Almost global existence of weak solutions for the nonlinear elastodynamics system with general strain energy

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Abstract

The aim of this paper is to prove the existence of almost global weak solutions for the unsteady nonlinear elastodynamics system in dimension $d = 2$ or 3, for a range of strain energy density functions satisfying some given assumptions. These assumptions are satisfied by the main strain energies generally considered. The domain is assumed to be bounded, and mixed boundary conditions are considered. Our approach is based on a nonlinear parabolic regularization technique, involving the p-Laplace operator. First we prove the existence of a local-in-time solution for the regularized system, by a fixed point technique. Next, using an energy estimate, we show that if the data are small enough, the maximal time of existence does not depend on the parabolic regularization parameter. The solution is thus obtained by passing this parameter to zero. The key point of our proof is due to the recent nonlinear Korn's inequality proven by Ciarlet & Mardare in $\mathrm{W}^{1,p}$ spaces, for $p > 2$.

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Contents

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1 Introduction

1.1 The model

The elastodynamics system we consider in this paper is a hyperbolic partial differential equation combined with boundary conditions – when the domain has a boundary – and initial conditions, whose unknown is the displacement inside a deformable body. We denote by $u(\cdot,t)$ the displacement field at time t with respect to the reference configuration represented by a bounded domain Ω of $\mathbb{R}^d$ (d = 2 or 3). It is assumed to obey the laws of elasticity (see [Cia88] for instance). The density of the body in the reference configuration is denoted by ρ. It is positive, and for a sake of simplicity, we assume it to be constant. We further assume that the boundary of the domain is split into two parts denoted by Γ_D and Γ_N. For mathematical convenience, we will consider that the displacement is null on Γ_D, and that the Lebesgue measure of this boundary is not equal to zero, namely: $|\Gamma_D| > 0$.

Mixed boundary conditions are considered on $\partial\Omega = \Gamma_D \cup \Gamma_N$. A homogeneous Dirichlet condition is imposed on Γ_D, and a non-homogeneous Neumann-type boundary condition is considered on Γ_N. For $0 < T \leq \infty$, the system governing the evolution of the displacement u is the following:

$$\left\{\begin{array}{rl} \rho\ddot{u} - \mathrm{div}((\mathrm{I}+\nabla u)\Sigma(u)) = f & \text{in } \Omega\times(0,T), \\ u = 0, & \text{on } \Gamma_D\times(0,T), \\ (\mathrm{I}+\nabla u)\Sigma(u)n = g & \text{on } \Gamma_N\times(0,T), \\ u(\cdot,0) = u_0, \quad \dot{u}(\cdot,0) = u_1 & \text{in } \Omega. \end{array}\right. \tag{1}$$

In this system, the symbol I denotes the identity matrix of $\mathbb{R}^{d\times d}$, and Σ denotes the so-called second Piola-Kirchhoff stress tensor, namely the derivative of the strain energy density function $\mathcal{W}$ with respect to the Green–St-Venant strain tensor E:

$$\Sigma(u) = \frac{\partial\mathcal{W}}{\partial E}(E(u)), \qquad E(u) = \frac{1}{2}\left((\mathrm{I}+\nabla u)^T(\mathrm{I}+\nabla u) - \mathrm{I}\right).$$

For the choice of the strain energy, we can consider, for instance, the example of the St-Venant–Kirchhoff model

$$\mathcal{W}(E) \quad = \quad \mu_L \mathrm{tr}(E^2) + \frac{\lambda_L}{2}\mathrm{tr}(E)^2,$$

where μ_L and λ_L denote the classical Lamé coefficients. The functions u_0, u_1, f and g are data of the problem.

1.2 Main result

The question of local-in-time existence for the elastodynamics system has been first addressed in [HKM76], for data reduced to initial conditions, and then in [ST88] for small Neumann data, both in the framework of strong solutions. A negative answer about the question of global existence has been given in [KP79], and blow-up of strong solutions has been proven in [GK08], under particular assumptions on the strain energy density function. The global existence of large rigid displacements (but in the context of linearized elasticity) has been obtained in [GMM02] for small data, and in [GMM07] for small strains. Almost global existence, that is to say finite time of existence with large bounds on the data, has been obtained in [JT08] for the St-Venant–Kirchhoff model, and in [LSZ15] in the context of incompressible materials. For incompressible materials, the literature for global existence is abundant. Let us mention, for instance, the works of [Ebi93, Ebi96, Tho03, ST05, ST07], in the case of small data, and more recently the result of [Yin16] for an elastodynamics system written in Eulerian formulation. Finally, we mention the paper of [ZY09] where a locally distributed dissipation is added to the model, in order to stabilize the system.

As far as we know, no result concerning the global-in-time existence of solutions for the elastodynamics system, in the case of general strain energy, has been obtained until now. Besides the complexity due to the nonlinearity of this system, the main difficulty lies in the control, by the total strain energy, of the gradient of the displacement. This difficulty can be now addressed thanks to the recent nonlinear Korn's inequalities proved in [CM15], and also in [Mus16]. More specifically, if $\det(\mathrm{I}+\nabla u) > 0$ almost everywhere in Ω, the inequality given in the part (b) of Theorem 3 of [CM15] yields in particular that

$$\|u\|^p_{[\mathrm{W}^{1,p}(\Omega)]^d} \quad \leq \quad C\|E(u)\|^{p/2}_{[L^{p/2}(\Omega)]^{d\times d}},$$

where $p > 2$, and where the constant $C > 0$ does not depend on $u \in \left[\mathrm{W}^{1,p}(\Omega)\right]^d$. In the example of the St-Venant–Kirchhoff model, the total strain energy on Ω controls the L^2-norm of the tensor E, and thus for this case the exponent $p = 4$ is well-chosen. This Korn's inequality is the key point leading to the main result of our work, namely:

Theorem 1. *Let Ω be a bounded domain of $\mathbb{R}^d$ with $d = 2$ or 3. Assume that its boundary $\partial\Omega = \Gamma_D \sqcup \Gamma_N$ is Lipschitz, and that Γ_D is non-empty and relatively open in $\partial\Omega$. Let be $p > 2$, $p \geq d$, and define $p' = p/(p-1)$. Assume that there exists $C > 0$ such that, for all $E \in [\mathrm{L}^{p/2}(\Omega)]^{d\times d}$, the total strain energy satisfies*

$$\int_\Omega \mathcal{W}(E)\,\mathrm{d}\Omega \quad \geq \quad C\|E\|^{p/2}_{[\mathrm{L}^{p/2}(\Omega)]^{d\times d}}.$$

Assume further that $\mathcal{W}$ is of class $\mathcal{C}^1$ on $\left[\mathrm{L}^{p/2}(\Omega)\right]^{d\times d}$. Denoting by $\check{\Sigma}$ its differential and by E the Green–St-Venant tensor, we assume that the tensor field $\check{\Sigma}\circ E$ is symmetric and locally α-Hölderian on $\mathrm{L}^p\left(0,T;[\mathrm{W}^{1,p}]^d\right)$ for all $T > 0$, with $\alpha = \min(1,(p-2)/2)$. Assume that $u_0 \in [\mathrm{W}^{1,p}(\Omega)]^d$, $u_{0|\Gamma_D} \equiv 0$ and that $\det(\mathrm{I} + \nabla u_0) > 0$ almost everywhere in Ω. Let be $T > 0$. Then there exists a constant $C(T) > 0$ such that, if

$$\int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega + \|u_1\|_{[\mathrm{L}^2(\Omega)]^d} + \|f\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^d} + \|g\|_{\mathrm{L}^2(0,T;[\mathrm{H}^{1/2}(\Gamma_N)']^d)} \quad \leq \quad C(T),$$

then system (1) *admits a solution u such that*

$$u \in \mathrm{L}^\infty(0,T;[\mathrm{W}^{1,p}(\Omega)]^d), \quad \dot{u} \in \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^d), \quad \ddot{u} \in \mathrm{L}^{p'}(0,T;[\mathrm{W}^{1,p}(\Omega)']^d).$$

The assumption $p \geq d$ is made only for giving a sense in a time continuous space, namely $\mathcal{C}([0,\infty);\mathrm{L}^1(\Omega))$, to the quantity $\det(\mathrm{I}+\nabla u)$, whose the positivity required for the Korn's inequality aforementioned. The smallness assumption on the data is also made in order to take into account this criteria. The assumptions made on the strain energy in this theorem are actually satisfied by three important families of strain energies, namely the St-Venant–Kirchhoff model, the Fung's model (at least a polynomial approximation of this model), and the Ogden's model in some cases. See section 2.3 for more details.

1.3 Strategy

The weak solution whose existence is proven in this paper is obtained by a parabolic regularization technique. The parabolic term we add to the elastodynamics system is the p-Laplace operator, in order to obtain the regularity of the time-derivative of the displacement in $[\mathrm{W}^{1,p}(\Omega)]^d$. The study of an evolutionary p-Laplace system enables us to define a mapping whose a fixed point is a weak solution of the so regularized elastodynamics system. For T small enough, and under assumptions on the differential of the strain energy density function, by the Schauder's theorem we prove that this mapping admits a fixed point, and thus the existence of a local-in-time solution follows for the regularized elastodynamics system. Next, an estimate on the energy of the regularized system is obtained. Assuming that the total strain energy can control the norm of the Green–St-Venant tensor in $[\mathrm{L}^{p/2}(\Omega)]^{d\times d}$, we can thus control the gradient of the displacement in $[\mathrm{L}^p(\Omega)]^d$, thanks to the aforementioned nonlinear Korn's inequality. Furthermore, the energy estimate then shows that the maximal time of existence of the weak solution of the regularized system does not depend on the regularization coefficient, provided that the data are small enough. We can thus allow this parameter to tend to zero, and extract a solution by weak-* convergence. The solution obtained by this particular means is unique for the original hyperbolic system, but not unique in general. The general uniqueness could perhaps be proven in some particular cases, like for the model of St-Venant–Kirchhoff, under some additional regularity property on the displacement (see the coerciveness result in [CM15], part (b) of Theorem 3). The question of uniqueness remains open in the general case.

