FEEDBACK STABILIZATION TO NON-STATIONARY SOLUTIONS OF A CLASS OF REACTION DIFFUSION EQUATIONS OF FITZHUGH-NAGUMO TYPE

TOBIAS BREITEN[†], KARL KUNISCH[† ‡], AND SÉRGIO S. RODRIGUES[‡]

Abstract. Stabilization to a trajectory for the monodomain equations, a coupled nonlinear PDE-ODE system, is investigated. The results rely on stabilization of linear first-order in time nonautonomous evolution equations combined with stabilizability results for the linearized monodomain equations and a fixed point argument to treat local stabilizability of the nonlinear system. Numerical experiments for feedback stabilization of reentry phenomena are included.

Key words. feedback stabilization, finite dimensional control, differential Riccati equation, reaction diffusion equation.

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, denote a bounded domain with smooth boundary $\Gamma = \partial\Omega$. Consider the following controlled coupled reaction-diffusion system

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \Delta v - I_{\text{ion}}(v,w) + f + Bu, \quad \text{in } \Omega \times (0,\infty),\\
\frac{\partial w}{\partial t} &= \gamma v - \delta w, \quad \text{in } \Omega \times (0,\infty),\\
\frac{\partial v}{\partial \nu} &= 0, \quad \text{on } \Gamma \times (0,\infty),\\
v(x,0) &= v_0(x) \text{ and } w(x,0) = w_0(x), \quad \text{in } \Omega,
\end{aligned} \tag{1.1}$$

where $f = f(x,t)$ is an external forcing term, $I_{\text{ion}}(v,w)$ is a non-monotone nonlinear function, $u = u(t) \in L^2((0,+\infty);\mathbb{R}^m)$ denotes a finite dimensional control and ν is the unit outward normal vector to Γ. In electrophysiology, system (1.1) is known as the *monodomain equations*, see e.g. [10, Section 12.3.3]. In this context, the variable $v = v(x,t)$ models the transmembrane electric potential of the human heart and $w = w(x,t)$ is a so-called gating variable. Some typical models for the ionic current include the FitzHugh-Nagumo model

$$I_{\text{ion}}^{FN}(v,w) = av^3 - bv^2 + cv + dw, \tag{1.2}$$

as well as the Rogers-McCulloch model

$$I_{\text{ion}}^{RM}(v,w) = av^3 - bv^2 + cv + dvw, \tag{1.3}$$

where a,b,c,d are positive real constants. Besides leading to different linearizations (see below), distinct dynamical behaviors can be observed for these two models. In particular, a typical solution waveform of the FHN system includes negative values for the potential v, see [20]. This unphysiological *undershoot* does not appear for the bilinear coupling used in the Rogers-McCulloch model.

[†]Institute for Mathematics and Scientic Computing, Karl-Franzens-Universität, Heinrichstr. 36, 8010 Graz, Austria (tobias.breiten@uni-graz.at).

[‡]Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria (karl.kunisch@uni-graz.at, sergio.rodrigues@oeaw.ac.at)

Our interest in studying (optimal) control problems for the monodomain equations has several reasons. The specific PDE-ODE structure of (1.1) poses a significant mathematical challenge on its own right. To some extent, this is due to rather unexpected phenomena such as reentry waves, where wave phenomena are usually attributed to hyperbolic equations. A further notable property concerns the linearized version of (1.1). As shown in [4], in contrast to other parabolic equations, the spectrum is no longer discrete and, as a consequence, the system is not exactly null controllable. Also from a practical point of view, the monodomain equations are of interest since (1.1) allows to model fibrillation processes of the human heart. The control $u(t)$ here can be interpreted as an external stimulus resembling a defibrillation process, see [13, 17].

With this in mind, assume that a desired heart rhythm is given as the solution of the uncontrolled system

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} &= \Delta \bar{v} - I_{\text{ion}}(\bar{v}, \bar{w}) + f, \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial \bar{w}}{\partial t} &= \gamma \bar{v} - \delta \bar{w}, \quad \text{in } \Omega \times (0, \infty), \\ \frac{\partial \bar{v}}{\partial \nu} &= 0, \quad \text{on } \Gamma \times (0, \infty), \\ \bar{v}(x, 0) &= \bar{v}_0(x) \text{ and } \bar{w}(x, 0) = \bar{w}_0(x), \quad \text{in } \Omega. \end{aligned} \tag{1.4}$$

The goal of this paper is to design a *feedback control law* of the form $u = k(v-\bar{v}, w-\bar{w})$ such that the solution (v, w) of (1.1) converges exponentially to the solution $(\bar{v}, \bar{w})$ of (1.4) provided that $\|(v_0, w_0) - (\bar{v}_0, \bar{w}_0)\|$ is *small enough*. For this, we consider the difference of (1.1) and (1.4) as an infinite dimensional time varying control system of the form

$$\dot{\vec{z}}(t) = \mathcal{A}(t)\vec{z}(t) + \mathcal{F}(\vec{z}) + \mathcal{B}u(t), \ \vec{z}(0) = \vec{z}_0, \tag{1.5}$$

where

$$\vec{z} := (z_v, z_w) = (v - \bar{v}, w - \bar{w}).$$

For the sake of illustration, let us consider the Rogers-McCulloch model (1.3). We obtain

$$\begin{aligned} \frac{\partial z_v}{\partial t} &= \Delta z_v - c z_v - a(v^3 - \bar{v}^3) + b(v^2 - \bar{v}^2) - d(vw - \bar{v}\bar{w}) + Bu, \\ \frac{\partial z_w}{\partial t} &= \gamma z_v - \delta z_w, \end{aligned} \tag{1.6}$$

which, using that

$$\begin{aligned} vw - \bar{v}\bar{w} &= (z_v + \bar{v})(z_w + \bar{w}) - \bar{v}\bar{w} = z_v z_w + \bar{v} z_w + \bar{w} z_v, \\ v^3 - \bar{v}^3 &= (y_v + \bar{v})^3 - \bar{v}^3 = z_v^3 + 3\bar{v} z_v^2 + 3\bar{v}^2 z_v, \\ v^2 - \bar{v}^2 &= (z_v + \bar{v})^2 - \bar{v}^2 = z_v^2 + 2\bar{v} z_v, \end{aligned}$$

leads to the time-varying control system

$$\dot{\vec{z}}(t) = \mathcal{A}^{RM}(t)\vec{z}(t) + \mathcal{F}^{RM}(t, \vec{z}) + \mathcal{B}u(t), \quad \vec{z}(0) = \vec{z}_0,$$

where the operators $\mathcal{A}^{RM}$ and $\mathcal{F}^{RM}$ are given as

$$\begin{aligned} \mathcal{A}^{RM}(t)\vec{z} &= \begin{pmatrix} \Delta - (3a\bar{v}^2 - 2b\bar{v} + c + d\bar{w}) & -d\bar{v} \\ \gamma & -\delta \end{pmatrix} \begin{pmatrix} z_v \\ z_w \end{pmatrix}, \\ \mathcal{F}^{RM}(t, \vec{z}) &= \begin{pmatrix} -az_v^3 - (-b + 3a\bar{v})z_v^2 - dz_v z_w \\ 0 \end{pmatrix}, \quad \mathcal{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}. \end{aligned} \tag{1.7}$$

Analogously, for the FitzHugh-Nagumo model we obtain

$$\begin{aligned} \mathcal{A}^{FN}(t)\vec{z} &= \begin{pmatrix} \Delta - (3a\bar{v}^2 - 2b\bar{v} + c) & -d \\ \gamma & -\delta \end{pmatrix} \begin{pmatrix} z_v \\ z_w \end{pmatrix}, \\ \mathcal{F}^{FN}(t, \vec{z}) &= \begin{pmatrix} -az_v^3 - (-b + 3a\bar{v})z_v^2 \\ 0 \end{pmatrix}, \quad \mathcal{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}. \end{aligned} \tag{1.8}$$

The feedback stabilization approach to (1.5) will mainly consist in two nested subproblems. In the first one, similar to the approach taken in [2, 3, 12], we focus on the linearized system, arising from (1.5), which is given by

$$\dot{\vec{z}}(t) = \mathcal{A}(t)\vec{z}(t) + \mathcal{B}u(t), \quad \vec{z}(0) = \vec{z}_0. \tag{1.9}$$

In the second, the inner, subproblem, we decouple the PDE part of the system, i.e. we consider the $(1, 1)$ block of (1.9), for which we study a stabilization problem together with an associated differential Riccati equation. In this way, we can compensate for the lack of null controllability of the coupled linear system (1.9), see [4, Section 2.2] and [8, Theorem 1.2 and Remark 1.3].

The feedback law we will be based on an infinite-horizon optimal control problem associated with (1.9), for which we shall use the cost functional

$$\mathcal{J}(u) = \frac{1}{2}\left(\int_0^\infty |\mathcal{M}\vec{z}|^2 + |\mathcal{R}^{\frac{1}{2}}u| \,\mathrm{d}t\right), \tag{1.10}$$

where the particular structure of the pair $(\mathcal{M}, \mathcal{R})$ will be specified subsequently.

The structure of the paper is as follows. In Section 2 we investigate stabilization to zero for a system of the form

$$\frac{\partial z}{\partial t} + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + Bu = 0, \tag{1.11a}$$

$$\frac{\partial z}{\partial \nu} = 0, \qquad z(0) = z_0. \tag{1.11b}$$

which can be seen to contain the linearizations of the Rogers-McCulloch and the FitzHugh-Nagumo nonlinearities as special cases. These results will provide the stabilization of the decoupled $(1, 1)$ block of (1.9) described above. In Section 3 it will be shown that under suitable assumptions on the system parameters, the obtained feedback formula is shown to stabilize the linearized PDE-ODE system, resulting from the Rogers-McCulloch and the FitzHugh-Nagumo nonlinearities. In Section 4, we show the local exponential stabilization of the full nonlinear system. The theoretical results are illustrated by means of different numerical examples in Section 5.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_a := (a, +\infty)$ for all $a \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}_0$, a bounded domain with a smooth boundary $\Gamma = \partial\Omega$.

Given a function $v\colon (t, x_1, x_2, \ldots, x_n) \mapsto v(t, x_1, x_2, \ldots, x_n) \in \mathbb{R}$, defined in an open subset of $\mathbb{R} \times \Omega$, its partial time derivative $\frac{\partial v}{\partial t}$ will be denoted by $\partial_t v$, and its normal derivative $\frac{\partial v}{\partial \nu}$ at the boundary will be denoted $\partial_\nu v|_\Gamma$.

We use the standard notation for Bochner spaces $L^p(I,\, X)$ where $I \subseteq \mathbb{R}$, and X is a Banach space. The Lebesgue spaces $L^p(\Omega)^m$ will be denoted by simply L^p whenever there is no ambiguity concerning the superscript $m \in \mathbb{N}_0$.

Given an open interval $I \subseteq \mathbb{R}$, and Banach spaces X and Y, then we write $W(I,\, X,\, Y) := \{f \in L^2(I,\, X) \mid \partial_t f \in L^2(I,\, Y)\}$, where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I,\, X,\, Y)} := \left(|f|^2_{L^2(I,\, X)} + |\partial_t f|^2_{L^2(I,\, Y)}\right)^{1/2}$. The space of continuous linear mappings from X into Y will be denoted by $\mathcal{L}(X, Y)$. In case $X = Y$ we write $\mathcal{L}(X) := \mathcal{L}(X, X)$ instead. If the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$; we write $X \overset{\mathrm{d}}{\hookrightarrow} Y$, respectively $X \overset{\mathrm{c}}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact. The kernel and range of a linear mapping $A\colon Z \to W$, between vector spaces Z and W, will be denoted $\operatorname{Ker} A := \{x \in Z \mid Ax = 0\}$ and $\operatorname{Ran} A := \{Ax \in x \in Z\}$, respectively.

$\overline{C}_{[a_1,\ldots,a_k]}$ denotes a function of nonnegative variables a_j that increases in each of its arguments, and C, C_i, $i = 1, 2, \ldots$, stand for positive constants.

2. Stabilization for parabolic equations with homogeneous Neumann boundary conditions.

The section is devoted to the stabilization to zero for systems of the form (1.11). We can take advantage of the results obtained in [2, 12] for Oseen–Burgers and Oseen–Stokes equations under homogeneous Dirichlet boundary conditions. Here we deal with homogeneous Neumann boundary conditions. Moreover in exploiting the relation between null controllability of (1.11) and observability of its adjoint we also follow a different procedure. While the one in [2, 12] is based on optimal control theoretic tools here we follow a functional analytic approach. Finally in Lemma 2.6 we give a property of global solutions of (1.11) which will allow us to take cost functionals different from those in [2, 12], with respect to the state.

2.1. Some regularity results.

We start by deriving some regularity results for

$$\partial_t z + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + f = 0, \tag{2.1a}$$

$$\partial_\nu z|_\Gamma = 0, \qquad z(0) = z_0. \tag{2.1b}$$

in the form which will be required further below.

For simplicity we denote $H = L^2$, and $V = H^1(\Omega)$. We consider H as the pivot space and define the operator $A\colon V \to V'$ by $\langle -Au,\, v\rangle_{V',\, V} := (u,\, v)_V$. We have that, $V \xhookrightarrow{\mathrm{d,\,c}} H \xhookrightarrow{\mathrm{d,\,c}} V'$, and if $(f,\, v) \in H \times V$ we have $\langle f,\, v\rangle_{V',\, V} = (f,\, v)_H$.

By the Lax-Milgram lemma (cf. [23, Section II.2.1, Theorem 2.1]), $A\colon V \to V'$ is a bijective isometry. The domain $\mathrm{D}(A)$ of A is defined as $\mathrm{D}(A) := \{z \in H \mid Az \in H\}$.

LEMMA 2.1. *We have that* $\mathrm{D}(A) = \{z \in H \mid (-\Delta + 1)z \in H$ *and* $\partial_\nu z|_\Gamma = 0\} = \{z \in H^2(\Omega) \mid \partial_\nu z|_\Gamma = 0\}$ *and the norms* $z \mapsto |z|_{H^2(\Omega)}$ *and* $z \mapsto |Az|_H = |(-\Delta + 1)z|_H$ *are equivalent in* $\mathrm{D}(A)$.

Proof. We can derive the above identities by following the arguments in [23, Section II.2.2, Example 2.5]. In order to check the equivalence of the norms, let us fix $z \in \mathrm{D}(A)$. Clearly $|(-\Delta + 1)z|_H \le C\,|z|_{H^2(\Omega)}$. From [21, Chapter 5, Proposition 7.2] we also have $|z|^2_{H^2(\Omega)} \le C(|\Delta z|^2_H + |z|^2_V) = C(|-\Delta z + z|^2_H - 2\,|\nabla z|^2_{L^2} - |z|^2_H + |z|^2_V) = C(|-\Delta z + z|^2_H - |\nabla z|^2_{L^2})$, which implies $|z|^2_{H^2(\Omega)} \le C\,|(-\Delta + 1)z|^2_H$. ∎

For $m \in \mathbb{N}_0$, in order to simplify the writing we denote

$$\begin{array}{rcl}\mathcal{W}^{J,\,m} &:=& L^\infty(J,\,L^\infty(\Omega,\,\mathbb{R}^m)) = L^\infty(J\times\Omega,\,\mathbb{R}^m),\\ \mathcal{W}^m &:=& L^\infty(\mathbb{R}_0\times\Omega,\,\mathbb{R}^m),\end{array} \tag{2.2}$$

where $J \subseteq (0,\,+\infty)$ is an open interval. In the case $m = 1$, we will omit the superscript 'm'.

The notation for the interval $I = (s_0,\,s_1)$ with $0 \le s_0 < s_1$ is fixed throughout the paper, and its length is denoted by $|I|$. We also fix ρ and σ, which may depend on time and space, and a constant $C_\mathcal{W} \ge 0$, satisfying

$$|\rho|_\mathcal{W} + |\sigma|_{\mathcal{W}^n} \le C_\mathcal{W}. \tag{2.3}$$

Lemma 2.2. *Given $f \in L^2(I,\,V')$ and $z_0 \in H$, there is a weak (variational) solution $z \in W(I,\,V,\,V')$ for (2.1). Moreover z is unique and depends continuously on the data:*

$$|z|^2_{W(I,\,V,\,V')} \le \overline{C}_{[|I|,\,C_\mathcal{W}]}\left(|z(s_0)|^2_H + |f|^2_{L^2(I,\,V')}\right).$$

Proof. While this result can be found in the literature we will provide a proof since the explicit estimates will be used later on.

Weak solutions for system (2.1) are understood in the variational sense. We restrict ourselves to the derivation of some a priori (like) estimates. In fact those estimates will also hold for Galerkin approximations of the system, for example using a basis of eigenfunctions of the operator $A = \Delta - 1$, thus the estimates can be used to precisely derive the existence of weak solutions. For more details on the procedure we refer to [15, Chapter 1, Section 6], [22, Chapter 1, Section 3], and [24, Chapter 3, Sections 1.3, 1.4, and 3.2].

