A convex analysis approach to optimal controls with switching structure for partial differential equations

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Convex relaxation of binary-continuous optimization problems and their numerical solution by semi-smooth Newton methods are discussed. The proposed framework is especially suited for optimal control problems governed by partial differential equations subject to a switching constraint.

1. Introduction

In the context of control of differential equations, switching control refers to problems with two controls of which only one should be active at every point in time. This is a challenging problem due to its hybrid discrete–continuous nature.

To partially set the stage, consider the parabolic partial differential equation $Ly = Bu$ on $\Omega_T := [0,T] \times \Omega$, where $L = \partial_t - A$ for an elliptic operator A defined on $\Omega \subset \mathbb{R}^n$, and B is defined by $(Bu)(t,x) = \chi_{\omega_1}(x)u_1(t) + \chi_{\omega_2}(x)u_2(t)$ for given control domains $\omega_1, \omega_2 \subset \overline{\Omega}$ (which may include controls acting on the boundary). To promote a switching structure, we propose to use the binary function

$$|v|_0 := \begin{cases} 1 & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases}$$

to construct a penalty that has the value 0 if and only if at most one control is active pointwise. We also add a quadratic term to model control costs for the active controls, i.e., we define for $v = (v_1, v_2) \in \mathbb{R}^2$ the pointwise penalty

$$g(v) = \frac{\alpha}{2}(v_1^2 + v_2^2) + \beta |v_1 v_2|_0$$

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and consider for some $\omega_T \subset \Omega_T$ the problem

$$
\begin{cases} \displaystyle\min_{u\in L^2(0,T;\mathbb{R}^2)} \frac{1}{2}\|y-z\|^2_{L^2(\omega_T)} + \int_0^T g(u(t))\,dt, \\ \qquad \text{s.t.} \quad Ly = Bu. \end{cases} \tag{1.1}
$$

Using the solution operator $S = L^{-1}B : u \mapsto y$, problem (1.1) can be expressed in reduced form as

$$
\min_u \mathcal{F}(u) + \mathcal{G}(u), \tag{1.2}
$$

where $\mathcal{F}$ is smooth and convex, and $\mathcal{G}$ is neither smooth nor convex nor, in fact, lower semi-continuous. This makes both its analysis and its numerical solution challenging; dor example, one cannot rely on standard techniques to guarantee existence of solutions. We therefore consider the relaxed problem

$$
\min_u \mathcal{F}(u) + \mathcal{G}^{**}(u), \tag{1.3}
$$

where $\mathcal{G}^{**}$ is the biconjugate of $\mathcal{G}$, which is always convex. Existence and optimality conditions for the relaxed problem can readily be obtained. However, as we shall see, these optimality conditions are not suitable for semi-smooth Newton techniques. For this reason we introduce a regularization of the form

$$
\min_u \mathcal{F}(u) + ((\mathcal{G}^*)_\gamma)^*(u), \tag{1.4}
$$

where $(\mathcal{G}^*)_\gamma$ is the Moreau–Yosida approximation of $\mathcal{G}^*$; see [16]. In Section 2, we shall provide the abstract existence results, derive optimality conditions, and prove the convergence of solutions to system (1.4) to those of system (1.3). Section 3 is dedicated to give the explicit pointwise characterization of the subdifferential and its Moreau–Yosida approximation to the operators arising from $\mathcal{G}^*$ in the concrete case of switching control; two other functionals involving $|v|_0$ (sparsity and multi-bang penalties) are discussed in Appendix A. These characterizations allow addressing the significant questions related to the relaxation (1.3) of (1.2) in Section 4: We clarify the relation between the value of the costs in (1.3) and in (1.2) in terms of the duality gap between $\mathcal{G}$ and $\mathcal{G}^*$, and show that in certain cases it can be guaranteed to be zero. If this is the case, then the solution to problem (1.3) is also a solution to problem (1.2). Moreover, we analyse to which extent the choice of the functional $(v_1, v_2) \mapsto |v_1 v_2|_0$, when used as a penalty in cost functionals, in fact leads to optimal solutions of switching type. We shall be able to give a sufficient condition on the relation of α and β for (1.3) that rule out free arcs – where $|v_1|$ and $|v_2|$ are both strictly positive but not equal, – whereas singular arcs – on which $|v_1| = |v_2| > 0$ – may remain. Section 5 is concerned with the numerical solution of (1.4) via a path-following semismooth Newton method. To guarantee convergence, a globalization is required. It guarantees superlinear convergence of the semi-smooth Newton algorithm in spite of the challenging cost, which combines continuous and discrete objectives. Finally, Section 6 contains numerical tests for switching controls in the context of an elliptic and a parabolic partial differential equation.

Let us put our work into perspective with the existing literature. Casting the problem of switching controls as a nonconvex optimization problem involving the binary functional $|v|_0$ is certainly new. Concerning the convex relaxation of nonconvex problems, we can draw from existing works. We only mention the monographs [2, 7], where, however, typically the whole problem is dualized, as opposed to only dualizing the nonconvex term as in this work. The partial (Moreau–Yosida) regularization of nonsmooth convex finite-dimensional problems for the purpose of efficiently applying first-order methods was investigated in [3]. Switching control has been studied mainly for ordinary differential equations; here we refer to [18] for a survey with emphasis on stability of switching systems. The Hamilton–Jacobi–Bellman equation for switching controls was extensively studied in [5] and [20]. Switching control in the context of partial differential equations was especially investigated with respect to their improved flexibility over nonswitching controls for stabilization [8, 15]. Controllability for systems with switching controls were studied in [21, 14]. The hybrid nature of continuous and discrete phenomena when the system switches among different modes is the focus of the work in [9, 10]. Altogether the problem of efficiently computing switching controls has received little attention in the literature.

2. Convex relaxation and regularization approach

In this section we introduce the abstract framework and recall relevant concepts from convex analysis. Consider the variational problem

$$\min_{u\in U} \mathcal{J}(u) = \min_{u\in U} \mathcal{F}(u) + \mathcal{G}(u), \tag{$\mathcal{P}$}$$

where U is a Hilbert space and $\mathcal{F} : U \to \mathbb{R}$ is convex. If moreover $\mathcal{G} : U \to \mathbb{R} \cup \{\infty\}$ is convex, any minimizer $\bar u \in U$ satisfies (under a regularity assumption stated below) the following necessary optimality conditions: There exists a $\bar p \in -\partial\mathcal{F}(\bar u) \in U$ such that $\bar p \in \partial\mathcal{G}(\bar u) \subset U^* \equiv U$, which holds if and only if $\bar u \in \partial\mathcal{G}^*(\bar p)$; see, e.g., [17, Proposition 4.4.4]. Here,

$$\mathcal{G}^*(p) = \sup_{u\in U} \langle u, p\rangle - \mathcal{G}(u)$$

denotes the Fenchel conjugate of the convex functional $\mathcal{G}$, and $\partial\mathcal{G}^*$ denotes its convex subdifferential. We thus obtain the primal-dual optimality system

$$\begin{cases} -\bar p \in \partial\mathcal{F}(\bar u), \\ \ \ \bar u \in \partial\mathcal{G}^*(\bar p), \end{cases} \tag{2.1}$$

which is well-defined even for nonconvex $\mathcal{G} : U \to \mathbb{R}_+ \cup \{\infty\}$ as in the situation we are interested in. To argue existence of a solution, we will show that the system (2.1) is the necessary optimality condition for

$$\min_u \mathcal{F}(u) + \mathcal{G}^{**}(u), \tag{2.2}$$

where $\mathcal{G}^{**} = (\mathcal{G}^*)^*$ is the biconjugate of $\mathcal{G}$, and make the following standard assumptions:

(A1) $\mathcal{F}$ is convex, weakly lower-semicontinuous, and radially unbounded.

Proposition 2.1. *Under assumption* (A1), *the system* (2.1) *admits a solution* $(\bar u,\bar p) \in U \times U$. *If* $\mathcal{F}$ *is strictly convex, this solution is unique.*

Proof. By assumption, $\mathcal{G} : U \to \mathbb{R}_+ \cup \{\infty\}$ is bounded from below by 0, which implies that $\mathcal{G}^{**} \geq 0$ as well, see, e.g. [2, Proposition 13.14]. Furthermore, Fenchel conjugates are always lower semicontinuous and convex, see, e.g. [2, Proposition 13.11]. Together with assumption (A1) this implies that $\mathcal{F} + \mathcal{G}^{**}$ is convex, weakly lower semicontinuous, and radially unbounded, and thus a standard subsequence argument yields existence of a minimizer $\bar u \in U$ to (2.2).

Since $\operatorname{dom} \mathcal{F} = U$ ensures that the stability condition

$$\bigcup_{\lambda \geq 0} (\operatorname{dom} \mathcal{F} - \operatorname{dom} \mathcal{G}^{**}) \text{ is a closed vector space,}$$

holds, we can apply the sum rule for the convex subdifferential from [1] and again appeal to [17, Proposition 4.4.4] for $\partial \mathcal{G}^{**}$ to arrive at the necessary optimality conditions (2.1). □

Problem (2.2) can be seen a convex relaxation of problem ($\mathcal{P}$). This approach is thus related to the Γ-regularization in the calculus of variations, see, e.g., [7, Chapter IX], although here we consider a more specific relaxation and pass to the biconjugate only in the nonconvex term rather than to the full biconjugate functional $\mathcal{J}^{**}$.

In general a solution to system (2.1) is not necessarily a minimizer of ($\mathcal{P}$), since for nonconvex $\mathcal{G}$ we cannot rely on equality in the Fenchel–Young inequality (which requires the characterization of the convex subdifferential). In fact, a solution to problem ($\mathcal{P}$) may not even exist. However, for the class of penalties we are interested in, it is possible to show that a solution to system (2.1) is *suboptimal* in the sense that the corresponding functional value is within a certain distance of the infimum. The proof will make use of the following lemma.

Lemma 2.2. *Let* $\mathcal{F}$ *satisfy* (A1), *and let* $(\bar u,\bar p)$ *satisfy* (2.1). *If there is an* $\varepsilon > 0$ *such that*

$$\mathcal{G}(\bar u) + \mathcal{G}^*(\bar p) - \langle \bar p, \bar u\rangle \leq \varepsilon, \tag{2.3}$$

then

$$\mathcal{J}(\bar u) \leq \mathcal{J}(u) + \varepsilon \quad \textit{for all } u \in U.$$

Proof. Assume that $(\bar u,\bar p)$ is a solution to system (2.1) and let $u \in U$ be arbitrary. Recall that the first relation of (2.1) then implies that

$$\mathcal{F}(u) - \mathcal{F}(\bar u) - \langle -\bar p, u - \bar u\rangle \geq 0.$$

Furthermore, inequality (2.3) and the Fenchel–Young inequality (which holds for any proper $\mathcal{G}$) imply that

$$\mathcal{G}(u) - \mathcal{G}(\bar u) - \langle \bar p, u - \bar u\rangle \geq \mathcal{G}(u) - \langle \bar p, u\rangle + \mathcal{G}^*(\bar p) - \varepsilon \geq -\varepsilon.$$

Hence,

$$\begin{aligned}\mathcal{J}(u) - \mathcal{J}(\bar u) &= (\mathcal{F}(u) + \mathcal{G}(u)) - (\mathcal{F}(\bar u) + \mathcal{G}(\bar u))\\ &= (\mathcal{F}(u) - \mathcal{F}(\bar u) - \langle -\bar p, u - \bar u\rangle) + (\mathcal{G}(u) - \mathcal{G}(\bar u) - \langle \bar p, u - \bar u\rangle) \geq -\varepsilon.\end{aligned}$$ □

Since the subdifferential $\partial\mathcal{G}^*$ is in general multivalued and not Lipschitz continuous, system (2.1) is not amenable to numerical solution. We therefore introduce the *Moreau–Yosida regularization* of $\partial\mathcal{G}^*$:

$$u = (\partial\mathcal{G}^*)_\gamma(p) := \frac{1}{\gamma}\left(p - \operatorname{prox}_{\gamma\mathcal{G}^*}(p)\right), \tag{2.4}$$

where

$$\operatorname{prox}_{\gamma f}(v) = \arg\min_w f(w) + \frac{1}{2\gamma}\|w - v\|^2$$

is the *proximal mapping* of f; see [16]. We recall the following properties of $\operatorname{prox}_{\gamma f}$ and $(\partial f)_\gamma$, e.g., from [2, Props. 12.29, 12.15, 23.10, 23.43, 12.9, 16.34]; see also [12, Chapter 4.4].

Proposition 2.3. *Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function on a Hilbert space H. Then,*

(i) $(\partial f)_\gamma = (f_\gamma)'$, *where*

$$f_\gamma(v) = f(\operatorname{prox}_{\gamma\partial f}(v)) + \frac{1}{2\gamma}\|\operatorname{prox}_{\gamma\partial f}(v) - v\|^2$$

is the Moreau-envelope *of f, which is real-valued and convex.*

(ii) $(\partial f)_\gamma$ *is single-valued, maximally monotone and Lipschitz-continuous with constant* γ^{-1},

(iii) $\|(\partial f)_\gamma(v)\|_H \leq \inf\|\partial f(v)\| := \inf_{q\in\partial f(v)}\|q\|_H$ *for all* $v \in H$,

(iv) $f\left(\operatorname{prox}_{\gamma f}(v)\right) \leq f_\gamma(v) \leq f(v)$ *for all* $\gamma > 0$ *and* $v \in H$,

(v) $\operatorname{prox}_{\gamma f} = (\mathrm{Id} + \gamma\partial f)^{-1}$ *(the* resolvent *of* ∂f*).*

From the last property, we can see that

$$(\partial f)_\gamma = \frac{1}{\gamma}\left(\mathrm{Id} - (\mathrm{Id} + \gamma\partial f)^{-1}\right) = \partial f \circ (\mathrm{Id} + \gamma\partial f)^{-1},$$

i.e., $(\partial f)_\gamma$ is indeed the Moreau–Yosida regularization of ∂f.

