial t}\|_{L^2(Q_T)} + \frac{1}{2}\int_\Omega (|\nabla y_0|^2 + a_0 y_0^2)\,dx \\
&\le \frac{1}{2}\big(\|u\|_{L^2(0,T;L^2(\omega))} + \|f(y)\|_{L^2(Q)}\big)^2 + \frac{1}{2}\|\frac{\partial y}{\partial t}\|^2_{L^2(Q_T)} + \frac{C}{2}\|y_0\|^2_{H^1(\Omega)},
\end{aligned}$$

hence

$$\begin{aligned}
&\|\frac{\partial y}{\partial t}\|^2_{L^2(Q_T)} + \int_\Omega (|\nabla y(T)|^2 + a_0 y^2(T))\,dx \\
&\le \big(\|u\|_{L^2(0,T;L^2(\omega))} + \|f(y)\|_{L^2(Q)}\big)^2 + C\|y_0\|^2_{H^1(\Omega)}.
\end{aligned}$$

Since $T>0$ is arbitrary, the above inequality concludes the proof of (2.7). Moreover, from the obtained estimates the bounds for the second two terms in (2.9) follow.

Proof of (2.8). Since $y \in H^1(0,T;L^2(\Omega))$ for every $T>0$, then the function $t \in [0,T] \to \|y(t)\|^2_{L^2(\Omega)}$ is absolutely continuous and

$$\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} = 2\int_\Omega y(t)\frac{\partial y}{\partial t}(t)\,dx;$$

see, for instance, [15, Proposition III-1.2]. Moreover, the fact that $y \in L^2(Q)$ implies the existence of a monotone increasing sequence of positive numbers $\{t_k\}_{k=1}^\infty$ such that $\|y(t_k)\|_{L^2(\Omega)} \to 0$ as $k \to \infty$. Then, given $T>0$ and taking $t_k > T$ and we get

$$\begin{aligned}
\|y(T)\|^2_{L^2(\Omega)} &= \|y(t_k)\|^2_{L^2(\Omega)} - 2\int_T^{t_k}\int_\Omega y(t)\frac{\partial y}{\partial t}(t)\,dx\,dt \\
&\le \|y(t_k)\|^2_{L^2(\Omega)} + 2\|y\|_{L^2(T,\infty;L^2(\Omega))}\|\frac{\partial y}{\partial t}\|_{L^2(T,\infty;L^2(\Omega))}
\end{aligned}$$

Taking the limit when $k \to \infty$ we get

$$\|y(T)\|^2_{L^2(\Omega)} \le \lim_{T\to\infty} 2\|y\|_{L^2(T,\infty;L^2(\Omega))}\|\frac{\partial y}{\partial t}\|_{L^2(T,\infty;L^2(\Omega))} = 0.$$

□

Theorem 2.4 suggests to introduce the following space of solutions:

$$Y = \{y \in H^1(Q) \cap BC([0,\infty); H^1(\Omega)) : \Delta y \in L^2(Q) \text{ and } \partial_n y = 0 \text{ on } \Sigma\},$$

where $BC([0,\infty); H^1(\Omega))$ is the Banach space of continuous and bounded functions $y : [0,\infty) \longrightarrow H^1(\Omega)$. The space Y endowed with the norm

$$\|y\|_Y = \|y\|_{H^1(Q)} + \|y\|_{L^\infty(0,\infty,H^1(\Omega))} + \|\Delta y\|_{L^2(Q)}$$

is a Banach space. Let us point out that any element $y \in H^1(0,\infty; L^2(\Omega))$ satisfies (2.8). This was proved in the last step of the proof of Theorem 2.4. Hence this property holds for every element $y \in Y$.

In the next theorem we prove that if $\|y_0\|_{L^2(\Omega)}$ is sufficiently small, then the solution of (1.2) associated to the null control $u \equiv 0$ is stable and it has an exponential decay. Let us introduce some notation to make precise how small $\|y_0\|_{L^2(\Omega)}$ must be. Let μ_0 and M_0 satisfy (2.12) and set $C_0' = \max_{\mu_0 \le |s| \le M_0} |f'(s)|$. Now we take

$$K_f = \frac{\mu_0 C_a^2}{(C_0' C_4)^2}, \tag{2.17}$$

where $C_a > 0$ and $C_4 > 0$ are taken so that

$$\begin{aligned} &\int_\Omega (|\nabla z|^2 + az^2)\,dx \ge C_a\|z\|^2_{H^1(\Omega)} \ge C_a\|z\|^2_{L^2(\Omega)} \\ &\|z\|_{L^4(\Omega)} \le C_4\|z\|_{H^1(\Omega)}, \quad \forall z \in H^1(\Omega). \end{aligned} \tag{2.18}$$

THEOREM 2.5. *Let us assume that $\|y_0\|_{L^2(\Omega)} < K_f$ and $u \equiv 0$. Then the solution of (1.2) belongs to $L^2(Q)$ and there exists $\lambda > 0$ such that*

$$\|y(t)\|_{L^2(\Omega)} \le \|y_0\|_{L^2(\Omega)} e^{-\lambda t} \quad \forall t > 0. \tag{2.19}$$

Together with (2.9) of Theorem 2.4 this theorem provides a sufficient condition for $y \in Y$.

Proof. Let us take

$$K_0 = \frac{1}{2}\big(\|y_0\|_{L^2(\Omega)} + K_f\big) \text{ and } T_0 = \sup\{T > 0 : \|y(t)\|_{L^2(\Omega)} < K_0,\ \forall t \in [0,T]\}.$$

Since $y : [0,\infty) \to L^2(\Omega)$ is a continuous function and $\|y(0)\|_{L^2(\Omega)} = \|y_0\|_{L^2(\Omega)} < K_0$, we have that $T_0 > 0$. We will prove that $T_0 = \infty$. For every $t \in (0, T_0)$ we define

$$\Omega_t = \{x \in \Omega : \mu_0 < |y(x,t)| < M_0\}.$$

Then we have

$$|\Omega_t| \le \frac{1}{\mu_0^2}\int_{\Omega_t} y^2(t)\,dx \le \frac{1}{\mu_0^2}\|y(t)\|^2_{L^2(\Omega)} < \frac{K_0^2}{\mu_0^2} \quad \forall t \in (0, T_0). \tag{2.20}$$

We will use the following interpolation inequality, see e.g. [4, p. 93].

$$\|z\|_{L^{8/3}(\Omega)} \le \|z\|^{1/2}_{L^2(\Omega)}\|z\|^{1/2}_{L^4(\Omega)} \le C_4^{1/2}\|z\|^{1/2}_{L^2(\Omega)}\|z\|^{1/2}_{H^1(\Omega)} \quad \forall z \in H^1(\Omega). \tag{2.21}$$