The paper is organized as follows. The functional framework and notation are introduced in section 2. In particular, assumptions are made on the type of strain energies we can consider in this paper, underlined by the study of classical examples. A preliminary result lies in the study of an evolutionary p-Laplace system in section 3. Section 4 is devoted to the proof of the existence of a local-in-time weak solution for the regularized elastodynamics system by a fixed point method. The question of global existence for system (1) is addressed in section 5.

2 Preliminaries

In the whole paper, we denote by Ω a bounded domain with Lipschitz boundary of $\mathbb{R}^d$, $d = 2$ or 3. We denote by Γ_D a non-empty relatively open subset of $\partial\Omega$, and we define $\Gamma_N := \partial\Omega \setminus \Gamma_D$. The first and second time-derivatives of a vector field v will be denoted by $\dot{v}$ and $\ddot{v}$, respectively. For all $q > 1$, we denote by $q' = q/(q-1)$ its conjugate number.

2.1 Functional settings

Throughout the paper, we will use the Hölder's inequality: Denoting by $\mathcal{S}$ a measure space, for $1 \leq p, q \leq \infty$, and $1/r = 1/p + 1/q$, if $f \in \mathrm{L}^p(\mathcal{S})$ and $g \in \mathrm{L}^q(\mathcal{S})$, then

$$\|fg\|_{\mathrm{L}^r(\mathcal{S})} \quad \leq \quad \|f\|_{\mathrm{L}^p(\mathcal{S})}\|g\|_{\mathrm{L}^q(\mathcal{S})}.$$

In particular, when the domain Ω is bounded, the embeddings $\mathrm{L}^r(\Omega) \hookrightarrow \mathrm{L}^s(\Omega)$ are continuous for $1 \leq s \leq r \leq \infty$. For $q > 1$, we consider multi-dimensional Sobolev spaces, by using the notation

$$\mathbf{L}^q(\Omega) = \left[\mathrm{L}^q(\Omega)\right]^d, \qquad \mathbb{L}^q(\Omega) = \left[\mathrm{L}^q(\Omega)\right]^{d\times d},$$
$$\mathbf{W}^q(\Omega) = \left[\mathrm{W}^q(\Omega)\right]^d, \qquad \mathbb{W}^q(\Omega) = \left[\mathrm{W}^q(\Omega)\right]^{d\times d}.$$

The classical inner product for tensors in $\mathbb{R}^{d\times d}$ is denoted by $A : B = \mathrm{tr}(A^T B)$, and the associated norm is given by $|A|^2 = \mathrm{tr}(A^T A)$. Recall that it satisfies $|AB| \leq |A||B|$ for all $A,\ B \in \mathbb{R}^d$. For $p > 1$, we define

$$\mathbf{W}^{1,p}_{0,D}(\Omega) \quad := \quad \left\{ v \in \mathbf{W}^{1,p}(\Omega),\ v_{|\Gamma_D} = 0 \right\}.$$

In order to define a specific norm for $\mathbf{W}^{1,p}_{0,D}(\Omega)$, we first write

$$\|v\|_{\mathbf{W}^{1,p}(\Omega)} \quad = \quad \|v\|_{\mathbf{L}^p(\Omega)} + \|\nabla v\|_{\mathbb{L}^p(\Omega)},$$

for $v \in \mathbf{W}^{1,p}(\Omega)$. From the Rellich-Kondrachov theorem, the embedding $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$ is compact, and since the operator $v \mapsto \nabla v$ is injective from $\mathbf{W}^{1,p}_{0,D}(\Omega)$ to $\mathbb{L}^p(\Omega)$, the Petree-Tartar lemma (see [EG04], Lemma A.38 page 469) enables us to endow the space $\mathbf{W}^{1,p}_{0,D}(\Omega)$ with the norm

$$\|v\|_{\mathbf{W}^{1,p}_{0,D}(\Omega)} \quad := \quad \|\nabla v\|_{\mathbb{L}^p(\Omega)},$$

for all $v \in \mathbf{W}^{1,p}_{0,D}(\Omega)$. Finally, for the boundary Γ_N we recall the trace inequality

$$\|v\|_{\mathbf{W}^{1-1/p,p}(\Gamma_N)} \quad \leq \quad C_{p,N}\|v\|_{\mathbf{W}^{1,p}(\Omega)}, \tag{2}$$

where the constant $C_{p,N} > 0$ does not depend on v. For the sake of brevity, we will use the notation

$$\mathbf{V}^p(\Omega) = \mathbf{W}^{1,p}_{0,D}(\Omega), \qquad \mathbf{V}^p(\Gamma_N) = \mathbf{W}^{1-1/p,p}(\Gamma_N).$$

Coerciveness of the Green - St-Venant strain tensor The Green – St-Venant strain tensor is defined by

$$E(v) \quad = \quad \frac{1}{2}\left((\mathrm{I} + \nabla v)^T(\mathrm{I} + \nabla v) - \mathrm{I}\right) = \frac{1}{2}\left(\nabla v + \nabla v^T + \nabla v^T \nabla v\right). \tag{3}$$

Part (b) of Theorem 3 of [CM15] is a Korn type inequality for this nonlinear tensor. It provides coerciveness for this tensor in the space $\mathbb{L}^p(\Omega)$, with respect to $\mathbf{V}^p(\Omega)$. In particular, given $p > 2$, there exists a positive constant $C_K > 0$ such that, for all $v \in \mathbf{V}^p(\Omega)$ satisfying $\det(\mathrm{I} + \nabla v) > 0$ almost everywhere in Ω, the following inequality holds

$$\|v\|^2_{\mathbf{V}^p(\Omega)} \leq \|v\|^2_{\mathbf{W}^{1,p}(\Omega)} \quad \leq \quad C_K\|E(v)\|_{\mathbb{L}^{p/2}(\Omega)}. \tag{4}$$

On the other hand, for $v_1,\ v_2 \in \mathbf{W}^{1,p}(\Omega)$, with the Cauchy-Schwarz inequality it is easy to get the estimate

$$\|E(v)\|_{\mathbb{L}^{p/2}(\Omega)} \quad \leq \quad C\left(1 + \|\nabla v\|_{\mathbb{L}^p(\Omega)}\right)\|\nabla v\|_{\mathbb{L}^p(\Omega)}. \tag{5}$$

Here, as in the rest of the paper, the notation C will define a generic positive constant, independent of T, the unknowns and the data of the problem, except u_0. But it may depend on Ω, Γ_D, Γ_N, $\|u_0\|_{\mathbf{V}^p(\Omega)}$, $\int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega$ and $\frac{\partial \mathcal{W}}{\partial E}(0)$.

2.2 Assumptions on the strain energy density function

Let $p > 2$. The strain energy

$$\mathcal{W}: \ \mathbb{L}^{p/2}(\Omega) \quad \rightarrow \quad \mathrm{L}^1(\Omega)$$

is a positive function of the Green – St-Venant strain tensor E. When this mapping is Gâteaux-differentiable, the derivative of $\mathcal{W} \circ E$ with respect to the displacement u in the direction v can be expressed as

$$\frac{\partial(\mathcal{W} \circ E)}{\partial u}.v \quad = \quad \frac{\partial \mathcal{W}}{\partial E}(E(u)) : (E'(u).v)$$

with $E'(u).v = \frac{1}{2}((\mathrm{I} + \nabla u)^T \nabla v + \nabla v^T (\mathrm{I} + \nabla u)^T)$. If furthermore the mapping $\frac{\partial \mathcal{W}}{\partial E}(E(u))$ defines a symmetric tensor, then this expression reduces to

$$\frac{\partial(\mathcal{W} \circ E)}{\partial u}.v \quad = \quad (\mathrm{I} + \nabla u)\frac{\partial \mathcal{W}}{\partial E}(E(u)) : \nabla v.$$

We assume the following set of hypotheses on the strain energy:

A1 There exists $C > 0$ such that for all $E \in \mathbb{L}^{p/2}(\Omega)$ we have

$$C \|E\|^{p/2}_{\mathbb{L}^{p/2}(\Omega)} \quad \leq \quad \int_\Omega \mathcal{W}(E) \, \mathrm{d}\Omega. \tag{6}$$

A2 The strain energy $\mathcal{W}$ is of class $\mathcal{C}^1$ on $\mathbb{L}^{p/2}(\Omega)$. We denote

$$\check{\Sigma}(E) = \frac{\partial \mathcal{W}}{\partial E}(E) \quad \in \quad \mathcal{L}\left(\mathbb{L}^{p/2}(\Omega); \mathrm{L}^1(\Omega)\right) \simeq \mathbb{L}^{(p/2)'}(\Omega) = \mathbb{L}^{p/(p-2)}(\Omega).$$

We assume that, for each symmetric tensor E, the tensor $\check{\Sigma}(E)$ is symmetric. When the tensor E is expressed as function of a vector field v, through the expressions (3), we will denote $\Sigma(v) := \check{\Sigma}(E(v))$.

