Bang-bang property of time optimal controls of Burgers equation

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Abstract

The bang-bang property of time optimal controls for Burgers equations in dimension up to three, with homogeneous Dirichlet boundary conditions and distributed controls acting on an open subset of the domain is established. This relies on an observability estimate from a measurable set in time for linear parabolic equations, with potentials depending on both space and time variables. The proof of the bang-bang property relies on a Kakutani fixed point argument.

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1 Introduction

While significant progress was made over the last years giving sufficient conditions for the bang-bang property of time optimal control problems of linear control systems, see e.g. [8] and [14], the bang-bang nature of controls for non-linear infinite dimensional control systems is much less understood, see, however [1], [17] and [22] . The purpose of this work is to analyze the bang-bang property of time optimal controls for a system which is not of global Lipschitzian nature.

Unless stated otherwise Ω is a bounded, convex domain in $\mathbb{R}^d$, if $d = 2, 3$, with boundary $\partial\Omega$ of class C^2, and it is a bounded interval if $d = 1$. Further ω is a non-trivial subdomain of Ω. We write χ_ω for the characteristic function of the set ω. For $q \geq 2$ and $\rho_0 > 0$, to be made precise later, we define the constraint set of controls to be

$$\mathcal{U} \equiv \{\vec{u} : [0, +\infty) \to (L^q(\Omega))^d \text{ is measurable} : \ \|\vec{u}(\cdot, t)\|_{(L^q(\Omega))^d} \leq \rho_0 \ \text{ for almost all } t > 0\}.$$

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The controlled Burgers equation under consideration is as follows:

$$\begin{cases} \vec{y}_t - \Delta\vec{y} + (\vec{y}\cdot\nabla)\vec{y} = \chi_\omega \vec{u} & \text{in } \Omega\times(0,+\infty), \\ \vec{y} = \vec{0} & \text{on } \partial\Omega\times(0,+\infty), \\ \vec{y}(\cdot,0) = \vec{y}_0(\cdot) & \text{in } \Omega. \end{cases} \tag{1.1}$$

This equation was developed by J.M.Burgers as a simplified fluid flow model, which describes the propagation of diffusive waves of finite amplitude (see. e.g. [5], [6] and [18]). While for $d = 1$ the function space setting for (1.1) is well-established this does not appear to be the case for higher dimensions. Therefore we consider well-posedness for (1.1) in a function space that is convenient for our analysis. Specifically for $\vec{y}_0(\cdot) \in W_q^{2-2/q}(\Omega) \cap W_0^{1,q}(\Omega)$ and $\vec{u} \in L^\infty(0,T; L^q(\Omega))$ we prove that (1.1) has a unique solution $\vec{y}(\cdot,\cdot;\vec{u}) \in \dot{W}_q^{2,1}(Q_T)$, see Proposition 2.1. Here for $s > 0$ and $T > 0$ fixed, the set

$$\{y \in L^s(\Omega\times(0,T)) : y_t \in L^s(\Omega\times(0,T)) \text{ and } y \in L^s(0,T; W^{2,s}(\Omega)\cap W_0^{1,s}(\Omega))\}$$

endowed with the usual $W_s^{2,1}(\Omega\times(0,T))$-norm is denoted by $\dot{W}_s^{2,1}(Q_T)$. For simplicity of notation we do not distinguish in notation between the space X and the vector-valued space X^d.

The set of admissible controls contains those which are bounded and which steer the state to the origin in finite time:

$$\mathcal{U}_{ad} \equiv \{\vec{u} \in \mathcal{U} : \ \vec{y}(\cdot,T;\vec{u}) = \vec{0} \ \text{ over } \ \Omega, \ \vec{y}(\cdot,\cdot;\vec{u}) \in \dot{W}_q^{2,1}(Q_T) \ \text{ for some } \ T > 0\}.$$

In Proposition 2.4 it will be proved that $\mathcal{U}_{ad}$ is not empty.

The time optimal control problem under consideration can now be stated as follows:

$$(P) \qquad \inf\{T : \ \vec{u} \in \mathcal{U}_{ad}\} \equiv T^*,$$

i.e., the minimal time needed to steer the system to $\vec{0}$ with controls in $\mathcal{U}_{ad}$. In this problem, the number T^* is called the optimal time; a control $\vec{u}^* \in \mathcal{U}_{ad}$, with $\vec{y}(\cdot,T^*;\vec{u}^*) = \vec{0}$ over Ω, is called a time optimal control (or optimal control for simplicity). *In Proposition 2.4 it will be proved that (P) allows optimal controls.*

We can now state the main result of this paper:

Theorem 1.1. *Assume that $q > 2$ for $d = 2$, and $q \in (3,6]$ for $d = 3$. Then there exists a nontrivial interval $\mathcal{I}$ of bounds ρ_0 such that the bang-bang property holds for (P): for $\rho_0 \in \mathcal{I}$ any optimal control $\vec{u}^*$ satisfies that $\|\vec{u}^*(\cdot,t)\|_{L^q(\Omega)} = \rho_0$ for a.e. $t \in (0,T^*)$. For $d = 1$ the assertion holds with $q = 2$ and all $\rho_0 > 0$.*

The bang-bang property is one of the most important and interesting properties of time optimal control problems. For abstract linear problems in Banach spaces, to the best of our best knowledge, this property was first established, via a smart construction manner, by H. O.