Then, multiplying (1.2) by $y(t)$, $t \in (0,T_0)$, integrating in Ω, taking into account that $f(y(t))y(t) \ge 0$ in $\Omega \setminus \Omega_t$ by (2.12), using (2.20) and (2.21), and Young's inequality we deduce for almost all $t \in (0,T_0)$ for some $\theta(x,t) \in [0,1]$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} + C_a\|y(t)\|^2_{H^1(\Omega)} \le \frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} + \int_\Omega (|\nabla y(t)|^2 + ay(t)^2)\,dx\\
&+ \int_{\Omega\setminus\Omega_t} f(y(t))y(t)\,dx = -\int_{\Omega_t} f(y(t))y(t))\,dx = -\int_{\Omega_t} f'(\theta(t)y(t))y^2(t)\,dx\\
&\le \|f'(\theta(t)y(t))\|_{L^4(\Omega_t)}\|y(t)\|^2_{L^{8/3}(\Omega)} \le C_0'|\Omega_t|^{1/4}C_4\|y(t)\|_{L^2(\Omega)}\|y(t)\|_{H^1(\Omega)}\\
&\le \frac{C_0'^2C_4^2|\Omega_t|^{1/2}}{2C_a}\|y(t)\|^2_{L^2(\Omega)} + \frac{C_a}{2}\|y(t)\|^2_{H^1(\Omega)}.
\end{aligned}$$

With (2.20) this leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} + \frac{C_a}{2}\|y(t)\|^2_{L^2(\Omega)} \le \frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} + \frac{C_a}{2}\|y(t)\|^2_{H^1(\Omega)}\\
&\le \frac{C_0'^2C_4^2|\Omega_t|^{1/2}}{2C_a}\|y(t)\|^2_{L^2(\Omega)} \le \frac{C_0'^2C_4^2K_0}{2C_a\mu_0}\|y(t)\|^2_{L^2(\Omega)},
\end{aligned}$$

hence

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} + \lambda\|y(t)\|^2_{L^2(\Omega)} \le 0, \tag{2.22}$$

where

$$\lambda = \frac{1}{2}\Big(C_a - \frac{C_0'^2C_4^2K_0}{C_a\mu_0}\Big).$$

From the choice of K_0 and (2.17) we infer that $\lambda > 0$. Then we have $\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} \le 0$ $\forall t \in (0,T_0)$, hence $T_0 = \infty$. Moreover, inequality (2.22) implies (2.19). □

Let us denote by ρ_- the biggest negative root of the polynomial f such that $f(\rho)$ changes the sign when ρ crosses ρ_-; i.e. ρ_- is a root of f of odd multiplicity. If f has no negative root with such a property, then we set $\rho_- = -\infty$. Analogously, we define ρ_+ as the smallest positive root of f with odd multiplicity, and we take $\rho_+ = +\infty$ if such a root does not exist. Then we have the following theorem.

THEOREM 2.6. *Let us assume that $\rho_- \le y_0(x) \le \rho_+$ for a.a. $x \in \Omega$, and $u \equiv 0$. Then the solution y of (1.2) belongs to $L^2(Q)$, $\rho_- \le y(x,t) \le \rho_+$ $\forall (x,t) \in Q$, and*

$$\|y(t)\|_{L^2(\Omega)} \le \|y_0\|_{L^2(\Omega)}e^{-C_at} \quad \forall t > 0, \tag{2.23}$$

where $C_a > 0$ is given by (2.18)

Proof. First we assume that $y_0 \in C(\bar\Omega)$ and $\rho_- < y_0(x) < \rho_+$ $\forall x \in \bar\Omega$. Let us set $\lambda_- = \min\{0, \min_{x\in\bar\Omega} y_0(x)\}$ and $\lambda_+ = \max\{0, \max_{x\in\bar\Omega} y_0(x)\}$. Then we have $\rho_- < \lambda_- \le 0 \le \lambda_+ < \rho_+$. Let y be the solution of (1.2) associated to $u \equiv 0$. Since $y_0 \in C(\bar\Omega)$, then $y \in C(\bar Q_T)$ $\forall T > 0$; see Remark 2.3. Then, $\lim_{t\to 0}\|y(t) - y_0\|_{C(\bar\Omega)} = 0$ holds. This implies that

$$T^* = \sup\{T > 0 : \rho_- < y(x,t) < \rho_+ \quad \forall (x,t) \in \bar Q_T\} > 0.$$

Let us proof that $T^* = +\infty$. We argue by contradiction. If $T^* < +\infty$, then there exists at least one point $x^* \in \bar{\Omega}$ such that $\rho_- < y(x,t) < \rho_+ \ \forall (x,t) \in \bar{\Omega} \times [0,T^*)$ and either $y(x^*,T^*) = \rho_-$ or $y(x^*,T^*) = \rho_+$. Let us assume that $y(x^*,T^*) = \rho_-$ and take $z(x,t) = (y(x,t) - \lambda_-)^- = \min\{y(x,t) - \lambda_-, 0\}$. It is clear that $\frac{\partial y}{\partial t} z = \frac{\partial z}{\partial t} z$, $\nabla y \nabla z = |\nabla z|^2$, and $yz \geq z^2$ almost everywhere in Q_{T^*}. Hence, multiplying (2.1) by z, integrating in Q_{T^*} and using that $z(0) = 0$ we get

$$\|z(T^*)\|^2_{L^2(\Omega)} + \int_{Q_{T^*}} (|\nabla z|^2 + az^2)\, dx\, dt + \int_{Q_{T^*}} f(y) z\, dx\, dt = 0.$$

Now, we observe that $f(s)s \geq 0 \ \forall s \in [\rho_-, \rho_+]$, hence $f(y(x,t))z(x,t) \geq 0 \ \forall (x,t) \in Q_{T^*}$. Using this in the above inequality we obtain with (2.18)

$$\|z(T^*)\|^2_{L^2(\Omega)} + C_a \|z\|^2_{L^2(Q_{T^*})} \leq 0,$$

therefore $z = 0$. This implies that $y(x,t) \geq \lambda_- > \rho_- \ \forall (x,t) \in Q_{T^*}$, which contradicts the equality $y(x^*,T^*) = \rho_-$.