For brevity, we set $H_\gamma := (\partial\mathcal{G}^*)_\gamma$ from here on and consider the regularized optimality system

$$\begin{cases} -p_\gamma \in \partial\mathcal{F}(u_\gamma), \\ \quad u_\gamma = H_\gamma(p_\gamma). \end{cases} \tag{2.5}$$

Arguing as in Proposition 2.1, existence of a solution follows from the fact that this system is the necessary optimality condition for the problem

$$\min_u \mathcal{F}(u) + ((\mathcal{G}^*)_\gamma)^*(u),$$

using that $(\mathcal{G}^*)_\gamma \leq \mathcal{G}^*$ implies that $0 \leq \mathcal{G}^{**} \leq ((\mathcal{G}^*)_\gamma)^*$ and that $H_\gamma = \partial(\mathcal{G}^*)_\gamma$ is single-valued by Proposition 2.3(i,ii).

Proposition 2.4. *Under assumption (A1), the system (2.5) admits a solution $(u_\gamma,p_\gamma) \in U \times U$. If $\mathcal{F}$ is strictly convex, this solution is unique.*

The convergence $(u_\gamma,p_\gamma) \to (\bar u,\bar p)$ as $\gamma \to 0$ requires additional assumptions on $\mathcal{F}$ and $\mathcal{G}$:

$$\begin{cases} \mathcal{F} \text{ is Fréchet differentiable, } \mathcal{F}' \text{ is weakly closed, and} \\ \{\mathcal{F}(u_\gamma)\}_{\gamma>0} \text{ bounded implies} \{\mathcal{F}'(u_\gamma)\}_{\gamma>0} \text{ bounded,} \end{cases} \tag{A2}$$

$$\{p_\gamma\}_{\gamma>0} \text{ bounded implies } \inf \|\partial\mathcal{G}^*(p_\gamma)\| \text{ bounded.} \tag{A3}$$

We point out that the first assumption is generically satisfied for functionals of the type $\mathcal{F}(u) = F(S(u))$, where

(i) $F : Y \to \mathbb{R}$ is radially unbounded on a Banach space Y,

(ii) F is Fréchet differentiable and F' is bounded on bounded sets,

(iii) $S : U \to Y$ is Fréchet differentiable and $S'(u)^*$ is uniformly bounded on U,

since in this case boundedness of $\mathcal{F}(u_\gamma)$ implies boundedness of $y_\gamma := S(u_\gamma)$ and hence boundedness of $\mathcal{F}'(u_\gamma) = S'(u_\gamma)^* F'(y_\gamma)$. In particular, it holds for many common tracking-type functionals F and bounded linear control-to-state mappings S. The second assumption is more restrictive but satisfied for the class of functionals we shall consider later on.

Proposition 2.5. *If $\mathcal{F}$ and $\mathcal{G}$ satisfy assumptions (A1)–(A3), the family $\{(u_\gamma,p_\gamma)\}_{\gamma>0}$ contains a subsequence converging weakly as $\gamma \to 0$ to a solution $(\bar u,\bar p)$ to system (2.1). If $\mathcal{F}$ is strictly convex, the whole sequence converges.*

Proof. First, observe that

$$((\mathcal{G}^*)_\gamma)^*(0) = \sup_{p\in U} -(\mathcal{G}^*)_\gamma(p) = \inf_{p\in U} (\mathcal{G}^*)_\gamma(p) \le \inf_{p\in U} \mathcal{G}^*(p)$$

by Proposition 2.3(iii). By the optimality of u_γ we thus have for any $\gamma > 0$ that

$$\mathcal{F}(u_\gamma) \le \mathcal{F}(u_\gamma) + ((\mathcal{G}^*)_\gamma)^*(u_\gamma) \le \mathcal{F}(0) + \inf_{p\in U} \mathcal{G}^*(p).$$

Hence, $\{\mathcal{F}(u_\gamma)\}_{\gamma>0}$ is bounded, and assumption (A2) yields that

$$\{p_\gamma\}_{\gamma>0} = \{-\mathcal{F}'(u_\gamma)\}_{\gamma>0}$$

is bounded. From assumption (A3) together with Proposition 2.3(iii) it then follows that for every $\gamma > 0$, we have that

$$\|u_\gamma\|_U = \|H_\gamma(p_\gamma)\|_U \le \inf_{q\in\partial\mathcal{G}^*(p_\gamma)} \|q\|_U \le C,$$

i.e., $\{H_\gamma(u_\gamma)\}_{\gamma>0}$ and $\{u_\gamma\}_{\gamma>0}$ are bounded. Hence, there exist subsequences $\{u_{\gamma_n}\}_{n\in\mathbb{N}}$, $\{p_{\gamma_n}\}_{n\in\mathbb{N}}$ and $\{H_{\gamma_n}(u_{\gamma_n})\}_{n\in\mathbb{N}}$ converging weakly in U to some $\hat u$, $\hat p$, and $\hat y$, respectively. The weak closedness of $\mathcal{F}'$ then yields

$$\hat p = -\mathcal{F}'(\hat u).$$

For the second relation of system (2.1), we first observe that due to the monotonicity of H_γ and $\mathcal{F}'$ and using both relations of system (2.5), we have for any $\gamma_1, \gamma_2 > 0$ that

$$0 \le \langle H_{\gamma_1}(p_{\gamma_1}) - H_{\gamma_2}(p_{\gamma_2}), p_{\gamma_1} - p_{\gamma_2} \rangle = -\langle u_{\gamma_1} - u_{\gamma_2}, \mathcal{F}'(u_{\gamma_1}) - \mathcal{F}'(u_{\gamma_2}) \rangle \le 0.$$

Hence, we deduce from [4, Lemma 1.3(e)] that $\hat{u} = \mathcal{G}^*(\hat{p})$, i.e., $(\hat{u}, \hat{p})$ satisfies system (2.1).

If $\mathcal{F}$ is strictly convex, the solution to system (2.1) is unique, and the claim follows from a subsequence-subsequence argument. □

To conclude this section, we compare the Moreau–Yosida regularization with the following complementarity formulation of the second relation of system (2.1): For any $\gamma > 0$, we have that

$$\begin{aligned} u \in \partial\mathcal{G}^*(p) &\Leftrightarrow p + \gamma u \in (\mathrm{Id} + \gamma\partial\mathcal{G}^*)(p) \\ &\Leftrightarrow p \in (\mathrm{Id} + \gamma\partial\mathcal{G}^*)^{-1}(p + \gamma u) \\ &\Leftrightarrow p = \mathrm{prox}_{\gamma\mathcal{G}^*}(p + \gamma u) \\ &\Leftrightarrow u = \frac{1}{\gamma}\Big((p + \gamma u) - \mathrm{prox}_{\gamma\mathcal{G}^*}(p + \gamma u)\Big) = (\partial\mathcal{G}^*)_\gamma(p + \gamma u) = ((\mathcal{G}^*)_\gamma)'(p + \gamma u), \end{aligned}$$

see also [12, Theorem 4.41]. The subdifferential inclusion can thus be equivalently expressed as a nonlinear equation. While the subdifferential inclusion is explicit with respect to u, the nonlinear equation is implicit. Moreover, the appearance of u in the proximal mapping rules out the effective use of semismooth Newton methods for the applications we have in mind. On the other hand, note that the Moreau–Yosida approximation (2.4) differs only in the absence of γu on the right hand side of the last equality. Hence semismooth Newton methods will be applicable.

3. Switching penalty

To make practical use of the proposed approach, we require an explicit, pointwise, characterization of $\partial\mathcal{G}^*$ and $(\partial\mathcal{G}^*)_\gamma$. For this, we exploit the integral nature of functionals of the type

$$\mathcal{G}(u) = \int_D g(u(x))\,dx$$

with $D \subset \mathbb{R}^d$, for some $d \ge 1$, which allows computing the Fenchel conjugate and its subdifferential pointwise as well; see, e.g., [7, Props. IV.1.2, IX.2.1].

Specifically, we consider here the switching penalty on $\mathbb{R}^2$,

$$g(v) = \frac{\alpha}{2}(v_1^2 + v_2^2) + \beta|v_1 v_2|_0\,. \tag{3.1}$$

Other penalties of this class are discussed in Appendix A. The use of the term $|v_1 v_2|_0$ enhances switching between the control variables v_1 and v_2 in such a manner that simultaneous nontriviality of both of them is penalized. We shall give sufficient conditions which guarantee that in fact v_1 and v_2 are not simultaneously nontrival except for a singular set of controls for which $|v_1| = |v_2| \le \sqrt{2\beta/\alpha}$.

3.1. Fenchel conjugate

To characterize

$$g^*(q) = \sup_{v\in\mathbb{R}^2} v \cdot q - g(v), \tag{3.2}$$

assume that the supremum is attained for some $\bar v \in \mathbb{R}^2$. Then we discriminate the following cases:

(i) $\bar v_1 = 0$, in which case $g(\bar v) = \frac{\alpha}{2}\bar v_2^2$. The supremum in (3.2) is attained if and only if the necessary optimality condition $q_2 - \alpha \bar v_2 = 0$ holds. Solving for $\bar v_2$ and inserting into (3.2) yields

$$g^*(q) = \frac{1}{2\alpha} q_2^2.$$

(ii) $\bar v_2 = 0$, in which case $g(\bar v) = \frac{\alpha}{2}\bar v_1^2$. By the same argument as in case (i) we obtain

$$g^*(q) = \frac{1}{2\alpha} q_1^2.$$

(iii) $\bar v_1, \bar v_2 \neq 0$, in which case $g(\bar v) = \frac{\alpha}{2}(\bar v_1^2 + \bar v_2^2) + \beta$. Again, using the necessary optimality condition for the supremum in (3.2) yields

$$g^*(q) = \frac{1}{2\alpha}(q_1^2 + q_2^2) - \beta.$$

It remains to decide which of these cases is attained based on the value of q. For this purpose, define

$$g_i^*(q) = \begin{cases} \frac{1}{2\alpha} q_i^2 & \text{if } i \in \{1,2\}, \\ \frac{1}{2\alpha}(q_1^2 + q_2^2) - \beta & \text{if } i = 0. \end{cases}$$

Since all g_i^* are finite, the supremum in (3.2) is attained at

$$g^*(q) = \max_{i\in} 0, 1, 2\} g_i^*(q).$$

From the definition, we have that $g_1^*(q) \geq g_2^*(q)$ if $|\bar v_1| \geq |\bar v_2|$ and $g_1^*(q) \geq g_0^*(q)$ if $|\bar v_2| \leq \sqrt{2\alpha\beta}$; similarly for $g_2^*(q)$. Conversely, $g_0^*(q) \geq g_i^*(q)$ if $|\bar v_j| \leq \sqrt{2\alpha\beta}$, $j = 1,2$. Summarizing the above, we have

$$g^*(q) = \begin{cases} \frac{1}{2\alpha} q_1^2 & \text{if } |q_1| \geq |q_2| \text{ and } |q_2| \leq \sqrt{2\alpha\beta}, \\ \frac{1}{2\alpha} q_2^2 & \text{if } |q_1| \leq |q_2| \text{ and } |q_1| \leq \sqrt{2\alpha\beta}, \\ \frac{1}{2\alpha}(q_1^2 + q_2^2) - \beta & \text{if } |q_1|, |q_2| \geq \sqrt{2\alpha\beta}. \end{cases}$$

3.2. Subdifferential

Since g^* is the maximum of a finite number of convex functions, its subdifferential is given by

$$\partial g^*(q) = \overline{\mathrm{co}}\left(\bigcup_{\{i:g^*(q)=g_i^*(q)\}} \left\{(g_i^*)'(q)\right\}\right),$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull; see, e.g., [11, Corollary 4.3.2]. We make a case distinction based on all possibilities for $g^*(q) = g_i^*(q)$, $i \in \{0, 1, 2\}$:

(i) $g^*(q) = g_1^*(q)$ only, which is the case if and only if

$$q \in Q_1 := \left\{q \in \mathbb{R}^2 : |q_1| > |q_2| \text{ and } |q_2| < \sqrt{2\alpha\beta}\right\}.$$

Here the subdifferential is single-valued and given by

$$\partial g^*(q) = \left(\left\{\tfrac{1}{\alpha}q_1\right\}, \{0\}\right).$$

(ii) $g^*(q) = g_2^*(q)$ only, which is the case if and only if

$$q \in Q_2 := \left\{q \in \mathbb{R}^2 : |q_2| > |q_1| \text{ and } |q_1| < \sqrt{2\alpha\beta}\right\}.$$

Here,

$$\partial g^*(q) = \left(\{0\}, \left\{\tfrac{1}{\alpha}q_1\right\}\right).$$

(iii) $g^*(q) = g_0^*(q)$ only, which is the case if and only if

$$q \in Q_0 := \left\{q \in \mathbb{R}^2 : |q_1|, |q_2| > \sqrt{2\alpha\beta}\right\}.$$

Here,

$$\partial g^*(q) = \left(\left\{\tfrac{1}{\alpha}q_1\right\}, \left\{\tfrac{1}{\alpha}q_2\right\}\right).$$

(iv) $g^*(q) = g_1^*(q) = g_0^*(q) \neq g_2^*(q)$, which is the case if and only if

$$q \in Q_{10} := \left\{q \in \mathbb{R}^2 : |q_1| > |q_2| = \sqrt{2\alpha\beta}\right\}.$$

Here, the subdifferential is given by the convex hull of $\{(g_1^*)'(q), (g_0^*)'(q)\}$, i.e.,

$$\partial g^*(q) = \left(\left\{\tfrac{1}{\alpha}q_1\right\}, \left[0, \tfrac{1}{\alpha}q_2\right]\right).$$

To keep the notation concise, we use the convention $[a, b] := [\min\{a, b\}, \max\{a, b\}]$ here and below.