Fattorini (see. e.g. [7]). But in the context of the distributed control of the heat equation, for example, these techniques only apply for the special case where the control is distributed everywhere in the domain, i.e. $\omega = \Omega$. Since then, bang-bang controls with $\omega = \Omega$ for time optimal problems related to linear and semilinear parabolic differential equations, were investigated in many papers, see e.g. [1], [2], [8], [13], [22] and the references therein. More recently the case $\omega \subsetneq \Omega$ was treated successfully for parabolic equations. In [20], after establishing null-controllability of the internally controlled heat equation with controls restricted to a product set of an open nonempty subset in Ω and a subset of positive measure in time, the author proved the bang-bang property of time optimal controls. Partially motivated by these results the authors in [17] realized that the bang-bang property can be obtained by combining a strategy based on null controllability of the system, where the control functions act on a measurable set, and a fixed point argument. When the target set is a ball, the bang-bang properties for time optimal control problems of differential equations can be also derived from the Pontryagin maximum principle and unique continuation properties for the corresponding equations. We mention [21], [10], and [11] in this respect.

Controllability and numerical methods for optimal control of the Burgers equation were investigated in e.g. [9] and [19]. However, the bang-bang property for time optimal control problems of Burgers equation, with controls restricted over a proper subset of Ω was not yet studied. To prove Theorem 1.1, we first establish an observability estimate from a measurable set in time for parabolic equations, and then use the Kakutani's fixed point theorem. It should be pointed out that compared with (1.1), the semilinear equation considered in [17] has good properties, such as global existence and uniqueness of the strong solution, and good regularity of potential in the linearized system. However, the Burgers equation (1.1) lacks these properties, see Proposition 2.1 and (2.34).

The observability estimate mentioned above, can be obtained in arbitrary dimension. For this purpose let $\hat{\Omega}$ be a bounded connected domain in $\mathbb{R}^d$, $d \geq 1$, with boundary $\partial\hat{\Omega}$ of class C^2, let $T > 0$ and m be a positive integer. We introduce the following parabolic equation:

$$\begin{cases} \partial_t \vec{\varphi} - \Delta\vec{\varphi} + A\vec{\varphi} + a\vec{\varphi} + (\vec{b}\cdot\nabla)\vec{\varphi} = \vec{0} & \text{in } \hat{\Omega}\times(0,T), \\ \vec{\varphi} = \vec{0} & \text{on } \partial\hat{\Omega}\times(0,T), \\ \vec{\varphi}(\cdot,0) = \vec{\varphi}_0 \in L^2(\hat{\Omega}), \end{cases} \tag{1.2}$$

where $\vec{\varphi} = (\varphi_1, \cdots, \varphi_m)^\top$, $A = (a_{ij})_{1\leq i,j\leq m} \in (L^\infty(0,T;L^{\hat{q}}(\hat{\Omega})))^{m\times m}$, $a \in L^\infty(0,T;L^{\hat{q}}(\hat{\Omega}))$ with $\hat{q} \geq 2$ for $d = 1$, and $\hat{q} > d$ for $d \geq 2$, $\vec{b} \in (L^\infty(\hat{\Omega}\times(0,T)))^d$ and $(\vec{b}\cdot\nabla)\vec{\varphi} = (\vec{b}\cdot\nabla\varphi_1, \cdots, \vec{b}\cdot\nabla\varphi_m)^\top$. Then we have the following result.

Theorem 1.2. *Let $E \subset (0,T)$ be a measurable set with a positive measure and let $\hat{\omega}$ be a nonempty subdomain of $\hat{\Omega}$. Then any solution of (1.2) satisfies the estimate*

$$\begin{aligned} \|\vec{\varphi}(\cdot,T)\|_{L^2(\hat{\Omega})} \leq\; & e^{C(\hat{\Omega},\hat{\omega},d,m,\hat{q},E)} \\ & \cdot e^{C(\hat{\Omega},\hat{\omega},d,m,\hat{q})[1+(\|A\|_\infty^2+\|a\|_\infty^2+\|\vec{b}\|_\infty^2)(T+1)+\|A\|_\infty^{4/(2-\hat{p})}+\|a\|_\infty^{4/(2-\hat{p})}]} \int_{\hat{\omega}\times E} |\vec{\varphi}(x,t)|\,dx\,dt, \end{aligned} \tag{1.3}$$

where $\|A\|_\infty \equiv \|A\|_{L^\infty(0,T;L^{\hat q}(\hat\Omega))}$, $\|a\|_\infty \equiv \|a\|_{L^\infty(0,T;L^{\hat q}(\hat\Omega))}$, $\|\vec b\|_\infty \equiv \|\vec b\|_{L^\infty(\hat\Omega\times(0,T))}$ *and*

$$\hat p = \begin{cases} \frac{2d}{\hat q} & \text{if } \; d < \hat q < 2d, \\ 1 & \text{if } \; 2d \le \hat q. \end{cases} \tag{1.4}$$

Here and throughout Section 3, $C(\cdots)$ *denotes a generic positive constant that only depends on what is enclosed in the brackets.*