A3 The mapping $\mathcal{W}$ is of class $\mathcal{C}^1$ on $\mathrm{L}^{p/2}(0,T;\mathbb{L}^{p/2}(\Omega))$. Moreover, the mapping $\Sigma = \frac{\partial \mathcal{W}}{\partial E} \circ E$ is locally α-sublinear with $\alpha = \min(1, (p-2)/2)$. More precisely, for $T > 0$ and $R(T) > 0$, there exists a positive constant $C_{R(T)} > 0$ such that

$$\|v\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \leq R(T) \quad \Rightarrow \quad \|\Sigma(v)\|_{\mathrm{L}^{(p/2)'}(0,T;\mathbb{L}^{(p/2)'}(\Omega))} \leq C + C_{R(T)} \|v\|^\alpha_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}. \tag{7}$$

Here R and C_R are assumed to be non-decreasing with respect to T and R, respectively.

Inequalities (4) and (5) imply that

$$\begin{aligned}
\|v\|^2_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \quad &\leq \quad C_K \|E(v)\|_{\mathrm{L}^{p/2}(0,T;\mathbb{L}^{p/2}(\Omega))}, \\
\|E(v)\|_{\mathrm{L}^{p/2}(0,T;\mathbf{V}^{p/2}(\Omega))} \quad &\leq \quad C\left(T^{1/p} + \|v\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}\right) \|v\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))},
\end{aligned}$$

for $v, \ v_1, \ v_2 \in \mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))$. Therefore, for $T \leq 1$ for instance (this assumption will be used only for proving the local-in-time result), assumption **A3** is implied by the following one.

A3' The mapping $\mathcal{W}$ is of class $\mathcal{C}^1$ on $\mathbb{L}^{p/2}(\Omega)$. Moreover, its derivative $\check{\Sigma}$ is locally α-sublinear on $\mathbf{V}^p(\Omega)$, with $\alpha = \min(1, (p-2)/2)$. Namely, for $R > 0$, there exists a positive constant $\check{C}_R > 0$, non-decreasing with respect to R such that

$$\|E\|_{\mathbb{L}^{p/2}(\Omega)} \leq R \quad \Rightarrow \quad \|\check{\Sigma}(E)\|_{\mathbb{L}^{(p/2)'}(\Omega)} \leq C + \check{C}_R \|E\|^\alpha_{\mathbb{L}^{p/2}(\Omega)}. \tag{8}$$

Finally, we sum up the hypotheses we make on the other data:

A4 We assume that $u_0 \in \mathbf{V}^p(\Omega)$ satisfies $\det(\mathrm{I} + \nabla u_0) > 0$ almost everywhere in Ω, and that

$$\int_\Omega \mathcal{W}(E(u_0)) \, \mathrm{d}\Omega < \infty, \quad u_1 \in \mathbf{L}^2(\Omega), \quad f \in \mathrm{L}^2_{loc}(0,\infty;\mathbf{L}^2(\Omega)), \quad g \in \mathrm{L}^2_{loc}(0,\infty;\mathbf{V}^2(\Gamma_N)').$$

Assumptions **A1** and **A2** are used in an essential manner in section 5 for energy estimates, as well as assumption **A4**. Assumption **A3** is mainly used in section 4, in the proof of the fixed point method.

Remark 1. *The regularity on Σ postulated in assumption* **A3** *is made in particular in order to have* $(\mathrm{I}+\nabla v)\Sigma(v)$ *in* $\mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))$ *for* $v \in \mathbf{V}^p(\Omega)$ *(see Lemma 2). The nonlinear Korn's inequality given in [CM04] in the case* $p=2$ *would enable us only to consider* $\Sigma(v)$ *in the space* $\mathbb{L}^\infty(\Omega)$*, which is not appropriate in view of the standard examples of strain energies, and leads to difficulties due to lack of reflexivity.*

Remark 2. *In the assumptions* **A3** *and* **A3′***, we distinguish two cases. For the sake of simplicity, let us focus our comments on assumption* **A3′***. First, when* $p \leq 4$*, that is to say* $\alpha = (p-2)/2$*, inequality (8) of assumption* **A3'** *is implied by*

$$\|\check{\Sigma}(E) - \check{\Sigma}(0)\|^{p/(p-2)}_{\mathbb{L}^{p/(p-2)}(\Omega)} \quad \leq \quad \check{C}_R \|E(v)\|^{p/2}_{\mathbb{L}^{p/2}(\Omega)},$$

and even more so it is satisfied when $\check{\Sigma}$ *is assumed to be locally* $(p-2)/2$*-Hölderian. Here we use that* $(p/2)/(p/2)' = \alpha$*. Secondly, when* $p \geq 4$*, and when the mapping* $\check{\Sigma}$ *is of class* $\mathcal{C}^1$*, with*

$$\frac{\partial \check{\Sigma}}{\partial E}(E) \quad \in \quad \mathcal{L}\left(\mathbb{L}^{p/2}(\Omega); \mathbb{L}^{(p/2)'}(\Omega)\right) \simeq \mathbb{L}^{p/(p-4)}(\Omega),$$

from the mean value theorem the mapping $\check{\Sigma}$ *is locally Lipschitz, and thus this assumption is automatically satisfied:*

$$\|\check{\Sigma}(E)\|_{\mathbb{L}^{(p/2)'}(\Omega)} \quad \leq \quad \|\check{\Sigma}(0)\|_{\mathbb{L}^{p/2)'}(\Omega)} + \sup_{\|E\|_{\mathbb{L}^{p/2}(\Omega)} \leq R} \left(\left\| \frac{\partial \check{\Sigma}}{\partial E}(E) \right\|_{\mathbb{L}^{p/(p-4)}(\Omega)} \right) \|E\|_{\mathbb{L}^{p/2}(\Omega)}.$$

2.3 Examples of strain energy density functions

Let us mention some models of strain energy, and see if the assumptions **A1** – **A3** are satisfied for these examples. We refer to [Cia88] (section 4.10, page 183) or [FFP79] for more comments on the models addressed below. Implicitly, in the expressions below we assume that the tensor E is symmetric.

The St-Venant – Kirchhoff. It corresponds to the following strain energy

$$\mathcal{W}_1(E) \quad = \quad \mu_L \mathrm{tr}\left(E^2\right) + \frac{\lambda_L}{2}\mathrm{tr}(E)^2,$$

where $\mu_L > 0$ and $\lambda_L \geq 0$ are the so-called Lamé coefficients. Here, the exponent $p = 4$ is well-fitted, because in this case $p/2 = p/(p-2) = 2$, and we can estimate easily

$$\begin{aligned}
\int_\Omega \mathcal{W}_1(E)\,\mathrm{d}\Omega \quad &\geq \quad C\|E\|^2_{\mathbb{L}^2(\Omega)}, \\
\check{\Sigma}_1(E) := \frac{\partial \mathcal{W}_1}{\partial E}(E) \quad &= \quad 2\mu_L E + \lambda_L \mathrm{tr}(E)\mathrm{I}, \\
\|\check{\Sigma}_1(E)\|_{\mathbb{L}^2(\Omega)} \quad &\leq \quad C\|E\|_{\mathbb{L}^2(\Omega)}.
\end{aligned}$$

Thus the assumptions **A1** – **A3** (and even **A3′**) are verified for this example.

The Fung's model. It corresponds to the following strain energy

$$\mathcal{W}_2(E) \quad = \quad \mathcal{W}_2(0) + \beta\left(\exp\left(\gamma\ \mathrm{tr}(E^2)\right) - 1\right),$$

where $\mathcal{W}_2(0) \geq 0$, $\beta > 0$ and $\gamma > 0$ are given coefficients. We only know that the space $\mathrm{W}^{s,q}(\Omega)$ is invariant under composition of the exponential function if $s \geq 1$, for certain values of q (see [BB74], Lemma A.2. page 359). Therefore, in our context where E is considered only in $\mathbb{L}^{p/2}(\Omega)$, we need to simplify this model. We approximate this energy by the following one

$$\mathcal{W}_2^N(E) \quad = \quad \mathcal{W}_2(0) + \beta \sum_{k=1}^{N} \frac{\gamma^k \mathrm{tr}(E^2)^k}{k!},$$

where $2 \leq N \in \mathbb{N}$ is the degree of approximation of the power series defining the exponential function. Choosing $p = 4N$, we have $(p-2)/2 \geq 1$, $p/(p-4) = N/(N-1)$ and the following estimate holds:

$$\int_\Omega \mathcal{W}_2^N(E)\,\mathrm{d}\Omega \quad \geq \quad C\|E\|^{2N}_{\mathbb{L}^{2N}(\Omega)}.$$

From the identities

$$\begin{aligned}
\check{\Sigma}_2^N(E) := \frac{\partial \mathcal{W}_2^N}{\partial E}(E) &= 2\beta\gamma\left(\sum_{k=0}^{N-1}\frac{\gamma^k\mathrm{tr}(E^2)^k}{k!}\right)E, \\
\frac{\partial \check{\Sigma}_2^N}{\partial E}(E) &= 2\beta\gamma\left(\sum_{k=0}^{N-1}\frac{\gamma^k\mathrm{tr}(E^2)^k}{k!}\right)\mathrm{I} + 4\beta\gamma^2\left(\sum_{k=0}^{N-2}\frac{\gamma^k\mathrm{tr}(E^2)^k}{k!}\right)E^2,
\end{aligned}$$

and from from Remark 2, for $R > 0$ large enough we can estimate

$$\begin{aligned}
\left\|\frac{\partial \check{\Sigma}_2^N}{\partial E}\right\|_{\mathrm{L}^{N/(N-1)}(0,T;\mathbb{L}^{N/(N-1)}(\Omega))} &\leq C\sum_{k=0}^{N-1}\|E\|^{2k}_{\mathrm{L}^{2N}(0,T;\mathbb{L}^{2N}(\Omega))} \leq C\sum_{k=0}^{N-1}R^{2k} \leq CR^{2N-2}, \\
\|\check{\Sigma}_2^N(E)\|_{\mathbb{L}^{2N/(2N-1)}(\Omega)} &\leq \|\check{\Sigma}_2^N(0)\|_{\mathbb{L}^{2N/(2N-1)}(\Omega)} + CR^{2N-2}\|E\|_{\mathbb{L}^{2N}(\Omega)}.
\end{aligned}$$

Assumptions **A1** – **A3′** are thus verified for this approximation of the Fung's model.