By standard arguments, multiplying (2.1a) by $2z$, formally we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2_H + 2|z|^2_V \le 2|\rho|_{L^\infty}|z|^2_H + 2|\sigma\cdot\nabla z|_H|z|_H + 2|f|_{V'}|z|_V\,,$$

and since $|\sigma\cdot\nabla z|^2_H \le 3|\sigma|^2_{L^\infty}|\nabla z|^2_{L^2}$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2_H + |z|^2_V \le 2|\rho|_{L^\infty}|z|^2_H + 6|\sigma|^2_{L^\infty}|z|^2_H + 2|f|^2_{V'}\,. \tag{2.4}$$

By the Gronwall inequality it follows that for all $s \in I$,

$$|z(s)|^2_H \le \mathrm{e}^{(2|\rho|_{\mathcal{W}^I}+6|\sigma|^2_{\mathcal{W}^{I,\,n}})(s-s_0)}\left(|z(s_0)|^2_H + 2|f|^2_{L^2(I,\,V')}\right) \tag{2.5}$$

and, integrating (2.4),

$$|z(s)|^2_H + |z|^2_{L^2((s_0,\,s),\,V)} \le |z(s_0)|^2_H + (2|\rho|_{\mathcal{W}^I} + 6|\sigma|^2_{\mathcal{W}^{I,\,n}})|z|^2_{L^2(I,\,H)} + 2|f|^2_{L^2(I,\,V')}\,. \tag{2.6}$$

From (2.1a) and $H \hookrightarrow V'$, with $|\cdot|_{V'} \le |\cdot|_H$, we also have

$$|\partial_t z|_{L^2(I,\,V')} \le |z|_{L^2(I,\,V)} + |\rho z + \sigma\cdot\nabla z|_{L^2(I,\,H)} + |f|_{L^2(I,\,V')}$$

from which, using (2.5) and (2.6) we can conclude that

$$|z|^2_{W(I,\,V,\,V')} \le \overline{C}_{\left[s_1-s_0,\,|\rho|_{\mathcal{W}^I},\,|\sigma|_{\mathcal{W}^{I,\,n}}\right]}\left(|z(s_0)|^2_H + |f|^2_{L^2(I,\,V')}\right).$$

Finally the uniqueness of z, follows from the fact that if $\tilde{z}$ is another weak solution, then $\delta z = z - \tilde{z}$, solves (2.1) with $\delta z(s_0) = 0$ and $f = 0$. From (2.5) it will follow that $|\delta z(s)|_H = 0$ for all $s \in I$. ∎

LEMMA 2.3. *Given $f \in L^2(I, H)$ and $z_0 \in V$, there is a strong solution $z \in W(I, \mathrm{D}(A), H)$ for system (2.1), which depends continuously on the data:*

$$|z|^2_{W(I,\,\mathrm{D}(A),\,H)} \le \overline{C}_{[|I|,\,C_{\mathcal{W}}]} \left(|z(s_0)|^2_V + |f|^2_{L^2(I,\,H)} \right).$$

Proof. Multiplying (2.1a) by $2(-\Delta + 1)z$, formally we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} |z|^2_V + 2\,|z|^2_{\mathrm{D}(A)} \le 2\,|\rho|_{L^\infty}\,|z|_H\,|z|_{\mathrm{D}(A)} + 2\,|\sigma \cdot \nabla z|_H\,|z|_{\mathrm{D}(A)} + 2\,|f|_H\,|z|_{\mathrm{D}(A)}\,,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} |z|^2_V + |z|^2_{\mathrm{D}(A)} \le 3\,|\rho|^2_{L^\infty}\,|z|^2_H + 9\,|\sigma|^2_{\mathcal{W}^{I,\,n}}\,|z|^2_V + 3\,|f|^2_H\,. \tag{2.7}$$

Thus, for all $s \in I$,

$$|z(s)|^2_H \le \mathrm{e}^{9|\sigma|^2_{\mathcal{W}^{I,\,n}}(s-s_0)} \left(|z(s_0)|^2_V + 3(s-s_0)\,|\rho|^2_{\mathcal{W}^I} + 3\,|f|^2_{L^2(I,\,H)} \right); \tag{2.8}$$

$$\begin{aligned} |z(s)|^2_V + |z|^2_{L^2((s_0,\,s),\,\mathrm{D}(A))} \le |z(s_0)|^2_V + 3\,|\rho|^2_{\mathcal{W}^I}\,|z|^2_{L^2((s_0,\,s),\,H)} \\ + 9\,|\sigma|^2_{\mathcal{W}^{I,\,n}}\,|z|^2_{L^2((s_0,\,s),\,V)} + 3\,|f|^2_{L^2(I,\,H)}\,. \end{aligned} \tag{2.9}$$

From (2.1a) we also have

$$|\partial_t z|_{L^2(I,\,H)} \le |z|_{L^2(I,\,\mathrm{D}(A))} + |\rho z + \sigma \cdot \nabla z|_{L^2(I,\,H)} + |f|_{L^2(I,\,H)}$$

and we can conclude that

$$|z|^2_{W(I,\,\mathrm{D}(A),\,H)} \le \overline{C}_{\left[s_1 - s_0,\,|\rho|_{\mathcal{W}^I},\,|\sigma|_{\mathcal{W}^{I,\,n}}\right]} \left(|z(s_0)|^2_V + |f|^2_{L^2(I,\,H)} \right),$$

which ends the proof. ∎

The next lemma shows a certain smoothing property of system (2.1).

LEMMA 2.4. *Given $f \in L^2(I, H)$ and $z_0 \in H$, let z be the weak solution for system (2.1). Then $y(t) \coloneqq (t - s_0)z(t)$ is in $W(I, \mathrm{D}(A), H)$ and satisfies the estimates*

$$\begin{aligned} |y|^2_{W(I,\,\mathrm{D}(A),\,H)} &\le \overline{C}_{[|I|,\,C_{\mathcal{W}}]} \left((s_1 - s_0)^2\,|f|^2_{L^2(I,\,H)} + |z|^2_{L^2(I,\,H)} \right) \\ &\le \overline{C}_{[|I|,\,C_{\mathcal{W}}]} \left(|z(s_0)|^2_H + |f|^2_{L^2(I,\,H)} \right). \end{aligned}$$

Proof. Notice that $y(t) = (t - s_0)z(t)$ solves (2.1a) with $g = g(t) = (t - s_0)f(t) + z(t)$ in place of f, and $y(s_0) = 0$. Hence by Lemma 2.3,

$$|y|^2_{W(I,\,\mathrm{D}(A),\,H)} \le \overline{C}_{[|I|,\,C_{\mathcal{W}}]}\,|g|^2_{L^2(I,\,H)}\,,$$

and using (2.5),

$$|z|^2_{L^2(I,\,H)} \le \overline{C}_{[|I|,\,C_{\mathcal{W}}]}(s_1 - s_0) \left(|z(s_0)|^2_H + |f|^2_{L^2(I,\,V')} \right),$$

which implies that $|y|^2_{W(I,\,\mathrm{D}(A),\,H)} \le \overline{C}_{[|I|,\,C_{\mathcal{W}}]}\left(|z(s_0)|^2_H + |f|^2_{L^2(I,\,H)}\right)$. □

DEFINITION 2.5. *For $f \in L^2_{\mathrm{loc}}(\mathbb{R}_{s_0},\,H)$ and $y_0 \in H$ the function z defined in $\mathbb{R}_{s_0} \times \Omega$ by the property that $z|_{(s_0,\,\tau)}$ coincides with the weak solution of (2.1) in $(s_0,\,\tau)$, for all $\tau > s_0$ is well defined. It is called the global weak solution of (2.1) in $\mathbb{R}_{s_0} \times \Omega$.*

We have the following property for the solutions of (2.1) on the infinite time interval $\mathbb{R}_{s_0} = (s_0,\,+\infty)$, $s_0 \ge 0$.

LEMMA 2.6. *For $f \in L^2(\mathbb{R}_{s_0},\,V')$ and $z_0 \in H$, let z be the global weak solution of (2.1) in $\mathbb{R}_{s_0}$, with $z(s_0) = z_0$. If $z \in L^2(\mathbb{R}_{s_0},\,H)$, then $z \in W(\mathbb{R}_{s_0},\,V,\,V')$, and*

$$|z|_{W(\mathbb{R}_{s_0},\,V,\,V')} \le \overline{C}_{[C_{\mathcal{W}}]}\left(|z(s_0)|^2_H + |f|^2_{L^2(\mathbb{R}_{s_0},\,V')} + |z|^2_{L^2(\mathbb{R}_{s_0},\,H)}\right). \tag{2.10}$$

Proof. Integrating (2.4) over $(s_0,\,\tau)$, we find

$$|z(\tau)|^2_H + |z|^2_{L^2((s_0,\,\tau),\,V)} \le |z(s_0)|^2_H + \overline{C}_{[C_{\mathcal{W}}]}\,|z|^2_{L^2((s_0,\,\tau),\,H)} + 2\,|f|^2_{L^2((s_0,\,\tau),\,V')}\,,$$

which leads us to

$$|z|^2_{L^2(\mathbb{R}_{s_0},\,V)} \le |z(s_0)|^2_H + \overline{C}_{[C_{\mathcal{W}}]}\,|z|^2_{L^2(\mathbb{R}_{s_0},\,H)} + 2\,|f|^2_{L^2(\mathbb{R}_{s_0},\,V')}\,. \tag{2.11}$$

Finally, from (2.1a) is follows also that

$$|\partial_t z|^2_{L^2(\mathbb{R}_{s_0},\,V')} \le \overline{C}_{[C_{\mathcal{W}}]}\left(|z(s_0)|^2_H + |z|^2_{L^2(\mathbb{R}_{s_0},\,V)} + |f|^2_{L^2(\mathbb{R}_{s_0},\,V')}\right),$$

which, together with (2.11), gives us (2.10). □

2.2. Null controllability. Here we recall the relation between null controllability of system (2.1) and a suitable observability inequality for the adjoint system.

Consider, in the bounded cylinder $I \times \Omega$, the controlled system

$$\partial_t z + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + Bu = 0, \tag{2.12a}$$
$$\partial_\nu z|_\Gamma = 0, \qquad z(s_0) = z_0, \tag{2.12b}$$

where $u \in L^2(I,\,H)$ and $B \in \mathcal{L}(H)$, with adjoint denoted by B^*. Let us also consider in $I \times \Omega$ the adjoint system

$$-\,\partial_t q + (-\Delta + 1)q + \rho q - \nabla \cdot (q\sigma) = 0, \tag{2.13a}$$
$$(q\sigma + \nabla q) \cdot \nu|_\Gamma = 0, \qquad q(s_1) = q_1 \in H, \tag{2.13b}$$

and let $z(z_0,\,u)(t) \coloneqq z(t)$ and $q(q_1)(t) \coloneqq q(t)$ denote the solutions of (2.12) and (2.13), for given data $(z_0,\,u)$ and q_1, respectively.

Weak solutions $q \in W(I,\,V,\,V')$ for system (2.13) are understood again in the variational sense as in [7]. In [7, Section 2] weak solutions are asked to be in $L^2(I,\,V) \cap C([s_0,\,s_1],\,H)$, but from $\rho \in L^2(I,\,H)$ and $\sigma q \in L^2(I,\,H^n)$ we can obtain that the variational solution is indeed in the space $W(I,\,V,\,V')$.

Let $z(\cdot) = z(z_0,\,u)(\cdot)$ and $q(\cdot) = q(q_1)(\cdot)$ solve (2.12) and (2.13), respectively.

DEFINITION 2.7. *(i) We say that (2.12) is null controllable in I if there exists a family $\{u(z_0) \mid z_0 \in H\} \subset L^2(I,\,H)$ such that $z(z_0,\,u(z_0))(s_1) = 0$, for $z_0 \in H$.*

(ii) We say that (2.13) *is observable in* I *if there exists a constant* $C_2 > 0$ *such that for all* $q_1 \in H$ *we have that the corresponding weak solution* q *satisfies the inequality*

$$|q(q_1)(s_0)|_H \le C_2 \left|B^* q(q_1)\right|_{L^2(I,\,H)}\,. \tag{2.14}$$

The constant C_2 in (2.14) depends, in general, on Ω, ω, I, B, and on the coefficients ρ and σ.

LEMMA 2.8. *System* (2.13) *is observable in* I *if, and only if, system* (2.12) *is null controllable in* I *and the family of controls* $\{u(z_0) \mid z_0 \in H\}$ *is a bounded linear function of* z_0*:*

$$|u(z_0)|_{L^2(I,\,H)} \le C_2 \, |z_0|_H\,, \text{ where } C_2 \text{ is as in (2.14).}$$

Proof. From [24, Chapter 3, Section 1.4, Lemma 1.2], we can write $\frac{\mathrm{d}}{\mathrm{d}t}(z,\,q)_H = \frac{1}{2}\left(|z+q|_H^2 - |z|_H^2 - |q|_H^2\right) = \langle \partial_t(z+q),\, z+q\rangle_{V',\,V} - \langle \partial_t z,\, z\rangle_{V',\,V} - \langle \partial_t q,\, q\rangle_{V',\,V}$, for a.e. $t \in I$, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(z,\,q)_H = \langle \partial_t z,\, q\rangle_{V',\,V} + \langle z,\, \partial_t q\rangle_{V,\,V'} = (-Bu,\, q)_H, \tag{2.15}$$

$$(z(s_1),\, q_1)_H - (z_0,\, q(s_0))_H = -\int_{s_0}^{s_1} (u(s),\, B^* q(s))_H \,\mathrm{d}s. \tag{2.16}$$

Thus if there is a family $u = u(z_0) \in L^2(I,\,H)$ with $|u(z_0)|_{L^2(I,\,H)} \le C_2\,|z_0|_H$ such that $z(z_0,\,u)(s_1) = 0$, then we find

$$(z_0,\, q(q_1)(s_0))_H \le C_2\,|z_0|_H \left|B^* q(q_1)\right|_{L^2(I,\,H)}\,, \text{ for all } z_0 \in H,$$

that is, $|q(q_1)(s_0)|_H \le C_2\,|B^* q|_{L^2(I,\,H)}$.

On the other hand if there is $C_2 > 0$ such that $|q(q_1)(s_0)|_H \le C_2\,|B^* q(q_1)|_{L^2(I,\,H)}$, then null controllability in I of (2.12) can be proven by the following arguments (see, e.g., [5, Chapter 2]). We note that the literature typically considers the case of autonomous systems but this does not change the proof (cf. [1, Section 2]). Let us define the mappings

$$\begin{array}{rcl} \mathcal{F}\colon L^2(I,\,H) & \to & H \\ u & \mapsto & z(0,\,u)(s_1) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{G}\colon H & \to & H \\ z_0 & \mapsto & z(z_0,\,0)(s_1). \end{array}$$

From (2.16), we have

$$(\mathcal{F}u,\, q_1)_H = -(u,\, B^* q(q_1))_{L^2(I,\,H)},$$
$$(\mathcal{G}z_0,\, q_1)_H = (z_0,\, q(q_1)(s_0))_H,$$

which show that the adjoints of $\mathcal{F}$ and $\mathcal{G}$ are given, respectively, by

$$\begin{array}{rcl} \mathcal{F}^*\colon H & \to & L^2(I,\,H) \\ q_1 & \mapsto & -B^* q(q_1) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{G}^*\colon H & \to & H \\ q_1 & \mapsto & q(q_1)(s_0) \end{array}.$$

Now we can write the observability inequality (2.14) as

$$|\mathcal{G}^* q_1|_H \le C_2\,|\mathcal{F}^* q_1|_{L^2(I,\,H)}\,, \tag{2.17}$$

and from $z(0,\, u(z_0))(s_1) = \mathcal{F}u(z_0) + \mathcal{G}z_0$ we can conclude that null controllability of system (2.12) holds if, and only if,

$$\operatorname{Ran}\mathcal{G} \subseteq \operatorname{Ran}\mathcal{F} \tag{2.18}$$

and by Lemma 2.48 in [5, Section 2.3.2] (cf. Theorem 2.2 in [25, Chapter 2]), we have that (2.18) is equivalent to (2.17).

It remains to prove that the family $\{u(z_0) \mid z_0 \in H\}$ can be chosen as a linear and continuous mapping of z_0, with $|u(z_0)|_{L^2(I,\,H)} \le C_2\, |z_0|_H$. This fact follows also from Lemma 2.48 in [5, Section 2.3.2], which states the existence of a mapping $\mathcal{F}^\bullet \in \mathcal{L}(H \to L^2(I,\, H))$ such that $\mathcal{G} = \mathcal{F}\mathcal{F}^\bullet$ and $|\mathcal{F}^\bullet|_{\mathcal{L}(H \to L^2(I,\,H))} \le C_2$ with C_2 as in (2.17). The family $\{u(z_0) \mid z_0 \in H\}$ is constructed by setting $u \in \mathcal{L}(H \to L^2(I,\, H))$:

$$u(z_0) \coloneqq -\mathcal{F}^\bullet z_0.$$

Notice that $z(z_0,\, u(z_0))(s_1) = z(z_0,\, 0)(s_1) + z(0,\, u(z_0))(s_1) = \mathcal{G}z_0 - \mathcal{F}\mathcal{F}^\bullet z_0 = 0$, and thus this choice of control (also) provides the desired null controllability, and we have the announced inequality $|u(z_0)|_{L^2(I,\,H)} \le C_2\, |z_0|_H$. $\square$

Controls supported in a subset. From now on, we will deal with controls supported in any given open subset $\omega \subseteq \Omega$. From [7] we know that in the case we take $B = 1_\omega \in \mathcal{L}(H)$ with

$$1_\omega u(x) \coloneqq \begin{cases} u(x), & \text{if } x \in \omega \\ 0, & \text{if } x \in \Omega \setminus \overline{\omega} \end{cases},$$

we have that (2.13) is observable and (2.12) is null controllable. More precisely we know (cf. [7, Theorem 2]) that the following theorem holds true

Theorem 2.9. *Let $B = 1_\omega$ and let $I = (s_0, s_1)$ be arbitrary, then, there exists a family $\{u(z_0) \mid z_0 \in H\} \subseteq L^2(I,\, H)$ such that the solutions $z(z_0,\, u(z_0))$ to (2.12) satisfy $z(z_0,\, u(z_0))(s_1) = 0$ and, for a constant $\widehat{C} = C(\omega,\, \Omega)$, we have that*

$$|u(z_0)|_{L^2(I,\,H)} \le \mathrm{e}^{\widehat{C}\Theta}\, |z_0|_H \quad \textit{with} \quad \Theta = \Theta(s_1 - s_0,\, |\rho|_{\mathcal{W}^I},\, |\sigma|_{\mathcal{W}^{I,\,n}})$$

given by

$$\Theta(\theta_1,\, \theta_2,\, \theta_3) = 1 + \theta_2^{\frac{2}{3}} + \theta_3^2 + \frac{1}{\theta_1} + \theta_1 \left(\theta_2 + \theta_3^2\right). \tag{2.19}$$

Notice that Theorem 2.9 and Lemma 2.8 imply that (2.14) holds with $C_2 = \mathrm{e}^{\widehat{C}\Theta}$ and $B = 1_\omega$. Proceeding as in [2, Section A.2] we can conclude that (2.14) also holds with $C_2 = C_\chi \mathrm{e}^{\widehat{C}_\chi\Theta} \le \mathrm{e}^{\widehat{D}\Theta}$ and $B^*q \coloneqq \chi 1_\omega q = 1_\omega \chi 1_\omega q$, where $\widehat{D} = \log(C_\chi) + \widehat{C}_\chi$ and $\chi \in C^\infty(\overline{\Omega})$ is any given smooth function with $\emptyset \ne \omega \cap \operatorname{supp}\chi$. Here $\widehat{D} = \widehat{D}(\chi,\, \omega,\, \Omega) > 0$ depends only on $(\chi,\, \omega,\, \Omega)$.

Corollary 2.10. *Theorem 2.9 holds in the more general case $B = 1_\omega\chi 1_\omega$, with $\widehat{D}$ in the place of $\widehat{C}$.*

Remark 2.11. We point out that the observability constants $C_2 = \mathrm{e}^{\widehat{C}\Theta}$ and $C_2 = \mathrm{e}^{\widehat{D}\Theta}$, in Theorem 2.9 and Corollary 2.10, do depend on the triple $(I,\, \rho,\, \sigma)$, but that dependence is in terms of the triple $(|I|,\, |\rho|_{\mathcal{W}^I},\, |\sigma|_{\mathcal{W}^{I,\,n}})$ only. This particular dependence on I is of crucial importance in this section. This dependence holds for the control operators $B = 1_\omega\chi 1_\omega$, but we do not know what happens for a general B.

2.3. Stabilization to zero by finite dimensional controls. Here we analyze the case when stabilization can be achieved by finite dimensional control action. Earlier related results are contained in [11, 12]. Let $\mathcal{C} = \{\Psi_i \in H \mid i \in \{1, 2, \ldots, M\}\}$ and denote by P_M the orthogonal projection in H onto $\mathcal{S}_\mathcal{C} \coloneqq \operatorname{span}\mathcal{C}$. Henceforth we also fix a positive constant $\lambda > 0$ and an open subset $\omega \subseteq \Omega$.