Estimate (1.3) is an observability inequality from a measurable set in time. It was established for the case $m = 1$ and assuming that $\hat\Omega$ is convex in [16], where the essential step consisted in a quantitative unique continuation at one point in time. Later, in [17], still for $m = 1$ the convexity assumption on $\hat\Omega$ was successfully dropped, but the potentials were assumed to be bounded. In our Theorem 1.2, the potentials still have the same regularity as in [16]. We prove (1.3) by using similar arguments as [17]. But compared with [17] and [16], the method of the present paper has the following merit: In [17] and [16], as $\varphi_0(\cdot) \neq 0$, the facts that $\varphi(\cdot,t) \neq 0$ in a small open subset of $\hat\Omega$ and $\varphi(\cdot,t) \neq 0$ in $\hat\Omega$ are the basis of the proofs, respectively. These properties can be guaranteed by the strong unique continuation property of parabolic equations with homogeneous boundary conditions and Théorème II.1 in [4], respectively. In this paper, the property $\varphi(\cdot,t) \neq 0$ is unnecessary. This is a consequence of the construction of a special frequency function, see Lemma 3.2. Moreover, the above-mentioned unique continuation property can be deduced by the result in this paper, see Remark 3.6. Finally let us remark that the results of this paper remain applicable if $-\Delta$ in (1.1) is replaced by $-\epsilon\Delta$ with ϵ a positive diffusion coefficient.

The rest of the paper is organized as follows: Section 2 contains the proof of Theorem 1.1. In Section 3 we give the proof of Theorem 1.2.

2 Time optimal control for the Burgers equation

The ultimate goal of this section is to give the proof for Theorem 1.1. Before address existence and uniqueness for (1.1), which is not readily available in the literature, and prove existence for the optimal control problem (P). The restrictions on the spatial dimension and on the range of q will be specified with each of these results. The case $d = 1$ will be considered at the end of this section. For convenience we first recall the definition of the space $W_q^{2-2/q}(\Omega)$. It is a Banach space consisting of the elements of $W^{1,q}(\Omega)$ with finite norm (see. e.g [12])

$$\|\varphi\|_{W_q^{2-2/q}(\Omega)} = \|\varphi\|_{W^{1,q}(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|D_x\varphi(x) - D_{\tilde x}\varphi(\tilde x)|^q}{|x-\tilde x|^q} \, d\tilde x \, dx \right)^{\frac{1}{q}}.$$

Proposition 2.1. *Let* $q \ge 2$ *for* $d = 2$, $q \in [2,6]$ *for* $d = 3$, *and* $q \in (2,4)$ *for* $d = 4$. *Then for any* $T > 0$ *and* $M > 0$, *there exists a positive constant* $\rho_1 = \rho_1(M,T)$, *such that for* $(\vec u, \vec y_0) \in L^\infty(0,T;L^q(\Omega)) \times W_q^{2-2/q}(\Omega) \cap W_0^{1,q}(\Omega)$ *satisfying*

$$\|\vec u\|_{L^\infty(0,T;L^q(\Omega))} + \|\vec y_0\|_{W_q^{2-2/q}(\Omega)\cap W_0^{1,q}(\Omega)} \le \rho_1,$$

the equation

$$\begin{cases} \partial_t \vec{y} - \Delta \vec{y} + (\vec{y}\cdot\nabla)\vec{y} = \chi_\omega \vec{u} & \textit{in } \Omega\times(0,T), \\ \vec{y} = \vec{0} & \textit{on } \partial\Omega\times(0,T), \\ \vec{y}(\cdot,0) = \vec{y}_0 & \textit{in } \Omega \end{cases} \tag{2.1}$$

has a unique solution $\vec{y}\in \dot{W}^{2,1}_q(Q_T)$. *Moreover,* $\|\vec{y}\|_{\dot{W}^{2,1}_q(Q_T)} \le M$.

Proof. The proof is based on the Schauder fixed point theorem. We set

$$\mathcal{K} = \{\vec{\xi}\in L^q(0,T;L^q(\Omega)) : \|\vec{\xi}\|_{\dot{W}^{2,1}_q(Q_T)} \le M\},$$

and consider for $\vec{\xi}\in\mathcal{K}$ the following linear equation

$$\begin{cases} \partial_t \vec{y} - \Delta \vec{y} + (\vec{y}\cdot\nabla)\vec{\xi} = \chi_\omega \vec{u} & \text{in } \Omega\times(0,T), \\ \vec{y} = \vec{0} & \text{on } \partial\Omega\times(0,T), \\ \vec{y}(\cdot,0) = \vec{y}_0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Multiplying the first equation of (2.2) by $-2\Delta\vec{y}$ and integrating it over $\Omega\times(0,t)$ we obtain using that $d\le 4$

$$\|\vec{y}(\cdot,t)\|^2_{H^1_0(\Omega)} \le C\left(\|\vec{y}_0\|^2_{H^1_0(\Omega)} + \|\vec{u}\|^2_{L^2(0,T;L^2(\Omega))}\right) + C\int_0^t \|\vec{y}\|^2_{H^1_0(\Omega)}\|\vec{\xi}\|^2_{H^2(\Omega)}\,ds, \quad \forall\, t\in[0,T].$$