The Ogden's model. The family of strain energies corresponding to this model are linear combinations of energies of the following form

$$\mathcal{W}_3(E) \quad = \quad \mathrm{tr}\left((2E+\mathrm{I})^\gamma - \mathrm{I}\right),$$

where $\gamma \in \mathbb{R}$. Since the tensor $2E+\mathrm{I}$ is real and symmetric, the expression $(2E+\mathrm{I})^\beta$ makes sense for all number $\beta \in \mathbb{R}$, and the energy $\mathcal{W}_3(E)$ can be expressed in terms of the eigenvalues of $2E+\mathrm{I}$. This general form of the strain energy includes the cases of the Neo-Hookean and Mooney-Rivlin models ($\gamma = 1$ and $\gamma \in \{-1,+1\}$ respectively). But here we only evoke the case $\gamma > 1$. First, since $2E(u)+\mathrm{I} = (\mathrm{I}+\nabla u)^T(\mathrm{I}+\nabla u)$, if $(\lambda_i)_{1\leq i\leq d}$ denote the singular values of $\mathrm{I}+\nabla u$, and $(\mu_i)_{1\leq i\leq d}$ denote those of $E(u)$, we have

$$\mathrm{tr}\left((2E+\mathrm{I})^\gamma - \mathrm{I}\right) = \sum_{i=1}^d\left(\lambda_i^{2\gamma}-1\right) = \sum_{i=1}^d\left((1+2\mu_i)^\gamma - 1\right) \geq \sum_{i=1}^d(2\mu_i)^\gamma \quad \geq \quad C\left(\sum_{i=1}^d\mu_i^2\right)^{\gamma/2} = C\,|E|^\gamma,$$

because of the equivalence of norms in $\mathbb{R}^d$. Thus, by choosing $p = 2\gamma > 2$, we have

$$\int_\Omega \mathcal{W}_3(E)\,\mathrm{d}\Omega = \|2E+\mathrm{I}\|^{p/2}_{\mathbb{L}^{p/2}(\Omega)} - d|\Omega| \quad \geq \quad C\|E\|^{p/2}_{\mathbb{L}^{p/2}(\Omega)},$$

that is to say **A1** holds, and **A2** can be easily checked. For the assumptions **A3** and **A3′**, since the derivative of $\mathcal{W}_3$ is given by $\check{\Sigma}_3(E) = 2\gamma(2E+\mathrm{I})^{\gamma-1}$, the case $\gamma \geq 2$ can be treated as previously. Turning to $1 < \gamma < 2$, we have that $(p-2)/2 = \gamma - 1 \in (0,1)$. Since the derivative of $\mathcal{W}_3$ writes $\check{\Sigma}_3(E) = 2\gamma(2E+\mathrm{I})^{\gamma-1}$. Since the function $x \mapsto x^{\gamma-1}$ is $(\gamma-1)$-Hölderian on $[0,1]$, we deduce that for E small enough in $\mathrm{L}^\gamma(0,T;\mathbb{L}^\gamma(\Omega))$, namely $R(T)$ small enough, the following estimate holds

$$\|\check{\Sigma}_3(E)\|_{\mathrm{L}^{\gamma/(\gamma-1)}(0,T;\mathbb{L}^{\gamma/(\gamma-1)}(\Omega)} \quad \leq \quad T^{1-1/\gamma}\|\check{\Sigma}_3(0)\|_{\mathrm{L}^{\gamma/(\gamma-1)}(0,T;\mathbb{L}^{\gamma/(\gamma-1)}(\Omega)} + C_{R(T)}\|E\|^{\gamma-1}_{\mathrm{L}^\gamma(0,T;\mathbb{L}^\gamma(\Omega))}.$$

We will see in section 4 (more precisely Lemma 1) that the constant $R(T)$ of assumption **A3** can be chosen small enough, provided that T is chosen small enough. Thus, for the Ogden's model with a coefficient $\gamma > 1$, assumptions **A1**–**A3** are satisfied.

3 A nonlinear parabolic system

In this section, we are interested in the following nonlinear parabolic system

$$\left\{\begin{array}{rll}\rho\dot{w}-\kappa\,\mathrm{div}(|\nabla w|^{p-2}\nabla w)=f & & \text{in } \Omega\times(0,T),\\ w=0 & & \text{on } \Gamma_D\times(0,T),\\ \kappa|\nabla w|^{p-2}\nabla w\, n=g & & \text{on } \Gamma_N\times(0,T),\\ w(\cdot,0)=u_1 & & \text{in } \Omega,\end{array}\right. \tag{9}$$

where $\kappa>0$, $p\geq 2$, and $T>0$ is fixed and arbitrary. System (9) is an evolutionary p-Laplace equation with mixed boundary conditions. Throughout it is assumed that

$$u_1\in\mathbf{L}^2(\Omega),\quad f\in\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)'),\quad g\in\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)').$$

Definition 1. *We say that w is a weak solution of system* (9) *if $w\in\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))$, $\dot{w}\in\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')$, $w(0)=u_1$, and for all $\varphi\in\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))$ we have*

$$\rho\langle\dot{w};\varphi\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)}+\kappa\int_\Omega|\nabla w|^{p-2}\nabla w:\nabla\varphi\,\mathrm{d}\Omega\quad=\quad\langle f;\varphi\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)}+\langle g;\varphi\rangle_{\mathbf{V}^p(\Gamma_N)';\mathbf{V}^p(\Gamma_N)},$$

almost everywhere in $(0,T)$.

Remark 3. *By definition, a weak solution of system* (9) *lies in the space $W(0,T;\mathbf{W}^{1,p}(\Omega))$ defined by*

$$w\in W(0,T;\mathrm{V}^p(\Omega))\quad\Leftrightarrow\quad\left\{\begin{array}{l}w\in\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))\\ \dot{w}\in\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')\end{array}\right.,$$

corresponding to the Gelfand triplet $V\hookrightarrow H\equiv H'\hookrightarrow V'$, with $H=\mathbf{L}^2(\Omega)$, $V=\mathbf{V}^p(\Omega)$, and dense embeddings. So it is well-known that such a solution lies also in $C([0,T];\mathbf{L}^2(\Omega))$, and thus the space $\mathbf{L}^2(\Omega)$ in which the initial condition is considered makes sense.

Proposition 1. *System* (9) *admits a unique weak solution w, in the sense of definition 1. Moreover, it satisfies the estimate*

$$\begin{aligned}\|\dot{w}\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')}+\|w\|^p_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}\quad\leq\quad & C\Big(\|u_1\|^2_{\mathbf{L}^2(\Omega)}+\\ & +\|f\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')}+\|g\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')}\Big),\end{aligned} \tag{10}$$

where the constant C depends only on ρ, κ and Ω.

Proof. System (9) is a p-Laplace evolution problem. Existence and uniqueness of a weak solution for this system have been proven in [BB69] for instance (see the example 1, p. 391-392, consequence of Theorem V.3. p. 387). Uniqueness is due to the convexity of the function $v\mapsto|v|^p$. Let us prove the announced estimate, which will be obtained with standard arguments. Taking the inner product of the first equation of (9) by w and integrating on Ω yields, with the Green formula

$$\begin{aligned}\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|^2_{\mathbf{L}^2(\Omega)}\right)+\kappa\|\nabla w\|^p_{\mathbf{L}^p(\Omega)}\quad&=\quad\langle f;w\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)}+\langle g;w\rangle_{\mathbf{V}^p(\Gamma_N)';\mathbf{V}^p(\Gamma_N)},\\ &\leq\quad\|f\|_{\mathbf{V}^p(\Omega)'}\|w\|_{\mathbf{V}^p(\Omega)}+\|g\|_{\mathbf{V}^p(\Gamma_N)'}\|w\|_{\mathbf{V}^p(\Gamma_N)}.\end{aligned}$$