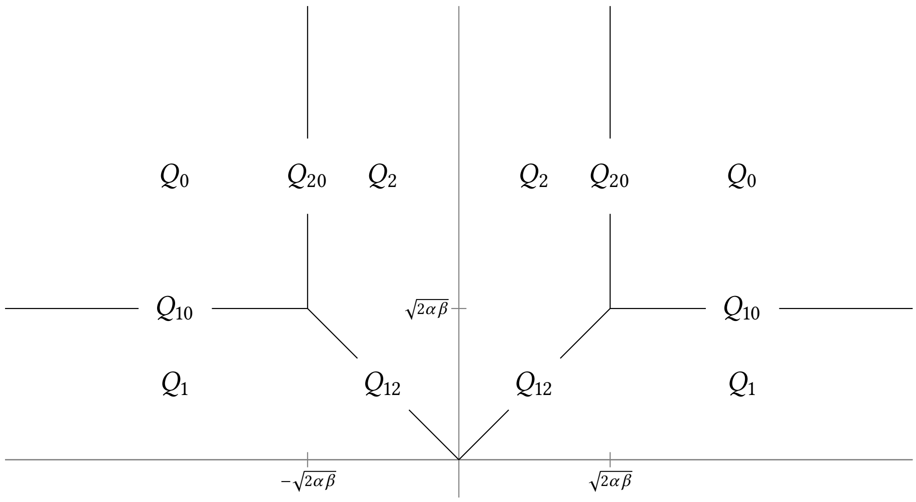


Figure 1: Subdomains $Q_i \subset \mathbb{R}^2$ for the definition of ∂g^*.

(v) $g^*(q) = g_2^*(q) = g_0^*(q) \neq g_1^*(q)$, which is the case if and only if

$$q \in Q_{20} := \left\{ q \in \mathbb{R}^2 : |q_2| > |q_1| = \sqrt{2\alpha\beta} \right\}.$$

Here,

$$\partial g^*(q) = \left(\left[0, \tfrac{1}{\alpha} q_1\right], \left\{\tfrac{1}{\alpha} q_2\right\} \right).$$

(vi) $g^*(q) = g_1^*(q) = g_2^*(q)$, which is the case if and only if

$$q \in Q_{12} := \left\{ q \in \mathbb{R}^2 : |q_1| = |q_2| \leq \sqrt{2\alpha\beta} \right\}.$$

Here,

$$\partial g^*(q) = \left(\left[0, \tfrac{1}{\alpha} q_1\right], \left[0, \tfrac{1}{\alpha} q_2\right] \right).$$

Note that this also includes the case $g^*(q) = g_1^*(q) = g_2^*(q) = g_0^*(q)$, since then $(g_0^*)'(q) \in \partial g^*(q)$.

Since $\mathbb{R}^2$ is the disjoint union of the sets Q_i defined above, see Figure 1, we thus obtain a complete characterization of the subdifferential $\partial g^*(q)$.

3.3. Proximal mapping

For the Moreau–Yosida regularization or the complementarity formulation, we need to compute the proximal mapping of g^* or, equivalently, the resolvent of ∂g^*. For given $\gamma > 0$ and $v \in \mathbb{R}$, the resolvent $w := (\mathrm{Id} + \gamma \partial g^*)^{-1}(v)$ is characterized by the subdifferential inclusion

$$v \in (\mathrm{Id} + \gamma \partial g^*)(w) = \{w\} + \gamma \partial g^*(w). \tag{3.3}$$

Note that this implies

$$v \in \left[w, (1+\tfrac{\gamma}{\alpha})w\right] \qquad \text{or equivalently that} \qquad w \in \left[\tfrac{\alpha}{\alpha+\gamma}v, v\right], \tag{3.4}$$

and hence that $\operatorname{sign}(v_j) = \operatorname{sign}(w_j)$, $j = 1, 2$. We now follow the case discrimination in the characterization of the subdifferential.

(i) $w \in Q_1$: In this case, the subdifferential inclusion (3.3) yields $v_1 = (1+\frac{\gamma}{\alpha})w_1$ and $v_2 = w_2$; solving for w_1, w_2 and inserting the result into the definition of Q_1 yields

$$w = \left(\tfrac{\alpha}{\alpha+\gamma}v_1, v_2\right) \quad \text{and} \quad |v_1| > (1+\tfrac{\gamma}{\alpha})|v_2|, \quad |v_2| < \sqrt{2\alpha\beta}.$$

(ii) $w \in Q_2$: In this case, $v_1 = w_1$ and $v_2 = (1+\frac{\gamma}{\alpha})w_2$, and as in case (i) we have that

$$w = \left(v_1, \tfrac{\alpha}{\alpha+\gamma}v_2\right) \quad \text{and} \quad |v_2| > (1+\tfrac{\gamma}{\alpha})|v_1|, \quad |v_1| < \sqrt{2\alpha\beta}.$$

(iii) $w \in Q_0$: In this case, $v_1 = (1+\frac{\gamma}{\alpha})w_1$ and $v_2 = (1+\frac{\gamma}{\alpha})w_2$, and hence

$$w = \left(\tfrac{\alpha}{\alpha+\gamma}v_1, \tfrac{\alpha}{\alpha+\gamma}v_2\right) \quad \text{and} \quad |v_1| > (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}, \quad |v_2| > (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}.$$

(iv) $w \in Q_{10}$: In this case, $v_1 = (1+\frac{\gamma}{\alpha})w_1$ and $v_2 \in [w_2, (1+\frac{\gamma}{\alpha})w_2]$. Since $\operatorname{sign}(w_2) = \operatorname{sign}(v_2)$, we have from the definition of Q_{10} that $w_2 = \operatorname{sign}(v_2)\sqrt{2\alpha\beta}$. Hence

$$w = \left(\tfrac{\alpha}{\alpha+\gamma}v_1, \operatorname{sign}(v_2)\sqrt{2\alpha\beta}\right) \quad \text{and} \quad \sqrt{2\alpha\beta} \le |v_2| \le (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}, \quad |v_1| > (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}.$$

(v) $w \in Q_{20}$: In this case, $v_2 = (1+\frac{\gamma}{\alpha})w_2$ and $v_1 \in [w_1, (1+\frac{\gamma}{\alpha})w_1]$. As in (iv), we have that

$$w = \left(\operatorname{sign}(v_1)\sqrt{2\alpha\beta}, \tfrac{\alpha}{\alpha+\gamma}v_2\right) \quad \text{and} \quad \sqrt{2\alpha\beta} \le |v_1| \le (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}, \quad |v_2| > (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}.$$

(vi) $w \in Q_{12}$: In this case, $[w_1, (1+\frac{\gamma}{\alpha})w_1]$ and $v_2 \in [w_2, (1+\frac{\gamma}{\alpha})w_2]$. This does not yield an explicit value for w, although the definition of Q_{12} implies that $|w_1| = |w_2| \le \sqrt{2\alpha\beta}$. We therefore turn to the equivalent characterization of w via the proximal mapping

$$w = \operatorname{prox}_{\gamma g^*}(v) = \operatorname*{argmin}_{|z_1|=|z_2|\le\sqrt{2\alpha\beta}} \frac{1}{2\gamma}|z-v|_2^2 + g^*(z).$$

First, assume that $z_1 = z_2 =: z$ (which implies $\operatorname{sign}(v_1) = \operatorname{sign}(z) = \operatorname{sign}(v_2)$). The minimizer of the reduced problem is then given by the projection of the unconstrained minimizer $z = \frac{\alpha}{2\alpha+\gamma}(v_1+v_2)$ to the (convex) feasible set $[0, \sqrt{2\alpha\beta}]$, i.e.,

$$w = \begin{cases} \left(\tfrac{\alpha}{2\alpha+\gamma}(v_1+v_2), \tfrac{\alpha}{2\alpha+\gamma}(v_1+v_2)\right) & \text{if } \tfrac{\alpha}{2\alpha+\gamma}|v_1+v_2| \le \sqrt{2\alpha\beta}, \\ \left(\operatorname{sign}(v_1)\sqrt{2\alpha\beta}, \operatorname{sign}(v_2)\sqrt{2\alpha\beta}\right) & \text{if } \tfrac{\alpha}{2\alpha+\gamma}|v_1+v_2| > \sqrt{2\alpha\beta}. \end{cases}$$

Inserting each of these values for w into the relation $v \in [w, (1+\frac{\gamma}{\alpha})w]$ yields (after some algebraic manipulations)

$$\tfrac{\alpha}{\alpha+\gamma}|v_2| \le |v_1| \le (1+\tfrac{\gamma}{\alpha})|v_2|$$

and

$$\sqrt{2\alpha\beta} \le |v_1|, |v_2| \le (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta},$$

respectively.

We argue similarly for $z_1 = -z_2$ (where $\operatorname{sign}(v_1) = \operatorname{sign}(z) = -\operatorname{sign}(v_2)$). Combining the two cases, we obtain

$$\begin{aligned} w = \Big(&\operatorname{sign}(v_1)\tfrac{\alpha}{2\alpha+\gamma}(|v_1|+|v_2|), \operatorname{sign}(v_2)\tfrac{\alpha}{2\alpha+\gamma}(|v_1|+|v_2|)\Big) \\ &\text{and}\quad \tfrac{\alpha}{\alpha+\gamma}|v_2| \le |v_1| \le (1+\tfrac{\gamma}{\alpha})|v_2|, \quad |v_1|+|v_2| \le (2+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}, \end{aligned}$$

and

$$\begin{aligned} w = \Big(&\operatorname{sign}(v_1)\sqrt{2\alpha\beta}, \operatorname{sign}(v_2)\sqrt{2\alpha\beta}\Big) \\ &\text{and}\quad \sqrt{2\alpha\beta} \le |v_1|, |v_2| \le (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}, \quad |v_1|+|v_2| > (2+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}. \end{aligned}$$

Inserting this into the definition of the Moreau–Yosida regularization

$$(\partial g^*)_\gamma(q) = \frac{1}{\gamma}\left(q - \operatorname{prox}_{\gamma g^*}(q)\right)$$

and simplifying yields

$$(\partial g^*)_\gamma(q) = \begin{cases} \left(\frac{1}{\alpha+\gamma}q_1, 0\right) & \text{if } q \in Q_1^\gamma, \\ \left(0, \frac{1}{\alpha+\gamma}q_2\right) & \text{if } q \in Q_2^\gamma, \\ \left(\frac{1}{\alpha+\gamma}q_1, \frac{1}{\alpha+\gamma}q_2\right) & \text{if } q \in Q_0^\gamma, \\ \left(\frac{1}{\alpha+\gamma}q_1, \frac{1}{\gamma}\left(q_2 - \operatorname{sign}(q_2)\sqrt{2\alpha\beta}\right)\right) & \text{if } q \in Q_{10}^\gamma, \\ \left(\frac{1}{\gamma}\left(q_1 - \operatorname{sign}(q_1)\sqrt{2\alpha\beta}\right), \frac{1}{\alpha+\gamma}q_2, \right) & \text{if } q \in Q_{20}^\gamma, \\ \left(\frac{1}{\gamma}\left(q_1 - \operatorname{sign}(q_1)\sqrt{2\alpha\beta}\right), \frac{1}{\gamma}\left(q_2 - \operatorname{sign}(q_2)\sqrt{2\alpha\beta}\right)\right) & \text{if } q \in Q_{00}^\gamma, \\ \left(\frac{1}{\gamma}\left(\frac{\alpha+\gamma}{2\alpha+\gamma}q_1 - \operatorname{sign}(q_1)\frac{\alpha}{2\alpha+\gamma}|q_2|\right), \frac{1}{\gamma}\left(\frac{\alpha+\gamma}{2\alpha+\gamma}q_2 - \operatorname{sign}(q_2)\frac{\alpha}{2\alpha+\gamma}|q_1|\right)\right) & \text{if } q \in Q_{12}^\gamma, \end{cases}$$

where

$$\begin{aligned} Q_1^\gamma &= \left\{q : |q_1| > (1+\tfrac{\gamma}{\alpha})|q_2| \text{ and } |q_2| < \sqrt{2\alpha\beta}\right\}, \\ Q_2^\gamma &= \left\{q : |q_2| > (1+\tfrac{\gamma}{\alpha})|q_1| \text{ and } |q_1| < \sqrt{2\alpha\beta}\right\}, \\ Q_0^\gamma &= \left\{q : |q_1|, |q_2| > (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right\}, \end{aligned}$$