Let us consider, in $\mathbb{R}_{s_0} \times \Omega$, the system:

$$\partial_t z + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + 1_\omega \chi P_M 1_\omega u = 0, \tag{2.20a}$$

$$\partial_\nu z|_\Gamma = 0, \qquad z(s_0) = z_0. \tag{2.20b}$$

Definition 2.12. *We say that* (2.20) *is exponentially stabilizable to zero, with rate* $\frac{\lambda}{2}$*, if there are a constant* $C > 0$ *and a family* $\{u = u(z_0) \mid z_0 \in H\} \subseteq L^2(\mathbb{R}_{s_0}, H)$ *such that the corresponding global solution* $z(t) = z(z_0, u(z_0))(t)$ *satisfies*

$$|z(t)|_H^2 \le C\,\mathrm{e}^{-\lambda(t-s_0)}|z_0|_H^2, \quad \textit{for all } t \ge s_0. \tag{2.21}$$

The stabilizing control ζ takes its values in the finite dimensional space $\operatorname{span}\{1_\omega\chi\Psi_i \in H \mid i \in \{1, 2, \ldots, M\}\}$, for all $t \in \mathbb{R}_{s_0}$. Henceforth we use the control operator

$$B_M = 1_\omega \chi P_M 1_\omega. \tag{2.22}$$

Further θ and $\widehat{D}$ are the constants of Theorem 2.9 and Corollary 2.10.

Let us consider the function $\Phi\colon (0, +\infty) \to (0, +\infty)$ defined by

$$\Phi\colon (0, +\infty) \to (0, +\infty), \quad \Phi(\tau) \coloneqq 2\mathrm{e}^{(2\left|\rho - \frac{\lambda}{2}\right|_\mathcal{W} + 6|\sigma|^2_{\mathcal{W}^n})\tau}\mathrm{e}^{2\widehat{D}\Theta(\tau, \left|\rho-\frac{\lambda}{2}\right|_\mathcal{W}, |\sigma|_{\mathcal{W}^n})},$$

which we can extend to a function $\Phi\colon [0, +\infty] \to (0, +\infty]$ setting

$$\Phi_\mathrm{e}(\tau) \coloneqq \begin{cases} \Phi(\tau), & \text{if } \tau \in (0, +\infty), \\ \lim\limits_{t \to \tau} \Phi(t), & \text{if } \tau \in \{0, +\infty\}. \end{cases}$$

The minimum and minimizer of Φ_e are denoted by Υ and T_*, respectively. From

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}\right|_{\tau = t} = \left(2\left|\rho - \tfrac{\lambda}{2}\right|_\mathcal{W} + 6\,|\sigma|^2_{\mathcal{W}^n} + 2\widehat{D}\left(-t^{-2} + \left|\rho - \tfrac{\lambda}{2}\right|_\mathcal{W} + |\sigma|^2_{\mathcal{W}^n}\right)\right)\Phi(t)$$

we can conclude that $T_* > 0$ can be defined by

$$T_*^2 = \frac{2\widehat{D}}{2\left|\rho - \frac{\lambda}{2}\right|_\mathcal{W} + 6\,|\sigma|^2_{\mathcal{W}^n} + 2\widehat{D}\left(\left|\rho - \frac{\lambda}{2}\right|_\mathcal{W} + |\sigma|^2_{\mathcal{W}^n}\right)}.$$

Further $T_* = +\infty$ if, and only if, both $\left|\rho - \frac{\lambda}{2}\right|_\mathcal{W}$ and $|\sigma|_{\mathcal{W}^n}$ vanish.

The following result gives us a sufficient condition on the family $\mathcal{C}$ for the existence of a stabilizing control.

Theorem 2.13. *Let us be given* $\chi \in C^\infty(\overline{\Omega})$ *satisfying* $\emptyset \ne \omega \cap \operatorname{supp}\chi$*. If*

$$T_* \in \mathbb{R}_0 \quad \textit{and} \quad |1_\omega\chi(1 - P_M)1_\omega|_{\mathcal{L}(H, V')} \le \Upsilon^{-1}, \tag{2.23}$$

then system (2.20) is stabilizable to zero with rate $\frac{\lambda}{2}$. Moreover, we can set the stabilizing control function $u = u(z_0)$ such that

$$\begin{aligned} |z(t)|_H^2 &\le \left(\Upsilon_0 + \Upsilon\, |B_M|_{\mathcal{L}(H,V')}^2\right) \mathrm{e}^{-\lambda(t-s_0)}\, |z_0|_H^2\,, \ \textit{for } t \ge s_0,\\ \left|\mathrm{e}^{\frac{\hat\lambda}{2}\cdot} u(z_0)\right|_{L^2(\mathbb{R}_{s_0},H)}^2 &\le \tfrac{1}{1-\mathrm{e}^{(\hat\lambda-\lambda)T_*}} \mathrm{e}^{2\widehat{D}\Theta_*}\, |z_0|_H^2\,, \ \textit{for } \hat\lambda < \lambda, \end{aligned}$$

with $\Upsilon_0 \coloneqq \mathrm{e}^{(2\left|\rho-\frac{\lambda}{2}\right|_{\mathcal{W}}+6|\sigma|_{\mathcal{W}^n}^2)T_*}$ *and* $\Theta_* \coloneqq \Theta(T_*,\, \left|\rho - \frac{\lambda}{2}\right|_{\mathcal{W}},\, |\sigma|_{\mathcal{W}^n})$.

If $T_* = +\infty$*, then setting* $u = u(z_0) = 0$ *the solution* z *of system* (2.20) *satisfies* $|z(t)|_H^2 \le \mathrm{e}^{-\lambda(t-s_0)}\, |z_0|_H^2$, *for* $t \ge s_0$.

Proof. We consider separately the two cases $T_* \in \mathbb{R}_0$ and $T_* = +\infty$.

(a) *The case* $T_* \in \mathbb{R}_0$. Let $I_0 \coloneqq (s_0,\, s_0 + T_*)$ and let z solve

$$\partial_t z + (-\Delta + 1)z + (\rho - \tfrac{\lambda}{2})z + \sigma\cdot\nabla z + 1_\omega \chi u = 0, \tag{2.24a}$$
$$\partial_\nu z|_\Gamma = 0, \qquad z(s_0) = z_0, \tag{2.24b}$$

in $I_0\times\Omega$, with $u = u(z_0)$ given by Corollary 2.10, sending the solution of (2.24) to zero at time $t = s_0 + T_*$ (notice that Corollary 2.10 holds true with $\rho - \frac{\lambda}{2}$ in the place of ρ), and let z_M be the solution of (2.24) with $u = P_M 1_\omega u(z_0)$ in the place of $u(z_0)$. Then, the difference $d \coloneqq z - z_M$ satisfies (2.24) with $d(s_0) = 0$ and $u = (1 - P_M)1_\omega u(z_0)$. The analogues to (2.5) for z_M and d read: for all $s \in I_0$,

$$\begin{aligned} |z_M(s)|_H^2 &\le \Upsilon_0 \left(|z_0|_H^2 + 2\, |B_M u(z_0)|_{L^2(I_0,\, V')}^2\right)\\ |d(s)|_H^2 &\le \Upsilon_0 2\, |1_\omega\chi(1 - P_M)1_\omega u(z_0)|_{L^2(I_0,\, V')}^2\,. \end{aligned}$$

From Corollary 2.10 it follows that

$$|z_M(s)|_H^2 \le (\Upsilon_0 + \Upsilon\, |B_M|_{\mathcal{L}(H,V')}^2)\, |z_0|_H^2\,; \tag{2.25}$$
$$|d(s)|_H^2 \le \Upsilon\, |1_\omega\chi(1 - P_M)1_\omega|_{\mathcal{L}(H,V')}^2\, |z_0|_H^2\,. \tag{2.26}$$

Then, from (2.23) we obtain

$$|z_M(s_0 + T_*)|_H^2 = |d(s_0 + T_*)|_H^2 \le |z_0|_H^2\,. \tag{2.27}$$

Repeating the argument in the time intervals $I_i \coloneqq (s_0 + iT_*,\, s_0 + (i+1)T_*)$ with initial state $z_0^i \coloneqq z(s_0 + iT_*) = z_M(s_0 + iT_*)$ in (2.24b), finding $u^i = P_M 1_\omega u(z_0^i) \in L^2(I_i,\, H)$, leads to the analogues to (2.25), (2.26), and (2.27): for all $s \in I_i$,

$$\begin{aligned} |z_M(s)|_H^2 &\le (\Upsilon_0 + \Upsilon\, |B_M|_{\mathcal{L}(H,V')}^2)\, \left|z_0^i\right|_H^2\\ |d(s)|_H^2 &\le \Upsilon\, |1_\omega\chi(1 - P_M)1_\omega|_{\mathcal{L}(H,V')}^2\, \left|z_0^i\right|_H^2\,;\\ |z_M(s_0 + (i+1)T_*)|_H^2 &\le \left|z_0^i\right|_H^2\,. \end{aligned}$$

Concatenating these controls we can see that the corresponding solution z_M will remain bounded: $|z_M|_{L^\infty(\mathbb{R}_{s_0},\, H)}^2 \le (\Upsilon_0 + \Upsilon\, |B_M|_{\mathcal{L}(H,V')}^2)\, |z_0|_H^2$.

Next, we notice that $\hat z(t) \coloneqq \mathrm{e}^{-\frac{\lambda}{2}(t-s_0)} z_M(t)$ solves (2.20) in $\mathbb{R}_{s_0}\times\Omega$, with the concatenated control $u = \hat u$ defined by $\hat u|_{I_i} \coloneqq \mathrm{e}^{-\frac{\lambda}{2}(\cdot - s_0)} u(z_0^i)$. Moreover, we have the estimates

$$|\hat z(t)|_H^2 \le \mathrm{e}^{-\lambda(t-s_0)}\, |z_M(t)|_H^2 \le (\Upsilon_0 + \Upsilon\, |B_M|_{\mathcal{L}(H,V')}^2) \mathrm{e}^{-\lambda(t-s_0)}\, |z_0|_H^2$$

and, using Corollary 2.10 and $\left|z_0^i\right|_H \le |z_0|_H$, for all $i \in \mathbb{N}_0$,

$$\begin{aligned}\left|\mathrm{e}^{\frac{\hat\lambda}{2}(\bullet-s_0)}\hat u\right|^2_{L^2(\mathbb{R}_{s_0},\,H)} &= \lim_{j\to+\infty}\sum_{i=0}^{j}\int_{I_i}\mathrm{e}^{(\hat\lambda-\lambda)(s-s_0)}\left|u(z_0^i)(s)\right|_H^2\,\mathrm{d}s\\ &\le \lim_{j\to+\infty}\sum_{i=0}^{j}\mathrm{e}^{(\hat\lambda-\lambda)iT_*}\int_{I_i}\left|u(z_0^i)(s)\right|_H^2\,\mathrm{d}s\\ &\le \mathrm{e}^{2\widehat{D}\Theta_*}\left|z_0^i\right|_H^2\lim_{j\to+\infty}\sum_{i=0}^{j}\mathrm{e}^{(\hat\lambda-\lambda)iT_*} \le \tfrac{1}{1-\mathrm{e}^{(\hat\lambda-\lambda)T_*}}\mathrm{e}^{2\widehat{D}\Theta_*}\left|z_0\right|_H^2\end{aligned}$$

which ends the proof in the case $T_* \in \mathbb{R}_0$.

(b) *The case* $T_* = +\infty$. In this case the solution of system (2.24) remains bounded with zero control $u = 0$. Indeed the analogue to (2.5) reads $|z(s)|_H^2 \le \mathrm{e}^0\,|z(s_0)|_H^2$, for all $s \ge s_0$. ∎

Now we give 2 examples of families $\mathcal{C}$ which satisfy (2.23). For simplicity we suppose that $\omega := \Pi_{j=1}^n(l_{j,1},\, l_{j,2}) \subset \Omega$ is an open nonempty rectangle.

Example 2.14. **Eigenfunctions of the Laplacian operator.** Here we choose $0 \ne \chi \in C^\infty(\overline{\Omega})$ such that $\operatorname{supp}\chi \subseteq \overline{\omega}$ and we let $\{\Psi_{R,\,i} \mid i \in \mathbb{N}_0\}$ be a complete system of eigenfunctions of the negative Laplacian in ω with homogeneous Dirichlet boundary conditions, which are ordered according to the increasing sequence of the (repeated) eigenvalues: $0 < \lambda_i \le \lambda_{i+1}$, $\lim_{i\to\infty} = \infty$. We define

$$\Psi_i(x) := \begin{cases}\Psi_{R,\,i}(x), & \text{if } x \in \omega\\ 0, & \text{if } x \in \Omega\setminus\overline{\omega},\end{cases}$$

and set $\mathcal{C} = \{\Psi_i \mid i \in \{1,\,2,\,\ldots,\,M\}\}$. Let $P_M\colon H \to \mathcal{S}_\mathcal{C}$ be the orthogonal projection in H onto $\mathcal{S}_\mathcal{C} = \operatorname{span}\mathcal{C}$. We observe that $1_\omega(1-P_M)\chi 1_\omega v$ and $(1-P_M^R)(\chi v)|_\omega$ coincide in ω. Here $P_M^R\colon L^2(\omega) \to \mathcal{S}_{\mathcal{C}_R^M}$ is the orthogonal projection in $L^2(\omega)$ onto $\mathcal{S}_{\mathcal{C}_R^M} := \operatorname{span}\{\Psi_{R,\,i} \mid i \in \{1,\,2,\,\ldots,\,M\}\}$. Thus we obtain

$$\begin{aligned}(1_\omega\chi(1-P_M)1_\omega z, v)_{V',\,V} &= (z|_\omega\,, (1-P_M^R)(\chi v)|_\omega)_{L^2(\omega)}\\ &\le |z|_H\left|1-P_M^R\right|_{\mathcal{L}(H^1(\omega),L^2(\omega))}|\chi v|_{H^1(\omega)}\end{aligned}$$

and, since by assumption $\chi|_{\partial\omega} = 0$, we arrive at

$$\begin{aligned}|1_\omega\chi(1-P_M)1_\omega|_{\mathcal{L}(H,V')} &\le \left|1-P_M^R\right|_{\mathcal{L}(H_0^1(\omega),L^2(\omega))}|\chi\cdot|_{\mathcal{L}(V,\,H_0^1(\omega))}\\ &\le 2\,|\chi|_{C^1(\overline{\Omega})}(\lambda_M+1)^{-\frac12}.\end{aligned}$$

Consequently condition (2.23) is satisfied provided that $\lambda_M + 1 \ge (2\,|\chi|_{C^1(\overline{\Omega})}\,\Upsilon)^2$. Furthermore, from the asymptotic behavior $\lambda_M \ge C_0 M^{\frac{2}{n}}$ (cf. [14, Corollary 1]) we can also arrive at the sufficient condition $M \ge C_0^{-\frac{n}{2}}(2\,|\chi|_{C^1(\overline{\Omega})}\,\Upsilon)^n$, which gives us an upper bound on the number M of controls which are needed to stabilize the system.

Example 2.15. **Piecewise constant controls.**

Here we consider a uniform partition of ω where each interval $(l_{j,\,1},\, l_{j,\,2})$ is divided into p_i intervals: $I_{j,\,k} = (l_{j,\,1}+k_j\frac{\bar l_j}{p_j},\, l_{j,\,1}+(k_j+1)\frac{\bar l_j}{p_j})$, with $k_j \in \{0,\,1,\,\ldots,\,p_j-1\}$ and $\bar l_j := l_{j,\,2}-l_{j,\,1}$. In this way, our rectangle is divided into $M = \Pi_{j=1}^n p_j$ sub-rectangles

$$\{R_i \mid i \in \{1,\,2,\,\ldots,\,M\}\} = \{\Pi_{j=1}^n I_{j,\,k_j} \mid k_j \in \{0,\,1,\,\ldots,\,p_j-1\}\}.$$

Let us set $\mathcal{C} = \left\{\Psi_i = \frac{1}{|1_{R_i}|_H} 1_{R_i} \mid i \in \{1, 2, \dots, M\right\} \in H$, and $\chi = 1$. For given $v \in V$ and $z \in H$ we find that

$$(1_\omega(1 - P_M)1_\omega z, v)_{V', V} = (z, 1_\omega(1 - P_M)1_\omega v)_H$$
$$= \left(z|_\omega, v|_\omega - \sum_{i=1}^{M} (v, \Psi_i)_H \Psi_i|_\omega\right)_{L^2(\omega)} = \sum_{i=1}^{M} \left(z|_{R_i}, \varphi_i\right)_{L^2(R_i)},$$

where $\varphi_i := v|_{R_i} - (v|_{R_i}, \Psi_i|_{R_i})_{L^2(R_i)} \Psi_i|_{R_i} = v|_{R_i} - \frac{1}{|1|^2_{L^2(R_i)}} (v|_{R_i}, 1)_{L^2(R_i)}$ has zero average in R_i. This allows to conclude that $|\nabla \varphi_i|^2_{L^2(R_i)^n} \ge \beta_i |\varphi_i|^2_{L^2(R_i)}$ where β_i is the smallest positive eigenvalue of the Laplace–Neumann problem in the rectangle R_i: $-\Delta\phi = \beta_i \phi$ in R_i, $\partial_\nu \phi = 0$ on ∂R_i. Since $\beta_i = \pi^2 \mu_M$ where $\mu_M := \min\left\{ \frac{p_j^2}{\bar{l}_j^2} \mid j \in \{1, 2, \dots, M\}\right\}$, we find for $z \in H$ and $v \in V$ with $|z|_H = 1$, $|v|_V = 1$ the estimates

$$(1_\omega(1 - P_M)1_\omega z, v)_{V', V} \le \sum_{i=1}^{M} (\mu_M \pi^2)^{-\frac{1}{2}} \left|z|_{R_i}\right|_{L^2(R_i)} \left|\nabla v|_{R_i}\right|_{L^2(R_i)} \le (\mu_M \pi^2)^{-\frac{1}{2}}.$$

Since $\mu_M \to \infty$ as the meshsize tends to 0, we conclude that condition (2.23) is satisfied provided that $\mu_M \ge \frac{\Upsilon^2}{\pi^2}$. Furthermore, in the case we take $p_j = p \in \mathbb{N}_0$, we arrive at the sufficient condition $M^{\frac{1}{n}} = p \ge \frac{\Upsilon}{\pi} \bar{l}$ with $\bar{l} := \max\{\bar{l}_j \mid j \in \{1, 2, \dots, M\}\}$, which gives us an upper bound on the number M of controls we need to stabilize the system: $M \ge \left(\frac{\Upsilon}{\pi} \bar{l}\right)^n$. For the treatment in dimension 1 we refer to [11, Section IV.A].

2.4. Feedback stabilizing rule and Riccati equation. From Theorem 2.13 we know that system (2.20) is stabilizable. Here we show that the control can be taken in feedback form, i.e.

$$u = K(t)z = B_M^* \Pi(t) z,$$

with B_M given in (2.22). To specify the structure of the feedback operator K a suitably defined optimal control problem together with the dynamical programming principle will be used. It will turn out that Π satisfies a differential Riccati equation.