Keep in mind the identity $\|\nabla w\|_{\mathbf{L}^p(\Omega)}=\|w\|_{\mathbf{W}^{1,p}(\Omega)}$, and recall the trace inequality $\|w\|_{\mathbf{V}^p(\Gamma_N)}\leq C_{p,N}\|w\|_{\mathbf{V}^p(\Omega)}$, where $C_{p,N}>0$ depends only on Γ_N, Ω and p. Next, integrating this inequality in time between 0 and T gives, with the Young's inequality involving some $\alpha>0$,

$$\begin{aligned}\frac{\rho}{2}\|w(T)\|^2_{\mathbf{L}^2(\Omega)}+\kappa\int_0^T\|\nabla w(t)\|^p_{\mathbf{L}^p(\Omega)}\mathrm{d}t+\quad\leq\quad&\frac{\rho}{2}\|u_1\|^2_{\mathbf{L}^2(\Omega)}+\\ &\frac{\alpha^{-p'}}{p'}\int_0^T\|f(t)\|^{p'}_{\mathbf{V}^p(\Omega)'}\mathrm{d}t+\frac{\alpha^p}{p}\int_0^T\|w(t)\|^p_{\mathbf{V}^p(\Omega)}\mathrm{d}t+\\ &\frac{\alpha^{-p'}}{p'}\int_0^T\|g(t)\|^{p'}_{\mathbf{V}^p(\Gamma_N)'}\mathrm{d}t+\frac{(\alpha C_{p,N})^p}{p}\int_0^T\|w(t)\|^p_{\mathbf{V}^p(\Omega)}\mathrm{d}t.\end{aligned}$$

Choose $\alpha > 0$ small enough, such that $(\alpha^p + (\alpha C_{p,N})^p)/p \leq \kappa/2$, and we obtain

$$\frac{\kappa}{2}\|w\|^p_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \leq \frac{\rho}{2}\|w(0)\|^2_{\mathbf{L}^2(\Omega)} + C\left(\|f\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|g\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')}\right).$$

For the estimate on the time-derivative, due to the equality $\rho\dot{w} = f + \kappa \operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right)$ it is sufficient to control the term $\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right)$ in the space $\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')$. First, it is easy to verify that $|\nabla w|^{p-2}\nabla w$ lies in $\mathrm{L}^{p'}(0,T;\mathbf{L}^{p'}(\Omega))$. More specifically, we have

$$\begin{aligned}
\||\nabla w|^{p-2}\nabla w\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{L}^{p'}(\Omega))} &= \|\nabla w\|^p_{\mathrm{L}^p(0,T;\mathbf{L}^p(\Omega))},\\
\||\nabla w|^{p-2}\nabla w\|_{\mathrm{L}^{p'}(0,T;\mathbf{L}^{p'}(\Omega))} &= \|w\|^{p-1}_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}.
\end{aligned}$$

Therefore, for all $\varphi \in \mathbf{V}^p(\Omega)$, from Hölder's inequality we have

$$\begin{aligned}
\left\langle \operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right);\varphi\right\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)} &= -\int_\Omega |\nabla w|^{p-2}\nabla w : \nabla\varphi \,\mathrm{d}\Omega + \langle g;\varphi\rangle_{\mathbf{V}^p(\Gamma_N)';\mathbf{V}^p(\Gamma_N)},\\
\left|\left\langle \operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right);\varphi\right\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)}\right| &\leq \||\nabla w|^{p-2}\nabla w\|_{\mathbf{L}^{p'}(\Omega))}\|\nabla\varphi\|_{\mathbf{L}^p(\Omega))} + C_{p,N}\|g\|_{\mathbf{V}^p(\Gamma_N)'}\|\varphi\|_{\mathbf{V}^p(\Omega)}\\
&\leq C\left(\|w\|^{p-1}_{\mathbf{V}^p(\Omega)} + \|g\|_{\mathbf{V}^p(\Gamma_N)'}\right)\|\varphi\|_{\mathbf{V}^p(\Omega)},
\end{aligned}$$

and thus, by Young's inequality

$$\begin{aligned}
\left\|\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right)\right\|^{p'}_{\mathbf{V}^p(\Omega)'} &\leq C\left(\|w\|^{p-1}_{\mathbf{V}^p(\Omega)} + \|g\|_{\mathbf{V}^p(\Gamma_N)'}\right)^{p'}\\
&\leq C\left(\|w\|^p_{\mathbf{V}^p(\Omega)} + \|g\|^{p'}_{\mathbf{V}^p(\Gamma_N)'}\right),\\
\left\|\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right)\right\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} &\leq C\left(\|w\|^p_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} + \|g\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')}\right),
\end{aligned}$$

which concludes the proof. $\square$

We now consider system (9) with particular additional right-hand-sides, namely

$$\left\{\begin{array}{rl}
\rho\dot{w} - \kappa\operatorname{div}(|\nabla w|^{p-2}\nabla w) = f + \operatorname{div} A & \text{in } \Omega\times(0,T),\\
w = 0 & \text{on } \Gamma_D\times(0,T),\\
\kappa|\nabla w|^{p-2}\nabla w\, n = g - An & \text{on } \Gamma_N\times(0,T),\\
w(\cdot,0) = u_1 & \text{in } \Omega,
\end{array}\right. \tag{11}$$

where A is a given tensor field.

Definition 2. *Let $A \in \mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))$. We say that w is a weak solution of system (11) if $w \in \mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))$, $\dot{w} \in \mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')$, $w(0) = u_1$, and for all $\varphi \in \mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))$ we have*

$$\begin{aligned}
\rho\langle\dot{w};\varphi\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)} + \kappa\int_\Omega |\nabla w|^{p-2}\nabla w : \nabla\varphi\,\mathrm{d}\Omega &= \langle f;\varphi\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)} + \langle g;\varphi\rangle_{\mathbf{V}^p(\Gamma_N)';\mathbf{V}^p(\Gamma_N)}\\
&\quad - \int_\Omega A : \nabla\varphi\,\mathrm{d}\Omega,
\end{aligned}$$

almost everywhere in $(0,T)$.

From Proposition 1 we can deduce the following result.

Corollary 1. *Let $A \in \mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))$. Then system (11) admits a unique weak solution w, in the sense of Definition 2. Moreover, it satisfies the following estimate*

$$\begin{aligned}
\|\dot{w}\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|w\|^p_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} &\leq C\Big(\|u_1\|^2_{\mathbf{L}^2(\Omega)} + \|f\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|g\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')}\\
&\quad + \|A\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))}\Big),
\end{aligned} \tag{12}$$

where the constant C depends only on ρ, κ and Ω.

Proof. In the variational formulation of system (11), the Green's formula reduces the terms involving A to

$$\langle \operatorname{div} A ; \varphi\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)} - \langle An ; \varphi\rangle_{\mathbf{V}^p(\Gamma_N)';\mathbf{V}^p(\Gamma_N)} \quad = \quad \int_\Omega A : \nabla\varphi \, \mathrm{d}\Omega.$$

From the Hölder's inequality, we have

$$\left| \int_\Omega A : \nabla\varphi \, \mathrm{d}\Omega \right| \quad \leq \quad \|A\|_{\mathbb{L}^{p'}(\Omega)} \|\varphi\|_{\mathbf{V}^p(\Omega)},$$

Therefore the result of Proposition 1 holds in this case. In particular, the steps of the proof of the announced estimate are the same as those given in the proof of Proposition 1. □

4 Local existence by parabolic regularization

We choose $\kappa > 0$, $p > 2$, $f \in \mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')$, $g \in \mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')$, and consider the following system

$$\left\{ \begin{array}{rl} \rho\ddot{u} - \kappa \operatorname{div}\left(|\nabla\dot{u}|^{p-2}\nabla\dot{u}\right) - \operatorname{div}((\mathrm{I}+\nabla u)\Sigma(u)) = f & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \Gamma_D \times (0,T), \\ \kappa\,|\nabla\dot{u}|^{p-2}\,\nabla\dot{u}\,n + (\mathrm{I}+\nabla u)\Sigma(u)\,n = g & \text{on } \Gamma_N\times(0,T), \\ u(\cdot,0) = u_0, \quad \dot{u}(\cdot,0) = u_1 & \text{in } \Omega. \end{array}\right. \tag{13}$$

The definition of a weak solution for this system is inspired by Definition 2, with $\dot{u}$ in the role of w, and $(\mathrm{I}+\nabla u)\Sigma(u)$ in the role of the tensor field A.

Definition 3. *We say that w is a weak solution of system* (13) *if $w \in \mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))$, $\dot{w} \in \mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')$, $w(0) = u_1$, and for all $\varphi \in \mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))$ we have*

$$\left\{ \begin{array}{l} u(\cdot,t) \;:=\; u_0(\cdot) + \displaystyle\int_0^t w(\cdot,s)\mathrm{d}s, \\ \rho\langle\dot{w};\varphi\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)} + \kappa\displaystyle\int_\Omega |\nabla w|^{p-2}\nabla w : \nabla\varphi\,\mathrm{d}\Omega + \int_\Omega (\mathrm{I}+\nabla u)\Sigma(u) : \nabla\varphi\,\mathrm{d}\Omega \;=\; \langle f;\varphi\rangle_{\mathbf{V}^p(\Omega)';\mathbf{V}^p(\Omega)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \langle g;\varphi\rangle_{\mathbf{V}^p(\Gamma_N)';\mathbf{V}^p(\Gamma_N)}, \end{array}\right.$$

almost everywhere in $(0,T)$.