In the case $y(x^*,t^*) = \rho_+$, we take $z(x,t) = \max\{y(x,t) - \lambda_+, 0\}$ and we argue similarly as above. Hence $T^* = \infty$ holds. If $\lambda_- = \rho_-$ or $\lambda_+ = \rho_+$, then we take $y_{0\varepsilon} = \mathrm{Proj}_{[\lambda_- + \varepsilon, \lambda_+ - \varepsilon]}(y_0(x))$. Then $y_{0\varepsilon} \to y_0$ in $C(\bar{\Omega}) \cap H^1(\Omega)$ holds. If we denote by y_ε the solution of (1.2) associated with $y_{0\varepsilon}$ and $u \equiv 0$, then we have that $y_\varepsilon \to y$ in $\bar{Q}_T$ for every $T > 0$. Since $\rho_- \leq y_\varepsilon(x,t) \leq \rho_+ \ \forall (x,t) \in Q$, we conclude that $\rho_- \leq y(x,t) \leq \rho_+ \ \forall (x,t) \in Q$ as well.

Since $f(y(x,t))y(x,t) \geq 0 \ \forall (x,t) \in Q$, multiplying the equation (1.2) by y and integrating in Ω we get

$$\frac{1}{2} \frac{d}{dt} \|y(t)\|^2_{L^2(\Omega)} + C_a \|y(t)\|^2_{L^2(\Omega)} \leq 0,$$

which implies (2.23), and consequently $y \in L^2(Q)$.

Finally, if $y_0 \in H^1(\Omega) \setminus C(\bar{\Omega})$ and $\rho_- \leq y_0(x) \leq \rho_+$ for a.a. $x \in \Omega$, then we take a sequence $\{z_k\}_{k=1}^\infty \subset H^1(\Omega) \cap C(\bar{\Omega})$ such that $z_k \to y_0$ in $H^1(\Omega)$ and $z_k(x) \to y_0(x)$ a.e. in Ω. Now, we take $y_{0k}(x) = \mathrm{Proj}_{[\rho_-, \rho_+]}(z_k(x))$, and we still have that $\{y_{0k}\}_{k=1}^\infty \subset H^1(\Omega) \cap C(\bar{\Omega})$, $y_{0k} \to y_0$ in $H^1(\Omega)$, and $y_{0k}(x) \to y_0(x)$ a.e. in Ω. The solution y_k of (1.2) corresponding to the initial condition y_{0k} belongs to $L^2(Q)$, $\rho_- \leq y_k(x,t) \leq \rho_+ \ \forall (x,t) \in Q$, and it satisfies

$$\|y_k(t)\|_{L^2(\Omega)} \leq \|y_{0k}\|_{L^2(\Omega)} \mathrm{e}^{-C_a t} \quad \forall t > 0.$$

Now, it is easy to prove the boundedness of $\{y_k\}_{k=1}^\infty$ in Y, and hence we pass to the limit in the above inequality as $k \to \infty$ and we deduce that y satisfies (2.23). □

Remark 2.7. *Let us come back to our original equation*

$$\frac{\partial y}{\partial t} - \Delta y + f(y) = u \chi_\omega, \tag{2.24}$$

where $a \equiv 0$. We consider ρ_- and ρ_+ as in Theorem 2.6. Then the proof of Theorem 2.6 and (2.23) *fail. However we can still stabilize the system by a feedback control. Indeed, let us take $a = \lambda \chi_\omega$, where λ is an arbitrary strictly positive constant. Then we are under the conditions of Theorem 2.6 and the theorem holds. Now, we take $u = -ay = -\lambda \chi_\omega y$, the equation* (2.24) *holds and the state y is stabilized.*

Remark 2.8. *We remark that the fact that the nonlinearity f in the equation* (1.2) *is a polynomial function played an essential role in the proofs of this section.*

However, if we assume that if $y_0 \in H^1(\Omega) \cap L^{2m+2}(\Omega)$, then all the results of this section are valid for a polynomial of arbitrary degree $2m+1$, with the obvious changes in the estimates. The assumption $m = 1$ in dimension 3 will be used in Theorem 3.4 below.

3. Analysis of the Control Problem. We divide this section into three parts. First we study the existence of an optimal control. Then, we address the sensitivity of the states with respect to the controls, and finally we analyze the adjoint state equation.

3.1. Existence of a solution to (P)**.** Before proving the existence of an optimal control we establish the following lemma.

LEMMA 3.1. *Let $\{u_k\}$ be a bounded sequence in $L^2(0,\infty;L^2(\omega))$ such that the corresponding states $\{y_k\}$ are bounded in $L^2(Q)$. Then, there exist subsequences such that $u_k \rightharpoonup \bar{u}$ in $L^2(0,\infty;L^2(\omega))$ and $y_k \rightharpoonup \bar{y}$ in $H^1(Q)$, where $\bar{y}$ is the state associated to $\bar{u}$.*

Proof. From the assumptions of the lemma and (2.9) we deduce the existence of subsequences such that $u_k \rightharpoonup \bar{u}$ in $L^2(0,\infty;L^2(\omega))$ and $y_k \rightharpoonup \bar{y}$ in $H^1(Q)$. We prove that $\bar{y}$ is the solution of (1.2) associated to $\bar{u}$. To this end we have to check Definition 2.1. First we observe that $\bar{y} \in H^1(Q) \subset C([0,\infty);L^2(\Omega))$. Now let $T > 0$ be arbitrary. From the compactness of the embedding $H^1(Q_T) \subset L^2(Q_T)$ we infer the existence of a subsequence such that

$$y_k \to \bar{y} \ \text{ in } L^2(Q_T) \ \text{ and } \ y_k(x,t) \to \bar{y}(x,t) \ \text{ a.e. in } \ Q_T.$$

Using again (2.9) we deduce from the above pointwise convergence that $f(y_k) \rightharpoonup f(\bar{y})$ in $L^2(Q_T)$. Then it is easy to pass to the limit weakly in the state equation (1.2) and to deduce that $\bar{y}$ satisfies the equation in the variational sense in Q_T with $\bar{u}$ on the right hand side. Moreover, from the continuity of the embedding $H^1(Q_T) \subset C([0,T];L^2(\Omega))$ we have $\bar{y}(0) = \lim_{k\to\infty} y_k(0) = y_0$. □

THEOREM 3.2. *Assume that there exists an element $u_0 \in L^2(0,\infty;L^2(\omega))$ such that $J(u_0) < \infty$. Then* (P) *admits at least one solution.*