We we shall require the spaces $\mathcal{X}_{s_0} := W(\mathbb{R}_{s_0}, V, V') \times L^2(\mathbb{R}_{s_0}, H)$, and

$$\mathcal{X}_{s_0}^\lambda := \{(z, u) \in \mathcal{X}_{s_0} \mid \mathrm{e}^{\frac{\lambda}{2}\cdot}(z, u) \in \mathcal{X}_{s_0}\},$$

and the cost functionals

$$\mathcal{J}_{s_0}^{(\lambda)}(z, u) := \frac{1}{2}\left(|\mathrm{e}^{\frac{\lambda}{2}\cdot} \mathcal{M} z|^2_{L^2(\mathbb{R}_{s_0}, H)} + |\mathrm{e}^{\frac{\lambda}{2}\cdot} \mathcal{R}^{\frac{1}{2}} u|^2_{L^2(\mathbb{R}_{s_0}, H)}\right),$$

where $\lambda \ge 0$ and we set $\mathcal{J}_{s_0} := \mathcal{J}_{s_0}^{(0)}$. For each $z_0 \in H$ and $s_0 \ge 0$, we consider

$$\textit{Minimize } \mathcal{J}_{s_0}^{(\lambda)}(z, u), \textit{ over } (z, u) \in \mathcal{X}_{s_0}^\lambda \textit{ satisfying } (2.20). \qquad (2.28, s_0)$$

Notice that (z, u) solves (2.20) if, and only if, $(y, v) = \mathrm{e}^{\frac{\lambda}{2}(\cdot - s_0)}(z, u)$ solves

$$\partial_t y + (-\Delta + 1)y + (\rho - \tfrac{\lambda}{2})y + \sigma\cdot\nabla y + B_M v = 0, \qquad (2.29a)$$
$$\partial_\nu y|_\Gamma = 0, \qquad y(s_0) = z_0. \qquad (2.29b)$$

Consequently $(\bar{z},\, \bar{u})$ is a minimizer for (2.28) if, and only if, $(\bar{y},\, \bar{v}) = \mathrm{e}^{\frac{\lambda}{2}(\cdot - s_0)}(\bar{z},\, \bar{u})$ is a minimizer for:

$$\textit{Minimize } \mathcal{J}_{s_0}(y,\, v),\ \textit{ over } (y,\, u) \in \mathcal{X}_{s_0} \textit{ satisfying } (2.29). \tag{2.30, s_0}$$

From now on we focus on problem (2.30). First of all, notice that from Theorem 2.13 both problems (2.28) and (2.30) are well-defined with $(\mathcal{M}, \mathcal{R}) = (1, 1)$ (for example, taking $(\lambda, 2\lambda)$ for $(\hat{\lambda}, \lambda)$). Subsequently, from Lemma 2.6 it follows that they are also well-defined for the choice $(\mathcal{M}, \mathcal{R}) = ((-\Delta + 1)^{\frac{1}{2}}, 1)$.

Let us denote

$$\widehat{\mathcal{X}} \coloneqq \{(z,\, u) \in \mathcal{X}_{s_0} \mid (z,\, u) \text{ satisfy } (2.29)\}$$

and observe that, from Theorem 2.13 the mapping $A_1 \in \mathcal{L}(\widehat{\mathcal{X}}, H)$, $A_1(y,\, v) \coloneqq y(s_0)$ is surjective. Moreover, for given $(z_0,\, c) \in H \times \mathbb{R}_0$ it follows that the set $S = \{(y,\, v) \in A_1^{-1}(\{z_0\}) \mid J(y,\, v) \le c\}$ is bounded in $\widehat{\mathcal{X}}$, if $(\mathcal{M},\, \mathcal{R}) = (1, 1)$ or $(\mathcal{M},\, \mathcal{R}) = ((-\Delta+1)^{\frac{1}{2}},\, 1)$. Hence, from [19, Lemma A.14 and Remark A.15], we know that Problem $(2.30, s_0)$ has a unique minimizer which we denote by $(y^*_{s_0},\, v^*_{s_0}) = (y^*_{s_0},\, v^*_{s_0})(z_0)$. Furthermore, the mapping $z_0 \mapsto (y^*_{s_0},\, v^*_{s_0})(z_0)$ is linear. From Theorem 2.13 and from the fact that $(y,\, v) \mapsto \mathcal{J}_{s_0}(y,\, v)$ is quadratic we can conclude that there exists an operator $\Pi_{s_0} \in \mathcal{L}(H)$ such that

$$\mathcal{J}_{s_0}(y^*_{s_0},\, v^*_{s_0}) = \frac{1}{2}(\Pi_{s_0} z_0,\, z_0)_H, \text{ with } |\Pi_{s_0}|_{\mathcal{L}(H)} \le \overline{C}_{\left[C_{\mathcal{W}},\, \lambda,\, \frac{1}{\lambda}\right]}, \tag{2.31, s_0}$$

with $\overline{C}_{\left[C_{\mathcal{W}},\, \lambda,\, \frac{1}{\lambda}\right]}$ independent of s_0, and where $C_{\mathcal{W}}$ is as in (2.3).

Motivated by the dynamical programming principle we define

$$\begin{aligned} \mathcal{X}_I &\coloneqq W(I,\, V,\, V') \times L^2(I,\, H) \\ \mathcal{I}_I(y,\, v) &\coloneqq \frac{1}{2}\left(|\mathcal{M}y|^2_{L^2(I,\, H)} + |\mathcal{R}^{\frac{1}{2}} v|^2_{L^2(I,\, H)} + (\Pi_{s_1} y(s_1),\, y(s_1))_H\right). \end{aligned}$$

For arbitrary $z_0 \in H$, we consider the finite horizon problem:

$$\textit{Minimize } \mathcal{I}_I(y,\, v),\ \textit{ over } (y,\, v) \in \mathcal{X}_I,\ \textit{ satisfying } (2.29). \tag{2.32, s_0, s_1}$$

Proceeding as above we can prove that Problem 2.32 has a unique minimizer we denote $(y^\bullet_I,\, v^\bullet_I) = (y^\bullet_I,\, v^\bullet_I)(z_0)$, with $z_0 \mapsto (y^\bullet_I,\, v^\bullet_I)(z_0)$ linear.

The next Lemma is the dynamical programming principle for problem $(2.30, s_0)$. Since the result is standard we omit the proof (cf. Lemma 3.10 in [2].)

Lemma 2.16. *The minimizers of Problems $(2.30, s_0)$ and $(2.32, s_0, s_1)$ have the following properties:*

$$(y^*_{s_0},\, v^*_{s_0})(z_0)\big|_I = (y^\bullet_I,\, v^\bullet_I)(z_0) \text{ and } (y^*_{s_0},\, v^*_{s_0})(z_0)\big|_{\mathbb{R}_{s_1}} = (y^*_{s_1},\, v^*_{s_1})(y^*_{s_0}(s_1)).$$

Next we describe how the optimal control $Bv^*_{s_0}$ can be expressed in feedback form. For this purpose we define

$$\widetilde{\mathcal{X}} \coloneqq \left\{ (y,\, v) \in \mathcal{X}_I \;\middle|\; \begin{array}{l} (y,\, v) \text{ satisfies } (2.29), \text{ with} \\ y(s_0) = y_0 \text{ for some } y_0 \in H \end{array} \right\}.$$

We observe that

$$\begin{aligned}&F\colon \widetilde{\mathcal{X}} \to \mathcal{Y} \coloneqq H \times L^2(I,\, V'),\\&(y,\, v) \mapsto (y(s_0) - z_0,\, \partial_t y + Ay + (\rho - \tfrac{\lambda}{2})y + \sigma\cdot\nabla y + B_M v)\end{aligned}$$

is a differentiable mapping and $\mathrm{d}F|_{(y_I^\bullet,\, v_I^\bullet)}\colon (z,\, u) \mapsto F(z,\, u) + (z_0,\, 0)$ is surjective. By the Karush–Kuhn–Tucker Theorem (e.g., see [2, Theorem A.1]) there is a Lagrange multiplier $(\mu^I,\, q^I) \in H \times L^2((s_0, s_1),\, V)$ such that

$$\mathrm{d}\mathcal{I}_I|_{(y_I^\bullet,\, v_I^\bullet)} + (\mu^I,\, q^I) \circ \mathrm{d}F|_{(y_I^\bullet,\, v_I^\bullet)} = 0.$$

That is, for all $(z,\, \xi) \in \mathcal{X}$, we have

$$\begin{aligned}0 &= (\Pi_{s_1} y_I^\bullet(s_1),\, z(s_1))_H + (\mu^I,\, z(s_0))_H + \int_{s_0}^{s_1} (\mathcal{M}^*\mathcal{M} y_I^\bullet,\, z)_H(t)\,\mathrm{d}t\\&\quad + \int_{s_0}^{s_1} \langle \partial_t z + (-\Delta + 1)z + (\rho - \tfrac{\lambda}{2})z + \sigma\cdot\nabla z,\, q^I\rangle_{V',V}(t)\,\mathrm{d}t, \qquad (2.33)\\0 &= \int_{s_0}^{s_1} (\mathcal{R} v_I^\bullet,\, \xi)_H(t)\,\mathrm{d}t + \int_{s_0}^{s_1} \langle B_M\xi,\, q^I\rangle_{V',V}(t)\,\mathrm{d}t. \qquad (2.34)\end{aligned}$$

Relation (2.33) implies that $q = q^I$ solves

$$\begin{aligned}&-\partial_t q + (-\Delta + 1)q + \rho q - \nabla\cdot(q\sigma) + \mathcal{M}^*\mathcal{M} y_I^\bullet = 0, && (2.35a)\\&(q\sigma + \nabla q)\cdot \nu|_\Gamma = 0, \qquad q(s_1) = -\Pi_{s_1} y_I^\bullet(s_1). && (2.35b)\end{aligned}$$

On the other hand (2.34) implies that $\mathcal{R} v_I^\bullet = -B_M^* q^I$. Using Lemma 2.16, we find $v_{s_0}^*(s_1) = -\mathcal{R}^{-1} B_M^* q^I(s_1) = \mathcal{R}^{-1} B_M^* \Pi_{s_1} y_{s_0}^*(s_1)$. That is, the optimal control $\zeta = B_M v_{s_0}^*$ is given in feedback form

$$\zeta(s) = B_M K(s) y_{s_0}^*(s), \text{ with } K(s) \coloneqq \mathcal{R}^{-1} B_M^* \Pi_s, \quad s > s_0 \qquad (2.36)$$

In particular, we observe that $K(s)$ does not depend on the past $t < s$.

Let us now consider the closed-loop system

$$\begin{aligned}&\partial_t y + (-\Delta + 1)y + (\rho - \tfrac{\lambda}{2})y + \sigma\cdot\nabla y + B_M K y = 0, && (2.37a)\\&\partial_\nu y|_\Gamma = 0, \qquad y(s_0) = z_0. && (2.37b)\end{aligned}$$

Theorem 2.17. *Let χ and P_M satisfy the conditions in Theorem 2.13, let $(\mathcal{M},\, \mathcal{R}) = (1, 1)$ or $(\mathcal{M},\, \mathcal{R}) = ((-\Delta + 1)^{\frac{1}{2}},\, 1)$, and let $z_0 \in H$. Then the solution y for (2.37) is defined for all $t \ge s_0$, and it satisfies*

$$|y|^2_{W(\mathbb{R}_{s_0},\, V,\, V')} \le \overline{C}_{\left[C_{\mathcal{W}},\, \lambda,\, \frac{1}{\lambda}\right]} |z_0|_H^2, \text{ and} \qquad (2.38a)$$

$$|y|^2_{C([s_0, +\infty),\, V)} + \sup_{\tau \ge s_0} |y|^2_{L^2((\tau,\, \tau+1),\, \mathrm{D}(A))} \le \overline{C}_{\left[C_{\mathcal{W}},\, \lambda,\, \frac{1}{\lambda}\right]} |z_0|_V^2, \qquad (2.38b)$$

if in addition $z_0 \in V$.

Proof. We know that $y_{s_0}^*(s)$ solves (2.37). From (2.31) we have the uniform boundedness of $|K(s)|_{\mathcal{L}(H)}$, in $s \ge 0$. Thus, proceeding as in the proof of Lemma 2.2 we can arrive to the estimate (cf. (2.5))

$$|y(s)|_H^2 \le \mathrm{e}^{\overline{C}_{\left[C_{\mathcal{W}},\, \lambda,\, \frac{1}{\lambda}\right]}(s - s_0)} |y(s_0)|_H^2,$$

from which we can, in particular, conclude the uniqueness of the solution $y^*_{s_0}(s)$. Estimate (2.38a) follows from $\mathcal{J}_{s_0}(y^*_{s_0},\, B_M K y^*_{s_0})(z_0) = \frac{1}{2}(\Pi_{s_0} z_0,\, z_0)_H \le \overline{C}_{\left[C_{\mathcal{W}},\, \lambda,\, \frac{1}{\lambda}\right]} |z_0|^2_H$ (cf. Lemma 2.6 and (2.31)). On the other hand, from Lemma 2.4, (2.31), and (2.38a), we can derive that

$$\begin{aligned}
&\left|y^*_{s_0}\right|^2_{W((\tau,\,\tau+1),\,\mathrm{D}(A),\,H)} \le \overline{C}_{\left[C_{\mathcal{W}},\,\lambda,\,\frac{1}{\lambda}\right]} \left(\left|y^*_{s_0}(\tau)\right|^2_H + \left|B_M K y^*_{s_0}\right|^2_{L^2((\tau,\,\tau+1),\,H)} \right)\\
&\le \overline{C}_{\left[C_{\mathcal{W}},\,\lambda,\,\frac{1}{\lambda}\right]} \left(\left|y^*_{s_0}(\tau)\right|^2_H + \left|y^*_{s_0}(s_0)\right|^2_H \right) \le \overline{C}_{\left[C_{\mathcal{W}},\,\lambda,\,\frac{1}{\lambda}\right]} |z_0|^2_H\,, \text{ for all } t \ge s_0.
\end{aligned}$$

Finally, from $W((\tau,\,\tau+1),\,\mathrm{D}(A),\,H) \hookrightarrow C([\tau,\,\tau+1],\,V)$ uniformly with respect to $\tau \ge 0$, we obtain the inequality $|y|^2_{C([s_0,\,+\infty),\,V)} \le C \sup_{\tau \ge s_0} |y|^2_{L^2((\tau,\,\tau+1),\,\mathrm{D}(A))} \le \overline{C}_{\left[C_{\mathcal{W}},\,\lambda,\,\frac{1}{\lambda}\right]} |z_0|^2_V$. ☐

The next Lemma can be derived following the arguments in [2, Remark 3.11(b) and proof of Lemma 3.8] and in [12, Section 3.4].

LEMMA 2.18. *The function $\Pi\colon s \mapsto \Pi(s) \coloneqq \Pi_s$, $s \ge 0$, belongs to*

$$\mathcal{P} \coloneqq \left\{ P \in L^\infty(\mathbb{R}_0,\, \mathcal{L}(H)) \;\middle|\; \begin{array}{l} P(t) \textit{ is self-adjoint positive definite for all } t \ge 0, \\ \textit{the family } \{P(t) \mid t \ge 0\} \textit{ is continuous in the} \\ \textit{weak operator topology} \end{array} \right\}$$

and satisfies the differential Riccati equation

$$\dot{\Pi} + \Pi\mathbb{A} + \mathbb{A}^*\Pi - \Pi B_M \mathcal{R}^{-1} B_M^* \Pi + \lambda\Pi + \mathcal{M}^*\mathcal{M} = 0, \tag{2.39}$$

with $\mathbb{A}y \coloneqq (\Delta - 1)y - (\rho - \frac{\lambda}{2})y - \sigma \cdot \nabla y$. Moreover, Π is the unique solution of (2.39) *in the class $\mathcal{P}$.*

Recall that y solves (2.37) if, and only if, $z = \mathrm{e}^{-(\cdot - s_0)\frac{\lambda}{2}} y(t)$ solves

$$\begin{aligned}
&\partial_t z + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + B_M \mathcal{R}^{-1} B_M^* \Pi z = 0, &\text{(2.40a)}\\
&\partial_\nu z|_\Gamma = 0, \qquad z(s_0) = z_0. &\text{(2.40b)}
\end{aligned}$$

Therefore we can conclude the next result.

COROLLARY 2.19. *Under the assumptions of Theorem 2.17 let $\Pi \in \mathcal{P}$ be the unique solution of* (2.39). *Then for any $z_0 \in H$, the solution z of* (2.40) *is defined globally and satisfies, for all $t \ge s_0$,*

$$\mathrm{e}^{(t-s_0)\lambda} |z(t)|^2_H + \int_{s_0}^t \mathrm{e}^{(\tau - s_0)\lambda} (|z(\tau)|^2_V + |\partial_t z(\tau)|^2_{V'})\, \mathrm{d}\tau \le \overline{C}_{\left[C_{\mathcal{W}},\,\lambda,\,\frac{1}{\lambda}\right]} |z_0|^2_H\,, \tag{2.41a}$$

and

$$|z(t)|^2_V + |z|^2_{L^2((t,\,t+1),\,\mathrm{D}(A))} \le \overline{C}_{\left[C_{\mathcal{W}},\,\lambda,\,\frac{1}{\lambda}\right]} \mathrm{e}^{-(t-s_0)\lambda} |z_0|^2_V\,, \tag{2.41b}$$

if $z_0 \in V$.

REMARK 2.20. We have shown that Theorem 2.17 holds for the two choices $(\mathcal{M},\, \mathcal{R}) = (1, 1)$ and $(\mathcal{M},\, \mathcal{R}) = ((-\Delta + 1)^{\frac{1}{2}},\, 1)$, for $B_M = 1_\omega \chi P_M 1\omega$. It may be of interest to investigate alternative triples $(\mathcal{M},\, \mathcal{R},\, B_M)$ for which Theorem 2.17 holds. One such example is the following: suppose ρ is constant and $\sigma = 0$. Then we can restrict ourselves to the subspaces $H_{\mathrm{av}} \subset H$ and $V_{\mathrm{av}} \subset V$ containing the functions with zero mean in Ω, and in that case we can take $\mathcal{M} = (-\Delta)^{\frac{1}{2}}$, since the norms $|\cdot|_V$ and $|\nabla \cdot|_{L^2}$ are equivalent in V_{av}.

3. Stabilization of the coupled system. Here we address the stabilization of the coupled linear system (1.9) where $\mathcal{A}$ is either $\mathcal{A}^{RM}$ or $\mathcal{A}^{FN}$.

3.1. Conditional stabilization of the coupled system. Let $\Pi(t) = \Pi_\alpha(t)$ be the solution of (2.39) with $\lambda = 2\alpha > 0$ and let $U_v(t,s)$ denote the evolution operator generated by $A_v(t) - BR^{-1}B_M^*\Pi_\alpha(t)$. Then, from (2.41a), we have that

$$\|U_v^\alpha\| \coloneqq \sup_{t\ge s_0\ge 0} |e^{\alpha(t-s_0)}U_v(t,s_0)|_{\mathcal{L}(H)} \le \overline{C}_{\left[C_{\mathcal{W}},\,\lambda,\,\frac{1}{\lambda}\right]} \tag{3.1}$$

Recall the parameters d, γ, δ, and the reference trajectory $\vec{\bar{z}} \coloneqq \begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix}$ in systems (1.7) and (1.8). To deal with these systems simultaneously we set the model indicator

$$\imath_{\mathrm{m}} \coloneqq \begin{cases} \bar{v} & \text{for system (1.7)},\\ 1 & \text{for system (1.8)}, \end{cases} \quad \text{and } \|\imath\| \coloneqq |\imath_{\mathrm{m}}|_{L^\infty(\mathbb{R}\times\Omega)}\,.$$

Though it will play no role hereafter, notice that systems (1.7) and (1.8) take the form (2.40) with $\sigma = 0$, therefore we have that $C_{\mathcal{W}} = \overline{C}_{[|\rho|_{\mathcal{W}}]} = \overline{C}_{\left[|\bar{v}|_{\mathcal{W}},|\bar{w}|_{\mathcal{W}}\right]}$, see (2.3).