We look for a weak solution of system (13) in the following set

$$\mathcal{B}_R(T) \quad = \quad \left\{ w \in \mathrm{L}^p(0,T;\mathbf{V}^p(\Omega)),\ \|w\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \leq R \right\},$$

where $R > 0$ will be chosen large, and $T > 0$ small enough. This weak solution can be seen as a fixed point of the following mapping

$$\begin{array}{rccc} \mathcal{N}: & \mathcal{B}_R(T) & \rightarrow & \mathrm{L}^p(0,T;\mathbf{L}^p(\Omega)) \\ & \tilde{w} & \mapsto & w \end{array}$$

where $\tilde{w}$ defines

$$\tilde{u}(x,t) \quad := \quad u_0(x) + \int_0^t \tilde{w}(x,s)\mathrm{d}s, \qquad x \in \Omega,$$

and w is the solution – in the sense of Definition 2 – of the following system

$$\left\{ \begin{array}{rl} \rho\dot{w} - \kappa\operatorname{div}(|\nabla w|^{p-2}\nabla w) = f + \operatorname{div}\left((\mathrm{I}+\nabla\tilde{u})\Sigma(\tilde{u})\right) & \text{in } \Omega\times(0,T), \\ w = 0 & \text{on } \Gamma_D\times(0,T), \\ \kappa|\nabla w|^{p-2}\nabla w n = g - (\mathrm{I}+\nabla\tilde{u})\Sigma(\tilde{u})n & \text{on } \Gamma_N\times(0,T), \\ w(\cdot,0) = u_1 & \text{in } \Omega. \end{array}\right. \tag{14}$$

System (14) is of the same type as system (11). From Corollary 1, the solution of this system is well-defined in $\mathrm{W}(0,T;\mathbf{W}^{1,p}(\Omega))$, provided that the right-hand-sides lie in the corresponding spaces, in particular $(\mathrm{I}+\nabla\tilde{u})\Sigma(\tilde{u})$ should lie in $\mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))$. This point is verified in Lemma 2 below.

4.1 Stability estimates

Lemma 1. *Let be $R > 0$, $0 < T \leq 1$ and $w \in \mathcal{B}_R(T)$. Then the function defined by*

$$v(\cdot, t) = u_0(\cdot) + \int_0^t w(\cdot, s)\mathrm{d}s \tag{15}$$

satisfies

$$\|v\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \quad \leq \quad T^{1/p}(\|u_0\|_{\mathbf{V}^p(\Omega)} + R). \tag{16}$$

Proof. From the triangle inequality leading to

$$\|v(t)\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \quad \leq \quad T^{1/p}\|u_0\|_{\mathbf{V}^p(\Omega)} + \left\| \int_0^t w(s)\mathrm{d}s \right\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))},$$

we deduce with the Hölder's inequality

$$\begin{aligned} \|v\|_{\mathrm{L}^p(0,T;\mathbf{V}(\Omega))} \quad &\leq \quad T^{1/p}\|u_0\|_{\mathbf{V}^p(\Omega)} + \left(\int_0^T t^{p/p'} \|w\|^p_{\mathrm{L}^p(0,t;\mathbf{V}^p(\Omega))}\mathrm{d}t \right)^{1/p} \\ &\leq \quad T^{1/p}\|u_0\|_{\mathbf{V}^p(\Omega)} + T\|w\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}, \end{aligned}$$

and then for $T \leq 1$ the result follows. □

In the remainder of this section it is assumed that $T \leq 1$. The following lemma shows that the mapping $\mathcal{N}$ is well-defined. As before $R(T)$ and C_R are non-decreasing with respect to T and R respectively.

Lemma 2. *Assume that* **A3** *is satisfied. Let be $R > 0$ and $0 < T \leq 1$. Then for all $w \in \mathcal{B}_R(T)$, the function v defined by* (15) *satisfies*

$$\|(\mathrm{I} + \nabla v)\Sigma(v)\|_{\mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))} \quad \leq \quad C\left(1 + C_{R(T)}T^{\alpha/p}\right), \tag{17}$$

where $\alpha = \min(1, (p-2)/2)$.

Proof. By Hölder's inequality, we have

$$\begin{aligned} \|(\mathrm{I} + \nabla v)\Sigma(v)\|_{\mathbb{L}^{p'}(\Omega)} \quad &\leq \quad \|(\mathrm{I} + \nabla v)\|_{\mathbb{L}^p(\Omega)}\|\Sigma(v)\|_{\mathbb{L}^{p/(p-2)}(\Omega)}, \\ \|(\mathrm{I} + \nabla v)\Sigma(v)\|_{\mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))} \quad &\leq \quad \|(\mathrm{I} + \nabla v)\|_{\mathrm{L}^p(0,T;\mathbb{L}^p(\Omega))}\|\Sigma(v)\|_{\mathrm{L}^{(p/2)'}(0,T;\mathbb{L}^{(p/2)'}(\Omega))}. \end{aligned}$$

From Lemma 1, we have $\|v\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \leq R(T)$, where $R(T) = T^{1/p}(\|u_0\|_{\mathbf{V}^p(\Omega)} + R)$. Hence, inequality (7) given in assumption **A3** enables us to deduce

$$\begin{aligned} \|(\mathrm{I} + \nabla v)\Sigma(v)\|_{\mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))} \quad &\leq \quad \|(\mathrm{I} + \nabla v)\|_{\mathrm{L}^p(0,T;\mathbb{L}^p(\Omega))}\left(C + C_{R(T)}\|v\|^\alpha_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}\right) \\ &\leq \quad \|(\mathrm{I} + \nabla v)\|_{\mathrm{L}^p(0,T;\mathbb{L}^p(\Omega))}\left(C + C_{R(T)}T^{\alpha/p}\left(\|u_0\|_{\mathbf{V}^p(\Omega)} + R\right)^\alpha\right), \end{aligned}$$

where $\alpha = \min(1, (p-2)/2)$. Moreover by (16) the same estimate holds for

$$\|(\mathrm{I} + \nabla v)\|_{\mathrm{L}^p(0,T;\mathbb{L}^p(\Omega))} \quad \leq \quad C + T^{1/p}\left(\|u_0\|_{\mathbf{V}^p(\Omega)} + R\right),$$

so that we can conclude the proof by noticing that $T \leq 1$ implies $T^{(1+\alpha)/p} \leq T^{1/p} \leq T^{\alpha/p}$. □

4.2 Invariance and relative compactness of $\mathcal{N}$ in $\mathcal{B}_R(T)$

Proposition 2. *There exists $T_0 > 0$ and $R_0 > 0$ such that, for all $T \leq T_0$ and $R \geq R_0$, the set $\mathcal{B}_R(T)$ is invariant under the mapping $\mathcal{N}$. Moreover, the set $\mathcal{N}(\mathcal{B}_R(T))$ is relatively compact in $\mathrm{L}^p(0,T;\mathbf{L}(\Omega))$.*

Proof. Let us begin by proving that $\mathcal{B}_R(T)$ is invariant under $\mathcal{N}$. For $\tilde{w} \in \mathcal{B}_R(T)$, if $\tilde{u}$ denotes the function defined by (15), and if $w = \mathcal{N}(\tilde{w})$, then estimate (12) of Corollary 1 applied to system (14) gives the inequality

$$\begin{aligned}\|\dot{w}\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|w\|^p_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \quad \leq \quad & C\Big(\|u_1\|^2_{\mathbf{L}^2(\Omega)} + \|f\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|g\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')} \\ & + \|(\mathrm{I}+\nabla\tilde{u})\Sigma(\tilde{u})\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbb{L}^{p'}(\Omega))}\Big).\end{aligned}$$

Furthermore, from the estimate (17) we have by the Young's inequality

$$\begin{aligned}\|\dot{w}\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|w\|^p_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \quad \leq \quad & C\Big(\|u_1\|^2_{\mathbf{L}^2(\Omega)} + \|f\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|g\|^{p'}_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')} \\ & + 1 + C_{R(T)}T^{\alpha/(p-1)}\Big).\end{aligned} \tag{18}$$

By choosing any R large enough, for instance

$$R \quad = \quad C\Big(\|u_1\|_{\mathbf{L}^2(\Omega)} + \|f\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|g\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma_N)')} + 3/2\Big),$$

and then T small enough in order to have $C_{R(T)}T^{\alpha/(p-1)} \leq 1/2$, we see that the function w lies in the set $\mathcal{B}_R(T)$, and thus $\mathcal{B}_R(T)$ is invariant under $\mathcal{N}$. Moreover, estimate (18) shows that if a w lies in $\mathcal{N}(\mathcal{B}_R(T))$, then w is bounded in $\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega)$, and $\dot{w}$ is bounded in $\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')$. Since the embedding $\mathbf{V}^p(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$ is compact, the hypotheses of Corollary 4 of [Sim87] (page 85) are satisfied, and then $\mathcal{N}(\mathcal{B}_R(T))$ is relatively compact in $\mathrm{L}^p(0,T;\mathbf{L}^p(\Omega))$. □

It is clear that the set $\mathcal{B}_R(T)$ is a closed convex subset of $\mathrm{L}^p(0,T;\mathbf{L}^p(\Omega))$. The consequence of Proposition 2 is the existence of a fixed point of $\mathcal{N}$, by the Schauder's theorem, which leads to the existence of a local-in-time solution for system (13). We sum up this result as follows.