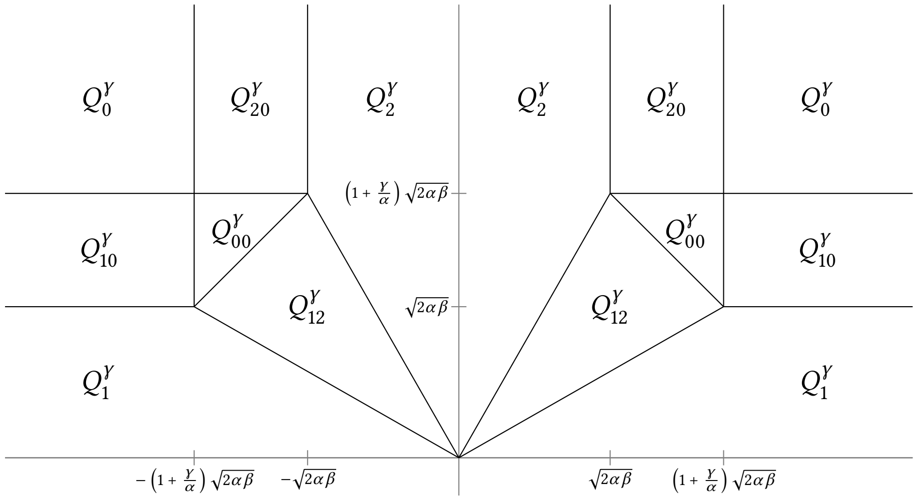


Figure 2: Subdomains $Q_i^\gamma \subset \mathbb{R}^2$ for the definition of $(\partial g^*)_\gamma$.

$$
\begin{aligned}
Q_{10}^\gamma &= \left\{q : |q_1| \in \left[\sqrt{2\alpha\beta}, (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right] \text{ and } |q_2| > (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right\},\\
Q_{20}^\gamma &= \left\{q : |q_2| \in \left[\sqrt{2\alpha\beta}, (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right] \text{ and } |q_1| > (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right\},\\
Q_{00}^\gamma &= \left\{q : |q_1|, |q_2| \in \left[\sqrt{2\alpha\beta}, (1+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right] \text{ and } |q_1| + |q_2| > (2+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right\},\\
Q_{12}^\gamma &= \left\{q : |q_1| \in \left[\tfrac{\alpha}{\alpha+\gamma}|q_2|, (1+\tfrac{\gamma}{\alpha})|q_2|\right] \text{ and } |q_1| + |q_2| \le (2+\tfrac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right\},
\end{aligned}
$$

see Figure 2.

This pointwise characterization allows obtaining expressions for the Moreau–Yosida approximation and the complementarity formulation of $u \in \partial\mathcal{G}^*(p)$.

4. Optimality conditions and structure

We now discuss the properties of solutions $(\bar{u}, \bar{p})$ to system (2.1). Specifically, let

$$
U = L^2(D; \mathbb{R}^2) \qquad \text{and} \qquad \mathcal{G} : U \to \mathbb{R}, \quad \mathcal{G}(u) = \int_D g(u(x))\, dx
$$

with g given by (3.1). The functional $\mathcal{F}$ will be assumed to be a tracking term of the form

$$
\mathcal{F}(u) = \frac{1}{2}\|Su - z\|_Y^2 \tag{4.1}
$$

for a Hilbert space $Y = Y^*$ (e.g., $Y = L^2([0,T] \times \Omega)$), given $z \in Y$, and a bounded linear control-to-observation mapping $S : U \to Y$. We further assume the existence of a Banach space $V \hookrightarrow L^r(D; \mathbb{R}^2)$ with $r > 2$ such that the adjoint $S^* : Y \to U$ maps continuously into V. The optimality system (2.1) is then given by

$$\text{(OS)} \qquad \begin{cases} \bar p = -S^*(S\bar u - z), \\ \bar u \in \partial \mathcal{G}^*(\bar p). \end{cases}$$

Since $\mathcal{F}$ given by (4.1) satisfies assumption (A1) (and is strictly convex if S is injective), Proposition 2.1 yields existence of a (unique) solution $(\bar u, \bar p) \in U \times U$.

Using the characterization from Section 3.2, the second relation in (OS) implies that for almost all $x \in D$,

$$(4.2) \quad \bar u(x) \in [\partial \mathcal{G}^*(p)](x) = \partial g^*(p(x))$$
$$= \begin{cases} \left(\left\{\frac{1}{\alpha}\bar p_1(x)\right\}, \{0\}\right) & \text{if } \bar p(x) \in Q_1 = \left\{q : |q_1| > |q_2| \text{ and } |q_2| < \sqrt{2\alpha\beta}\right\}, \\ \left(\left\{\{0\}, \{\frac{1}{\alpha}\bar p_2(x)\right\}\right) & \text{if } \bar p(x) \in Q_2 = \left\{q : |q_2| > |q_1| \text{ and } |q_1| < \sqrt{2\alpha\beta}\right\}, \\ \left(\left\{\frac{1}{\alpha}\bar p_1(x)\right\}, \left\{\frac{1}{\alpha}\bar p_2(x)\right\}\right) & \text{if } \bar p(x) \in Q_0 = \left\{q : |q_1|, |q_2| > \sqrt{2\alpha\beta}\right\}, \\ \left(\left\{\frac{1}{\alpha}\bar p_1(x)\right\}, \left[0, \frac{1}{\alpha}\bar p_2(x)\right]\right) & \text{if } \bar p(x) \in Q_{10} = \left\{q : |q_1| > |q_2| \text{ and } |q_2| = \sqrt{2\alpha\beta}\right\}, \\ \left(\left[0, \frac{1}{\alpha}\bar p_1(x)\right], \left\{\frac{1}{\alpha}\bar p_2(x)\right\}\right) & \text{if } \bar p(x) \in Q_{20} = \left\{q : |q_2| > |q_1| \text{ and } |q_1| = \sqrt{2\alpha\beta}\right\}, \\ \left(\left[0, \frac{1}{\alpha}\bar p_1(x)\right], \left[0, \frac{1}{\alpha}\bar p_2(x)\right]\right) & \text{if } \bar p(x) \in Q_{12} = \left\{q : |q_1| = |q_2| \text{ and } |q_1| \leq \sqrt{2\alpha\beta}\right\}. \end{cases}$$

We define the *switching arc* (where at most one control is active, i.e., nonzero)

$$\mathcal{A} = \{x \in D : \bar p(x) \in Q_1 \cup Q_2 \cup \{(0,0\}\},$$

the *free arc* (where both controls are active)

$$\mathcal{I} = \{x \in D : \bar p(x) \in Q_0 \cup Q_{10} \cup Q_{20}\},$$

and the *singular arc*

$$\mathcal{S} = \{x \in D : \bar p(x) \in Q_{12} \setminus \{(0,0)\}\}.$$

In a slight abuse of notation, we also introduce

$$\partial\mathcal{I} = \{x \in D : \bar p(x) \in Q_{10} \cup Q_{20}\}.$$

Clearly,

$$D = \mathcal{A} \cup \mathcal{I} \cup \mathcal{S}.$$

Let us address the question when the solution to system (OS) will be optimal. For this purpose, we first estimate the gap in the Fenchel–Young inequality.

Lemma 4.1. *If $(\bar u, \bar p) \in U \times U$ satisfies $\bar u \in \partial \mathcal{G}^*(\bar p)$, then*

$$\mathcal{G}(\bar u) + \mathcal{G}^*(\bar p) - \langle \bar p, \bar u\rangle \leq \beta|\partial\mathcal{I}| + 2\beta|\mathcal{S}|.$$

Proof. We discriminate pointwise based on the value of $\bar p(x)$ for almost every $x \in D$.

(i) $\bar{p}(x) \in Q_1$. In this case, the relation (4.2) yields $\bar{u}_1(x) = \frac{1}{\alpha}\bar{p}_1(x)$ and $\bar{u}_2(x) = 0$, and thus

$$g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x) \cdot \bar{u}(x) = \frac{1}{2\alpha}\bar{p}_1(x)^2 + \frac{1}{2\alpha}\bar{p}_1(x)^2 - \frac{1}{\alpha}\bar{p}_1(x)^2 = 0.$$

(ii) $\bar{p}(x) \in Q_2$. In this case, the relation (4.2) yields $\bar{u}_1(x) = 0$ and $\bar{u}_2(x) = \frac{1}{\alpha}\bar{p}_2(x)$, and thus

$$g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x) \cdot \bar{u}(x) = \frac{1}{2\alpha}\bar{p}_2(x)^2 + \frac{1}{2\alpha}\bar{p}_2(x)^2 - \frac{1}{\alpha}\bar{p}_2(x)^2 = 0.$$

(iii) $\bar{p}(x) \in Q_0$. In this case, the relation (4.2) yields $\bar{u}_1(x) = \frac{1}{\alpha}\bar{p}_1(x)$ and $\bar{u}_2(x) = \frac{1}{\alpha}\bar{p}_2(x)$, and thus

$$\begin{aligned} g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x) \cdot \bar{u}(x) = \frac{1}{2\alpha}(\bar{p}_1(x)^2 + \bar{p}_2(x)^2) + \beta + \frac{1}{2\alpha}(\bar{p}_1(x)^2 + \bar{p}_2(x)^2) \\ - \beta - \frac{1}{\alpha}(\bar{p}_1(x)^2 + \bar{p}_2(x)^2) = 0. \end{aligned}$$

(iv) $\bar{p}(x) \in Q_{10}$. In this case, the relation (4.2) yields $\bar{u}_1(x) = \frac{1}{\alpha}\bar{p}_1(x)$ and $\bar{u}_2(x) \in [0, \frac{1}{\alpha}\bar{p}_2(x)]$. Assume first that $\bar{p}_2(x)$ is positive, and that $0 < \bar{u}_2(x) < \frac{1}{\alpha}\bar{p}_2(x)$ (otherwise argue as in case (i) or (iii)). Then,

$$\begin{aligned} g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x) \cdot \bar{u}(x) &= \frac{1}{2\alpha}\bar{p}_1(x)^2 + \frac{\alpha}{2}\bar{u}_2(x)^2 + \beta + \frac{1}{2\alpha}\bar{p}_1(x)^2 \\ &\quad - \frac{1}{\alpha}\bar{p}_1(x)^2 - \bar{p}_2(x)\bar{u}_2(x) \\ &= \frac{\alpha}{2}u_2(x)^2 - \bar{p}_2(x)\bar{u}_2(x) + \beta. \end{aligned}$$

A simple calculus argument shows that the right-hand side is a monotonically decreasing function of $\bar{u}_2(x)$ on $(0, \frac{1}{\alpha}\bar{p}_2(x))$ and hence attains its supremum for $\bar{u}_2(x) = 0$, which implies that

$$g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x)\bar{u}(x) < \beta$$

for all $\bar{u}_2(x) \in (0, \frac{1}{\alpha}\bar{p}_2(x))$. For $\bar{q}_2(x)$ negative, we argue similarly.

(v) $\bar{p}(x) \in Q_{20}$. In this case, the relation (4.2) yields $\bar{u}_1(x) \in [0, \frac{1}{\alpha}\bar{p}_1(x)]$ and $\bar{u}_2(x) = \frac{1}{\alpha}\bar{p}_2(x)$. Proceeding as in case (iv) yields

$$g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x)\bar{u}(x) < \beta.$$

(vi) $\bar{p}(x) \in Q_{12}$. In this case, the relation (4.2) yields $\bar{u}_1(x) \in [0, \frac{1}{\alpha}\bar{p}_1(x)]$ and $\bar{u}_2(x) \in [0, \frac{1}{\alpha}\bar{p}_2(x)]$. Furthermore, we have that $|\bar{p}_1(x)| = |\bar{p}_2(x)| \leq \sqrt{2\alpha\beta}$.

First, if $\bar{p}(x) = (0, 0) \in Q_{12}$, this implies that $\bar{u}(x) = (0, 0)$ and hence

$$g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x)\bar{u}(x) = 0.$$

For $\bar{p}(x) \neq (0,0)$, we assume again that $\bar{u}_1(x) \in (0, \frac{1}{\alpha}\bar{p}_1(x))$ and $\bar{u}_2(x) \in (0, \frac{1}{\alpha}\bar{p}_2(x))$ and deduce that

$$\begin{aligned} g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x) \cdot \bar{u}(x) &= \frac{\alpha}{2}\bar{u}_1(x)^2 + \frac{\alpha}{2}\bar{u}_2(x)^2 + \beta + \frac{1}{2\alpha}\bar{p}_1(x)^2 \\ &\quad - \bar{p}_1(x)\bar{u}_1(x) - \bar{p}_2(x)\bar{u}_2(x) \\ &= \left(\frac{\alpha}{2}\bar{u}_1^2 - \bar{u}_1(x)\bar{p}_1(x)\right) + \left(\frac{\alpha}{2}\bar{u}_2^2 - \bar{u}_2(x)\bar{p}_2(x)\right) \\ &\quad + \frac{1}{2\alpha}\bar{p}_1(x)^2 + \beta. \end{aligned}$$

Again, both expressions in parentheses are decreasing functions of $\bar{u}_1(x)$ and $\bar{u}_2(x)$, respectively, and hence attain their supremum at 0. The bound on $|\bar{p}_1(x)|$ then implies that

$$g(\bar{u}(x)) + g^*(\bar{p}(x)) - \bar{p}(x)\bar{u}(x) \leq 2\beta.$$

Integrating over D now yields the claim. □

From Lemma 2.2 we obtain the following characterization of (sub)optimality of solutions.