THEOREM 3.1. *Let us be given $0 < \varepsilon < \min\{\alpha,\,\delta\}$. If*

$$\|U_v^\alpha\|\|\imath\|\gamma d < (\alpha-\varepsilon)(\delta-\varepsilon), \tag{3.2}$$

then for the evolution operator $\mathcal{U}(t,s_0)$ generated by $\mathcal{A}(t) - \mathcal{B}_{\mathcal{M}}R^{-1}B_M^*\Pi_\alpha(t)$ it holds that $|e^{\varepsilon(t-s_0)}\mathcal{U}(t,s_0)|_{\mathcal{L}(H\times H)} \le \widetilde{C}_1$ for all $t \ge s_0 \ge 0$. For a constant $\widetilde{C}_1$ depending on the bound $C_{\mathcal{W}}$ and on the parameters in (3.2).

Proof. Notice that $\mathrm{e}^{\varepsilon(t-s_0)}\mathcal{U}(t,s_0)$ is the evolution operator generated by $\mathcal{A}(t) - \mathcal{B}_{\mathcal{M}}R^{-1}B_M^*\Pi_\alpha(t) + \varepsilon$. Therefore we want to show that, for all $\vec{z}_{\varepsilon,0} = \begin{pmatrix}z_{\varepsilon,v,0}\\ z_{\varepsilon,w,0}\end{pmatrix} \in H\times H$, the global solution $\vec{z}_\varepsilon(t) = \begin{pmatrix}z_{\varepsilon,v}\\ z_{\varepsilon,w}\end{pmatrix}$ of the system

$$\dot{\vec{z}}_\varepsilon(t) = \mathcal{A}(t)\vec{z}_\varepsilon(t) + \varepsilon\vec{z}_\varepsilon(t) - \mathcal{B}_{\mathcal{M}}R^{-1}B_M^*\Pi_\alpha(t)\vec{z}_\varepsilon(t), \quad \vec{z}_\varepsilon(s_0) = \vec{z}_{\varepsilon,0},$$

is bounded. Using Duhamel (variation of constants) formula we integrate the equation $\dot{z}_{\varepsilon,w} = -\delta z_{\varepsilon,w} + \gamma z_{\varepsilon,v} + \varepsilon z_{\varepsilon,w}$ and obtain, for $t \ge s_0$,

$$z_{\varepsilon,w}(t) = e^{(\varepsilon-\delta)(t-s_0)}z_{\varepsilon,w,0} + \gamma\int_{s_0}^t \mathrm{e}^{(\varepsilon-\delta)(t-s)}z_{\varepsilon,v}(s)\,\mathrm{d}s, \tag{3.3a}$$

$$\dot{z}_{\varepsilon,v}(t) = L_\varepsilon z_{\varepsilon,v} - d\imath_{\mathrm{m}} z_{\varepsilon,w}(t), \quad z_{\varepsilon,v}(s_0) = z_{\varepsilon,v,0}, \tag{3.3b}$$

where for simplicity we have denoted $L_\varepsilon \coloneqq A_v(t) - B_M R^{-1}B_M^*\Pi_\alpha(t) + \varepsilon$. Notice that for the evolution operator $U_{v,\varepsilon}(t,s) = \mathrm{e}^{\varepsilon(t-s)}U_v(t,s)$ generated by L_ε it holds $|U_{v,\varepsilon}(t,s)|_{\mathcal{L}(H)} \le \|U_v^\alpha\|\mathrm{e}^{(\varepsilon-\alpha)(t-s)}$.

Therefore, with $t \ge s_0$, we arrive to

$$\begin{aligned}
&|z_{\varepsilon,v}(t)|_H\\
&\le \|U_v^\alpha\|\mathrm{e}^{(\varepsilon-\alpha)(t-s_0)}\left|z_{\varepsilon,v,0}\right|_H + \|U_v^\alpha\|\|\imath\|d\int_{s_0}^t \mathrm{e}^{(\varepsilon-\alpha)(t-s)}|z_{\varepsilon,w}(s)|_H\,\mathrm{d}s \qquad (3.4)\\
&\le \|U_v^\alpha\|\mathrm{e}^{(\varepsilon-\alpha)(t-s_0)}\left|z_{\varepsilon,v,0}\right|_H + \|U_v^\alpha\|\|\imath\|d\int_{s_0}^t \mathrm{e}^{(\varepsilon-\alpha)(t-s)}\mathrm{e}^{(\varepsilon-\delta)(s-s_0)}\left|z_{\varepsilon,w,0}\right|_H\,\mathrm{d}s\\
&\quad + \|U_v^\alpha\|\|\imath\|\gamma d\int_{s_0}^t \mathrm{e}^{(\varepsilon-\alpha)(t-s)}\int_{s_0}^s \mathrm{e}^{(\varepsilon-\delta)(s-\tau)}\left|z_{\varepsilon,v}(\tau)\right|_H\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}$$

That is,

$$
\begin{aligned}
&|z_{\varepsilon,v}(t)|_H \\
&\le \|U_v^\alpha\| \mathrm{e}^{(\varepsilon-\alpha)(t-s_0)} |z_{\varepsilon,v,0}|_H + \|U_v^\alpha\| \|\imath\| d \mathrm{e}^{(\delta-\varepsilon)s_0} \int_{s_0}^t \mathrm{e}^{(\varepsilon-\alpha)t} \mathrm{e}^{(\alpha-\delta)s} |z_{\varepsilon,w,0}|_H \, \mathrm{d}s \\
&\quad + \|U_v^\alpha\| \|\imath\| \gamma d \int_{s_0}^t \mathrm{e}^{(\varepsilon-\alpha)t} \mathrm{e}^{(\delta-\varepsilon)\tau} |z_{\varepsilon,v}(\tau)|_H \, \mathrm{d}\tau \int_\tau^t \mathrm{e}^{(\alpha-\delta)s} \, \mathrm{d}s, \\
&= \|U_v^\alpha\| \mathrm{e}^{(\varepsilon-\alpha)(t-s_0)} |z_{\varepsilon,v,0}|_H + \|U_v^\alpha\| \|\imath\| d \mathrm{e}^{(\delta-\varepsilon)s_0} \int_{s_0}^t \mathrm{e}^{(\varepsilon-\alpha)t} \mathrm{e}^{(\alpha-\delta)s} |z_{\varepsilon,w,0}|_H \, \mathrm{d}s \\
&\quad + \|U_v^\alpha\| \|\imath\| \gamma d \frac{1}{\alpha-\delta} \int_{s_0}^t \left(\mathrm{e}^{(\varepsilon-\delta)t} \mathrm{e}^{(\delta-\varepsilon)\tau} - \mathrm{e}^{(\varepsilon-\alpha)t} \mathrm{e}^{(\alpha-\varepsilon)\tau} \right) |z_{\varepsilon,v}(\tau)|_H \, \mathrm{d}\tau,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&|z_{\varepsilon,v}|_{L^1(\mathbb{R}_{s_0},\,H)} \\
&\le \frac{\|U_v^\alpha\|}{\alpha-\varepsilon} |z_{\varepsilon,v,0}|_H + \|U_v^\alpha\| \|\imath\| d \mathrm{e}^{(\delta-\varepsilon)s_0} |z_{\varepsilon,w,0}|_H \int_{s_0}^{+\infty} \mathrm{e}^{(\alpha-\delta)s} \, \mathrm{d}s \int_s^{+\infty} \mathrm{e}^{(\varepsilon-\alpha)t} \, \mathrm{d}t \\
&\quad + \frac{\|U_v^\alpha\| \|\imath\| d\gamma}{\alpha-\delta} \int_{s_0}^{+\infty} |z_{\varepsilon,v}(\tau)|_H \, \mathrm{d}\tau \int_\tau^{+\infty} \left(\mathrm{e}^{(\varepsilon-\delta)(t-\tau)} - \mathrm{e}^{(\varepsilon-\alpha)(t-\tau)} \right) \mathrm{d}t.
\end{aligned}
$$

For the last single integral we have

$$
\int_\tau^{+\infty} \left(\mathrm{e}^{(\varepsilon-\delta)(t-\tau)} - \mathrm{e}^{(\varepsilon-\alpha)(t-\tau)} \right) \mathrm{d}t = \frac{1}{\delta-\varepsilon} - \frac{1}{\alpha-\varepsilon} = \frac{\alpha-\delta}{(\delta-\varepsilon)(\alpha-\varepsilon)}
$$

which leads us to

$$
\begin{aligned}
|z_{\varepsilon,v}|_{L^1(\mathbb{R}_{s_0},\,H)} \le{}& \frac{\|U_v^\alpha\|}{\alpha-\varepsilon} |z_{\varepsilon,v,0}|_H + \frac{\|U_v^\alpha\| \|\imath\| d}{(\alpha-\varepsilon)(\delta-\varepsilon)} |z_{\varepsilon,w,0}|_H \\
&+ \frac{\|U_v^\alpha\| \|\imath\| d\gamma}{(\delta-\varepsilon)(\alpha-\varepsilon)} |z_{\varepsilon,v}|_{L^1(\mathbb{R}_{s_0},\,H)},
\end{aligned}
$$

and from (3.2) it follows that, with $\xi := \frac{\|U_v^\alpha\| \|\imath\| d\gamma}{(\delta-\varepsilon)(\alpha-\varepsilon)} < 1$,

$$
|z_{\varepsilon,v}|_{L^1(\mathbb{R}_{s_0},\,H)} \le \frac{1}{1-\xi} \left(\frac{\|U_v^\alpha\|}{\alpha-\varepsilon} |z_{\varepsilon,v,0}|_H + \frac{\|U_v^\alpha\| \|\imath\| d}{(\alpha-\varepsilon)(\delta-\varepsilon)} |z_{\varepsilon,w,0}|_H \right).
$$

Therefore, from (3.3a), it follows that $|z_{\varepsilon,v}|_{L^\infty(\mathbb{R}_{s_0},\,H)} \le C_1(|z_{\varepsilon,v,0}|_H + |z_{\varepsilon,w,0}|_H)$, and consequently from (3.4), it follows that $|z_{\varepsilon,w}|_{L^\infty(\mathbb{R}_{s_0},\,H)} \le C_2(|z_{\varepsilon,v,0}|_H + |z_{\varepsilon,w,0}|_H)$, for suitable constants C_1 and C_2. ◻

Corollary 3.2. *If $0 < \varepsilon < \min\{\alpha, \delta\}$, $\vec{z}_0 \in H \times H$ and (3.2) holds true, then the solution of the system $\dot{\vec{z}} = \mathcal{A}(t)\vec{z} - \mathcal{B}_\mathcal{M} R^{-1} B_M^* \Pi_\alpha(t)\vec{z}$, $\vec{z}(s_0) = \vec{z}_0$ satisfies $|\vec{z}(t)|_{H\times H} \le \widetilde{C}_1 \mathrm{e}^{-\varepsilon(t-s_0)} |\vec{z}_0|_{H\times H}$, for all $t \ge s_0 \ge 0$.*

3.2. Remarks on the conditional result. Lack of null controllability of the coupled system. Let us consider the system

$$
\dot{z} = Zz, \; z(0) \in \mathbb{R}^2, \quad \text{with } Z = \begin{bmatrix} -(\alpha-\varepsilon) & d \\ \gamma & -(\delta-\varepsilon) \end{bmatrix}, \quad t \ge 0.
$$

This system is stable if, and only if, the eigenvalues of Z have a nonpositive real part. Since those eigenvalues are the solutions of $(\alpha-\varepsilon+\lambda)(\delta-\varepsilon+\lambda)-\gamma d=0$, then the stability holds if, and only if, the real part of each of the two values $-(\alpha+\delta)\pm\sqrt{(\alpha+\delta)^2+4\gamma d-4(\alpha-\varepsilon)(\delta-\varepsilon)}$ is nonpositive, which is equivalent to the inequality $\gamma d-(\alpha-\varepsilon)(\delta-\varepsilon)\leq 0$.

We see that, for given γ, d, δ, and ε, we can choose α big enough such that the condition $\gamma d-(\alpha-\varepsilon)(\delta-\varepsilon)<0$ holds true. This condition may look like (3.2) where α is also at our disposal. However the *transient bound* $\|U_v^\alpha\|$ does depend on α. From [12, Theorem 3.4 and Figure 10(a)] we can also guess that this dependence could be like $C_1\mathrm{e}^{C_2\alpha^{\frac{2}{3}}}$ with suitable constants C_1 and C_2, for big α. See also [9].

In other words we expect to have $\inf\limits_{\alpha>0}\frac{\|U_v^\alpha\|\|\imath\|}{\alpha}>0$. In that case (3.2) is truly conditional on the parameters γ, d, δ, and ε. In particular, the necessity of the condition (3.2), in Theorem 3.1, would mean that the parameters γ, d, δ in systems (1.7) and (1.8) cannot be taken arbitrarily. However, we do not know whether (3.2) is necessary, we have only proven its sufficiency.

We would also like to remark that though null controllability holds for the uncoupled linearized system, it does not hold for the coupled one (cf. [4, Section 2.2]).

4. Local stabilization of the nonlinear system. Here we show that the feedback rule $-\mathcal{B}_\mathcal{M}R^{-1}B_M^*\Pi_\alpha(t)$ constructed to stabilize exponentially the linear system (1.9) to zero, with rate $\alpha=\frac{\lambda}{2}$, also stabilizes the nonlinear system (1.5) to zero, with the same rate, provided $\vec{z}_0$ is small enough.

Again in order to deal with the FitzHugh–Nagumo and Rogers–McCulloch models simultaneously we define another model indicator:

$$\jmath_\mathrm{m}\coloneqq\begin{cases}1 & \text{for system (1.7),}\\ 0 & \text{for system (1.8).}\end{cases}$$

System (1.5), under the feedback control becomes the closed loop system

$$\dot{\vec{z}}=\mathcal{A}_{\Pi_\alpha}\vec{z}+\mathcal{F}(\vec{z}),\qquad \vec{z}(0)=\vec{z}_0, \tag{4.1}$$

with the operator $\mathcal{A}_{\Pi_\alpha}\coloneqq\begin{pmatrix}\Delta-(3a\bar{v}^2-2b\bar{v}+c+\jmath_\mathrm{m}d\bar{w}) & -d\imath_\mathrm{m}\\ \gamma & -\delta\end{pmatrix}-\mathcal{B}_\mathcal{M}R^{-1}B_M^*\Pi_\alpha$ and the nonlinear function $\mathcal{F}(\vec{z})=\begin{pmatrix}-az_v^3-(-b+3a\bar{v})z_v^2-\jmath_\mathrm{m}dz_vz_w\\ 0\end{pmatrix}$.

4.1. Local stabilization for strong regularity. To derive the result for the nonlinear system we will need more regularity for the solutions. Thus we will ask more regularity for the initial conditions. Here we consider initial conditions in $V\times H$, instead of in $H\times H$ as in Corollary 3.2.

Theorem 4.1. *If $0<\varepsilon<\min\{\alpha,\,\delta\}$ and (3.2) holds true, then there is $\epsilon>0$ with the following property: if $|\vec{z}_0|_{V\times H}\leq\epsilon$, then there exists a solution for the system (4.1), in $\mathbb{R}_0\times\Omega$, which belongs to $L^2_\mathrm{loc}(\mathbb{R}_0,\,\mathrm{D}(A)\times H)\cap C([0,\,+\infty),\,V\times H)$, is unique, and satisfies*

$$|\vec{z}(t)|_{V\times H}\leq C\mathrm{e}^{-\varepsilon(t-s_0)}|\vec{z}_0|_{V\times H},\quad \textit{for all } t\geq 0, \tag{4.2}$$

for a suitable constant C independent of $(\epsilon,\,\vec{z}_0)$.

To prove Theorem 4.1 we will use a fixed point argument, following the procedure in [2, Section 4]. We start with a more regular version of Corollary 3.2.

Corollary 4.2. *If $0 < \varepsilon < \min\{\alpha, \delta\}$, $\vec{z}_0 \in V \times H$, and (3.2) holds true, then for a suitable constant $\widetilde{C}_2$, independent of $\vec{z}_0$, the solution of the system*

$$\dot{\vec{z}} = \mathcal{A}_{\Pi_\alpha} \vec{z}, \quad \vec{z}(s_0) = \vec{z}_0$$

satisfies $\sup_{t \ge s_0} \left| \mathrm{e}^{\varepsilon(\cdot - s_0)} \vec{z}(\cdot) \right|^2_{W((t,t+1),\,\mathrm{D}(A),\,H) \times H^1((t,t+1),\,H)} \le \widetilde{C}_2 \left| \vec{z}_0 \right|^2_{V \times H}$.

Proof. As in the proof of Theorem 3.1 we denote $\begin{pmatrix} z_{\varepsilon,v} \\ z_{\varepsilon,w} \end{pmatrix} := \mathrm{e}^{\varepsilon(\cdot - s_0)} \vec{z}(\cdot)$.

For $t = s_0$ we have, from Lemma 2.3,

$$|z_{\varepsilon,v}|^2_{W((s_0,\,s_0+1),\,\mathrm{D}(A),\,H)} \le \overline{C}_{[C_{\mathcal{W}}]} \left(|z_{\varepsilon,v}(s_0)|^2_V + \left| \widehat{f} \right|^2_{L^2((s_0,\,s_0+1),\,H)} \right)$$

with $f = \widehat{f} := B_M R^{-1} B_M^* \Pi_\alpha z_{\varepsilon,v} - \varepsilon z_{\varepsilon,v} + d\imath_{\mathrm{m}} z_{\varepsilon,w}$ and $\overline{C}_{[C_{\mathcal{W}}]} = \overline{C}_{[|\bar{v}|_{\mathcal{W}},\,|\bar{w}|_{\mathcal{W}}]}$. Thus, from Corollary 3.2

$$|z_{\varepsilon,v}|^2_{W((s_0,\,s_0+1),\,\mathrm{D}(A),\,H)} \le C_1 \left(|z_{\varepsilon,v}(s_0)|^2_V + |\vec{z}_0|^2_{H \times H} \right) \le C_2 \left| \vec{z}_0 \right|^2_{V \times H}. \tag{4.3}$$

On the other hand, from Lemma 2.4 it follows that for all $t \ge s_0$

$$|z_{\varepsilon,v}(t+1)|^2_V \le \overline{C}_{[C_{\mathcal{W}}]} \left(|z_{\varepsilon,v}(t)|^2_H + \left| \widehat{f} \right|^2_{L^2((t,\,t+1),\,H)} \right)$$

and, again by using Corollary 3.2,

$$|z_{\varepsilon,v}(t+1)|^2_V \le C_3 \left| \vec{z}_0 \right|^2_{H \times H}. \tag{4.4}$$

Finally, (4.3), (4.4), Lemma 2.3, and Corollary 3.2, give us

$$\begin{aligned} |z_{\varepsilon,v}|^2_{W((t,\,t+1),\,\mathrm{D}(A),\,H)} &\le \overline{C}_{[C_{\mathcal{W}}]} \left(|z_{\varepsilon,v}(t)|^2_V + \left| \widehat{f} \right|^2_{L^2((t,\,t+1),\,H)} \right) \\ &\le C_4 \left(|\vec{z}_0|^2_{V \times H} + |\vec{z}_0|^2_{H \times H} \right) \le 2C_4 \left| \vec{z}_0 \right|^2_{V \times H}, \quad \text{for all} \quad t \ge s_0. \end{aligned}$$

Further from $\dot{z}_{\varepsilon,w} = -\delta z_{\varepsilon,w} + \gamma z_{\varepsilon,v} + \varepsilon z_{\varepsilon,w}$ and Corollary 3.2 we also have

$$|z_{\varepsilon,w}|^2_{H^1((t,\,t+1),\,H)} \le C_5 \left| \vec{z}_0 \right|^2_{H \times H}.$$

Notice that C_4 and C_5 can be taken independent of t. The proof is complete. ☐

Inspired from Corollary 4.2, taking $s_0 = 0$, we define the Banach space

$$\mathcal{Z}^\varepsilon := \left\{ \vec{z} \in L^2_{\mathrm{loc}}(\mathbb{R}_0,\, H \times H) \;\middle|\; |\vec{z}|_{\mathcal{Z}^\varepsilon} < \infty \right\}$$

endowed with the norm $|\vec{z}|_{\mathcal{Z}^\varepsilon} := \sup_{r \ge 0} \left| \mathrm{e}^{\varepsilon \cdot} \vec{z} \right|_{W((r,\,r+1),\,\mathrm{D}(A) \times H,\,H \times H)}$. We also set $\mathcal{Z}^\varepsilon_{\mathrm{loc}} := \left\{ \vec{z} \in L^2_{\mathrm{loc}}(\mathbb{R}_0,\, H \times H)) \;\middle|\; \left| \mathrm{e}^{\varepsilon \cdot} \vec{z} \right|_{W((r,\,r+1),\,\mathrm{D}(A) \times H,\,H \times H)} < \infty, \text{ for all } r \ge 0 \right\}$.