Here and below C denotes a generic constant. Using Gronwall's inequality we find

$$\|\vec{y}(\cdot,t)\|^2_{H^1_0(\Omega)} \le C\left(\|\vec{y}_0\|^2_{H^1_0(\Omega)} + \|\vec{u}\|^2_{L^2(0,T;L^2(\Omega))}\right) e^{C\|\vec{\xi}\|^2_{L^2(0,T;H^2(\Omega))}}, \quad \forall\, t\in[0,T].$$

By Sobolev's embedding theorem, it can be checked that for the choice of dimensions and range of q values the following estimate holds:

$$\int_\Omega |(\vec{y}\cdot\nabla)\vec{\xi}|^q\,dx \le C\|\vec{y}\|^q_{H^1_0(\Omega)}\|\vec{\xi}\|^q_{W^{2,q}(\Omega)}$$

for a constant C independent of $\vec{y}\in H^1_0(\Omega)$ and $\vec{\xi}\in W^{2,q}(\Omega)$. Here we could still use $q\in[2,4)$ for $d=4$. Combining these estimates we obtain

$$\int_0^T\int_\Omega |(\vec{y}\cdot\nabla)\vec{\xi}|^q\,dx\,dt \le C(M,T)\left(\|\vec{y}_0\|^2_{W^{2-2/q}_q(\Omega)\cap W^{1,q}_0(\Omega)} + \|\vec{u}\|^2_{L^\infty(0,T;L^q(\Omega))}\right)^{\frac{q}{2}}. \tag{2.3}$$

From (2.2), (2.3) and L^p-theory for parabolic equations (see Theorem 9.1 of Chapter 4 in [12]), it follows that

$$\|\vec{y}\|_{\dot{W}^{2,1}_q(Q_T)} \le C(M,T)\left(\|\vec{y}_0\|_{W^{2-2/q}_q(\Omega)\cap W^{1,q}_0(\Omega)} + \|\vec{u}\|_{L^\infty(0,T;L^q(\Omega))}\right). \tag{2.4}$$

By (2.4) we obtain that there exists a constant $\rho_1 = \rho_1(M,T) > 0$, such that if

$$\|\vec{y}_0\|_{W^{2-2/q}_q(\Omega)\cap W^{1,q}_0(\Omega)} + \|\vec{u}\|_{L^\infty(0,T;L^q(\Omega))} \le \rho_1,$$

then $\|\vec{y}\|_{\dot{W}_q^{2,1}(Q_T)} \le M$.

Now we define the mapping $\Phi : \mathcal{K} \to \mathcal{K}$ by $\Phi(\vec{\xi}) = \vec{y}$, $\vec{\xi} \in \mathcal{K}$, where $\vec{y}$ is the solution to (2.2) and verify the conditions of the Schauder fixed point theorem. This consists of two steps.

Step 1. The fact that

$$\mathcal{K} \textit{ is a compact convex subset of } L^q(0,T;L^q(\Omega)),$$

follows from Sobolev's embedding theorems.

Step 2. $\Phi : \mathcal{K} \to \mathcal{K}$ *is continuous, i.e., if* $\vec{\xi}_n \in \mathcal{K}, \vec{\xi}_n \to \vec{\xi}$ *strongly in* $L^q(0,T;L^q(\Omega))$*, then* $\Phi(\vec{\xi}_n) \to \Phi(\vec{\xi})$ *strongly in* $L^q(0,T;L^q(\Omega))$.

Proceeding by a contradiction argument, assume that there exist a constant $\varepsilon_0 > 0$ and a subsequence of $\{\Phi(\vec{\xi}_n)\}_{n\ge1}$, denoted by $\{\Phi(\vec{\xi}_{n_k})\}_{k\ge1}$, such that

$$\|\Phi(\vec{\xi}_{n_k}) - \Phi(\vec{\xi})\|_{L^q(0,T;L^q(\Omega))} \ge \varepsilon_0. \tag{2.5}$$

Since $\Phi(\vec{\xi}_{n_k}) \in \mathcal{K}$, there exist a subsequence of $\{n_k\}_{k\ge1}$, still denoted by the same notation, and $\vec{z} \in \mathcal{K}$, such that

$$\Phi(\vec{\xi}_{n_k}) \to \vec{z},\ \vec{\xi}_{n_k} \to \vec{\xi} \text{ weakly in } \dot{W}_q^{2,1}(Q_T) \text{ and strongly in } L^q(0,T;W_0^{1,q}(\Omega)), \tag{2.6}$$

and

$$\begin{cases} \partial_t\Phi(\vec{\xi}_{n_k}) - \Delta\Phi(\vec{\xi}_{n_k}) + (\Phi(\vec{\xi}_{n_k})\cdot\nabla)\vec{\xi}_{n_k} = \chi_\omega\vec{u} & \text{in } \Omega\times(0,T), \\ \Phi(\vec{\xi}_{n_k}) = \vec{0} & \text{on } \partial\Omega\times(0,T), \\ \Phi(\vec{\xi}_{n_k})(\cdot,0) = \vec{y}_0 & \text{in } \Omega. \end{cases} \tag{2.7}$$