Theorem 4.2. *If $(\bar{u},\bar{p}) \in U \times U$ satisfies (OS), then for any $u \in U$,*

$$\mathcal{J}(\bar{u}) \leq \mathcal{J}(u) + \beta(|\partial\mathcal{I}| + 2|\mathcal{S}|).$$

Hence if $\partial\mathcal{I}$ and $\mathcal{S}$ are sets of Lebesgue measure zero, $\bar{u}$ is a solution to ($\mathcal{P}$).

We next investigate the behavior of $\mathcal{I}$ and $\mathcal{S}$ as $\beta \to \infty$. For this purpose, we denote by (u_β,p_β) the solution to (OS) for given $\beta > 0$, with corresponding free arc $\mathcal{I}_\beta$ and singular arc $\mathcal{S}_\beta$. Note that the value of β does not appear in the relation (3.4) except as part of the case distinction, and hence $\beta \to \infty$ does not necessarily imply that $u_\beta \to 0$.

Theorem 4.3. *Let $\alpha > 0$ be fixed and let (u_β,p_β) satisfy (OS). Then, $|\mathcal{I}_\beta| \to 0$ as $\beta \to \infty$. Furthermore, for every $\varepsilon > 0$, we have that*

$$\mathcal{S}_\beta^\varepsilon := \mathcal{S}_\beta \cap \left\{x \in D : |u_{\beta,1}(x)|, |u_{\beta,2}(x)| \geq \varepsilon\right\}$$

satisfies $|\mathcal{S}_\beta^\varepsilon| \to 0$ as $\beta \to \infty$ as well.

Proof. We use the minimizing properties of u_β. This requires computation of $\mathcal{G}^{**}$. As in Section 3.1 we proceed by a casewise maximization based on the definition of g^* and obtain

$$\begin{aligned} g_1^{**}(v) &= \sup_{q\in\mathbb{R}^2} v \cdot q - g_1^*(q) = \frac{\alpha}{2}v_1^2 + \sqrt{2\alpha\beta}|v_2|, \\ g_2^{**}(v) &= \sup_{q\in\mathbb{R}^2} v \cdot q - g_2^*(q) = \frac{\alpha}{2}v_2^2 + \sqrt{2\alpha\beta}|v_1|, \\ g_0^{**}(v) &= \sup_{q\in\mathbb{R}^2} v \cdot q - g_0^*(q) = \frac{\alpha}{2}(v_1^2 + v_2^2) + \beta. \end{aligned}$$

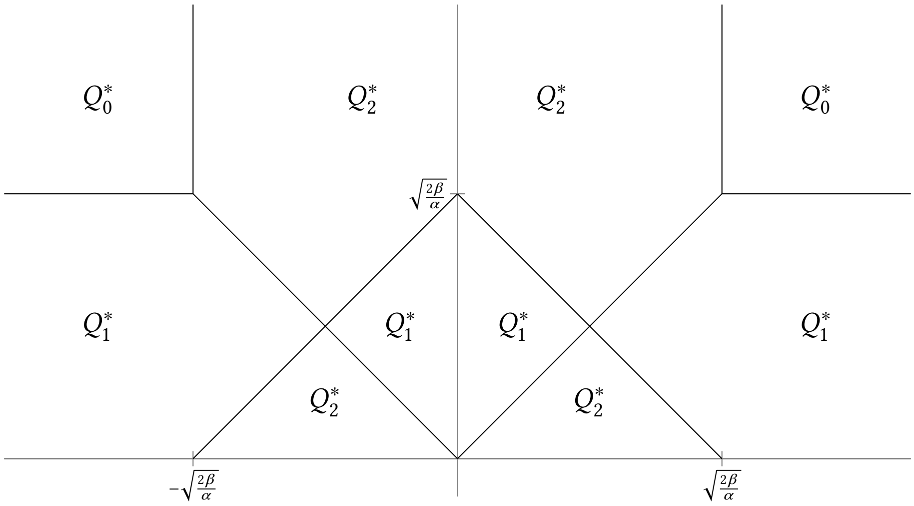


Figure 3: Subdomains $Q_i^* \subset \mathbb{R}^2$ for the definition of g^{**}.

From $g^{**}(v) = \max\{g_1^{**}(v), g_2^{**}(v), g_0^{**}(v)\}$ we then deduce that

$$g^{**}(v) = \begin{cases} \frac{\alpha}{2}v_1^2 + \sqrt{2\alpha\beta}|v_2| & \text{if } v \in Q_1^*, \\ \frac{\alpha}{2}v_2^2 + \sqrt{2\alpha\beta}|v_1| & \text{if } v \in Q_2^*, \\ \frac{\alpha}{2}(v_1^2 + v_2^2) + \beta & \text{if } v \in Q_0^*, \end{cases}$$

where

$$\begin{aligned} Q_1^* &= \left\{ v : |v_1|, |v_2| \le \sqrt{\frac{2\beta}{\alpha}} \text{ and } |v_1| \left(\frac{|v_1|}{2} - \sqrt{\frac{2\beta}{\alpha}} \right) \ge |v_2| \left(\frac{|v_2|}{2} - \sqrt{\frac{2\beta}{\alpha}} \right) \right\} \\ &\quad \cup \left\{ v : |v_1| \ge \sqrt{\frac{2\beta}{\alpha}}, |v_2| \le \sqrt{\frac{2\beta}{\alpha}} \right\}, \\ Q_2^* &= \left\{ v : |v_1|, |v_2| \le \sqrt{\frac{2\beta}{\alpha}} \text{ and } |v_2| \left(\frac{|v_2|}{2} - \sqrt{\frac{2\beta}{\alpha}} \right) \ge |v_1| \left(\frac{|v_1|}{2} - \sqrt{\frac{2\beta}{\alpha}} \right) \right\} \\ &\quad \cup \left\{ v : |v_1| \le \sqrt{\frac{2\beta}{\alpha}}, |v_2| \ge \sqrt{\frac{2\beta}{\alpha}} \right\}, \\ Q_0^* &= \left\{ v : |v_1|, |v_2| \ge \sqrt{\frac{2\beta}{\alpha}} \right\}, \end{aligned}$$

see Figure 3. Note that from the subdifferential inclusion (4.2), we can see that $u_\beta(x) \in Q_0^*$ if and only if $p_\beta(x) \in \overline{Q_0}$, and similarly for Q_1^* and Q_2^*.

Since $g^{**}(0) = 0$, we have that

$$\mathcal{G}^{**}(u_\beta) \le \mathcal{F}(u_\beta) + \mathcal{G}^{**}(u_\beta) \le \mathcal{F}(0) =: K,$$

i.e., the family $\{\mathcal{G}^{**}(u_\beta)\}_{\beta>0}$ is bounded. Consider first the free arc

$$\mathcal{I}_\beta = \left\{ x \in D : |p_{\beta,1}(x)|, |p_{\beta,2}(x)| \ge \sqrt{2\alpha\beta} \right\} = \left\{ x \in D : |u_{\beta,1}(x)|, |u_{\beta,2}(x)| \ge \sqrt{\frac{2\beta}{\alpha}} \right\}.$$

Then we have that

$$K \geq \int_D g^{**}(u_\beta(x))\,dx \geq \int_{\mathcal{I}_\beta} \frac{\alpha}{2}|u_\beta(x)|^2 + \beta\,dx \geq \beta|I_\beta|, \tag{4.3}$$

where the right-hand side remains bounded as $\beta \to \infty$ if and only if the second term goes to zero as claimed.

Similarly, we have for the singular arc that

$$\begin{aligned}\mathcal{S}_\beta &= \left\{x \in D : 0 < |p_{\beta,1}(x)| = |p_{\beta,2}(x)| \leq \sqrt{2\alpha\beta}\right\}\\ &= \left\{x \in D : 0 \leq |u_{\beta,1}(x)|, |u_{\beta,2}(x)| \leq \sqrt{\tfrac{2\beta}{\alpha}}\right\},\end{aligned}$$

and hence for any $\varepsilon > 0$ that

$$\mathcal{S}_\beta^\varepsilon = \left\{x \in D : \varepsilon \leq |u_{\beta,1}(x)|, |u_{\beta,2}(x)| \leq \sqrt{\tfrac{2\beta}{\alpha}}\right\}.$$

Setting $D_1 := \left\{x \in D : u_\beta(x) \in Q_1^*\right\}$ and $D_2 := \left\{x \in D : u_\beta(x) \in Q_2^*\right\}$, we have that $\mathcal{S}_\beta^\varepsilon \subset (D_1 \cup D_2)$, which implies that

$$\begin{aligned}K &\geq \int_D g^{**}(u_\beta(x))\,dx \geq \int_{\mathcal{S}_\beta^\varepsilon} g^{**}(u_\beta(x))\,dx\\ &= \int_{\mathcal{S}_\beta^\varepsilon\cap D_1} \frac{\alpha}{2}u_{\beta,1}(x)^2 + \sqrt{2\alpha\beta}|u_{\beta_2}(x)|\,dx + \int_{\mathcal{S}_\beta^\varepsilon\cap D_2} \frac{\alpha}{2}u_{\beta,2}(x)^2 + \sqrt{2\alpha\beta}|u_{\beta_1}(x)|\,dx\\ &\geq \varepsilon\sqrt{2\alpha\beta}|\mathcal{S}_\beta^\varepsilon|.\end{aligned}$$

For arbitrary fixed $\varepsilon > 0$, the right-hand side remains bounded as $\beta \to \infty$ if and only if $|\mathcal{S}_\beta^\varepsilon| \to 0$. □

Note that $\partial\mathcal{I}_\beta \subset \mathcal{I}_\beta$ and hence, from the estimate (4.3), the corresponding optimality gap $\beta|\partial\mathcal{S}|$ remains bounded for $\beta \to \infty$.

If p_β is uniformly bounded pointwise almost everywhere, we can deduce that $\mathcal{I}_\beta$ must vanish for some sufficiently large (finite) value of β.

Theorem 4.4. *If $V \hookrightarrow L^\infty(D)$, then there exists a $\beta_0 > 0$ such that $|\mathcal{I}_\beta| = 0$f or all $\beta \geq \beta_0$.*

Proof. Due to the estimate (4.3) and the definition of $\mathcal{G}^{**}$, the family $\{u_\beta\}_{\beta>0}$ is bounded in U. Hence $\{Su_\beta\}_{\beta>0}$ and thus $\{F'(Su_\beta)\}_{\beta>0}$ are bounded in Y and Y^*, respectively. Since S^* maps continuously to $L^\infty(D)$, this implies that $\{p_\beta\}_{\beta>0} = \{-S^*F'(Su_\beta)\}_{\beta>0}$ is uniformly bounded pointwise almost everywhere by a constant $M > 0$. Choosing β_0 such that $M > \sqrt{2\alpha\beta_0}$, we obtain from the subdifferential inclusion (4.2) that $Q_0 = Q_{10} = Q_{20} = \emptyset$, which yields the claim. □

Remark 1. *It appears difficult to give a sufficient condition for the singular set $\mathcal{S}$ to be empty, since on this set neither $\mathcal{F}(u)$ nor $\mathcal{G}(u)$ yield enough information to decide* which *component of u should be active. On the other hand, since $\bar{p}_1(x) = \bar{p}_2(x)$ on the active set, we can expect $|\mathcal{S}|$ to be small. We shall comment on the cardinality of $\mathcal{S}$ for the numerical examples.*

5. Numerical solution

We return to the Moreau–Yosida regularization of the optimality system (OS): For given $\gamma > 0$, find $(u_\gamma, p_\gamma) \in U \times U$ satisfying

$$\begin{cases} p_\gamma = -S^*(Su_\gamma - z), \\ u_\gamma = H_\gamma(p_\gamma). \end{cases} \tag{OS$_\gamma$}$$

Since $\mathcal{F}'(u) = S^*(Su - z)$ is linear and bounded, assumption (A2) is clearly satisfied; in addition, the explicit characterization of $\partial\mathcal{G}^*$ in Section 3 immediately yields that $\inf \|\partial\mathcal{G}^*(p)\| \leq \frac{1}{\alpha}\|p\|_U$, and hence assumption (A3) holds. From Proposition 2.4 and Proposition 2.5, we thus obtain existence of a solution (which is unique if S is injective) and convergence to a solution of (OS) as $\gamma \to 0$. For later reference, we note that the mapping properties of S^* imply that $p_\gamma \in V$.

The solution to (OS$_\gamma$) can be computed using a semismooth Newton method. We first show that H_γ is Newton-differentiable. Recall that H_γ is defined pointwise almost everywhere by

$$[H_\gamma(p)](x) = h_\gamma(p(x)) := (\partial g^*)_\gamma(p(x)),$$

and that h_γ is globally Lipschitz continuous with constant γ^{-1} by Proposition 2.3(iii). Hence, h_γ is directionally differentiable almost everywhere. In addition, h_γ is piecewise differentiable, and hence its directional derivative

$$h'_\gamma(q;\delta q) := \lim_{t\to 0} \frac{1}{t}(h_\gamma(q + t\delta q) - h_\gamma(q))$$

at q in direction δq satisfies

$$\lim_{|\delta q|\to 0} \frac{1}{|\delta q|}|h'_\gamma(q + \delta q;\delta q) - h'_\gamma(q;\delta q)| = 0 \qquad \text{for almost all } q.$$

Together we obtain that h_γ is semismooth; see, e.g., [12, Theorem 8.2] or [19, Proposition 2.7]; see also [19, Proposition 2.26].