For a given constant $\varrho > 0$ and $\vec{z}_0 \in V \times H$ we define the subset

$$\mathcal{Z}^\varepsilon_\varrho := \left\{ \vec{z} \in \mathcal{Z}^\varepsilon \mid |\vec{z}|^2_{\mathcal{Z}^\varepsilon} \le \varrho |\vec{z}_0|^2_{V \times H} \right\},$$

and the mapping $\Psi \colon \mathcal{Z}^\varepsilon_\varrho \to \mathcal{Z}^\varepsilon_{\mathrm{loc}}$, $\vec{\bar{z}} \mapsto \vec{z}$, taking a given vector $\vec{\bar{z}}$ to the solution $\vec{z}$ of

$$\dot{\vec{z}} = \mathcal{A}_{\Pi_\alpha} \vec{z} + \mathcal{F}(\vec{\bar{z}}), \qquad \vec{z}(0) = \vec{z}_0. \tag{4.5}$$

LEMMA 4.3. *Under the hypotheses of Theorem 4.1, there exists $\varrho > 0$ such that the following property holds: for any $\gamma \in (0,\,1)$ one can find a constant $\epsilon = \epsilon_\gamma > 0$ such that, for any $\vec{z}_0$ satisfying and $|\vec{z}_0|_{V\times H} \le \epsilon$, the mapping Ψ takes the set $\mathcal{Z}^\varepsilon_\varrho$ into itself and satisfies the inequality*

$$|\Psi(\vec{\bar{z}}_1) - \Psi(\vec{\bar{z}}_2)|_{\mathcal{Z}^\varepsilon} \le \gamma|\vec{\bar{z}}_1 - \vec{\bar{z}}_2|_{\mathcal{Z}^\varepsilon} \quad \textit{for all} \quad \vec{\bar{z}}_1,\,\vec{\bar{z}}_2 \in \mathcal{Z}^\varepsilon_\varrho. \tag{4.6}$$

Proof. We divide the proof into 3 main steps:

ⓢ Step 1: *a preliminary estimate.* Consider the system

$$\dot{\vec{z}} = \mathcal{A}_{\Pi_\alpha}\vec{z} + f, \qquad \vec{z}(0) = \vec{z}_0, \tag{4.7}$$

where $f \in L^2_{\mathrm{loc}}(\mathbb{R}_0,\,H)$. If $\vec{z}$ is the solution of system (4.7) with $f = 0$, by Corollary 4.2,

$$\sup_{r\ge 0} |\mathrm{e}^{\varepsilon\,\boldsymbol{\cdot}}\,\vec{z}(\cdot)|^2_{W((r,\,r+1),\,\mathrm{D}(A)\times H,\,H\times H)} \le \widetilde{C}_2|\vec{z}(0)|^2_{V\times H}. \tag{4.8}$$

We are going to derive a version of this estimate for suitable nonzero $f = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$.

We denote by $\mathcal{S}^f_{0,\,t}\vec{z}_0$ the solution $\vec{z}$ of (4.7). In the case $f = 0$, the operator $\mathcal{S}^0_{0,\,t}$ is linear; by the Duhamel formula we can write

$$\vec{z}(t) = \mathcal{S}^f_{0,\,t}\vec{z}_0 = \mathcal{S}^0_{0,\,t}\vec{z}_0 + \int_0^t \mathcal{S}^0_{s,\,t}f(s)\,\mathrm{d}s \tag{4.9}$$

where $\mathcal{S}^f_{s,\,t}w$ denotes the solution of the system (4.7), with the initial time moved to $t = s$, and the initial condition $\mathcal{S}^f_{s,\,s}w = w$. Further, from Corollary 3.2, it follows in particular that $|\mathrm{e}^{\varepsilon(t-s)}\mathcal{S}^0_{s,\,t}w|^2_{H\times H} \le \widetilde{C}_1^2|w|^2_{H\times H}$; then

$$\begin{aligned} |\vec{z}(t)|^2_{H\times H} &\le 2\left|\mathcal{S}^0_{0,\,t}\vec{z}_0\right|^2_{H\times H} + 2\left|\int_0^t \mathcal{S}^0_{s,\,t}f(s)\,\mathrm{d}s\right|^2_{H\times H} \\ &\le 2\widetilde{C}_1^2\mathrm{e}^{-2\varepsilon t}\left(|\vec{z}_0|^2_{H\times H} + \left(\int_0^t \mathrm{e}^{\varepsilon s}\,|f(s)|_{H\times H}\,\mathrm{d}s\right)^2\right) \end{aligned} \tag{4.10}$$

Now we can find, denoting by $\lfloor t\rfloor \in \mathbb{N}$ the integer satisfying $\lfloor t\rfloor \le t < \lfloor t\rfloor + 1$,

$$\begin{aligned} &\int_0^t \mathrm{e}^{\varepsilon s}|f(s)|_{H\times H}\,\mathrm{d}s \le \sum_{k=0}^{\lfloor t\rfloor}\int_k^{k+1} \mathrm{e}^{-\varepsilon s}\mathrm{e}^{2\varepsilon s}|f(s)|_{H\times H}\,\mathrm{d}s \\ &\le \sum_{k=0}^{\lfloor t\rfloor}\left(\int_k^{k+1}\mathrm{e}^{-2\varepsilon s}\,\mathrm{d}s\right)^{\frac12}\left(\int_k^{k+1}\mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right)^{\frac12} \\ &\le \sup_{\substack{j\in\mathbb{N}\\ 0\le j\le\lfloor t\rfloor}}\left(\int_j^{j+1}\mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right)^{\frac12}\sum_{k=0}^{\lfloor t\rfloor}\left(\int_k^{k+1}\mathrm{e}^{-2\varepsilon s}\,\mathrm{d}s\right)^{\frac12}. \end{aligned}$$

For the sum of the series, a direct computation gives us $\sum_{k=0}^{\lfloor t\rfloor}\left(\int_k^{k+1}\mathrm{e}^{-2\varepsilon s}\,\mathrm{d}s\right)^{\frac12} \le \sum_{k=0}^{\infty}\left(\int_k^{k+1}\mathrm{e}^{-2\varepsilon s}\,\mathrm{d}s\right)^{\frac12} = \left(-\frac{1}{2\varepsilon}(\mathrm{e}^{-2\varepsilon}-1)\right)^{\frac12}\sum_{k=0}^{\infty}\mathrm{e}^{-\varepsilon k} = \frac{(1-\mathrm{e}^{-2\varepsilon})^{\frac12}}{(2\varepsilon)^{\frac12}(1-\mathrm{e}^{-\varepsilon})} = \overline{C}_{\left[\frac{1}{\varepsilon}\right]}$.

So, $\int_0^t \mathrm{e}^{\varepsilon s}|f(s)|_{H\times H}\,\mathrm{d}s \le \overline{C}_{\left[\frac{1}{\varepsilon}\right]} \sup_{\substack{j\in\mathbb{N}\\ 0\le j\le \lfloor t\rfloor}} \left(\int_j^{j+1} \mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right)^{\frac{1}{2}}$ and, using (4.10),

$$\mathrm{e}^{2\varepsilon t}|\vec{z}(t)|^2_{H\times H} \le C_1\left(|\vec{z}_0|^2_{H\times H} + \sup_{\substack{j\in\mathbb{N}\\ 0\le j\le \lfloor t\rfloor}} \int_j^{j+1} \mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right) \tag{4.11}$$

for all $t \ge 0$.

Next, denoting $\begin{pmatrix} z_v \\ z_w \end{pmatrix} \coloneqq \vec{z}$ and using Lemma 2.4, we can obtain for all $r \ge 0$

$$|z_v(r+1)|^2_V \le \overline{C}_{[C_{\mathcal{W}}]}\left(|z_v(r)|^2_H + \left|\widehat{f}\right|^2_{L^2((r,\,r+1),\,H)}\right)$$

with $\widehat{f} \coloneqq -B_M R^{-1} B_M^* \Pi_\alpha z_v + d\imath z_w + f_1$, which implies, using (2.31),

$$|z_v(r+1)|^2_V \le C_2\left(\sup_{t\in(r,\,r+1)} |\vec{z}(t)|^2_{H\times H} + |f_1|^2_{L^2((r,\,r+1),\,H)}\right)$$

and, from (4.11),

$$|z_v(r+1)|^2_V \le C_3 \mathrm{e}^{-2\varepsilon r}\left(|\vec{z}_0|^2_{H\times H} + \sup_{\substack{k\in\mathbb{N}\\ 0\le k\le \lfloor r+1\rfloor}} \int_k^{k+1} \mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right) \tag{4.12}$$

since we have $|f_1|^2_{L^2((r,\,r+1),\,H)} = |f|^2_{L^2((r,\,r+1),\,H\times H)} \le \mathrm{e}^{-2\varepsilon r}\int_r^{r+1} \mathrm{e}^{2\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s \le \mathrm{e}^{-2\varepsilon r}\int_{\lfloor r\rfloor}^{\lfloor r+1\rfloor+1} \mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s \le 2\mathrm{e}^{-2\varepsilon r}\sup_{\substack{k\in\mathbb{N}\\ 0\le k\le \lfloor r+1\rfloor}} \int_k^{k+1} \mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s$.

For $t \in (0,\,1)$, from Lemma 2.3 and (4.11), we can also obtain

$$\begin{aligned}\sup_{t\in[0,\,1]} |z_v(t)|^2_V &\le \overline{C}_{[C_{\mathcal{W}}]}\left(|z_v(0)|^2_V + |\widehat{f}|^2_{L^2((0,\,1),\,H)}\right) \\ &\le C_4\left(|z_v(0)|^2_V + \sup_{t\in[0,\,1]} |\vec{z}(t)|^2_{H\times H} + |f_1|^2_{L^2((0,\,1),\,H)}\right) \\ &\le C_5\left(|\vec{z}_0|^2_{V\times H} + |f|^2_{L^2((0,\,1),\,H\times H)}\right).\end{aligned} \tag{4.13}$$

Equations (4.12) and (4.13) allow us to conclude that

$$|z_v(t)|^2_V \le C_6 \mathrm{e}^{-2\varepsilon t}\left(|\vec{z}_0|^2_{V\times H} + \sup_{\substack{k\in\mathbb{N}\\ 0\le k\le \lfloor t\rfloor}} \int_k^{k+1} \mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right), \quad \text{for all } t \ge 0.$$

Using again Lemma 2.3 and proceeding as above, we can now derive

$$
\begin{aligned}
|z_v|^2_{W((r,r+1),\,\mathrm{D}(A),\,H)} &\le \overline{C}_{[C_{\mathcal{W}}]}\left(|z_v(r)|^2_V + |\widehat{f}|^2_{L^2((r,r+1),\,H)}\right)\\
&\le C_7\left(|z_v(r)|^2_V + \sup_{t\in(r,\,r+1)}|\vec{z}(t)|^2_{H\times H} + |f_1|^2_{L^2((r,r+1),\,H)}\right)\\
&\le C_8\mathrm{e}^{-2\varepsilon r}\left(|\vec{z}_0|^2_{V\times H} + \sup_{\substack{k\in\mathbb{N}\\ 0\le k\le\lfloor r\rfloor}}\int_k^{k+1}\mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right);
\end{aligned}
$$

which implies, since $\partial_t(\mathrm{e}^{\varepsilon\bullet}z_v(\cdot)) = \varepsilon\mathrm{e}^{\varepsilon\bullet}z_v(\cdot) + \mathrm{e}^{\varepsilon\bullet}\partial_t z_v(\cdot)$, that

$$
\sup_{r\ge0}|\mathrm{e}^{\varepsilon\bullet}z_v(\cdot)|^2_{W((r,r+1),\,\mathrm{D}(A),\,H)} \le C_9\left(|\vec{z}_0|^2_{V\times H} + \sup_{k\in\mathbb{N}}\int_k^{k+1}\mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right). \tag{4.14}
$$

Finally, for $\begin{pmatrix} z_{\varepsilon,v}\\ z_{\varepsilon,w}\end{pmatrix} := \mathrm{e}^{\varepsilon\bullet}\begin{pmatrix} z_v(\cdot)\\ z_w(\cdot)\end{pmatrix} = \mathrm{e}^{\varepsilon\bullet}\vec{z}(\cdot)$ we have $\dot{z}_{\varepsilon,w} = -\delta z_{\varepsilon,w} + \gamma z_{\varepsilon,v} + \varepsilon z_{\varepsilon,w}$ and, after integration, $z_{\varepsilon,w}(t) = e^{(\varepsilon-\delta)t}z_{\varepsilon,w}(0) + \gamma\int_0^t \mathrm{e}^{(\varepsilon-\delta)(t-s)}z_{\varepsilon,v}(s)\,\mathrm{d}s$. Therefore, using (4.14), we arrive to

$$
\begin{aligned}
|z_{\varepsilon,w}(t)|^2_H &\le 2\,|z_{\varepsilon,w}(0)|^2_H + 2\gamma^2\left(\sup_{r\ge0}|z_{\varepsilon,v}(r)|_H\int_0^t \mathrm{e}^{(\varepsilon-\delta)(t-s)}\,\mathrm{d}s\right)^2\\
&\le C_{10}\left(|\vec{z}_0|^2_{V\times H} + \sup_{k\in\mathbb{N}}\int_k^{k+1}\mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right).
\end{aligned} \tag{4.15}
$$

Then, from $\dot{z}_{\varepsilon,w} = -\delta z_{\varepsilon,w} + \gamma z_{\varepsilon,v} + \varepsilon z_{\varepsilon,w}$, (4.14), and (4.15), we obtain

$$
|\dot{z}_{\varepsilon,w}(t)|^2_H \le C_{11}\left(|\vec{z}_0|^2_{V\times H} + \sup_{k\in\mathbb{N}}\int_k^{k+1}\mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right). \tag{4.16}
$$

Finally (4.14), (4.15), and (4.16), imply that

$$
\sup_{r\ge0}|\mathrm{e}^{\varepsilon\bullet}\vec{z}(\cdot)|^2_{W((r,r+1),\,\mathrm{D}(A)\times H,\,H\times H)} \le C_{12}\left(|\vec{z}_0|^2_{V\times H} + \sup_{k\in\mathbb{N}}\int_k^{k+1}\mathrm{e}^{4\varepsilon s}|f(s)|^2_{H\times H}\,\mathrm{d}s\right), \tag{4.17}
$$

as desired.

ⓢ Step 2: Ψ *maps* $\mathcal{Z}^\varepsilon_\varrho$ *into itself, if* $|\vec{z}_0|_{V\times H}$ *is small.* Denoting $\begin{pmatrix}\vec{\bar{z}}_v\\ \vec{\bar{z}}_w\end{pmatrix} := \vec{\bar{z}}$, we will replace f by $\mathcal{F}(\vec{\bar{z}}) = \begin{pmatrix}-a\vec{\bar{z}}_v^3 - (-b+3a\bar{v})\vec{\bar{z}}_v^2 - \jmath_{\mathrm{m}} d\vec{\bar{z}}_v\vec{\bar{z}}_w\\ 0\end{pmatrix}$ in (4.17). First we derive suitable estimates for the nonlinear term. We focus on the 3D case, that is $\Omega\subset\mathbb{R}^3$, however, the estimates also hold for the 2D case. We recall the inequalities

$$
|u|_{L^\infty(\Omega)} \le C|u|^{\frac12}_{H^1(\Omega)}|u|^{\frac12}_{H^2(\Omega)} \quad\text{and}\quad |u|_{L^6(\Omega)} \le C|u|_{H^1(\Omega)}
$$

which are given by the Agmon inequality and the Sobolev embedding theorem (see [23, Chapter II, Section 1.4]) and [18, Chapter 2, Theorem 3.6]).

Now, we observe that

$$|\mathcal{F}(\vec{\bar{z}})(s)|_{H\times H}^2 \le C_{13}\left(|\vec{\bar{z}}_v(s)|_{L^6(\Omega)}^6 + |\vec{\bar{z}}_v(s)|_{L^4(\Omega)}^4 + |\vec{\bar{z}}_v(s)|_{L^\infty(\Omega)}^2|\vec{\bar{z}}_w(s)|_{L^2(\Omega)}^2\right)$$
$$\le C_{14}\left(|\vec{\bar{z}}_v(s)|_{H^1(\Omega)}^6 + |\vec{\bar{z}}_v(s)|_{H^1(\Omega)}^4 + |\bar{z}_v(s)|_{H^2(\Omega)}^2|\vec{\bar{z}}_w(s)|_{L^2(\Omega)}^2\right)$$

which implies,

$$\begin{aligned}\sup_{k\in\mathbb{N}}\int_k^{k+1} \mathrm{e}^{4\varepsilon s}|\mathcal{F}(\vec{\bar{z}})(s)|_{H\times H}^2\,\mathrm{d}s &\le \sup_{k\in\mathbb{N}}\sup_{s\in[k,\,k+1]} C_{14}\left(|\mathrm{e}^{\varepsilon s}\vec{\bar{z}}_v(s)|_V^6 + |\mathrm{e}^{\varepsilon s}\vec{\bar{z}}_v(s)|_V^4\right)\\ &\quad + \sup_{k\in\mathbb{N}}\sup_{s\in[k,\,k+1]} C_{15}|\mathrm{e}^{\varepsilon s}\vec{\bar{z}}_w(s)|_H^2\int_k^{k+1}|\mathrm{e}^{\varepsilon s}\vec{\bar{z}}_v(s)|_{\mathrm{D}(A)}^2\,\mathrm{d}s\\ &\le C_{16}\left(|\vec{\bar{z}}|_{\mathcal{Z}^\varepsilon}^6 + |\vec{\bar{z}}|_{\mathcal{Z}^\varepsilon}^4\right).\end{aligned}$$

Thus, inequality (4.17) with $f = \mathcal{F}(\vec{\bar{z}})$ gives us

$$|\Psi(\vec{\bar{z}})|_{\mathcal{Z}^\varepsilon}^2 \le C_{17}\left(|\vec{z}_0|_{V\times H}^2 + |\vec{\bar{z}}|_{\mathcal{Z}^\varepsilon}^6 + |\vec{\bar{z}}|_{\mathcal{Z}^\varepsilon}^4\right). \tag{4.18}$$

If $\vec{\bar{z}} \in \mathcal{Z}_\varrho^\varepsilon$, then

$$|\Psi(\vec{\bar{z}})|_{\mathcal{Z}^\varepsilon}^2 \le C_{17}(1 + \varrho^3|\vec{z}_0|_{V\times H}^4 + \varrho^2|\vec{z}_0|_{V\times H}^2)|\vec{z}_0|_{V\times H}^2 \tag{4.19}$$

and if we set $\varrho = 3C_{17}$ and $\epsilon < \min\left\{1,\ \frac{1}{\varrho}\right\}$, then we obtain $C_{17}(1+\varrho^3\epsilon^4+\varrho^2\epsilon^2) \le \varrho$ if $|\vec{z}_0|_{V\times H} \le \epsilon$, which means that $\Psi(\vec{\bar{z}}) \in \mathcal{Z}_\varrho^\varepsilon$.