Proof. Since the set of feasible controls is nonempty, we can take a minimizing sequence $\{u_k\}$. From the inequality $J(u_k) \le J(u_0)$ for every k large (unless u_0 is already an optimal control), we deduce the boundedness of $\{(u_k, y_k)\}$ in $L^2(0,\infty;L^2(\omega)) \cap L^1(0,\infty,L^2(\omega)) \times L^2(Q)$, where y_k denotes the state associated with u_k. Let $(\bar{u},\bar{y})$ be a weak limit in $L^2(0,\infty;L^2(\omega)) \times L^2(Q)$ of a subsequence, denoted in the same way. Lemma 3.1 implies that $\bar{y}$ is the solution of (1.2) corresponding to $\bar{u}$. To prove that $\bar{u}$ is a solution to (P), we consider the following inequality for every $T > 0$

$$\frac{1}{2}\int_{Q_T} \bar{y}^2\,dx\,dt + \frac{\nu}{2}\int_0^T\int_\omega \bar{u}^2\,dx\,dt + \alpha\int_0^T \Big(\int_\omega \bar{u}^2\,dx\Big)^{1/2} dt \le \liminf_{k\to\infty} J(u_k) = \inf(\mathrm{P}),$$

which follows from the convexity of the objective functional with respect to pair (y,u) and the continuity of the embedding $L^2(0,T;L^2(\omega)) \subset L^1(0,T;L^2(\omega))$. Now we have

$$J(\bar{u}) = \sup_{T>0}\Big\{\frac{1}{2}\int_{Q_T} \bar{y}^2\,dx\,dt + \frac{\nu}{2}\int_0^T\int_\omega \bar{u}^2\,dx\,dt + \alpha\int_0^T \Big(\int_\omega \bar{u}^2\,dx\Big)^{1/2} dt\Big\} = \inf(\mathrm{P}),$$

which concludes the proof. □

Remark 3.3. *Concerning the feasibility assumption of Theorem 3.2, Theorems 2.5 and 2.6 provide sufficient conditions on y_0 to assure that $u_0 \equiv 0$ is a feasible*

control for (P). *If y_0 does not satisfy the required assumptions, but equation* (1.2) *is approximately controllable to zero, again we can rely on the above theorems to guarantee existence of a feasible control. In particular, for the case $\omega = \Omega$ we can prove approximate controllability as follows. First we solve the equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + (a+\Lambda)y + f(y) &= 0 \quad \text{in } Q, \\ \partial_n y &= 0 \quad \text{on } \Sigma, \\ y(0) &= y_0 \quad \text{in } \Omega, \end{cases}$$

where Λ is given by in (1.3). *Then we have*

$$\Lambda s^2 + f(s)s = (\Lambda + f'(\theta s))s^2 \geq 0 \quad \forall s \in \mathbb{R},$$

where $\theta = \theta(s) \in [0,1]$.
Hence, we infer the estimate

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} + C_a\|y(t)\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)} + \int_\Omega [|\nabla y(t)|^2 + (a+\Lambda)y(t)^2]\,dx + \int_\Omega f(y(t))y(t)\,dx = 0. \end{aligned}$$

Therefore we can argue as in the proofs of Theorems 2.2 and 2.4 to deduce the existence of a unique solution $y \in Y$. Moreover, from the above inequality we infer

$$\|y(t)\|_{L^2(\Omega)} \leq \|y_0\|_{L^2(\Omega)} e^{-C_a t}.$$

Now, it is enough to take $u = -\Lambda y$ in (1.2) *and then $y_u = y$ and $u \in U$. Indeed, since $Y \subset L^2(Q)$ we have that $u \in L^2(Q)$. Furthermore from the last inequality we get*

$$\int_0^\infty \int_\Omega \|u(t)\|_{L^2(\Omega)}\,dt \leq \Lambda\|y_0\|_{L^2(\Omega)} \int_0^\infty e^{-C_a t}\,dt < \infty.$$

Thus u is a feasible control for (P).

3.2. Sensitivity of the relationship control-to-state. We define $\mathcal{U}$ as the subset of the elements $u \in L^2(0,\infty;L^2(\omega))$ for which there exists a solution $y_u \in Y$ as well as the mapping $G : \mathcal{U} \longrightarrow Y$ by $G(u) = y_u$. We remark that for every $u \in L^2(0,\infty;L^2(\omega))$ there exists a unique solution y_u in the sense of Definition 2.1. Further, due to Theorem 2.4, an element $u \in L^2(0,\infty;L^2(\omega))$ belongs to $\mathcal{U}$ if and only if $y_u \in L^2(Q)$.

THEOREM 3.4. *The set $\mathcal{U}$ is open in $L^2(0,\infty;L^2(\omega))$ and $G : \mathcal{U} \longrightarrow Y$ is of class C^1. Furthermore, for every $(u,v) \in \mathcal{U} \times L^2(0,\infty;L^2(\omega))$ the derivative $z_v = DG(u)v$ is the unique solution to*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta z + az + f'(y_u)z &= v\chi_\omega \quad \text{in } Q, \\ \partial_n z &= 0 \quad \text{on } \Sigma, \\ z(0) &= 0 \quad \text{in } \Omega. \end{cases} \tag{3.1}$$

Proof. The proof is based on the implicit function theorem applied to the mapping

$$\begin{aligned} &\mathcal{F} : Y \times L^2(0,\infty;L^2(\omega)) \longrightarrow L^2(Q) \times H^1(\Omega), \\ &\mathcal{F}(y,u) = \big(\frac{\partial y}{\partial t} - \Delta y + ay + f(y) - u\chi_\omega, y(0) - y_0\big). \end{aligned}$$

The mapping $\mathcal{F}$ is well defined. Indeed, the only delicate issue concerns the membership $f(y) \in L^2(Q)$. To prove this we use that $f(y)^2 \le C_f^2(y^2 + y^{4m+2})$, as established in the proof of Theorem 2.4. Thus, it suffices to verify that $y^{4m+2} \in L^1(Q)$. Recalling that $4m + 2 \le 6$ for $n = 3$, this is obtained as follows

$$\int_Q y^{4m+2}\,dx\,dt \le C\int_0^\infty \|y(t)\|_{H^1(\Omega)}^{4m+2}\,dt \le C\|y\|_{L^\infty(0,\infty;H^1(\Omega))}^{4m}\|y\|_{L^2(0,\infty;H^1(\Omega))}^2 < \infty.$$

The above argument, in particular, implies that Y is continuously embedded in $L^{4m+2}(Q)$. It is easy to check that the mapping $y \in L^{4m+2}(Q) \cap L^2(Q) \to f(y) \in L^2(Q)$ is of class C^1. Hence, we have that $\mathcal{F}$ is also of class C^1, and the first partial derivative of $\mathcal{F}$ with respect to y at (y, u) in a direction $z \in Y$ is given by

$$\frac{\partial \mathcal{F}}{\partial y}(y,u)z = \big(\frac{\partial z}{\partial t} - \Delta z + az + f'(y)z, z(0)\big).$$