This implies that the superposition operator H_γ is Newton-differentiable from $V \hookrightarrow L^r(D;\mathbb{R}^2)$ to $L^2(D;\mathbb{R}^2)$ for any $r > 2$; see, e.g., [12, Example 8.12] or [19, Theorem 3.49]. Its Newton derivative will be denoted by $D_N H_\gamma : V \to U$, and it is given pointwise almost everywhere at p in direction δp by a measurable selection

$$[D_N H_\gamma(p)\delta p](x) \in \partial_C h_\gamma(p(x))\delta p(x),$$

where $\partial_C h_\gamma(q)$ is the Clarke derivative, which for piecewise differentiable functions is given by the convex hull of the piecewise derivatives at each point. Specifically, for h_γ given in Section 3.3, a Newton derivative $D_N h_\gamma(q) \in \partial_C h_\gamma(q)$ is given by

$$D_N h_\gamma(q) = \begin{cases} \operatorname{diag}\left(\frac{1}{\alpha+\gamma}, 0\right) & \text{if } q \in Q_1^\gamma, \\ \operatorname{diag}\left(0, \frac{1}{\alpha+\gamma}\right) & \text{if } q \in Q_2^\gamma, \\ \operatorname{diag}\left(\frac{1}{\alpha+\gamma}, \frac{1}{\alpha+\gamma}\right) & \text{if } q \in Q_0^\gamma, \\ \operatorname{diag}\left(\frac{1}{\alpha+\gamma}, \frac{1}{\gamma}\right) & \text{if } q \in Q_{10}^\gamma, \\ \operatorname{diag}\left(\frac{1}{\gamma}, \frac{1}{\alpha+\gamma}\right) & \text{if } q \in Q_{20}^\gamma, \\ \operatorname{diag}\left(\frac{1}{\gamma}, \frac{1}{\gamma}\right) & \text{if } q \in Q_{00}^\gamma, \\ \frac{1}{\gamma(2\alpha+\gamma)} \begin{pmatrix} (\alpha+\gamma) & \operatorname{sign}(q_1 q_2)\alpha \\ \operatorname{sign}(q_1 q_2)\alpha & (\alpha+\gamma) \end{pmatrix} & \text{if } q \in Q_{12}^\gamma, \end{cases}$$

where $\operatorname{diag}(\cdot,\cdot)$ denotes the 2×2 diagonal matrix with the given entries.

In the sequel, we shall require the following two properties of the Newton derivative.

Lemma 5.1. *For all $p \in V$ and $\delta p \in V$, we have*

$$\begin{aligned} \langle D_N H_\gamma(p)\delta p, \delta p\rangle_U &\geq 0, \\ \|D_N H_\gamma(p)\delta p\|_U &\leq \frac{1}{\gamma}\|\delta p\|_U. \end{aligned}$$

Proof. Recall from Proposition 2.3 that h_γ is the derivative of the convex functional $(g^*)_\gamma$ and hence is monotone. Therefore we have for all $t > 0$, almost all q, and all δq that

$$0 \leq (h_\gamma(q + t\delta q) - h_\gamma(q)) \cdot (q + t\delta q - q) = \frac{1}{t}\left(h(q + t\delta q) - h_\gamma(q)\right) \cdot (t^2 \delta q).$$

Dividing by $t^2 > 0$ and taking the limit as $t \to 0$ yields

$$h_\gamma'(q; \delta q) \cdot \delta q \geq 0. \tag{5.1}$$

Similarly, since h_γ is globally Lipschitz with constant γ^{-1}, we have for all $t > 0$, almost all q, and all δq that

$$\frac{1}{t}|h_\gamma(q + t\delta q) - h_\gamma(q)| \leq \frac{1}{\gamma}|\delta q|.$$

Taking again the limit as $t \to 0$ yields

$$|h_\gamma'(q; \delta q)| \leq \frac{1}{\gamma}|\delta q|. \tag{5.2}$$

As a consequence, all elements in the Clarke derivative satisfy the inequalities (5.1) and (5.2). Since $D_N H_\gamma(p)$ is taken as a measurable selection from $\partial_C h_\gamma(p(\cdot))$, the claim follows by substitution and integration over D. ◻

To apply a semismooth Newton method to (OS_γ), we first introduce the state $y_\gamma := S(u_\gamma) \in Y$ and eliminate u_γ, thus obtaining the equivalent optimality system

$$\begin{cases} y_\gamma = SH_\gamma(p_\gamma), \\ p_\gamma = -S^*(y_\gamma - z). \end{cases} \tag{5.3}$$

Considering the system (5.3) as an operator equation from $Y \times V$ to $Y \times V$, a semismooth Newton step for its solution consists in computing $(\delta y, \delta p) \in Y \times V$ for given $(y^k, p^k) \in Y \times V$ such that

$$\begin{cases} \delta y - SD_N H_\gamma(p^k)\delta p = -y^k + SH_\gamma(p^k), \\ \qquad\qquad \delta p + S^*\delta y = -p^k - S^*(y^k - z), \end{cases} \tag{5.4}$$

and setting $y^{k+1} = y^k + \delta y$ and $p^{k+1} = p^k + \delta p$.

To show superlinear convergence of this iteration, it remains to show uniform solvability of each Newton step.

Proposition 5.2. *For any $(y,p) \in Y \times V$ and $(w_1, w_2) \in Y \times V$, system*

$$\begin{cases} \delta y + SD_N H_\gamma(p) = w_1, \\ \qquad \delta p - S^*\delta y = w_2, \end{cases} \tag{5.5}$$

has a solution $(\delta y, \delta p) \in Y \times V$ which satisfies

$$\|\delta y\|_Y + \|\delta p\|_V \le C(\|w_1\|_Y + \|w_2\|_V).$$

Proof. Eliminating $\delta p = S^*\delta y + w_2 \in V$, we obtain that (5.5) is equivalent to

$$\delta y + SD_N H_\gamma(p) S^*\delta y = w_1 + SD_N H_\gamma(p) w_2. \tag{5.6}$$

Since S^* is linear and bounded from Y to V and $D_N H_\gamma$ is monotone on V from Lemma 5.1, the operator $SD_N H_\gamma(p)S^*$ is maximally monotone from Y to Y; see, e.g., [2, Propositions 20.10, 20.24]. Minty's theorem thus yields existence of a solution $\delta y \in Y$ and hence of a corresponding $\delta p \in V$; see, e.g., [2, Proposition 21.1].

Taking the inner product of equation (5.6) with δy and using Lemma 5.1 with $S^*\delta y \in V \hookrightarrow U$ implies that

$$\begin{aligned} \|\delta y\|_Y^2 &\le \langle \delta y, \delta y\rangle_Y + \langle D_N H_\gamma(p)(S^*\delta y), S^*\delta y\rangle_U \\ &= \langle w_1, \delta y\rangle_Y + \langle D_N H_\gamma(p) w_2, S^*\delta y\rangle_U \\ &\le \|w_1\|_Y \|\delta y\|_Y + \|D_N H_\gamma(p) w_2\|_U \|S^*\delta y\|_U \\ &\le \left(\|w_1\|_Y + \frac{C}{\gamma}\|w_2\|_V\right)\|\delta y\|_Y, \end{aligned}$$

using the boundedness of S^* from Y to V and Lemma 5.1 with $w_2 \in V \hookrightarrow U$. The second equation of (5.5) then yields

$$\|\delta p\|_V \le C\|w_1\|_Y + \left(1 + \frac{C^2}{\gamma}\right)\|w_2\|_V. \qquad \square$$

As a consequence of the Newton differentiability of H_γ and of Proposition 5.2, we obtain the following result; see, e.g., [12, Theorem 8.16], [19, Chapter 3.2].

Theorem 5.3. *The semismooth Newton iteration* (5.4) *converges locally superlinearly in* $Y \times V$.

Since the right-hand side of the Newton system (5.4) is linear apart from the term $H_\gamma(p^k)$, we can use the following termination criterion for the Newton iteration: If all active sets $A_i(p) = \left\{x \in \Omega : p(x) \in Q_i^\gamma\right\}$ coincide for p^k and p^{k+1}, and the control is computed as $u^{k+1} = H_\gamma(p^{k+1})$, then (u^{k+1}, p^{k+1}) satisfies (OS_γ); see, e.g., [12, Remark 7.1.1].

This can be used as part of a continuation strategy to deal with the local convergence behavior of Newton methods: Starting with γ^0 large and $(y^0, p^0) = (0,0)$, we solve the regularized optimality system (OS_γ) using the semismooth Newton iteration (5.4). If the iteration converges for some γ^m (in the sense that all active sets coincide), we reduce $\gamma^{m+1} = \frac{1}{10}\gamma^m$ and solve the system (OS_γ) again with the solution for γ^m as the starting point. This procedure is terminated if the Newton iteration converges in a single step (assuming that the corresponding iterate then satisfies the system for smaller values of γ as well) or if the Newton iteration failes to converge within a given number of steps (assuming that the system has then become too ill-conditioned for a stable numerical solution). In any case, the continuation is stopped when $\gamma^m \leq 10^{-16}$ is reached.

While this strategy has proved robust for problems with scalar L^1- and L^0-type penalties, see e.g. [13, 6], the situation is more delicate for the vector penalty considered here; this is in particular the case when the singular arc $\mathcal{S}$ is non-negligible and $D_N H_\gamma$ is not a diagonal matrix. We thus combine the semi-smooth Newton method with a backtracking line search along the Newton direction. In principle, this requires computation of $((\mathcal{G}^*)_\gamma)^*$ (or $\mathcal{F}^*$ and $(\mathcal{G}^*)_\gamma$); however, if the tracking term $\mathcal{F}$ is strictly convex (as will be the case in the examples considered below), the system (OS_γ) is a sufficient as well as necessary condition and hence we can equivalently backtrack according to the residual norm of (OS_γ). This was sufficient to achieve a robust and superlinear convergence in all examples.

6. Numerical examples

We illustrate the behavior of the proposed approach and the structure of the resulting controls with two numerical examples. First, we consider an elliptic problem where the two control components each act along a strip in one coordinate direction. Specifically, we set $\Omega = [0,1]^2$, $D = [0,1]$,

$$\omega_1 = \left\{(x_1, x_2) \in \Omega : x_2 < \tfrac{1}{4}\right\}, \qquad \omega_2 = \left\{(x_1, x_2) \in \Omega : x_2 > \tfrac{3}{4}\right\},$$

and consider the control-to-state mapping $S : u \mapsto y \in Y = L^2(\Omega)$ satisfying

$$-\Delta y = Bu = \chi_{\omega_1}(x_1, x_2)u_1(x_1) + \chi_{\omega_2}(x_1, x_2)u_2(x_1).$$

The target is

$$z(x) = x_1 \sin(2\pi x_1)\sin(2\pi x_2),$$

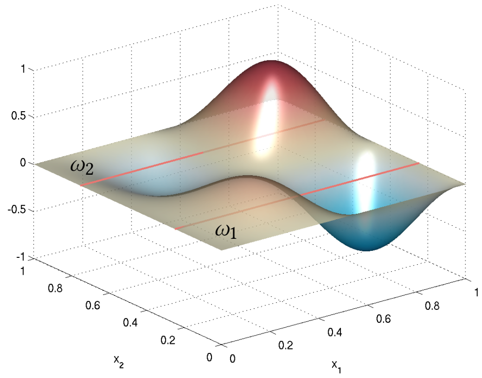
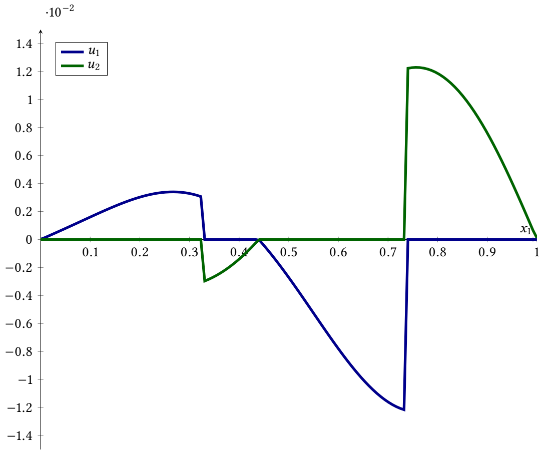


Figure 4: Elliptic problem, target z and control domains ω_1, ω_2

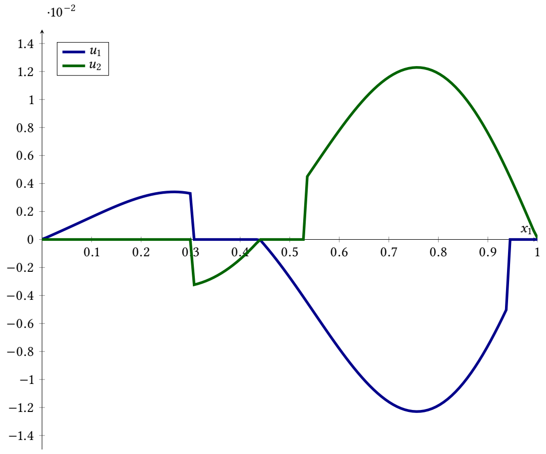
see Figure 4.