Now we claim that there exists a subsequence of $\{n_k\}_{k\ge1}$, still denoted in the same manner, such that

$$(\Phi(\vec{\xi}_{n_k})\cdot\nabla)\vec{\xi}_{n_k} \to (\vec{z}\cdot\nabla)\vec{\xi} \text{ weakly in } L^q(0,T;L^q(\Omega)). \tag{2.8}$$

On one hand,

$$\|(\Phi(\vec{\xi}_{n_k})\cdot\nabla)\vec{\xi}_{n_k}\|_{L^q(0,T;L^q(\Omega))} = \|\partial_t\Phi(\vec{\xi}_{n_k}) - \Delta\Phi(\vec{\xi}_{n_k}) - \chi_\omega\vec{u}\|_{L^q(0,T;L^q(\Omega))} \le C(M,T). \tag{2.9}$$

On the other hand, for any $\vec{h} \in L^\infty(\Omega\times(0,T))$, by (2.6), we have

$$\begin{aligned} & \Big|\int_0^T\int_\Omega [(\Phi(\vec{\xi}_{n_k})\cdot\nabla)\vec{\xi}_{n_k} - (\vec{z}\cdot\nabla)\vec{\xi}]\vec{h}\,dx\,dt\Big| \\ = & \Big|\int_0^T\int_\Omega ((\Phi(\vec{\xi}_{n_k}) - \vec{z})\cdot\nabla)\vec{\xi}_{n_k}\vec{h}\,dx\,dt + \int_0^T\int_\Omega (\vec{z}\cdot\nabla)(\vec{\xi}_{n_k} - \vec{\xi})\vec{h}\,dx\,dt\Big| \\ \le & C\|\Phi(\vec{\xi}_{n_k}) - \vec{z}\|_{L^2(0,T;L^2(\Omega))} + C\|\nabla(\vec{\xi}_{n_k} - \vec{\xi})\|_{L^2(0,T;L^2(\Omega))} \to 0. \end{aligned} \tag{2.10}$$

It follows from (2.9) and (2.10) that (2.8) holds. Then, passing to the limit for $k \to +\infty$ in (2.5) and (2.7), by (2.6) and (2.8), we obtain that

$$\|\vec{z} - \Phi(\vec{\xi})\|_{L^q(0,T;L^q(\Omega))} \geq \varepsilon_0 \ \text{ and } \ \vec{z} = \Phi(\vec{\xi}),$$

which lead to a contradiction.

By Step 1 and Step 2 the Schauder fixed point theorem implies the existence of $\vec{y} \in \mathcal{K}$ such that $\Phi(\vec{y}) = \vec{y}$.

Finally we prove uniqueness. Let $\vec{y}_1, \vec{y}_2 \in \dot{W}_q^{2,1}(Q_T)$ be two solutions to (2.1). Then

$$\begin{cases} \partial_t(\vec{y}_1 - \vec{y}_2) - \Delta(\vec{y}_1 - \vec{y}_2) = ((\vec{y}_2 - \vec{y}_1) \cdot \nabla)\vec{y}_2 + (\vec{y}_1 \cdot \nabla)(\vec{y}_2 - \vec{y}_1) & \text{in } \ \Omega \times (0,T), \\ \vec{y}_1 - \vec{y}_2 = \vec{0} & \text{on } \ \partial\Omega \times (0,T), \\ (\vec{y}_1 - \vec{y}_2)(\cdot, 0) = \vec{0} & \text{in } \ \Omega. \end{cases} \tag{2.11}$$

Multiplying the first equation of (2.11) by $2(\vec{y}_1 - \vec{y}_2)$, and integrating over Ω, we obtain by Hölder inequality and the Sobolev's embedding theorem that

$$\begin{aligned} & \frac{d}{dt}\|(\vec{y}_1 - \vec{y}_2)(\cdot,t)\|^2_{L^2(\Omega)} + 2\|\nabla(\vec{y}_1 - \vec{y}_2)(\cdot,t)\|^2_{L^2(\Omega)} \\ \leq \ & C\|\nabla(\vec{y}_1 - \vec{y}_2)(\cdot,t)\|_{L^2(\Omega)}(\|\vec{y}_1(\cdot,t)\|_{W^{2,q}(\Omega)} + \|\vec{y}_2(\cdot,t)\|_{W^{2,q}(\Omega)})\|(\vec{y}_1 - \vec{y}_2)(\cdot,t)\|_{L^2(\Omega)}, \end{aligned}$$

where we use that $W^{2,q}(\Omega)$ embeds continuously into $C(\bar{\Omega})$ if $2q > d$. This implies that

$$\frac{d}{dt}\|(\vec{y}_1 - \vec{y}_2)(\cdot,t)\|^2_{L^2(\Omega)} \leq C(\|\vec{y}_1(\cdot,t)\|^2_{W^{2,q}(\Omega)} + \|\vec{y}_2(\cdot,t)\|^2_{W^{2,q}(\Omega)})\|(\vec{y}_1 - \vec{y}_2)(\cdot,t)\|^2_{L^2(\Omega)}.$$

Integrating the latter inequality over $(0,t)$, $t \in [0,T]$, and using Gronwall's inequality, we obtain that $\vec{y}_1 = \vec{y}_2$.