Now for any $u \in \mathcal{U}$ we have that $\mathcal{F}(y_u, u) = \mathcal{F}(G(u), u) = (0, 0)$. To complete the verification of the assumptions of the implicit function theorem it remains to prove that $\frac{\partial \mathcal{F}}{\partial y}(y_u, u) : Y \longrightarrow L^2(Q) \times H^1(\Omega)$ is an isomorphism, or equivalently that the equation

$$\left\{\begin{array}{rcll} \dfrac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z &=& g & \text{in } Q,\\ \partial_n z &=& 0 & \text{on } \Sigma,\\ z(0) &=& z_0 & \text{in } \Omega. \end{array}\right. \tag{3.2}$$

has a unique solution in Y for every $(g, z_0) \in L^2(Q) \times H^1(\Omega)$. This is done in Lemma 3.5 below. □

LEMMA 3.5. *For every $(g, z_0) \in L^2(Q) \times H^1(\Omega)$ equation (3.2) has a unique solution $z \in Y$. Moreover, there exists a constant independent of (g, z_0) such that*

$$\|z\|_Y \le C\big(\|g\|_{L^2(Q)} + \|z_0\|_{H^1(\Omega)}\big). \tag{3.3}$$

Proof. From our assumptions on f we infer the existence of a polynomial p of degree $2m - 2$ such that

$$f'(s) = (2m + 1)a_{2m+1}s^{2m} + a_1 + p(s)s \ge p(s)s \quad \text{because} \quad a_{2m+1} > 0 \text{ and } a_1 > 0.$$

Moreover there exists a constant $C_f' > 0$ such that $|f'(s)| \le C_f'(s^{2m} + 1)$. We observe that $m = 1$ in dimension $n = 3$ and consequently p is a constant in that case.

Since $y_u \in Y$, given $\varepsilon > 0$ we can use (2.8) to deduce the existence of $T_\varepsilon > 0$ so that

$$\|y_u(t)\|_{L^2(\Omega)} < \varepsilon \quad \forall t \ge T_\varepsilon. \tag{3.4}$$

Let us take $T > T_\varepsilon$ arbitrary. From (2.9) we have that $y_u^{2m+1} \in L^2(Q)$, hence $f'(y_u) \in L^{2+\frac{1}{m}}(Q_T)$. The the classical theory for linear parabolic equations (see, for instance, [11, Chapter III]) we deduce the existence of a unique solution $z \in W(0, T)$ of (3.2) with

$$W(0,T) = \Big\{z \in L^2(0,T;H^1(\Omega)) : \frac{\partial z}{\partial t} \in L^2(0,T;H^1(\Omega)^*)\Big\}.$$

Moreover we have that

$$\|z\|_{W(0,T)} \le C_T\big(\|g\|_{L^2(Q_T)} + \|z_0\|_{L^2(\Omega)}\big). \tag{3.5}$$

To obtain the estimates for z in Q we introduce for every $t > 0$ the following sets

$$\Omega_t = \{x \in \Omega : |y_u(x,t)| < M_0\},$$

where M_0 was given in (2.12). Set $C_p = \max_{|s|\le M_0} |p(s)|$. Now, multiplying (3.2) by z, making integration by parts in Q_T and using that $f'(y_u)z^2 \ge p(y_u)y_u z^2$ in Q and $f'(y_u(x,t)) \ge 0$ for every $x \in \Omega \setminus \Omega_t$ we get

$$\begin{aligned}
&\frac{1}{2}\|z(T)\|^2_{L^2(\Omega)} + C_a\|z\|^2_{L^2(0,T;H^1(\Omega))}\\
&\le \frac{1}{2}\|z_0\|^2_{L^2(\Omega)} + \int_{Q_T} gz\,dx\,dt - \int_0^T\int_{\Omega_t} p(y_u)y_u z^2\,dx\,dt\\
&\le \frac{1}{2}\|z_0\|^2_{L^2(\Omega)} + \|g\|_{L^2(Q_T)}\|z\|_{L^2(Q_T)}\\
&+ \int_0^{T_\varepsilon}\int_{\Omega_t} |p(y_u)y_u|z^2\,dx\,dt + \int_{T_\varepsilon}^T\int_{\Omega_t} |p(y_u)y_u|z^2\,dx\,dt\\
&\le \frac{1}{2}\|z_0\|^2_{L^2(\Omega)} + \|g\|_{L^2(Q_T)}\|z\|_{L^2(0,T;H^1(\Omega))}\\
&+ C_pM_0\|z\|^2_{L^2(Q_{T_\varepsilon})} + C_p\int_{T_\varepsilon}^T \|y_u(t)\|_{L^2(\Omega)}\|z(t)\|^2_{L^4(\Omega)}\,dt\\
&\le \frac{1}{2}\|z_0\|^2_{L^2(\Omega)} + \frac{1}{C_a}\|g\|^2_{L^2(Q)} + \frac{C_a}{4}\|z\|^2_{L^2(0,T;H^1(\Omega))}\\
&+ C_pM_0\|z\|^2_{L^2(Q_{T_\varepsilon})} + C_p\varepsilon\|z\|^2_{L^2(0,T;H^1(\Omega))},
\end{aligned}$$

where C_a is the constant that appeared in (2.4). Taking $\varepsilon = \frac{C_a}{4C_p}$ we infer from the above inequality and (3.5) that

$$\begin{aligned}
&\|z\|_{L^2(0,T;H^1(\Omega))} \le \frac{1}{\sqrt{C_a}}\Big(\|z_0\|_{L^2(\Omega)} + \frac{\sqrt{2}}{\sqrt{C_a}}\|g\|_{L^2(Q)} + \sqrt{2C_pM_0}\|z\|_{L^2(Q_{T_\varepsilon})}\Big)\\
&\le C'\big(\|g\|_{L^2(Q)} + \|z_0\|_{L^2(\Omega)}\big) \quad \forall T > T_\varepsilon.
\end{aligned} \tag{3.6}$$

Hence $z \in L^2(0,\infty;H^1(\Omega))$ holds. Next we prove that $z \in Y$. To this end we first establish that $f'(y_u)z \in L^2(Q)$. Since

$$f'(y_u)^2z^2 \le {C'_f}^2(y_u^{2m}+1)^2z^2 \le 2{C'_f}^2(y_u^{4m}+1)z^2,$$

it is enough to prove that $y^{4m}z^2 \in L^1(Q)$. Using Hölder inequality with $p = \frac{4m+2}{4m}$ and $p' = 2m+1$, this is obtained as follows

$$\begin{aligned}
&\int_Q y_u^{4m}z^2\,dx\,dt \le \int_0^\infty \|y_u(t)\|^{4m}_{L^{4m+2}(\Omega)}\|z(t)\|^2_{L^{4m+2}(\Omega)}\,dt\\
&\le C\int_0^\infty \|y_u(t)\|^{4m}_{H^1(\Omega)}\|z(t)\|^2_{H^1(\Omega)}\,dt \le C\|y_u\|^{4m}_{L^\infty(0,\infty;H^1(\Omega))}\|z\|^2_{L^2(0,\infty;H^1(\Omega))}.
\end{aligned}$$