ⓢ Step 3: Ψ *is a contraction, if* $|\vec{z}_0|_{V\times H}$ *is smaller.* It remains to prove (4.6). Let us take two functions $\vec{\bar{z}}_1, \vec{\bar{z}}_2 \in \mathcal{Z}_\varrho^\varepsilon$ and let $\Psi(\vec{\bar{z}}_1)$ and $\Psi(\vec{\bar{z}}_2)$ be the corresponding solutions for (4.5). Set $e = \vec{\bar{z}}_1 - \vec{\bar{z}}_2$ and $d^\Psi = \Psi(\vec{\bar{z}}_1) - \Psi(\vec{\bar{z}}_2)$. Then d^Ψ solves (4.7) with $d^\Psi(0) = 0$ and $f = \mathcal{F}(\vec{\bar{z}}_1) - \mathcal{F}(\vec{\bar{z}}_2)$. Therefore, by inequality (4.17), we have

$$|\Psi(\vec{\bar{z}}_1) - \Psi(\vec{\bar{z}}_2)|_{\mathcal{Z}^\varepsilon}^2 \le C_{12}\sup_{t\ge 0}\int_t^{t+1} \mathrm{e}^{4\varepsilon s}|\mathcal{F}(\vec{\bar{z}}_1)(s) - \mathcal{F}(\vec{\bar{z}}_2)(s)|_{H\times H}^2\mathrm{d}s. \tag{4.20}$$

Denoting

$$\begin{pmatrix}\bar{z}_{1v}\\ \bar{z}_{1w}\end{pmatrix} := \vec{\bar{z}}_1, \quad \begin{pmatrix}\bar{z}_{2v}\\ \bar{z}_{2w}\end{pmatrix} := \vec{\bar{z}}_2, \quad \text{and} \quad \begin{pmatrix}e_v\\ e_w\end{pmatrix} := e = \begin{pmatrix}\bar{z}_{1v} - \bar{z}_{2v}\\ \bar{z}_{1w} - \bar{z}_{2w}\end{pmatrix},$$

we find that

$$\begin{aligned}\bar{z}_{1v}^3 - \bar{z}_{2v}^3 &= e_v(\bar{z}_{1v}^2 + \bar{z}_{1v}\bar{z}_{2v} + \bar{z}_{2v}^2), \qquad \bar{z}_{1v}^2 - \bar{z}_{2v}^2 = e_v(\bar{z}_{1v} + \bar{z}_{2v}),\\ \bar{z}_{1v}\bar{z}_{1w} - \bar{z}_{2v}\bar{z}_{2w} &= e_v\bar{z}_{1w} + \bar{z}_{2v}e_w.\end{aligned} \tag{4.21}$$

from which we can obtain

$$\begin{aligned}&|\mathcal{F}(\vec{\bar{z}}_1)(s) - \mathcal{F}(\vec{\bar{z}}_2)(s)|_{H\times H}^2\\ &\le C_{18}|e_v|_H^2\left(|\bar{z}_{1v}|_{L^\infty(\Omega)}^4 + |\bar{z}_{2v}|_{L^\infty(\Omega)}^4 + |\bar{z}_{1v}|_{L^\infty(\Omega)}^2 + |\bar{z}_{2v}|_{L^\infty(\Omega)}^2\right)\\ &\quad + C_{18}\left(|e_v|_{L^\infty(\Omega)}^2|\bar{z}_{1w}|_{L^2(\Omega)}^2 + |\bar{z}_{2v}|_{L^\infty(\Omega)}^2|e_w|_{L^2(\Omega)}^2\right)\\ &\le C_{19}|e_v|_H^2\left(|\bar{z}_{1v}|_{\mathrm{D}(A)}^2 + |\bar{z}_{2v}|_{\mathrm{D}(A)}^2\right)\left(|\bar{z}_{1v}|_V^2 + |\bar{z}_{2v}|_V^2 + 1\right)\\ &\quad + C_{19}\left(|e_v|_{\mathrm{D}(A)}^2|\bar{z}_{1w}|_{L^2(\Omega)}^2 + |\bar{z}_{2v}|_{\mathrm{D}(A)}^2|e_w|_{L^2(\Omega)}^2\right)\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{e}^{4\varepsilon s}|\mathcal{F}(\vec{\bar{z}}_1)(s) - \mathcal{F}(\vec{\bar{z}}_2)(s)|^2_{H\times H}\\
&\le C_{19}|\mathrm{e}^{\varepsilon s}e_v|^2_H\left(|\mathrm{e}^{\varepsilon s}\bar{z}_{1v}|^2_{\mathrm{D}(A)} + |\mathrm{e}^{\varepsilon s}\bar{z}_{2v}|^2_{\mathrm{D}(A)}\right)\left(|\mathrm{e}^{\varepsilon s}\bar{z}_{1v}|^2_V + |\mathrm{e}^{\varepsilon s}\bar{z}_{2v}|^2_V + 1\right)\\
&\quad + C_{19}\left(|\mathrm{e}^{\varepsilon s}e_v|^2_{\mathrm{D}(A)}|\mathrm{e}^{\varepsilon s}\bar{z}_{1w}|^2_{L^2(\Omega)} + |\mathrm{e}^{\varepsilon s}\bar{z}_{2v}|^2_{\mathrm{D}(A)}|\mathrm{e}^{\varepsilon s}e_w|^2_{L^2(\Omega)}\right)\\
&\le C_{19}|e|^2_{\mathcal{Z}^\varepsilon}\left(|\vec{\bar{z}}_1|^2_{\mathcal{Z}^\varepsilon} + |\vec{\bar{z}}_2|^2_{\mathcal{Z}^\varepsilon} + 1\right)\left(|\mathrm{e}^{\varepsilon s}\bar{z}_{1v}|^2_{\mathrm{D}(A)} + |\mathrm{e}^{\varepsilon s}\bar{z}_{2v}|^2_{\mathrm{D}(A)}\right)\\
&\quad + C_{19}\left(|\vec{\bar{z}}_1|^2_{\mathcal{Z}^\varepsilon}|\mathrm{e}^{\varepsilon s}e_v|^2_{\mathrm{D}(A)} + |e|^2_{\mathcal{Z}^\varepsilon}|\mathrm{e}^{\varepsilon s}\bar{z}_{2v}|^2_{\mathrm{D}(A)}\right)
\end{aligned}$$

Therefore, from (4.20), it follows

$$|\Psi(\vec{\bar{z}}_1) - \Psi(\vec{\bar{z}}_2)|^2_{\mathcal{Z}^\varepsilon} \le C_{20}|e|^2_{\mathcal{Z}^\varepsilon}\left(|\vec{\bar{z}}_1|^2_{\mathcal{Z}^\varepsilon} + |\vec{\bar{z}}_2|^2_{\mathcal{Z}^\varepsilon} + 1\right)\left(|\vec{\bar{z}}_1|^2_{\mathcal{Z}^\varepsilon} + |\vec{\bar{z}}_2|^2_{\mathcal{Z}^\varepsilon}\right),$$

and since $\vec{\bar{z}}_1$ and $\vec{\bar{z}}_2$ are both in $\mathcal{Z}^\varepsilon_\varrho$, we arrive to

$$|\Psi(\vec{\bar{z}}_1) - \Psi(\vec{\bar{z}}_2)|^2_{\mathcal{Z}^\varepsilon} \le C_{20}\left(2\varrho|\vec{z}_0|^2_{V\times H} + 1\right)2\varrho|\vec{z}_0|^2_{V\times H}|\vec{\bar{z}}_1 - \vec{\bar{z}}_2|^2_{\mathcal{Z}^\varepsilon} \tag{4.22}$$

Choosing $\epsilon > 0$ as in Step 2, that is $\epsilon < \min\left\{1,\, \frac{1}{\varrho}\right\}$, we find $2\varrho|\vec{z}_0|^2_{V\times H} + 1 < 3$. So choosing $\epsilon > 0$ still smaller so that $\epsilon < \min\left\{1,\, \frac{1}{\varrho},\, \frac{\gamma}{\sqrt{6C_{20}\varrho}}\right\}$, we see that (4.6) holds, provided $|\vec{z}_0|^2_{V\times H} \le \epsilon$.

The proof of Lemma 4.3 is complete. □

Proof of Theorem 4.1. From Lemma 4.3 and the contraction mapping principle it follows that if $\vec{z}_0 \in V\times H$ is sufficiently small, $|\vec{z}_0|_{V\times H} < \epsilon$, then there exists a unique fixed point $\vec{z} = \Psi(\vec{\bar{z}}) = \vec{\bar{z}} \in \mathcal{Z}^\epsilon_\varrho$ for Ψ. It follows from the definitions of Ψ and Z^ϵ_ϱ that $\vec{z}$ solves the system (4.5), with $\vec{\bar{z}} = \vec{z}$. We can conclude that $\vec{z}$ solves (1.5) with the feedback control $\mathcal{B}u = -\mathcal{B}_\mathcal{M}R^{-1}B^*_M\Pi_\alpha\vec{z}$.

Further inequality (4.2) can be concluded from (4.19).

Finally, it remains to prove the uniqueness of the solution for (4.1) in the space $Z := L^2_{\mathrm{loc}}(\mathbb{R}_0,\, \mathrm{D}(A)\times H)\cap C([0,+\infty),\, V\times H) \supset \mathcal{Z}^\epsilon_\varrho$. Let $\vec{z}_1$ and $\vec{z}_2$ be two solutions, in Z, for (4.1) and denote

$$\begin{pmatrix} z_{1v}\\ z_{1w}\end{pmatrix} := \vec{z}_1, \quad \begin{pmatrix} z_{2v}\\ z_{2w}\end{pmatrix} := \vec{z}_2, \quad \text{and} \quad \begin{pmatrix} e_v\\ e_w\end{pmatrix} := e := \begin{pmatrix} z_{1v} - z_{2v}\\ z_{1w} - z_{2w}\end{pmatrix}.$$

It turns out that e solves (4.7) with $f = \mathcal{F}(\vec{z_1}) - \mathcal{F}(\vec{z_2})$, that is,

$$\dot{e} = (\overline{\mathcal{A}} - \mathcal{B}_\mathcal{M}R^{-1}B^*_M\Pi_\alpha)e + \mathcal{F}(\vec{z_1}) - \mathcal{F}(\vec{z_2})$$

with $\overline{\mathcal{A}} := \begin{pmatrix}\Delta - 1 + 1 - (3a\bar{v}^2 - 2b\bar{v} + c + \jmath_\mathrm{m} d\bar{w}) & -d\imath_\mathrm{m}\\ \gamma & -\delta\end{pmatrix}$. Using (2.31) and (4.21), we can obtain

$$\begin{aligned}
\langle\overline{\mathcal{A}}e,\, e\rangle_{V'\times H,\, V\times H} &\le -|e_v|^2_V + \overline{C}_{\left[|(\bar{v},\,\bar{w})|_{L^\infty(\Omega,\,\mathbb{R}^2)}\right]}|e|^2_{H\times H},\\
\langle\mathcal{B}_\mathcal{M}R^{-1}B^*_M\Pi_\alpha e,\, e\rangle_{V'\times H,\, V\times H} &\le \overline{C}_{\left[|(\bar{v},\,\bar{w})|_{L^\infty(\Omega,\,\mathbb{R}^2)},\,\alpha,\,\frac{1}{\alpha}\right]}|e_v|^2_H,\\
\langle\mathcal{F}(\vec{z_1}) - \mathcal{F}(\vec{z_2}),\, e\rangle_{V'\times H,\, V\times H} &\le C_1(|z_{1v}|^2_{L^\infty(\Omega)} + |z_{2v}|^2_{L^\infty(\Omega)} + 1)\,|e_v|^2_H\\
&\quad + \left|e_v^2 z_{1w} + z_{2v}e_w e_v\right|_{L^1(\Omega)},\\
\left|e_v^2 z_{1w} + z_{2v}e_w e_v\right|_{L^1(\Omega)} &\le |e_v|^2_{L^4(\Omega)}\,|z_{1w}|_{L^2(\Omega)} + |z_{2v}|_{L^\infty(\Omega)}\,|e|^2_{H\times H}.
\end{aligned}$$

Now, from the continuity of the inclusion $H^{\frac{3}{4}}(\Omega) \subset L^4(\Omega)$ (cf. [6, Chapter 4, Section 4.4, Corollary 4.53]) and the fact $H^{\frac{3}{4}}(\Omega)$ can be seen as an interpolation space $H^{\frac{3}{4}}(\Omega) = [H^1(\Omega),\, L^2(\Omega)]_{\frac{1}{4}}$ (cf. [16, Chapter 1, Theorem 9.6 and Remark 9.1]), we can arrive to

$$|e_v|^2_{L^4(\Omega)}\, |z_{1w}|_{L^2(\Omega)} \le C_2\, |e_v|_H^{\frac{1}{2}}\, |e_v|_V^{\frac{3}{2}}\, |z_{1w}|_H \le C_3\, |e_v|^2_H\, |z_{1w}|^4_H + |e_v|^2_V\, .$$

Therefore, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\, |e|^2_{H\times H} &= 2\langle(\overline{\mathcal{A}} - \mathcal{B}_{\mathcal{M}} R^{-1} B^*_M \Pi_\alpha)e + \mathcal{F}(\vec{z_1}) - \mathcal{F}(\vec{z_2}),\, e\rangle_{V'\times H,\, V\times H}\\ &\le C_4 \left(\overline{C}_{\left[|(\bar{v},\, \bar{w})|_{L^\infty(\Omega,\, \mathbb{R}^2)},\, \alpha,\, \frac{1}{\alpha}\right]} + |z_{1v}|^2_{\mathrm{D}(A)} + |z_{2v}|^2_{\mathrm{D}(A)} + |z_{1w}|^4_H + 1\right) |e|^2_{H\times H}\, .\end{aligned}$$

Observe that $\psi \coloneqq C_3 \left(\overline{C}_{\left[|(\bar{v},\, \bar{w})|_{L^\infty(\Omega,\, \mathbb{R}^2)},\, \alpha,\, \frac{1}{\alpha}\right]} + |z_{1v}|^2_{\mathrm{D}(A)} + |z_{2v}|^2_{\mathrm{D}(A)} + |z_{1w}|^4_H + 1\right)$ is a locally integrable function, because $\vec{z_1}$ and $\vec{z_2}$ are in Z. Thus, by the Gronwall lemma we find

$$|e(t)|^2_{H\times H} \le \mathrm{e}^{\int_0^t \psi(s)\,\mathrm{d}s}\, |e(0)|^2_{H\times H} = 0,$$

that is, $\vec{z}_1 = \vec{z}_2$. ▯

4.2. Local stabilization to trajectories. As a straightforward consequence of Theorem 4.1, we have our main result on stabilization to trajectories for system (1.1).

COROLLARY 4.4. *If* $0 < \varepsilon < \min\{\alpha,\, \delta\}$ *and* (3.2) *hold true, then there is* $\epsilon > 0$ *with the following properties: if*

$$\bar{y} = (\bar{v},\, \bar{w}) \in W_{\mathrm{loc}}(\mathbb{R}_0,\, V \times H,\, V' \times H) \cap L^\infty(\mathbb{R}_0,\, L^\infty(\Omega))^2$$

is a solution for system (1.4)*, with* $\bar{y}_0 = (\bar{v}_0,\, \bar{w}_0) \in H \times H$*, and if* $y_0 = (v_0,\, w_0) \in H \times H$ *is such that*

$$(v_0 - \bar{v}_0,\, w_0 - \bar{w}_0) \in V \times H \quad \textit{and} \quad |(v_0 - \bar{v}_0,\, w_0 - \bar{w}_0)|_{V\times H} < \epsilon,$$

then the solution $y = (v,\, w)$ *of the system* (1.1) *with the feedback control* $Bu = -B_M R^{-1} B^*_M \Pi_\alpha(v - \bar{v})$ *goes exponentially to* $\bar{y}$ *with rate* ε*, that is,*

$$|y(t) - \bar{y}(t)|_{V\times H} \le C\mathrm{e}^{-\varepsilon(t-s_0)} |y_0 - \bar{y}_0|_{V\times H}, \quad \textit{for all } t \ge 0,$$

for a suitable constant C *independent of* $(\epsilon,\, y_0 - \bar{y}_0)$*, and the solution* $(v,\, w)$ *is, and is unique, in the affine space* $(\bar{v},\, \bar{w}) + L^2_{\mathrm{loc}}(\mathbb{R}_0,\, \mathrm{D}(A) \times H) \cap C([0,\, +\infty),\, V \times H)$.

5. Numerical examples. We consider the following version of the monodomain equations

$$\begin{aligned} \partial_t v &= (\varkappa\Delta - c1)v - dw - av^3 + bv^2 + Bu + f_1 + f_2, && \text{in } \Omega \times (0,T),\\ \partial_t w &= \gamma v - \delta w, && \text{in } \Omega \times (0,T),\\ \partial_\nu v|_\Gamma &= 0, && \text{on } \Gamma \times (0,T),\\ v(x,0) &= v_0(x) \text{ and } w(x,0) = w_0(x), && \text{in } \Omega, \end{aligned} \tag{5.1}$$

where $\Omega = (0,1) \times (0,1)$, and the parameters are chosen as $\varkappa = 1.5 \cdot 10^{-3}, a = 1.2 \cdot 10^{-3}, b = 0.1304, c = 1.5, d = 215.6, \gamma = 1.2 \cdot 10^{-4}$ and $\delta = 1.2 \cdot 10^{-3}$. For the control we take piecewise constants as described in Example 2.15. Figure 5.1 visualizes the corresponding control domains.

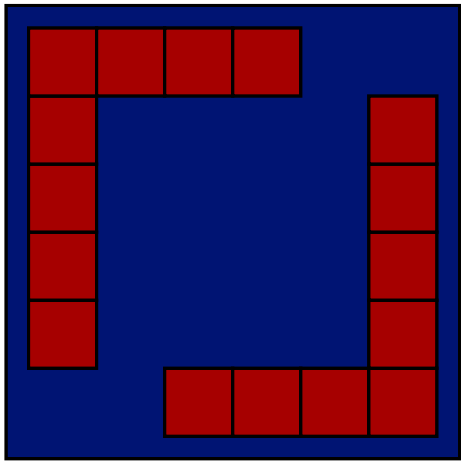

FIG. 5.1. *Control domains with piecewise constants.*

5.1. Termination of a reentry wave. As a test case we consider the termination of a reentry wave modeling cardiac arrhythmia. For this purpose we initialize the system by stimulating the lower boundary of the domain. As a result, a traveling wave is obtained. Placing an external stimulus f_1 within a critical time window leads to a reentry wave as shown in Figure 5.6 (top). Outside of this time window, the stimulus only results in an excitation that immediately starts to collapse (see Figure 5.6 (bottom)). With this in mind, our setup is as follows. We assume that the desired trajectory $\vec{y}_d = (v_d, w_d)$ is obtained from a typical heart rhythm starting at $\vec{y}_d(0) = \vec{y}_{d,0}$, such that the external stimulus is applied before the critical time window is reached. After the external stimulus has collapsed the natural heart rhythm restarts and a second traveling wave is stimulated by means of f_2, see also Figure 5.7 at time $t = 180$. Considering now a perturbation of the initial condition $\vec{y}(0) = \vec{y}_d(0) + \xi$ (postpone initial time), the external stimulus is shifted into the critical time window and causes the excitation of a reentry wave. The desired effect of the feedback law then is to stabilize the perturbed system around the natural heart beat.