Theorem 2. *Assume that assumption* **A3** *is satisfied by the strain energy, and that* $u_0 \in \mathbf{V}^p(\Omega)$, $u_1 \in \mathbf{L}^2(\Omega)$, $f \in \mathrm{L}^{p'}(0,T_0;\mathbf{V}^p(\Omega)')$ *and* $g \in \mathrm{L}^{p'}(0,T_0;\mathbf{V}^p(\Gamma)')$ *for some* $T_0 > 0$*. Then, if* T_0 *is small enough, system* (13) *admits a weak solution* w*, in the sense of Definition 3 with* T_0 *in place of* T*. It defines* $u(\cdot,t) = u_0 + \int_0^t w(\cdot,s)\mathrm{d}s$ *which satisfies*

$$u \in \mathrm{W}^{1,p}(0,T_0;\mathbf{V}^p(\Omega)), \qquad \ddot{u} \in \mathrm{L}^{p'}(0,T_0;\mathbf{V}^p(\Omega)').$$

5 Global existence of a weak solution

Throughout this section, we assume that the assumptions **A1**–**A4** are satisfied for the strain energy and the data. Then, in particular, the hypotheses of Theorem 2 hold.

Lemma 3. *For* $0 < T < +\infty$*, if* $v \in \mathrm{W}^{1,p}(0,T;\mathbf{V}^p(\Omega))$ *with* $p \geq d$*, then the function* $\chi : t \mapsto \det(\mathrm{I} + \nabla v(\cdot,t))$ *admits a (uniformly) continuous representative function on* $[0,T]$*, with values in* $\mathrm{L}^1(\Omega)$*. More precisely, for* $t,\ t' \in [0,T]$*, we have*

$$\|\chi(t)-\chi(t')\|_{\mathrm{L}^1(\Omega)} \quad \leq \quad C|t-t'|^{1-1/p}\Big(1+\|\nabla u_\kappa\|^{d-1}_{\mathrm{L}^\infty(0,T_0(\kappa);\mathbb{L}^p(\Omega))}\Big)\|\nabla\dot{u}_\kappa\|_{\mathrm{L}^p(0,T_0(\kappa);\mathbb{L}^p(\Omega))}.$$

In particular, the limit $\lim\limits_{t\mapsto T}\det(\mathrm{I}+\nabla v(\cdot,t))$ *exists in* $\mathrm{L}^1(\Omega)$*.*

Proof. Since $p \geq d$, we have $\chi \in \mathrm{L}^\infty(0,T_0(\kappa);\mathrm{L}^{p/d}(\Omega))$. Furthermore, if $\mathrm{cof}(A)$ denotes the cofactor matrix of a matrix A, we have

$$\begin{aligned}\dot{\chi} \quad &= \quad \mathrm{cof}(\mathrm{I}+\nabla v) : \nabla\dot{v}, \\ \|\dot{\chi}\|_{\mathrm{L}^p(0,T;\mathrm{L}^{p/d}(\Omega))} \quad &\leq \quad \|\mathrm{cof}(\mathrm{I}+\nabla v)\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^{p/(d-1)}(\Omega))}\|\nabla\dot{v}\|_{\mathrm{L}^p(0,T;\mathbb{L}^p(\Omega))} \\ &\leq \quad C\left(1+\|\nabla v\|\right)^{d-1}_{\mathrm{L}^\infty(0,T;\mathbb{L}^p(\Omega))}\|\nabla\dot{v}\|_{\mathrm{L}^p(0,T;\mathbb{L}^p(\Omega))}.\end{aligned}$$

For $t,\ t' \in [0,T_0(\kappa))$, since we have

$$\|\chi(t)-\chi(t')\|_{\mathrm{L}^1(\Omega)} \quad \leq \quad |t-t'|^{1-1/p}\|\dot{\chi}\|_{\mathrm{L}^p(0,T_0(\kappa);\mathrm{L}^1(\Omega))},$$

the result follows. □

Proposition 3. *Let $T_0 > 0$ be the time of existence provided by Theorem 2, of a local-in-time solution $\dot{u}$ for system* (13) *on $(0, T_0)$, in the sense of Definition 3. Assume that there exists $\eta \in \mathrm{L}^1(\Omega)$, such that for all $t \in [0, T_0]$*

$$\det(\mathrm{I} + \nabla u(\cdot, t)) \geq \eta > 0, \qquad \textit{almost everywhere in } \Omega. \tag{19}$$

Then, for all $t \in [0, T_0]$, the following energy estimate holds:

$$\begin{aligned} & \|\dot{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|u(t)\|^p_{\mathbf{V}^p(\Omega)} + \kappa \int_0^t \|\dot{u}(s)\|^p_{\mathbf{V}^p(\Omega)} \mathrm{d}s \\ \leq \quad & C_0 \exp(C_0 t) \left(\int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega + \|u_1\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|f(s)\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}s + \int_0^t \|g(s)\|^2_{\mathbf{V}^2(\Gamma_N)'} \mathrm{d}s \right), \end{aligned} \tag{20}$$

where in particular the constant $C_0 > 0$ does not depend on κ.

Proof. Taking the inner product in $\mathbf{L}^2(\Omega)$ by $\dot{u}$ of the first equation of system (13) leads to, after integration by parts

$$\begin{aligned} \frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\dot{u}\|^2_{\mathbf{L}^2(\Omega)}\right) + \kappa\|\nabla\dot{u}\|^p_{\mathbb{L}^p(\Omega)} + \int_\Omega (\mathrm{I}+\nabla u)\Sigma(u) : \nabla\dot{u}\,\mathrm{d}\Omega \quad = & \quad \int_\Omega f\cdot\dot{u}\,\mathrm{d}\Omega + \\ & \quad \int_{\Gamma_N} \langle g;\dot{u}\rangle_{\mathbf{V}^2(\Gamma_N)';\mathbf{V}^2(\Gamma_N)}\,\mathrm{d}\Gamma_N. \end{aligned} \tag{21}$$

Recall from assumption **A2** that the derivative $\check{\Sigma}$ of $\mathcal{W}$ with respect to E defines a symmetric tensor, and satisfies $\check{\Sigma}(E(u)) = \Sigma(u)$. Therefore

$$\begin{aligned} \int_\Omega (\mathrm{I}+\nabla u)\Sigma(u):\nabla\dot{u}\,\mathrm{d}\Omega \quad & = \quad \int_\Omega \Sigma(u) : \frac{1}{2}\left((\mathrm{I}+\nabla u)^T\nabla\dot{u} + \nabla\dot{u}^T(\mathrm{I}+\nabla u)\right)\mathrm{d}\Omega \\ & = \quad \int_\Omega \check{\Sigma}(E(u)) : (E'(u).\dot{u})\,\mathrm{d}\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{W}(E(u))\,\mathrm{d}\Omega. \end{aligned}$$

Then, integrating (21) on $(0,t)$ yields

$$\begin{aligned} & \frac{\rho}{2}\|\dot{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega \mathcal{W}(E(u(t)))\,\mathrm{d}\Omega + \kappa\int_0^t \|\dot{u}(s)\|^p_{\mathbf{V}^p(\Omega)}\mathrm{d}s \\ = \quad & \frac{\rho}{2}\|u_1\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega + \int_0^t\int_\Omega f(s)\cdot\dot{u}(s)\,\mathrm{d}\Omega\mathrm{d}s + \int_0^t\int_{\Gamma_N}\langle g(s);\dot{u}(s)\rangle_{\mathbf{V}^2(\Gamma_N)';\mathbf{V}^2(\Gamma_N)}\,\mathrm{d}\Gamma_N\mathrm{d}s, \\ \leq \quad & \frac{\rho}{2}\|u_1\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega + \frac{1}{2}\int_0^t\left(\|f(s)\|^2_{\mathbf{L}^2(\Omega)} + \|g(s)\|^2_{\mathbf{V}^2(\Gamma_N)'}\right)\mathrm{d}t + C\int_0^t \|\dot{u}(s)\|^2_{\mathbf{L}^2(\Omega)}\mathrm{d}t. \end{aligned}$$

From inequality (6) of assumption **A1**, combined with (4) which holds in particular because of (19), we get

$$CC_K\|u(t)\|^p_{\mathbf{V}^p(\Omega)} \leq C\|E(u(t))\|^{p/2}_{\mathbb{L}^{p/2}(\Omega)} \leq \int_\Omega \mathcal{W}(E(u(t)))\,\mathrm{d}\Omega,$$

where $C_0 \in \mathbb{R}$. The constant C_K does not depend on time, because of (19). Combined with these inequalities, the energy estimate above then becomes

$$\begin{aligned} & \|\dot{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|u(t)\|^p_{\mathbf{V}^p(\Omega)} + \kappa\int_0^t \|\dot{u}(s)\|^p_{\mathbf{V}^p(\Omega)}\mathrm{d}s \\ \leq \quad & C\left(\|u_1\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega + \int_0^t \|f(s)\|^2_{\mathbf{L}^2(\Omega)}\mathrm{d}s + \int_0^t \|g(s)\|^2_{\mathbf{V}^2(\Gamma_N)'}\mathrm{d}t\right) + C\int_0^t \|\dot{u}(s)\|^2_{\mathbf{L}^2(\Omega)}\mathrm{d}s. \end{aligned}$$

The proof can be concluded with the Grönwall's lemma. ☐

A weak solution $w = \dot{u}$ for system (1) can be defined as in Definition 3, with $\kappa = 0$. Without ambiguity, we still call u a weak solution, determined by $\dot{u}$ and u_0 through

$$u(\cdot,t) \quad = \quad u_0 + \int_0^t \dot{u}(\cdot,s)\mathrm{d}s.$$

The energy estimate of Proposition 3 enables us to prove the main result of the paper.