□

The next proposition is concerned with the local null controllability of (2.1).

Proposition 2.2. *Let $q > 2$ for $d = 2$, or $q \in (3,6]$ for $d = 3$. Then for any $T > 0$, there exist positive constants $\rho_2 = \rho_2(T)$ and $\rho_3 = \rho_3(T)$ such that if $\|\vec{y}_0\|_{W_q^{2-2/q}(\Omega) \cap W_0^{1,q}(\Omega)} \leq \rho_3$ then there exists a control $\vec{u}$ with $\|\vec{u}\|_{L^\infty(0,T;L^q(\Omega))} \leq \rho_2\|\vec{y}_0\|_{L^2(\Omega)}$, such that the solution of (2.1) satisfies $\vec{y}(\cdot,T) = \vec{0}$ in Ω.*

Proof. We shall use Kakutani's fixed point theorem (see e.g. [1]) for the proof. For this purpose, define

$$\mathcal{K} = \{\vec{\xi} \in L^q(0,T;L^q(\Omega)) : \|\vec{\xi}\|_{\dot{W}_q^{2,1}(Q_T)} \leq 1\}.$$

For each $\vec{\xi} \in \mathcal{K}$, we consider the linear control system

$$\begin{cases} \partial_t\vec{y} - \Delta\vec{y} + (\vec{y} \cdot \nabla)\vec{\xi} = \chi_\omega\vec{u} & \text{in } \ \Omega \times (0,T), \\ \vec{y} = \vec{0} & \text{on } \ \partial\Omega \times (0,T), \\ \vec{y}(\cdot,0) = \vec{y}_0 & \text{in } \ \Omega. \end{cases} \tag{2.12}$$

Its adjoint system is

$$\begin{cases} \partial_t \vec{\psi} + \Delta \vec{\psi} - (\nabla \vec{\xi}) \vec{\psi} = \vec{0} & \text{in } \Omega \times (0,T), \\ \vec{\psi} = \vec{0} & \text{on } \partial\Omega \times (0,T), \\ \vec{\psi}(\cdot, T) \in L^2(\Omega), \end{cases}$$

and hence by Theorem 1.2, and the equivalence of observability and controllability there exist a positive constant $\rho_2 = \rho_2(T)$, and a control $\vec{u} \in L^\infty(0,T; L^q(\Omega))$ such that

$$\vec{y}(\cdot, T) = \vec{0}, \tag{2.13}$$

and

$$\|\vec{u}\|_{L^\infty(0,T;L^q(\Omega))} \le \rho_2 \|\vec{y}_0\|_{L^2(\Omega)}. \tag{2.14}$$

Now we define a multivalued mapping $\Phi : \mathcal{K} \to L^q(0,T; L^q(\Omega))$ by

$$\Phi(\vec{\xi}) = \{\vec{y} : \text{ there exists a control } \vec{u} \text{ such that } (2.12) - (2.14) \text{ hold}\}, \quad \text{where } \vec{\xi} \in \mathcal{K}.$$

From the above arguments it follows that $\Phi(\vec{\xi}) \neq \emptyset$ for each $\vec{\xi} \in \mathcal{K}$.

Next we shall check in three steps the conditions of Kakutani's fixed point theorem.

Step 1. It is straightforward to verify that

$\mathcal{K}$ is a convex, compact subset of $L^q(0,T; L^q(\Omega))$ and
$\Phi(\vec{\xi})$ is a convex subset of $L^q(0,T; L^q(\Omega))$ for each $\vec{\xi} \in \mathcal{K}$.

Step 2. $\Phi(\mathcal{K}) \subset \mathcal{K}$.

In fact, for any $\vec{\xi} \in \mathcal{K}$, there exists a control $\vec{u} \in L^\infty(0,T; L^q(\Omega))$ satisfying

$$\|\vec{u}\|_{L^\infty(0,T;L^q(\Omega))} \le \rho_2 \|\vec{y}_0\|_{L^2(\Omega)} \tag{2.15}$$

and such that $\vec{y} = \vec{y}(u)$ satisfies

$$\begin{cases} \partial_t \vec{y} - \Delta \vec{y} + (\vec{y} \cdot \nabla) \vec{\xi} = \chi_\omega \vec{u} & \text{in } \Omega \times (0,T), \\ \vec{y} = \vec{0} & \text{on } \partial\Omega \times (0,T), \\ \vec{y}(\cdot, 0) = \vec{y}_0 & \text{in } \Omega, \\ \vec{y}(\cdot, T) = \vec{0} & \text{in } \Omega. \end{cases}$$

By (2.15) and the same arguments that led to (2.4), we have

$$\|\vec{y}\|_{\dot{W}_q^{2,1}(Q_T)} \le C(T) \left(\|\vec{y}_0\|_{W_q^{2-2/q}(\Omega) \cap W_0^{1,q}(\Omega)} + \|\vec{u}\|_{L^\infty(0,T;L^q(\Omega))} \right) \le C(T) \|\vec{y}_0\|_{W_q^{2-2/q}(\Omega) \cap W_0^{1,q}(\Omega)},$$

from which, we obtain that there exists a positive constant $\rho_3 = \rho_3(T)$, such that if

$$\|\vec{y}_0\|_{W_q^{2-2/q}(\Omega) \cap W_0^{1,q}(\Omega)} \le \rho_3,$$

then $\vec{y} \in \mathcal{K}$.