5.2. Discretization and the differential Riccati equation. All simulations are generated on an Intel®Xeon(R) CPU E31270 @ 3.40 GHz x 8, 16 GB RAM, Ubuntu Linux 14.04, MATLAB® Version 8.0.0.783 (R2012b) 64-bit (glnxa64).

For the spatial discretization of (5.1) we use a finite difference scheme on a uniform 32×32 grid. The resulting ODE system then reads

$$\begin{aligned} \partial_t v_n &= A_n v_n - d\,1_n w_n + I_{\text{ion}}(v_n) + B_n u + f_1 + f_2, \quad && v_n(0) = v_{d,n}(0) + \xi_v, \\ \partial_t w_n &= \gamma\,1_n v_n - \delta\,1_n w_n, && w_n(0) = w_{d,n}(0) + \xi_w, \end{aligned} \tag{5.2}$$

where the nonlinearity is evaluated pointwise such that $I_{\text{ion}}(v_n) = -a v_n^3 + b v_n^2$. We further have $A_n, 1_n \in \mathbb{R}^{n\times n}$ and $B_n \in \mathbb{R}^{n\times m}$, with $n = 1024$ and $m = 16$. The desired trajectory $(v_{d,n}, w_{d,n})$ is computed as a solution to the uncontrolled system

$$\begin{aligned} \partial_t v_{d,n} &= A_n v_{d,n} - d\,1_n w_{d,n} + I_{\text{ion}}(v_{d,n}) + f_1 + f_2, \\ \partial_t w_{d,n} &= \gamma\,1_n v_{d,n} - \delta\,1_n w_{d,n}. \end{aligned}$$

The solutions of the ODE systems are always obtained by the MATLAB routine `ode45`. The feedback control law $u(t) = -\mathcal{R}^{-1} B_n^* \Pi_n(t)(v_n(t) - v_{d,n}(t))$ is computed by

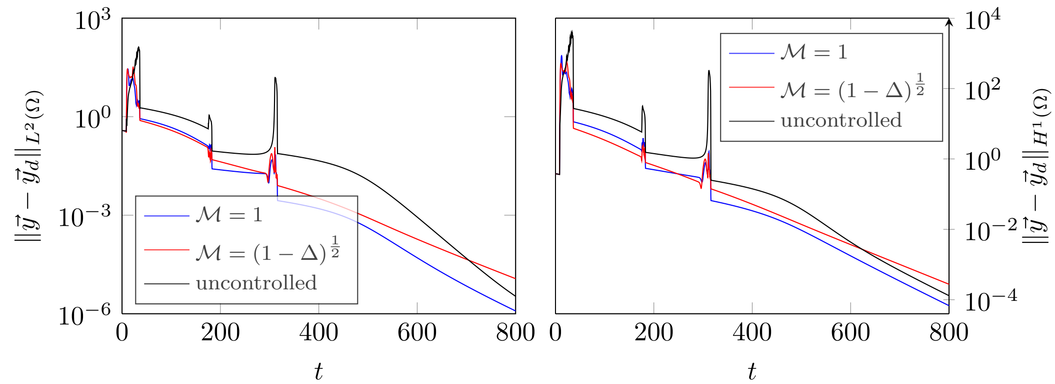

FIG. 5.2. *Linearized system. Comparison of $L^2(\Omega)$ and $H^1(\Omega)$ error for $\mathcal{R}=1$.*

solving the matrix differential Riccati equation associated with the decoupled system, i.e.,

$$\dot{\Pi}_n + (\mathbb{A}(v_{d,n}))^*\Pi_n + \Pi_n\mathbb{A}(v_{d,n}) - \Pi_n B_n \mathcal{R}^{-1} B_n^* \Pi_n + \lambda \Pi_n + \mathcal{M}^*\mathcal{M} = 0, \qquad (5.3)$$

where $\mathbb{A}(v_{d,n}) = A_n - \operatorname{diag}(3\,a\,v_{d,n}^2) + \operatorname{diag}(2\,b\,v_{d,n})$. Following the suggested methodology in [12], we exploit the fact that the desired trajectory is approaching a stationary state (zero). Hence, we solve (5.3) backwards in time using the initialization $\Pi_n(t_f) = \tilde{\Pi}_n$, where $\tilde{\Pi}_n$ solves the algebraic matrix Riccati equation

$$A_n^*\tilde{\Pi}_n + \tilde{\Pi}_n A_n - \tilde{\Pi}_n B_n \mathcal{R}^{-1} B_n^* \tilde{\Pi}_n + \lambda \tilde{\Pi}_n + \mathcal{M}^*\mathcal{M} = 0.$$

The solution of the resulting initial value problem (5.3) is determined by the MATLAB routine `ode45` rather than the Crank-Nicolson inspired scheme proposed in [12]. In this way we only need to evaluate the Riccati operator rather than solving an algebraic Riccati equation in each time step. While the latter approach generally allows for bigger time steps, in our case the performance of `ode45` was better.

5.3. The linearized system. Let us consider the effect of the feedback law when applied to the linearized system, i.e.,

$$\begin{aligned}
\partial_t y_{n,v} &= \left(\mathbb{A}(v_{d,n}) - B_n\mathcal{R}^{-1}B_n^*\Pi_n(t)\right) y_{n,v} - d\,1_n y_{n,w}, \qquad && y_{n,v}(0) = \xi_v,\\
\partial_t y_{n,w} &= \gamma\,1_n y_{n,v} - \delta\,1_n y_{n,w}, && y_{n,w}(0) = \xi_w,
\end{aligned}$$

where $y_{n,v} = v_n - v_{d,n}$ $y_{n,w} = w_n - w_{d,n}$. The shift λ for the desired exponential decay rate of the decoupled system is chosen as $\lambda = 1$. Figure 5.2 shows the decay of the closed loop system for $t \in [0, 800]$ and two different choices of $\mathcal{M}$. We also include a comparison with the uncontrolled solution. In this context, we remark that the system is asymptotically stable when linearized in the zero state. Since the desired trajectory $(v_{d,n}, w_{d,n})$ approaches zero, this implies that the same holds true for the uncontrolled solution. As is reflected in Figure 5.2 the controlled system performs better than the uncontrolled system. We further obtain a better performance with respect to both the $L^2(\Omega)$-norm as well as the $H^1(\Omega)$-norm in the case $\mathcal{M} = 1$. The characteristic "peaks" within the error plots can be explained as follows. The first excitation f_1 modeling the undesired external stimulus happens at $t = 9.52$. At $t = 176.39$ the regular heart rhythm restarts and causes a traveling wave (due to f_2) evolving from the center of

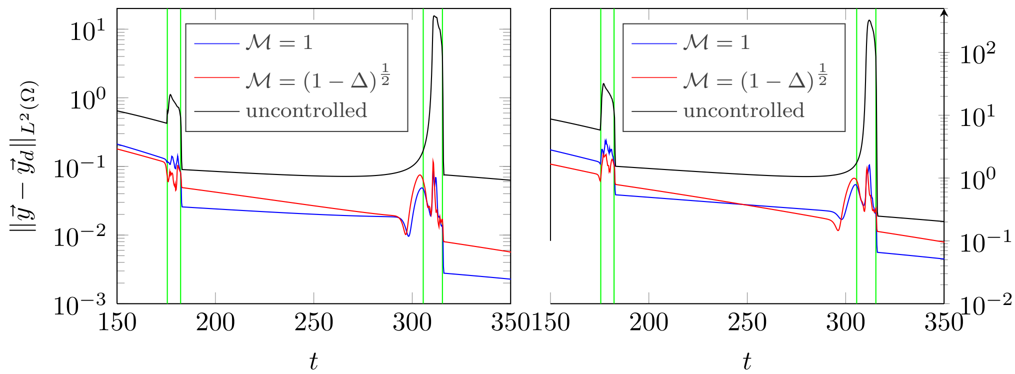

FIG. 5.3. *Linearized system. Comparison of $L^2(\Omega)$ and $H^1(\Omega)$ error for $\mathcal{R} = 1$ (zoom).*

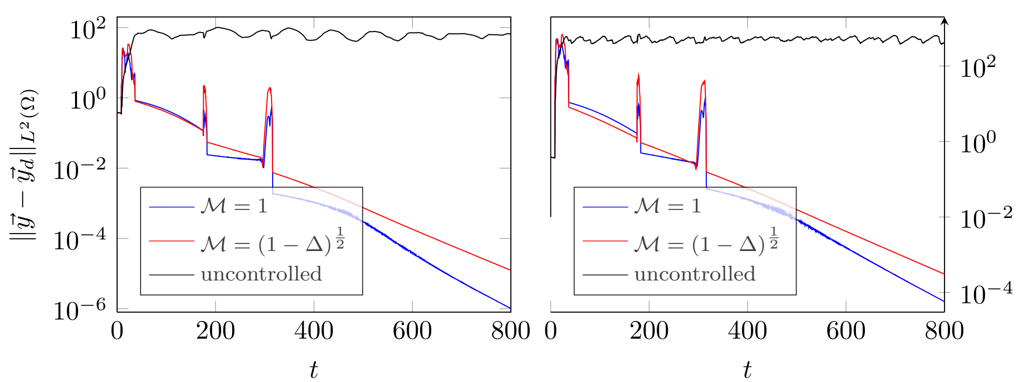

FIG. 5.4. *Nonlinear system. Comparison of $L^2(\Omega)$ and $H^1(\Omega)$ error for $\mathcal{R} = 1$.*

the domain, see again Figure 5.7 at $t = 180$. The third peak corresponds to the sudden collapse of the traveling wave at $t \approx 305$. Figure 5.3 shows the time span between the excitation and the collapse of the traveling wave, respectively. Here, we additionally include (green axis) the time interval in which at least one of the eigenvalues of the system matrix $\mathbb{A}_n(v_{d,n})$ has a positive real part. While it is well-known that for linear time-varying systems there is no one-to-one correspondence between spectral abscissa and stability of the system, it is still worthwhile to mention that the most significant differences to the uncontrolled system appear when $\mathbb{A}(v_{d,n})$ is unstable. This also concerns the relation between the quality of the solutions for $\mathcal{M} = 1$ and $\mathcal{M} = (1-\Delta)^{\frac{1}{2}}$.

5.4. The nonlinear system. We now focus on the full nonlinear system (5.2). Again, the results of the simulations for two different choices of $\mathcal{M}$ are compared with the uncontrolled solutions, see Figure 5.4. Note that the uncontrolled solution now exhibits a periodic behavior and, in particular, does not decay at all. On the other hand, both feedback control laws result in a successful termination of the reentry wave. As already indicated by the results for the linearized system, the choice $\mathcal{M} = 1$ shows a better performance than $\mathcal{M} = (1-\Delta)^{\frac{1}{2}}$. We also include a comparison for the weight matrix $\mathcal{R} = \frac{1}{5}\mathbb{1}$ rather than $\mathcal{R} = \mathbb{1}$. Since this decreases the amount of the control costs within the cost functional, we expect the control to have more influence.

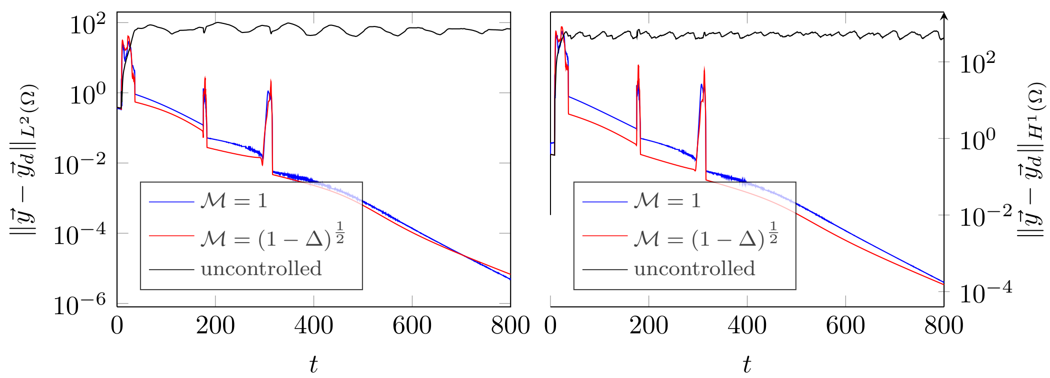

FIG. 5.5. *Nonlinear system. Comparison of* $L^2(\Omega)$ *and* $H^1(\Omega)$ *error for* $\mathcal{R} = \frac{1}{5}$.

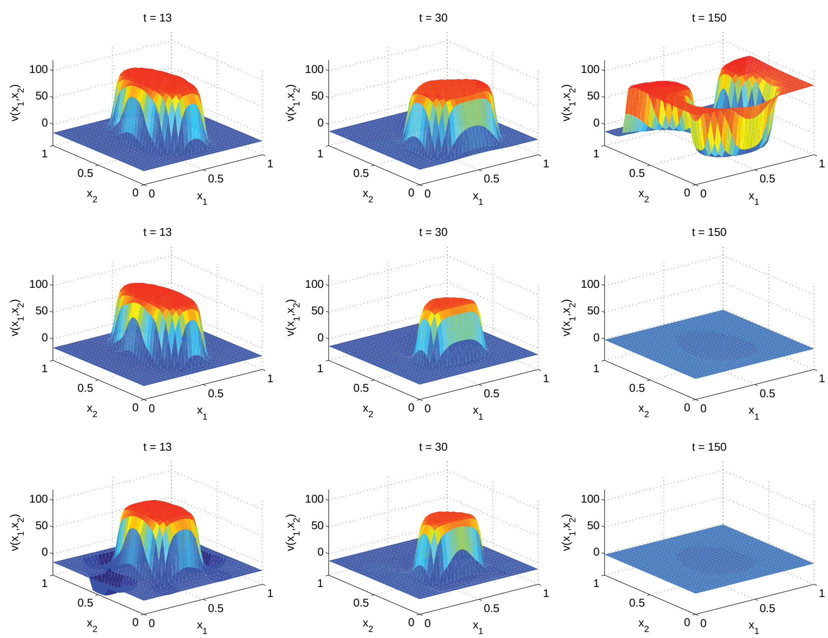

FIG. 5.6. *Evolution of* $v(x,t)$ *for uncontrolled (top), desired (center) and controlled (bottom) system with* $(\mathcal{M}, \mathcal{R}) = (1, 1)$.

Indeed, Figure 5.5 underlines this expectation. Here, the results corresponding to $\mathcal{M} = (1-\Delta)^{\frac{1}{2}}$ are better than those obtained for $\mathcal{M} = 1$. In Figure 5.6 and Figure 5.7 the temporal evolution of $v_{d,n}(x)$ for the uncontrolled, desired and controlled system is shown. While for $t = 13$, the difference between desired and controlled solution is clearly visible, for larger time instances the controlled solution approaches the desired solution. Finally, Figure 5.8 visualizes the action of the piecewise constant control functions. The largest magnitude can be observed after the external stimulus has been applied (see $t = 13$.) As expected, for increasing t, the feedback law approaches

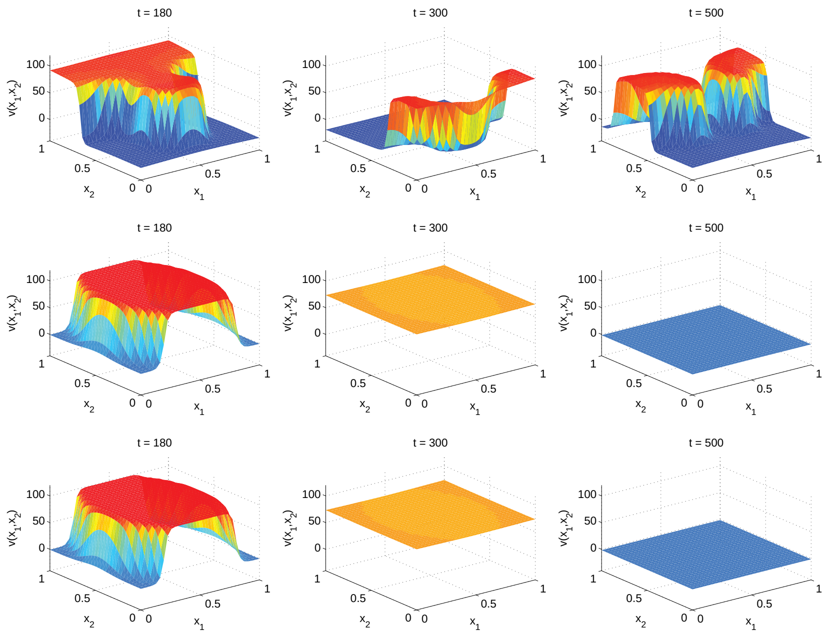

Fig. 5.7. *Evolution of* $v(x,t)$ *for uncontrolled (top), desired (center) and controlled (bottom) system with* $(\mathcal{M},\mathcal{R}) = (1,1)$.

zero (see $t = 500$.)

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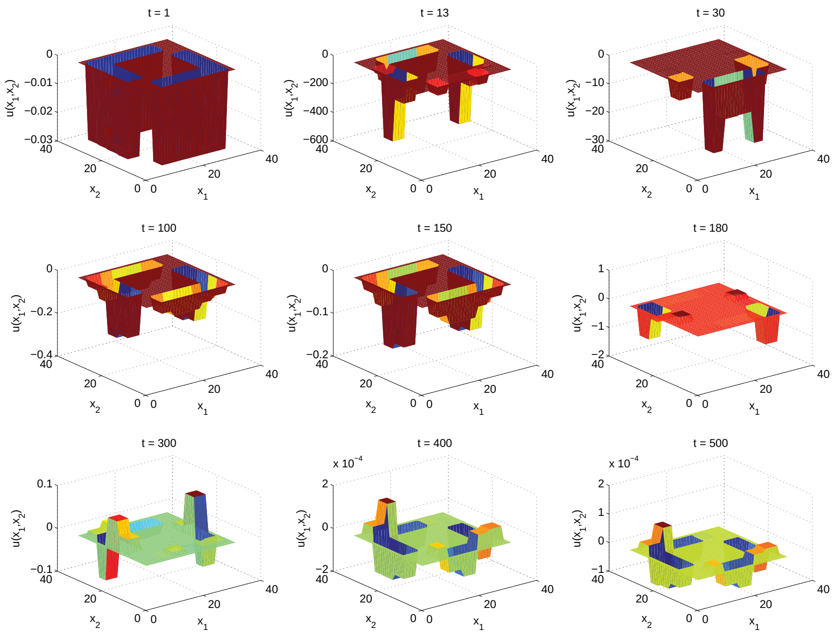

Fig. 5.8. *Control action for different time steps.*

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