Additionally, with (3.6) we get the estimate

$$\|f'(y_u)z\|_{L^2(Q)} \le C''\big(\|y_u\|^{2m}_{L^\infty(0,\infty;H^1(\Omega))} + 1\big)\big(\|g\|_{L^2(Q)} + \|z_0\|_{L^2(\Omega)}\big). \tag{3.7}$$

Finally, the regularity $\frac{\partial z}{\partial t} \in L^2(Q)$ and $z \in L^\infty(0,\infty;H^1(\Omega))$ and the corresponding estimates are proved as in the last two steps of the proof of Theorem 2.4, just taking into account the obtained a priori estimates (3.6) and (3.7). □

Remark 3.6. *Let us note that the assumption $m = 1$ in the case $n = 3$ was crucial in the proof of Theorem 3.4 to deduce that $\|y\|_{L^{4m+2}(Q)} < \infty$.*

3.3. Adjoint state equation. Let $u \in \mathcal{U}$ and y_u be the associated state. We denote by φ_u the adjoint state to y_u, which is the solution in Y to

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \varphi}{\partial t} - \Delta\varphi + a\varphi + f'(y_u)\varphi &=& y_u & \text{in } Q, \\ \partial_n \varphi &=& 0 & \text{on } \Sigma. \end{array} \right. \tag{3.8}$$

We observe that for any element $\varphi \in Y$ we have that $\frac{\partial \varphi}{\partial t}, \Delta\varphi \in L^2(Q)$, hence the first equation in (3.8) is interpreted in $L^2(Q)$, and the second in the $L^2(0,\infty;H^{-\frac{1}{2}}(\Gamma))$ sense. Moreover, since $\varphi \in Y$ we recall that $\lim_{t\to\infty} \|\varphi(t)\|_{L^2(\Omega)} = 0$.

THEOREM 3.7. *Equation* (3.8) *has a unique solution $\varphi_u \in Y$ and*

$$\|\varphi_u\|_Y \le C\|y_u\|_{L^2(Q)} \tag{3.9}$$

for some constant independent of $u \in \mathcal{U}$. Moreover the following identity holds

$$\int_Q \varphi_u \Big(\frac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z\Big)\,dx\,dt = \int_Q y_u z\,dx\,dt \quad \forall z \in Y_0, \tag{3.10}$$

where $Y_0 = \{z \in Y : z(0) = 0 \text{ in } \Omega\}$.

Proof. For every $T > 0$ we consider the auxiliary equation

$$\left\{ \begin{array}{rcll} \dfrac{\partial z_T}{\partial t} - \Delta z_T + az_T + f'(y_T)z_T &=& y_T & \text{in } Q_T, \\ \partial_n z_T &=& 0 & \text{on } \Sigma_T, \\ z_T(0) &=& 0 & \text{in } \Omega, \end{array} \right.$$

where $y_T(x,t) = y_u(x,T-t)\ \forall (x,t) \in Q_T$. As in Lemma 3.5 we have that this equation has a unique solution $z_T \in W(0,T)$ and the estimates (3.5)-(3.7) become in this case for every T

$$\|z_T\|_{W(0,T)} + \|f'(y_T)z_T\|_{L^2(Q_T)} \le C\|y_T\|_{L^2(Q_T)} \le C\|y_u\|_{L^2(Q)}.$$

From these estimates and arguing as in the proof of Theorem 2.4 we get

$$\|z_T\|_{L^\infty(0,T;H^1(\Omega))} + \|\frac{\partial z_T}{\partial t}\|_{L^2(Q_T)} \le C'\|y_u\|_{L^2(Q)}.$$

Now we take

$$\varphi_T(x,t) = \left\{ \begin{array}{cl} z_T(x,T-t) & \text{if } t \in [0,T], \\ 0 & \text{if } t > T. \end{array} \right.$$

From the above estimates for z_T we deduce that $\varphi_T \in Y$ and $\|\varphi_T\|_Y \le C''\|y_u\|_{L^2(Q)}$. Moreover φ_T satisfies

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \varphi_T}{\partial t} - \Delta\varphi_T + a\varphi_T + f'(y_u)\varphi_T &=& y_u & \text{in } Q_T, \\ \partial_n \varphi_T &=& 0 & \text{on } \Sigma_T, \\ \varphi_T(T) &=& 0 & \text{in } \Omega. \end{array} \right.$$

Now we take a sequence $T_k \to \infty$ such that $\varphi_{T_k} \rightharpoonup \varphi$ in Y. It is immediate to pass to the limit in the equations satisfied by the functions φ_{T_k} and to deduce that φ is a solution to equation (3.8) and (3.9) is satisfied. To verify (3.10) let us note that

$$\int_Q \big(\frac{\partial \varphi_T}{\partial t} - \Delta\varphi_T + a\varphi_T + f'(y_u)\varphi_T\big) z\,dx\,dt = \int_Q y_u z\,dx\,dt \quad \forall z \in Y_0,$$

which implies that

$$\int_0^T \int_\Omega \varphi_T\big(\frac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z\big)\,dx\,dt = \int_0^T \int_\Omega y_u z\,dx\,dt \quad \forall z \in Y_0.$$

Passing to the limit $T \to \infty$ we obtain (3.10). Finally if φ_u^1 and φ_u^2 are two solutions of the adjoint equation then

$$\int_Q (\varphi_u^1 - \varphi_u^2)\big(\frac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z\big)\,dx\,dt = 0 \quad \forall z \in Y_0.$$

Since the mapping of $z \to \frac{\partial z}{\partial t} - \Delta z + az + f'(y_u)z$ from Y_0 to $L^2(Q)$ is surjective by Lemma 3.5, we obtain that $\varphi_u^1 = \varphi_u^2$. This concludes the proof. □

4. Optimality Conditions and Sparsity. Before establishing the optimality conditions we analyze the cost functional J. We distinguish two terms in the functional: $J(u) = F(u) + \alpha j(u)$ with

$$F(u) = \frac{1}{2}\int_0^\infty \int_\Omega y_u^2\,dx\,dt + \frac{\nu}{2}\int_0^\infty \int_\omega u^2\,dx\,dt \text{ and}$$
$$j(u) = \int_0^\infty \Big(\int_\omega u^2\,dx\Big)^{1/2} dt = \int_0^\infty \|u(t)\|_{L^2(\omega)}\,dt.$$