Step 3. The map Φ is upper semicontinuous in $L^q(0,T;L^q(\Omega))$, i.e., if $\vec{\xi}_n \to \vec{\xi}$ strongly in $L^q(0,T;L^q(\Omega))$, $\vec{y}_n \in \Phi(\vec{\xi}_n)$ and $\vec{y}_n \to \vec{z}$ strongly in $L^q(0,T;L^q(\Omega))$, then $\vec{z} \in \Phi(\vec{\xi})$.

Since $\vec{y}_n \in \Phi(\vec{\xi}_n)$, there exists $\vec{u}_n \in L^\infty(0,T;L^q(\Omega))$ satisfying (2.15) and

$$\begin{cases} \partial_t \vec{y}_n - \Delta \vec{y}_n + (\vec{y}_n \cdot \nabla)\vec{\xi}_n = \chi_\omega \vec{u}_n & \text{in } \Omega \times (0,T), \\ \vec{y}_n = \vec{0} & \text{on } \partial\Omega \times (0,T), \\ \vec{y}_n(\cdot,0) = \vec{y}_0 & \text{in } \Omega, \\ \vec{y}_n(\cdot,T) = \vec{0} & \text{in } \Omega. \end{cases} \tag{2.16}$$

By Step 2 we have that $\{\vec{y}_n\}_{n\geq 1} \subset \mathcal{K}$. By (2.15) with $\vec{u}$ replaced by $\vec{u}_n$ there exist a subsequence of $\{n\}_{n\geq 1}$, still denoted in the same manner, and $\vec{u} \in L^\infty(0,T;L^q(\Omega))$, such that

$$\begin{aligned} \vec{y}_n \to \vec{z},\ \vec{\xi}_n \to \vec{\xi} \quad & \text{weakly in } \dot{W}_q^{2,1}(Q_T), \\ & \text{strongly in } L^q(0,T;W_0^{1,q}(\Omega)) \cap C([0,T];L^q(\Omega)), \end{aligned} \tag{2.17}$$

$$\vec{u}_n \to \vec{u} \ \text{ weakly star in } L^\infty(0,T;L^q(\Omega)) \tag{2.18}$$

and

$$\|\vec{u}\|_{L^\infty(0,T;L^q(\Omega))} \leq \rho_2 \|\vec{y}_0\|_{L^2(\Omega)}. \tag{2.19}$$

From (2.17) and the same arguments that led to (2.8) it follows that there exists a subsequence of $\{n\}_{n\geq 1}$, still denoted by the same notation, such that

$$(\vec{y}_n \cdot \nabla)\vec{\xi}_n \to (\vec{z} \cdot \nabla)\vec{\xi} \ \text{ weakly in } L^q(0,T;L^q(\Omega)). \tag{2.20}$$

Passing to the limit for $n \to +\infty$ in (2.16), we obtain from (2.17)-(2.20) that $\vec{z} \in \Phi(\vec{\xi})$.

Kakutani's fixed point theorem now implies the existence of $\vec{y} \in \mathcal{K}$ such that $\vec{y} \in \Phi(\vec{y})$. This completes the proof. $\square$

From Proposition 2.1 and Proposition 2.2 we deduce the following corollary, in which C_0 denotes the embedding constant of $W_q^{2-\frac{2}{q}}(\Omega)$ into $L^2(\Omega)$, and $M_0 > 0$ and $T_0 > 0$ are arbitrarily fixed constants.

Corollary 2.3. *Choose $\vec{y}_0 \in W_q^{2-2/q}(\Omega) \cap W_0^{1,q}(\Omega)$ satisfying*

$$0 < \|\vec{y}_0\|_{W_q^{2-2/q}(\Omega)\cap W_0^{1,q}(\Omega)} < min\left\{ \frac{\rho_1(M_0,T_0)}{\rho_2(T_0)C_0 + 1}, \rho_3(T_0) \right\}, \tag{2.21}$$

and let

$$\rho \in \left[\rho_2(T_0), \frac{1}{C_0}\left(\frac{\rho_1(M_0,T_0)}{\|\vec{y}_0\|_{W_q^{2-2/q}(\Omega)\cap W_0^{1,q}(\Omega)}} - 1 \right) \right].$$