The state y and adjoint p are discretized using piecewise linear finite elements based on a uniform triangulation $\mathcal{T}_h$ of the domain Ω with $N_h = 128 \times 128$ nodes. Since the control is eliminated, this can be interpreted as a variational discretization. Integration over the piecewise defined functions $H_\gamma(p_h)$ and $D_N H_\gamma(p_h)\delta p_h$ in the weak formulation of (5.4) is approximated by applying the mass matrix to the vector of nodal values; see [6]. The control operator B is approximated by forming the tensor product of the discrete indicator function of ω_i with the nodal values of u_i; the adjoint operator B^* is approximated by the transpose of this matrix in order to preserve symmetry. The "globalized" semismooth Newton method with continuation and line searches described above is applied to the discretized system. The continuation is started at $\gamma^0 = 1$ and the backtracking is performed in steps of $\tau_i = 2^{-i}$ for $i = 0, \dots, 40$; if $\tau_i < 10^{-12}$, the Newton iteration is restarted with reduced γ. Since we no longer perform full Newton steps, we augment the termination criterion for the Newton iteration with an additional check for the residual norm in the optimality system, i.e., we terminate if all active sets coincide and the residual is smaller than 10^{-6}. A Matlab implementation of the described algorithm can be downloaded from `http://www.uni-graz.at/~clason/codes/switchingcontrol.zip`.

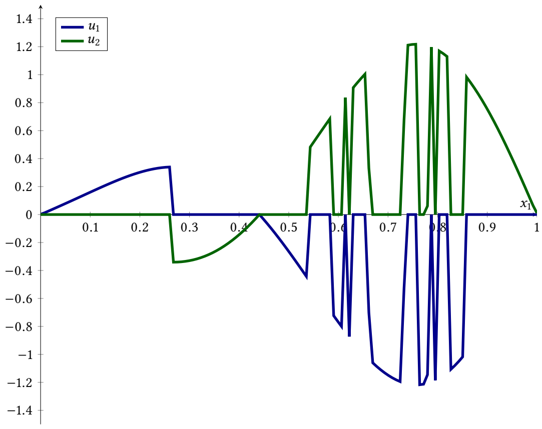
We begin by illustrating the effects of the values of α and β on the structure of the resulting controls. Figure 5 shows the final computed controls u_γ for the same target z and different combinations of control costs. For the choice $\alpha = \beta = 10^{-3}$ (Figure 5a), the control has a pure switching structure, with 80 nodes (out of 128) having values in the active set Q_1^γ and 48 nodes in the set Q_2^γ (the remaining sets being empty); in particular, the singular arc $\mathcal{S}$ is empty. Furthermore, the effect of the L^2 penalty on the active control components can be observed clearly. Decreasing β to 10^{-8} results in a control that is no longer purely switching (Figure 5b), although some switching behavior still obtains in parts of D; the resulting active sets have 51 nodes in Q_1^γ, 25 nodes in Q_2^γ, and 52 nodes in the regularized free arc Q_0^γ. Since α is unchanged, the magnitude of the active controls is the same as before. Decreasing α, on the other hand, allows for controls of larger magnitude, but results in the appearance of singular arcs. For $\alpha = 10^{-5}$ and $\beta = 10^{-3}$ (Figure 5c), we observe a control which is almost purely switching



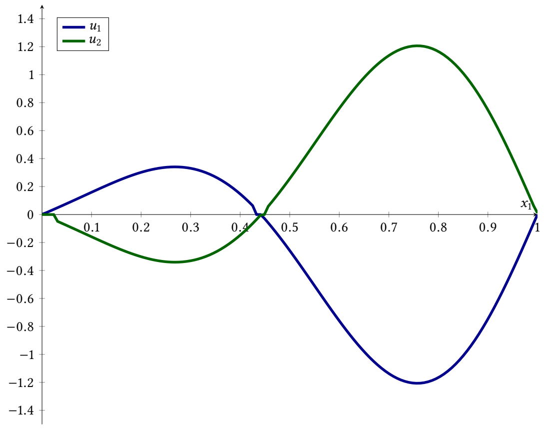
(a) $\alpha = 10^{-3}$, $\beta = 10^{-3}$



(b) $\alpha = 10^{-3}$, $\beta = 10^{-8}$



(c) $\alpha = 10^{-5}$, $\beta = 10^{-3}$



(d) $\alpha = 10^{-5}$, $\beta = 10^{-8}$

Figure 5: Elliptic problem, effect of α, β on structure of control u_γ (left: switching, right: no switching)

(66 and 59 nodes in Q_1^γ and Q_2^γ, respectively) but still has a non-negligible singular arc with 3 nodes in Q_{12}^γ. The control shows a chittering behavior on part of the switching arc, which can be attributed to the weak but not pointwise convergence of the regularized controls. For the smaller value of β (Figure 5d), the singular arc disappears at the expense of the appearance of a large free arc (5 nodes in Q_1^γ, 3 nodes in Q_2^γ, and 120 nodes in Q_0^γ).

Let us briefly comment on the convergence behavior of the "globalized" Newton method. For $\gamma > 10^{-9}$, the semismooth Newton iteration shows the typical superlinear behavior, converging within two or three (full) steps to a solution of the system (OS_γ). For smaller values of γ, backtracking becomes necessary after one full step, but, depending on the presence of singular arcs, often enters into a superlinear phase again where full steps are taken to convergence. Specifically, in the case of $\alpha = \beta = 10^{-3}$, the iteration terminates successfully at $\gamma = 10^{-12}$ with only a few reduced steps necessary. For $\alpha = 10^{-5}$ and $\beta = 10^{-3}$, more line searches are

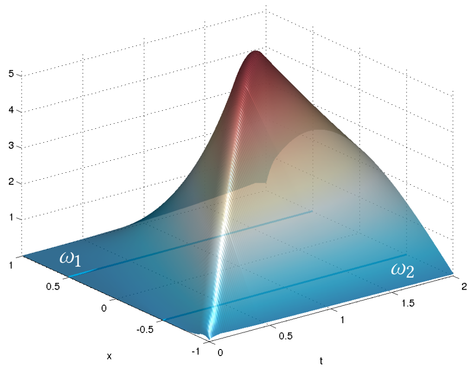


Figure 6: Parabolic problem, target z and control domains ω_1, ω_2

performed, but the final superlinear phase is still observed for $\gamma > 10^{-13}$, after which the Newton iteration terminated since no sufficient decrease in the residual was possible. However, restarting with smaller γ still allowed some successful steps before terminating again, which continued until the specified terminal value of $\gamma = 10^{-16}$ was reached. For $\beta = 10^{-8}$, no backtracking was necessary, and the algorithm showed the typical behavior of a semismooth Newton method with continuation (terminating successfully at $\gamma = 10^{-9}$ for $\alpha = 10^{-3}$ and at $\gamma = 10^{-10}$ for $\alpha = 10^{-5}$).

To demonstrate the applicability of the proposed approach to switching control of parabolic equations, we also show results for the one-dimensional heat equation, where $S : u \mapsto y$ satisfying

$$y_t - \Delta y = Bu = \chi_{\omega_1}(x)u_1(t) + \chi_{\omega_2}(x)u_2(t)$$

with $\Omega = [-1, 1]$, $D = [0, 2]$, $\Omega_T = D \times \Omega$,

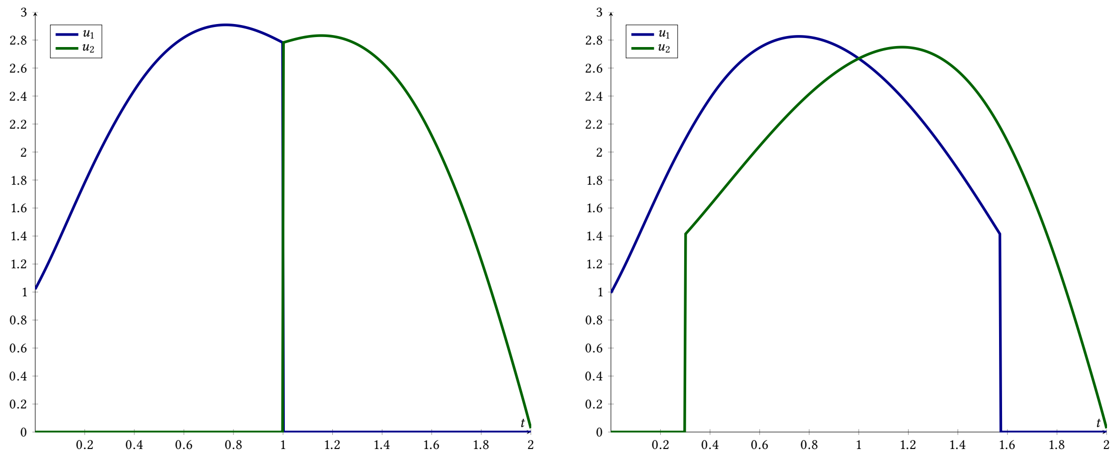
$$\omega_1 = \left\{x \in \Omega : x < -\tfrac{1}{2}\right\}, \qquad \omega_2 = \left\{x \in \Omega : x > \tfrac{1}{2}\right\}.$$

As a target, we choose the trajectory of the heat equation with the right-hand side

$$f(t, x) = \begin{cases} 63 & |t - 1 - x| < \frac{1}{10}, \\ 0 & \text{otherwise}, \end{cases}$$

see Figure 6. The discretization is similar as in the elliptic case, using a full space-time discontinuous Galerkin discretization corresponding to a backward Euler method with $N_h = 128$ spatial grid points and $N_t = 512$ time steps.

The resulting controls for $\alpha = 10^{-1}$ are shown in Figure 7. For $\beta = 1$ (Figure 7a), the control is again of purely switching type with 256 nodes each in Q_1^γ and Q_2^γ. No backtracking was necessary, and the continuation terminated successfully at $\gamma = 10^{-9}$. The control for $\beta = 10^{-1}$ (Figure 7b) shows a free arc, with 77 nodes in Q_1^γ, 110 nodes in Q_2^γ, and 325 nodes in Q_0^γ. The convergence behavior is now different due to the intermittent appearance of singular arcs:



(a) $\alpha = 10^{-1}$, $\beta = 10^{0}$ (b) $\alpha = 10^{-1}$, $\beta = 10^{-1}$

Figure 7: Parabolic problem, effect of α, β on structure of control u_γ (left: switching, right: no switching)

Although the first continuation step with $\gamma = 10^{-2}$ shows the usual superlinear convergence with full steps, the resulting iterate contains nodes in Q_{10}^{γ} and Q_{20}^{γ}. Subsequently, the iterations for $\gamma > 10^{-5}$ suffer from progressively smaller steps until no sufficient decrease is possible. At $\gamma = 10^{-5}$, however, the corresponding singular arc $\partial\mathcal{I}$ is empty and the iteration returns to superlinear convergence with full steps, terminating successfully at $\gamma = 10^{-9}$. The difference to the elliptic case can be attributed to the lower regularity of the adjoint state p with respect to the control dimension (here: time) and the corresponding smaller norm gap in the regularized subdifferential $H_\gamma(p)$.

7. Conclusion

A framework for optimal control problems was presented that promotes controls of switching type. While switching is promoted by a sparsity-enhancing part of the cost functional, the active controls are weighted with quadratic cost. Analysis of the proposed approach is carried out by techniques from convex analysis, while its numerical solution is achieved using a semismooth Newton method with continuation and line searches. Numerical results support the theoretical findings.

There are many interesting follow-up topics, including the treatment of problems with nonlinear control-to-state mappings, a more detailed analysis of the influence of the control cost parameters on the structure of the controls, and problems with multiple controls exhibiting generalized switching structures.

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A. Application to other binary penalties

This appendix demonstrates the application of the approach of Section 3 to other functionals involving the binary functional $|v|_0$. While the Fenchel conjugates and subdifferentials have already been obtained in the previous works cited below, the proximal mappings and corresponding Moreau–Yosida regularizations and complementarity formulations are new.