PROPOSITION 4.1. *The function $F : \mathcal{U} \longrightarrow \mathbb{R}$ is of class C^1 and*

$$F'(u)v = \int_0^\infty \int_\omega (\varphi_u + \nu u)v\,dx\,dt \quad \forall u \in \mathcal{U} \quad \text{and} \quad \forall v \in L^2(0,\infty, L^2(\omega)). \tag{4.1}$$

Proof. With the notation of Theorem 3.4 we have that

$$F(u) = \frac{1}{2}\|G(u)\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(0,\infty;L^2(\omega))}.$$

Hence, we deduce from the chain rule and Theorem 3.4 that F is of class C^1, and

$$F'(u)v = \int_Q y_u z_v\,dx\,dt + \nu\int_0^\infty \int_\omega uv\,dx\,dt.$$

Then, taking $\varphi_u \in Y$ as the solution of (3.8), noting that $z_v \in Y_0$, and using (3.1) and (3.10) we obtain (4.1). □

Now we study the functional $j : L^1(0,\infty;L^2(\Omega)) \longrightarrow \mathbb{R}$. This functional is not differentiable at every point of the domain, but it is convex and Lipschitz. Therefore there exist the directional derivatives $j'(u;v)$ for every $u, v \in L^1(0,\infty;L^2(\Omega))$ and the subdifferential $\partial j(u)$ is nonempty for every u. Let us characterize these objects.

Given an element $u \in L^1(0,\infty;L^2(\omega))$, we denote

$$I_u = \{t \in (0,\infty) : \|u(t)\|_{L^2(\omega)} \neq 0\} \text{ and } I_u^0 = (0,\infty) \setminus I_u.$$

PROPOSITION 4.2. *The following statements hold.*

1. $\lambda \in \partial j(u)$ *is equivalent to* $\lambda \in L^\infty(0,\infty;L^2(\omega))$ *and*

$$\begin{cases} \|\lambda(t)\|_{L^2(\omega)} \le 1 & \text{for a.a. } t \in I_u^0, \\ \lambda(x,t) = \dfrac{u(x,t)}{\|u(t)\|_{L^2(\omega)}} & \text{for a.a. } (x,t) \in \omega \times I_u. \end{cases} \tag{4.2}$$

2. For every $u, v \in L^1(0,\infty;L^2(\omega))$

$$j'(u;v) = \int_{I_u^0} \|v(t)\|_{L^2(\omega)}\,dt + \int_{I_u} \frac{1}{\|u(t)\|_{L^2(\omega)}} \left(\int_\omega uv\,dx \right) dt. \tag{4.3}$$

The reader is referred to [6, Proposition 2.8] for the proof of this result where the role of x and t are reversed.

Now we are prepared to establish the optimality conditions for a local solution of (P) in the sense of $L^2(0,\infty;L^2(\omega))$.

THEOREM 4.3. *Let* $\bar u$ *be a local solution of* (P). *Then there exists* $\bar\lambda \in \partial j(\bar u)$ *such that*

$$\bar\varphi + \nu\bar u + \alpha\bar\lambda = 0 \quad \text{in} \quad \omega \times (0,\infty), \tag{4.4}$$

where $\bar\varphi$ *is the adjoint state associated with* $\bar y = y_{\bar u}$.

Proof. For arbitrary $u \in U = L^2(0,\infty;L^2(\omega)) \cap L^1(0,\infty;L^2(\omega))$ we have with (4.1) and the convexity of j

$$\begin{aligned} 0 &\le \lim_{\rho\searrow 0} \frac{J(\bar u + \rho(u-\bar u)) - J(\bar u)}{\rho} \\ &= \lim_{\rho\searrow 0} \frac{F(\bar u + \rho(u-\bar u)) - F(\bar u)}{\rho} + \alpha \lim_{\rho\searrow 0} \frac{j(\bar u + \rho(u-\bar u)) - j(\bar u)}{\rho} \\ &\le F'(\bar u)(u-\bar u) + \alpha[j(u) - j(\bar u)] = \int_0^\infty \int_\omega (\bar\varphi + \nu\bar u)(u-\bar u) + \alpha[j(u) - j(\bar u)]. \end{aligned}$$

We set $\bar\lambda = -\frac{1}{\alpha}(\bar\varphi + \nu\bar u)\chi_\omega \in L^2(0,\infty;L^2(\omega))$. Then the above inequality implies that

$$\int_0^\infty \int_\omega \bar\lambda(u-\bar u)\,dx\,dt + j(\bar u) \le j(u) \quad \forall u \in U. \tag{4.5}$$

Let us check that $\bar\lambda \in \partial j(\bar u)$. To this end we need to prove that $\bar\lambda \in L^\infty(0,\infty;L^2(\omega))$. We define

$$E = \{t \in (0,\infty) : \|\bar\lambda(t)\|_{L^2(\omega)} > 1\}.$$

We will prove that $|E| = 0$. Since $\bar\lambda \in L^2(0,\infty;L^2(\omega))$ we have that $|E| < \infty$. Set

$$v(x,t) = \begin{cases} \bar\lambda(x,t) & \text{if } t \in E, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad u = \bar u + v \in U.$$

Putting this u in (4.5) we get

$$\begin{aligned} &\int_E \|\bar\lambda(t)\|^2_{L^2(\omega)}\,dt = \int_0^\infty \int_\omega \bar\lambda v\,dx\,dt \\ &= \int_0^\infty \int_\omega \bar\lambda(u-\bar u)\,dx\,dt \le j(u) - j(\bar u) \le j(v) \\ &= \int_0^\infty \|v(t)\|_{L^2(\omega)}\,dt = \int_E \|\bar\lambda(t)\|_{L^2(\omega)}\,dt, \end{aligned}$$

which is a contradiction to the definition of E unless $|E| = 0$. Thus $\|\bar\lambda\|_{L^\infty(0,\infty;L^2(\omega))} \le 1$ holds. Finally, it is enough to use the density of U in $L^1(0,\infty;L^2(\omega))$ to deduce that (4.5) holds $\forall u \in L^1(0,\infty;L^2(\omega))$, and hence $\bar\lambda \in \partial j(\bar u)$. □