Theorem 3. *Let be $p > 2$, $p \geq d$. Assume that the assumptions* **A1**–**A4** *are satisfied. For all $T > 0$, there exists a constant $C > 0$ such that, if*

$$\int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega + \|u_1\|_{\mathbf{L}^2(\Omega))} + \|f\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|g\|_{\mathrm{L}^2(0,T;\mathbf{V}^2(\Gamma_N)')} \quad \leq \quad C\exp(-CT), \tag{22}$$

then system (1) *admits a weak solution $\dot{u}$ such that*

$$u \in \mathrm{L}^\infty(0,T;\mathbf{V}^p(\Omega)), \quad \dot{u} \in \mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega)), \quad \ddot{u} \in \mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)').$$

Proof. Let $T > 0$ be arbitrary. For $\kappa > 0$, we denote by $\dot{u}_\kappa$ the solution provided on $[0, T_0]$ by Theorem 2.

Step 1. Consider $\eta \in \mathrm{L}^1(\Omega)$ such that $0 < \eta < \det(\mathrm{I}+\nabla u_0)$ almost everywhere in Ω. For instance, we choose $\eta = \frac{1}{2}\det(\mathrm{I}+\nabla u_0)$. We define

$$\begin{aligned} T_{max}(\kappa) \quad = \quad & \sup\{T_0 > 0, \text{ such that } \dot{u}_\kappa \text{ satisfies (13) on } [0,T_0], \text{ in the sense of Definition 3, and} \\ & \text{for all } t \in [0,T_0),\ \det(\mathrm{I}+\nabla u_\kappa(\cdot,t)) \geq \eta \text{ almost everywhere in } \Omega\}. \end{aligned}$$

Since $\det(\mathrm{I}+\nabla u_0) > 0$, Lemma 3 paired with Theorem 2 shows that $T_{max}(\kappa) > 0$. Assume that $T_{max}(\kappa) < T$. We will show that this leads to a contradiction, under the announced smallness assumption on the data. Estimate (20) of Proposition 3 – used for $t = T_{max}(\kappa)$ – shows that the functions $u_\kappa(T_{max}(\kappa))$ and $\dot{u}_\kappa(T_{max}(\kappa))$ are in $\mathbf{V}^p(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively. Hence, Theorem 2 enables us to extend u_κ on the interval $[T_{max}(\kappa), T_{max}(\kappa)+\tau(\kappa))$ for some $\tau(\kappa) > 0$. On the other hand, estimate (20) shows also that, for all $\varepsilon > 0$, the data can be chosen small enough, namely

$$\int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega + \|u_1\|_{\mathbf{L}^2(\Omega))} + \|f\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|g\|_{\mathrm{L}^2(0,T;\mathbf{V}^2(\Gamma_N)')} \quad \leq \quad \frac{\varepsilon^d}{C_0}\exp(-C_0T)$$

in order to have $\|u_\kappa(T_{max}(\kappa)\|_{\mathbf{V}^p(\Omega)} \leq \varepsilon$. Since there exists $\hat{C} > 0$ independent of $v \in \mathbf{V}^p(\Omega)$ such that

$$\det(\mathrm{I}+\nabla v) \geq 1 - \hat{C}\|v\|^d_{\mathbf{V}^p(\Omega)} \quad \text{and} \quad \det(\mathrm{I}+\nabla v) \leq 1 + \hat{C}\|v\|^d_{\mathbf{V}^p(\Omega)},$$

we can choose $\varepsilon > 0$ small enough, and further decrease $\|u_0\|_{\mathbf{V}^p(\Omega)}$ if necessary in order to have

$$\begin{aligned} \det(\mathrm{I}+\nabla u_\kappa(T_{max}(\kappa))) - \frac{2}{3}\det(\mathrm{I}+\nabla u_0) \geq \frac{1}{3} - \hat{C}\left(\frac{2}{3}\|u_0\|^d_{\mathbf{V}^p(\Omega)} + \varepsilon^d\right) \quad & \geq \quad 0, \\ \det(\mathrm{I}+\nabla u_\kappa(T_{max}(\kappa))) \quad & \geq \quad \frac{2}{3}\det(\mathrm{I}+\nabla u_0). \end{aligned}$$

Note that smallness for $\|u_0\|_{\mathbf{V}^p(\Omega)}$ is implied by $\int_\Omega \mathcal{W}(E(u_0))\,\mathrm{d}\Omega$ small, due to assumption **A1** and the Korn's inequality (4). Then, by continuity (see Lemma 3), we can choose $\tau(\kappa) > 0$ small enough in order to have

$$\det(\mathrm{I}+\nabla u_\kappa(t)) \quad \geq \quad \frac{1}{2}\det(\mathrm{I}+\nabla u_0), \qquad \text{for all } t \in [T_{max}(\kappa), T_{max}(\kappa)+\tau(\kappa)).$$

This contradicts the definition of $T_{max}(\kappa)$ as an upper bound. Thus, under the hypothesis (22), one can assume that $T_{max}(\kappa) \geq T$, for all $\kappa > 0$.

Step 2. In order to make κ tend to zero, estimate (20) gives us a κ-independent bound on u_κ in $\mathrm{L}^\infty(0,T;\mathbf{V}^p(\Omega))$, and on $\dot{u}_\kappa$ in $\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))$. We still need a bound on $\ddot{u}_\kappa$ in $\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')$. The variational formulation of system (13) – given in Definition 3 – shows that we have

$$\rho\|\ddot{u}_\kappa\|_{\mathbf{V}^p(\Omega)'} \quad \leq \quad \kappa\||\nabla\dot{u}_\kappa|^{p-1}\|_{\mathrm{L}^{p'}(\Omega)} + \|f\|_{\mathbf{V}^p(\Omega)'} + \|g\|_{\mathbf{V}^p(\Gamma)'} + \|(\mathrm{I}+\nabla u_\kappa)\Sigma(u_\kappa)\|_{\mathbb{L}^{p'}(\Omega)}.$$

As in the proof of Lemma 2, we can estimate the last term of the right-hand-side with the use of the estimate (7) of assumption **A3**, as follows

$$\begin{aligned} \rho\|\ddot{u}_\kappa\|_{\mathbf{V}^p(\Omega)'} \quad & \leq \quad \kappa\|\dot{u}_\kappa\|^{p-1}_{\mathbf{V}^p(\Omega)} + \|f\|_{\mathbf{V}^p(\Omega)'} + \|g\|_{\mathbf{V}^p(\Gamma)'} + \|(\mathrm{I}+\nabla u_\kappa)\|_{\mathbb{L}^p(\Omega)}\|\Sigma(u_\kappa)\|_{\mathbb{L}^{(p/2)'}(\Omega)}, \\ \rho\|\ddot{u}_\kappa\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} \quad & \leq \quad \kappa\|\dot{u}_\kappa\|^{p-1}_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} + \|f\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|g\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma)')} \\ & \qquad + \|(\mathrm{I}+\nabla u_\kappa)\|_{\mathrm{L}^p(0,T;\mathbb{L}^p(\Omega))}\|\Sigma(u_\kappa)\|_{\mathrm{L}^{(p/2)'}(0,T;\mathbb{L}^{(p/2)'}(\Omega))} \\ & \leq \quad \kappa\|\dot{u}_\kappa\|^{p-1}_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} + \|f\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')} + \|g\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Gamma)')} \\ & \qquad + C(T)\left(1 + \|u_\kappa\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}\right)\left(1 + \|u_\kappa\|^\alpha_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}\right), \end{aligned} \tag{23}$$

where $\alpha = \min(1, (p-2)/2$, and where the constant $C(T)$ is non-decreasing with respect to T, depends only on the bound of $\|u_\kappa\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}$, which is controlled by $\|u_\kappa\|_{\mathrm{L}^\infty(0,T;\mathbf{V}^p(\Omega))}$ as follows

$$\|u_\kappa\|_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \quad \leq \quad T^{1/p}\|u_\kappa\|_{\mathrm{L}^\infty(0,T;\mathbf{V}^p(\Omega))}.$$

From (20), the sequence $\left\{\|u_\kappa\|_{\mathrm{L}^\infty(0,T;\mathbf{V}^p(\Omega))}; \kappa > 0\right\}$ is bounded independently of κ. Since we have

$$\kappa\|\dot{u}_\kappa\|^{p-1}_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))} \quad = \quad \kappa^{1/p}\left(\kappa\|\dot{u}_\kappa\|^p_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}\right)^{1-1/p},$$

the sequence $\left\{\kappa\|\dot{u}_\kappa\|^{p-1}_{\mathrm{L}^p(0,T;\mathbf{V}^p(\Omega))}; \kappa \in (0,1]\right\}$ is bounded as well. Thus, from (23), the sequence $\left\{\|\ddot{u}_\kappa\|_{\mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)')}; \kappa \in (0,1]\right\}$ is also bounded. By the Banach-Alaoglu theorem (see for instance [Rud91], section 3.17), up to extraction of a subsequence, when κ goes to zero the sequence $\{u_\kappa; \kappa \in (0,1]\}$ converges weakly-* to some u such that

$$u \in \mathrm{L}^\infty(0,T;\mathbf{V}^p(\Omega)), \quad \dot{u} \in \mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega)), \quad \ddot{u} \in \mathrm{L}^{p'}(0,T;\mathbf{V}^p(\Omega)').$$

Step 3. By passing to the limit in the variational formulation of Definition 3, we see that u is a weak solution of system (1). □

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