A.1. Sparse control

We first consider the functional

$$\mathcal{G}(u) = \frac{\alpha}{2}\|u\|_{L^2}^2 + \beta \int_\Omega |u(x)|_0 \, dx,$$

which promotes sparsity in optimal control and, contrary to L^1-type penalties, allows separate penalization of magnitude and support; see [13]. Setting

$$g(v) = \frac{\alpha}{2}v^2 + \beta|v|_0 := \begin{cases} \frac{\alpha}{2}v^2 + \beta & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases}$$

we compute the Fenchel conjugate

$$g^*(q) = \sup_{v\in\mathbb{R}} v \cdot q - g(v) \tag{A.1}$$

by case distinction. Assume that the supremum is attained for some $\bar{v} \in \mathbb{R}$. Then we discriminate the following two cases:

(i) $\bar{v} = 0$, in which case $g(\bar{v}) = 0$ and hence $g^*(q) = 0$;

(ii) $\bar{v} \neq 0$, in which case $g(\bar{v}) = \frac{\alpha}{2}\bar{v}^2 + \beta$. Since g is differentiable at $\bar{v}$, the necessary condition for $\bar{v}$ to attain the maximum is $q = \alpha\bar{v}$. Solving for $\bar{v}$ and inserting in (A.1) yields

$$g^*(q) = \frac{1}{2\alpha}q^2 - \beta.$$

It remains to decide which of these cases is attained for a given q, i.e., whether

$$g_0^*(q) := 0 < \frac{1}{2\alpha}q^2 - \beta =: g_1^*(q).$$

This directly yields

$$g^*(q) = \max_{i\in\{0,1\}} g_i^*(q) = \begin{cases} 0 & \text{if } |q| \leq \sqrt{2\alpha\beta}, \\ \text{if } \frac{1}{2\alpha}q^2 - \beta & \text{if } |q| > \sqrt{2\alpha\beta}. \end{cases}$$

as well as

$$\partial g^*(q) = \overline{\text{co}}\left(\bigcup_{\{i:g^*(q)=g_i^*(q)\}} \left\{(g_i^*)'(q)\right\}\right) = \begin{cases} 0 & \text{if } |q| < \sqrt{2\alpha\beta}, \\ \left[0, \frac{1}{\alpha}q\right] & \text{if } |q| = \sqrt{2\alpha\beta}, \\ \frac{1}{\alpha}q & \text{if } |q| > \sqrt{2\alpha\beta}. \end{cases} \tag{A.2}$$

We now turn to the computation for given $\gamma > 0$ and $v \in \mathbb{R}$ of the proximal mapping $w = \text{prox}_{\gamma g^*}(v)$ of g^* or, equivalently, the resolvent of ∂g^*, which is characterized by the relation $v \in (\text{Id} + \gamma\partial g^*)(w)$. We now distinguish all possible cases in (A.2):

(i) $|w| < \sqrt{2\alpha\beta}$: In this case $v = w$, which implies that $|v| < \sqrt{2\alpha\beta}$.

(ii) $|w| > \sqrt{2\alpha\beta}$: In this case $v = (1 + \frac{\gamma}{\alpha})w$, which implies that $|v| > (1 + \frac{\gamma}{\alpha})\sqrt{2\alpha\beta}$.

(iii) $|w| = \sqrt{2\alpha\beta}$: In this case $v \in [w, (1 + \frac{\gamma}{\alpha})w]$, which implies that $\sqrt{2\alpha\beta} \leq |v| \leq (1 + \frac{\gamma}{\alpha})\sqrt{2\alpha\beta}$.

Inserting this into the definition of the Moreau–Yosida regularization and simplifying yields

$$(\partial g^*)_\gamma(q) = \begin{cases} 0 & \text{if } |q| < \sqrt{2\alpha\beta}, \\ \frac{1}{\gamma}\left(q - \sqrt{2\alpha\beta}\,\text{sign}(q)\right) & \text{if } |q| \in \left[\sqrt{2\alpha\beta}, (1 + \frac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right], \\ \frac{1}{\alpha+\gamma}q & \text{if } |q| > (1 + \frac{\gamma}{\alpha})\sqrt{2\alpha\beta}, \end{cases}$$

which can be interpreted as a soft-thresholding operator.

Since $h_\gamma := (\partial g^*)_\gamma$ is Lipschitz continuous and piecewise differentiable, it is semismooth, and its Newton-derivative at q in direction δq is given by

$$D_N h_\gamma(q)\partial q = \begin{cases} 0 & \text{if } |q| < \sqrt{2\alpha\beta}, \\ \frac{1}{\gamma}\delta q & \text{if } |q| \in \left[\sqrt{2\alpha\beta}, (1 + \frac{\gamma}{\alpha})\sqrt{2\alpha\beta}\right], \\ \frac{1}{\alpha+\gamma}\delta q & \text{if } |q| > (1 + \frac{\gamma}{\alpha})\sqrt{2\alpha\beta}. \end{cases}$$

A.2. Multi-bang control

We now consider the *multi-bang* penalty

$$g(v) = \frac{\alpha}{2}v^2 + \beta\prod_{i=1}^{d}|v - u_i|_0 + \delta_{[u_1,u_d]}(v),$$

where $u_1, \ldots, u_d$ are given desired control states and δ_C denotes the indicator function of the convex set C. In optimal control problems, the binary term (together with the pointwise constraints) promotes controls which, for β sufficiently large, take on only the desired values almost everywhere except possibly on a singular set; see [6].

Proceeding as in Appendix A.1 yields the Fenchel conjugate

$$g^*(q) = \begin{cases} qu_1 - \frac{\alpha}{2}u_1^2 & \text{if } q - \alpha u_1 \le \sqrt{2\alpha\beta} \quad \text{and} \quad q \le \frac{\alpha}{2}(u_1 + u_2), \\ qu_i - \frac{\alpha}{2}u_i^2 & \text{if } |q - \alpha u_i| \le \sqrt{2\alpha\beta} \quad \text{and} \quad \frac{\alpha}{2}(u_{i-1} + u_i) \le q \le \frac{\alpha}{2}(u_i + u_{i+1}), 1 < i < d, \\ qu_d - \frac{\alpha}{2}u_d^2 & \text{if } q - \alpha u_d \ge \sqrt{2\alpha\beta} \quad \text{and} \quad \frac{\alpha}{2}(u_d + u_{d-1}) \le q, \\ \frac{1}{2\alpha}q^2 - \beta & \text{if } |q - \alpha u_j| \le \sqrt{2\alpha\beta} \ \text{ for all } \ j \in \{1, \dots, d\} \quad \text{and} \quad \alpha u_1 \le q \le \alpha u_d, \end{cases}$$

whose subdifferential is

$$\partial g^*(q) = \begin{cases} \{u_i\} & \text{if } q \in Q_i, \ 1 \le i < d, \\ \{\frac{1}{\alpha}q\} & \text{if } q \in Q_0, \\ \left[u_i, \frac{1}{\alpha}q\right] & \text{if } q \in Q_{i0}, \ 1 \le i \le d, \\ [u_i, u_{i+1}] & \text{if } q \in Q_{i,i+1}, \ 1 \le i < d, \end{cases}$$

where

$$\begin{aligned} Q_1 &= \left\{q : q - \alpha u_1 < \sqrt{2\alpha\beta} \quad \text{and} \quad q < \tfrac{\alpha}{2}(u_1 + u_2)\right\}, \\ Q_i &= \left\{q : |q - \alpha u_i| < \sqrt{2\alpha\beta} \quad \text{and} \quad \tfrac{\alpha}{2}(u_{i-1} + u_i) < q < \tfrac{\alpha}{2}(u_i + u_{i+1})\right\} \qquad \text{for } 1 < i < d, \\ Q_d &= \left\{q : q - \alpha u_d > \sqrt{2\alpha\beta} \quad \text{and} \quad \tfrac{\alpha}{2}(u_d + u_{d-1}) < q\right\}, \\ Q_0 &= \left\{q : |q - \alpha u_j| > \sqrt{2\alpha\beta} \quad \text{for all } j \in \{1, \dots, d\} \quad \text{and} \quad \alpha u_1 < q < \alpha u_d\right\} \\ Q_{i0} &= \left\{q : |q - \alpha u_i| = \sqrt{2\alpha\beta}\right\} \qquad \text{for } 1 \le i \le d, \\ Q_{i,i+1} &= \left\{q : q = \tfrac{\alpha}{2}(u_i + u_{i+1})\right\} \qquad \text{for } 1 \le i < d, \end{aligned}$$

Note that some of these sets can be empty. In fact, for β sufficiently large, Q_0 and hence Q_{i0}, $i = 1, \dots, d$, can be guaranteed to vanish; see [6, § 2.3].

To compute for given $\gamma > 0$ and $v \in \mathbb{R}$ the resolvent $w = (\mathrm{Id} + \gamma\partial g^*)^{-1}(v)$ of ∂g^*, we again use the relation $v \in \{w\} + \gamma\partial g^*(w)$ and follow the case differentiation in the subdifferential.

(i) $w \in Q_i$ for some $i \in \{1, \dots, d\}$: In this case, $v = w + \gamma u_i$, which implies that

$$|v - (\alpha + \gamma)u_i| \le \sqrt{2\alpha\beta}$$

and

$$\tfrac{\alpha}{2}\left(u_{i-1} + \left(1 + \tfrac{2\gamma}{\alpha}\right)u_i\right) < v < \tfrac{\alpha}{2}\left(\left(1 + \tfrac{2\gamma}{\alpha}\right)u_i + u_{i+1}\right)$$

(with the first and last condition being void for $i = 1$ and $i = d$, respectively).

(ii) $w \in Q_0$: In this case, $v = \left(1 + \frac{\gamma}{\alpha}\right)w$, which implies that

$$|\tfrac{\alpha}{\alpha+\gamma}v - \alpha u_j| > \sqrt{2\alpha\beta} \qquad \text{for all } j \in \{1, \dots, d\}$$

and

$$(\alpha+\gamma)u_1 < v < (\alpha+\gamma u_d).$$

(iii) $w \in Q_{i0}$ for some $i \in \{1,\dots,d\}$: In this case, $v \in [w,(1+\frac{\gamma}{\alpha})w]$ and $w = \alpha u_i + \sqrt{2\alpha\beta}$, which implies that

$$\sqrt{2\alpha\beta} \leq v - (\alpha+\gamma)u_i \leq \left(1+\frac{\gamma}{\alpha}\right)\sqrt{2\alpha\beta}.$$

(iv) $w \in Q_{i,i+1}$ for some $i \in \{1,\dots,d-1\}$: In this case, $v \in [w+\gamma u_i, w+\gamma u_{i+1}]$ and $w = \frac{\alpha}{2}(u_i+u_{i+1})$, which implies that

$$\tfrac{\alpha}{2}\left(\left(1+\tfrac{2\gamma}{\alpha}\right)u_i + u_{i+1}\right) \leq v \leq \tfrac{\alpha}{2}\left(u_i + \left(1+\tfrac{2\gamma}{\alpha}\right)u_{i+1}\right).$$

Inserting this into the definition of the Moreau–Yosida regularization and simplifying, we obtain

$$(\partial g^*)_\gamma(q) = \begin{cases} u_i & \text{if } q \in Q_i^\gamma \quad \text{for some } i \in \{1,\dots,d\}, \\ \frac{1}{\alpha+\gamma}q & \text{if } q \in Q_0^\gamma, \\ \frac{1}{\gamma}\left(q - (\alpha u_i + \sqrt{2\alpha\beta})\right) & \text{if } q \in Q_{i0}^\gamma \quad \text{for some } i \in \{1,\dots,d\}, \\ \frac{1}{\gamma}\left(q - \frac{\alpha}{2}(u_i+u_{i+1})\right) & \text{if } q \in Q_{i,i+1}^\gamma \quad \text{for some } i \in \{1,\dots,d-1\}, \end{cases}$$

where

$$\begin{aligned} Q_1^\gamma &= \left\{q : q - (\alpha+\gamma)u_1 < \sqrt{2\alpha\beta} \quad \text{and} \quad q < \tfrac{\alpha}{2}\left(\left(1+\tfrac{2\gamma}{\alpha}\right)u_1 + u_2\right)\right\}, \\ Q_i^\gamma &= \Big\{q : |q-(\alpha+\gamma)u_i| < \sqrt{2\alpha\beta} \quad \text{and} \\ &\qquad \tfrac{\alpha}{2}\left(u_{i-1} + \left(1+\tfrac{2\gamma}{\alpha}\right)u_i\right) < q < \tfrac{\alpha}{2}\left(\left(1+\tfrac{2\gamma}{\alpha}\right)u_i + u_{i+1}\right)\Big\} \quad \text{for } 1 < i < d, \\ Q_d^\gamma &= \left\{q : q - (\alpha+\gamma)u_d > \sqrt{2\alpha\beta} \quad \text{and} \quad \tfrac{\alpha}{2}\left(u_{d-1} + \left(1+\tfrac{2\gamma}{\alpha}\right)u_d\right) < q\right\}, \\ Q_0^\gamma &= \left\{q : |q-(\alpha+\gamma)u_j| > \sqrt{2\alpha\beta} \text{ for all } j \in \{1,\dots,d\} \text{ and } (\alpha+\gamma)u_1 < q < (\alpha+\gamma)u_d\right\}, \\ Q_{i0}^\gamma &= \left\{q : \sqrt{2\alpha\beta} \leq q - (\alpha+\gamma)u_i \leq \left(1+\frac{\gamma}{\alpha}\right)\sqrt{2\alpha\beta}\right\} \quad \text{for } 1 \leq i \leq d, \\ Q_{i,i+1}^\gamma &= \left\{q : \tfrac{\alpha}{2}\left(\left(1+\tfrac{2\gamma}{\alpha}\right)u_i + u_{i+1}\right) \leq q \leq \tfrac{\alpha}{2}\left(u_i + \left(1+\tfrac{2\gamma}{\alpha}\right)u_{i+1}\right)\right\} \quad \text{for } 1 \leq i < d. \end{aligned}$$

Since $h_\gamma := (\partial g^*)_\gamma$ is Lipschitz continuous and piecewise differentiable, it is semismooth, and its Newton-derivative at q in direction δq is given by

$$D_N h_\gamma(q)\delta q = \begin{cases} 0 & \text{if } q \in Q_i^\gamma \quad \text{for some } i \in \{1,\dots,d\}, \\ \frac{1}{\alpha+\gamma}\delta q & \text{if } q \in Q_0^\gamma, \\ \frac{1}{\gamma}\delta q & \text{if } q \in Q_{i0}^\gamma \quad \text{for some } i \in \{1,\dots,d\}, \\ \frac{1}{\gamma}\delta q & \text{if } q \in Q_{i,i+1}^\gamma \quad \text{for some } i \in \{1,\dots,d-1\}. \end{cases}$$

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