COROLLARY 4.4. *Let $(\bar u, \bar\varphi, \bar\lambda)$ be as in Theorem 4.3. Then the following properties hold*

$$\|\bar u(t)\|_{L^2(\omega)} = 0 \Leftrightarrow \|\bar\varphi(t)\|_{L^2(\omega)} \le \alpha, \tag{4.6}$$

$$\bar\lambda(x,t) = \begin{cases} -\dfrac{1}{\alpha}\bar\varphi(x,t) & \text{if } t \in I^0_{\bar u}, \\ \dfrac{\bar u(x,t)}{\|\bar u(t)\|_{L^2(\omega)}} & \text{if } t \in I_{\bar u}, \end{cases} \tag{4.7}$$

$$\bar u(x,t) = -\frac{\|\bar u(t)\|_{L^2(\omega)}\bar\varphi(x,t)}{\nu\|\bar u(t)\|_{L^2(\omega)} + \alpha}, \tag{4.8}$$

for almost every $x \in \omega$. Moreover $\bar\lambda, \bar u \in C([0,\infty);L^2(\omega))$ and

$$\|\bar u\|_{L^\infty(0,\infty;L^2(\omega))} \le \frac{1}{\nu}\|\bar\varphi\|_{L^\infty(0,\infty;L^2(\omega))}. \tag{4.9}$$

Proof. First we observe that (4.7) is an immediate consequence of (4.2) and (4.4). Combining (4.4) and (4.7) we infer

$$\bar\varphi(x,t) + \bar u(x,t)\Big[\nu + \frac{\alpha}{\|\bar u(t)\|_{L^2(\omega)}}\Big] = 0 \quad \text{for a.a. } (x,t) \in \omega \times I_{\bar u}. \tag{4.10}$$

Taking norms in this identity we get

$$\|\bar\varphi(t)\|_{L^2(\omega)} = \nu\|\bar u(t)\|_{L^2(\omega)} + \alpha \quad \text{for a.a. } t \in I_{\bar u}. \tag{4.11}$$

From (4.7) and (4.2) we have that

$$\|\bar\varphi(t)\|_{L^2(\omega)} = \alpha\|\bar\lambda(t)\|_{L^2(\omega)} \le \alpha \quad \text{for a.a. } t \in I^0_{\bar u}.$$

Since $\bar u$ is zero in $I^0_{\bar u}$ we get from above relationship and (4.11)

$$\|\bar u(t)\|_{L^2(\Omega)} = \frac{1}{\nu}\max\{0, \|\bar\varphi(t)\|_{L^2(\omega)} - \alpha\} \quad \text{for a.a. } t \in (0,\infty). \tag{4.12}$$

From here (4.6) follows. Moreover, since $\bar\varphi \in C([0,\infty;L^2(\Omega))$ we deduce that the function $t \in [0,\infty) \to \|\bar u(t)\|_{L^2(\omega)} \in \mathbb{R}$ is continuous. Now, from (4.10) we obtain that (4.8) holds in $I_{\bar u}$. But, taking into account that $\bar u$ is zero in $I^0_{\bar u}$, we conclude that the identity (4.8) holds a.e. in $[0,\infty)$. Additionally, the continuity of $t \in [0,\infty) \to \|\bar u(t)\|_{L^2(\omega)} \in \mathbb{R}$ and the property $\bar\varphi \in C([0,\infty);L^2(\Omega))$ imply that $\bar u \in C([0,\infty);L^2(\omega))$. From (4.4) the same regularity follows for $\bar\lambda$. Finally, (4.9) is an immediate consequence of (4.8). □

Remark 4.5. *Let us observe that $\bar\varphi \in Y$ and consequently*

$$\lim_{t\to\infty}\|\bar\varphi(t)\|_{L^2(\omega)} \le \lim_{t\to\infty}\|\bar\varphi(t)\|_{L^2(\Omega)} = 0.$$

Hence there exists $T_\alpha > 0$ such that $\|\bar\varphi(t)\|_{L^2(\omega)} \le \alpha$ for all $t \ge T_\alpha$. Then (4.6) implies that $\bar u(x,t) = 0$ at least for all $t \ge T_\alpha$. This proves the sparsity of the optimal control.

REFERENCES

[1] V. Barbu and G. Wang, *Internal stabilization of semilinear parabolic systems*, J. Math. Anal. Appl., 285 (2003), pp. 387–407.

[2] ———, *Feedback stabilization of semilinear heat equations*, Abstr. Appl. Anal., (2003), pp. 697–714.

[3] A. J. V. Brandão, E. Fernández-Cara, P. M. D. Magalhães, and M. A. Rojas-Medar, *Theoretical analysis and control results for the FitzHugh-Nagumo equation*, Electron. J. Differential Equations, (2008), pp. No. 164, 20.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[5] E. Casas, C. Clason, and K. Kunisch, *Parabolic control problems in measure spaces with sparse solutions*, SIAM J. Control Optim., 51 (2013), pp. 28–63.

[6] E. Casas, R. Herzog, and G. Wachsmuth, *Analysis of spatio-temporally sparse optimal control problems of semilinear parabolic equations*, ESAIM Control Optim. Calc. Var., to appear (2015).

[7] E. Casas and K. Kunisch, *Parabolic Control Problems in Space-Time Measure Spaces*, ESAIM Control Optim. Calc. Var., 22 (2016). pp. 355–370.

[8] E. Casas, C. Ryll, and F. Tröltzsch, *Second order and stability analysis for optimal sparse control of the FitzHugh-Nagumo equation*, SIAM J. Control Optim., 53 (2015), pp. 2168–2202.

[9] R. Herzog, G. Stadler, and G. Wachsmuth, *Directional sparsity in optimal control of partial differential equations*, SIAM J. Control Optim., 50 (2012), pp. 943–963.

[10] K. Kunisch, K. Pieper, and B. Vexler, *Measure valued directional sparsity for parabolic optimal control problems*, SIAM J. Control Optim., 52 (2014), pp. 3078–3108.

[11] O. Ladyzhenskaya, V. Solonnikov, and N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, 1988.

[12] J. D. Murray, *Mathematical biology. I. , An introduction*, Interdisciplinary applied mathematics, Springer, New York, 2002.

[13] C. Nagaiah, K. Kunisch, and G. Plank, *Optimal control approach to termination of re-entry waves in cardiac electrophysiology*, J. Math. Biol., 67 (2013), pp. 359–388.

[14] F. Schlögl, *A characteristic critical quantity in nonequilibrium phase transitions*, Z. Phys. B: Condens. Matter, 52 (2004), pp. 51–60.

[15] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, vol. 49 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1997.

[16] M. Sîrbu, *Infinite time horizon optimal control of the semilinear heat equation*, Nonlinear Funct. Anal. Appl., 7 (2002), pp. 69–83.