Then for any $\vec{u} \in L^\infty(0,+\infty; L^q(\Omega))$ *with* $\|\vec{u}\|_{L^\infty(0,+\infty;L^q(\Omega))} \le \rho_0 := \rho\|\vec{y}_0\|_{L^2(\Omega)}$, *the equation*

$$\begin{cases} \partial_t \vec{y} - \Delta \vec{y} + (\vec{y}\cdot\nabla)\vec{y} = \chi_\omega \vec{u} & \text{in } \Omega\times(0,T_0), \\ \vec{y} = \vec{0} & \text{on } \partial\Omega\times(0,T_0), \\ \vec{y}(\cdot,0) = \vec{y}_0 & \text{in } \Omega \end{cases} \tag{2.22}$$

has a unique solution $\vec{y} \in \dot{W}_q^{2,1}(Q_{T_0})$ *with* $\|\vec{y}\|_{\dot{W}_q^{2,1}(Q_{T_0})} \le M_0$. *Moreover, there exists a control* $\vec{v} \in L^\infty(0,+\infty; L^q(\Omega))$ *with* $\|\vec{v}\|_{L^\infty(0,+\infty;L^q(\Omega))} \le \rho_0$, *such that the solution* $\vec{y}(\cdot,\cdot;\vec{v})$ *of (2.22) corresponding to* $\vec{v}$ *satisfies* $\vec{y}(\cdot,T_0;\vec{v}) = \vec{0}$ *in* Ω.

Henceforth we fix $\vec{y}_0$ satisfying (2.21) and $\rho_0 = \rho\|\vec{y}_0\|_{L^2(\Omega)}$.

Proposition 2.4. *Problem* (P) *has at least one solution.*

Proof. From Corollary 2.3 it follows that problem (P) has an admissible control. Let $T^* = \inf(P)$. It is obvious that $0 \le T^* \le T_0$.

If $T^* = T_0$, then the proof is complete. Otherwise $T^* < T_0$, and there exist sequences $\{T_n\}_{n\ge1}$ and $\{\vec{u}_n\}_{n\ge1} \subset \mathcal{U}$ such that

$$T^* = \lim_{n\to+\infty} T_n \tag{2.23}$$

and

$$\begin{cases} \partial_t \vec{y}_n - \Delta \vec{y}_n + (\vec{y}_n\cdot\nabla)\vec{y}_n = \chi_\omega \vec{u}_n & \text{in } \Omega\times(0,T_n), \\ \vec{y}_n = \vec{0} & \text{on } \partial\Omega\times(0,T_n), \\ \vec{y}_n(\cdot,0) = \vec{y}_0 & \text{in } \Omega, \\ \vec{y}_n(\cdot,T_n) = \vec{0} & \text{in } \Omega, \end{cases} \tag{2.24}$$

where $\vec{y}_n(\cdot,\cdot) = \vec{y}(\cdot,\cdot;\vec{u}_n) \in \dot{W}_q^{2,1}(Q_{T_n})$. By (2.23) and (2.24), we can assume that $0 < T_n < T_0$.

Set

$$\vec{v}_n(\cdot,t) = \begin{cases} \vec{u}_n, & t \in (0,T_n), \\ \vec{0}, & t \in [T_n,+\infty) \end{cases} \quad \text{and} \quad \vec{z}_n(\cdot,t) = \begin{cases} \vec{y}_n, & t \in (0,T_n), \\ \vec{0}, & t \in [T_n,T_0]. \end{cases} \tag{2.25}$$

From the fact that $\{\vec{u}_n\}_{n\ge1} \subset \mathcal{U}$, (2.24) and (2.25) it follows that

$$\{\vec{v}_n\}_{n\ge1} \subset \mathcal{U} \tag{2.26}$$

and $\vec{z}_n(\cdot,\cdot) \in \dot{W}_q^{2,1}(Q_{T_0})$ satisfies

$$\begin{cases} \partial_t \vec{z}_n - \Delta \vec{z}_n + (\vec{z}_n\cdot\nabla)\vec{z}_n = \chi_\omega \vec{v}_n & \text{in } \Omega\times(0,T_0), \\ \vec{z}_n = \vec{0} & \text{on } \partial\Omega\times(0,T_0), \\ \vec{z}_n(\cdot,0) = \vec{y}_0 & \text{in } \Omega, \\ \vec{z}_n(\cdot,T_n) = \vec{0} & \text{in } \Omega. \end{cases} \tag{2.27}$$

By (2.26), (2.27) and Corollary 2.3, we obtain

$$\|\vec{z}_n\|_{\dot{W}_q^{2,1}(Q_{T_0})} \le M_0, \qquad \forall\, n \ge 1,$$

which, combined with (2.26), implies that there exist a subsequence of $\{n\}_{n\geq 1}$, still denoted in the same manner, $\vec{z} \in \dot{W}_q^{2,1}(Q_{T_0})$ and $\vec{v} \in \mathcal{U}$, such that

$$\begin{array}{lll} \vec{z}_n \to \vec{z} & \text{weakly in } \dot{W}_q^{2,1}(Q_{T_0}) \text{ and strongly in } L^q(0,T_0;W_0^{1,q}(\Omega)) \cap C([0,T_0];L^q(\Omega)), \\ \vec{v}_n \to \vec{v} & \text{weakly star in } L^\infty(0,+\infty;L^q(\Omega)). \end{array} \tag{2.28}$$

By (2.28) and the same arguments as for (2.8) we have that there exists a subsequence of $\{n\}_{n\geq 1}$, still denoted by the same notation, such that

$$(\vec{z}_n \cdot \nabla)\vec{z}_n \to (\vec{z} \cdot \nabla)\vec{z} \text{ weakly in } L^q(0,T_0;L^q(\Omega)). \tag{2.29}$$

Passing to the limit for $n \to +\infty$ in (2.27), by (2.28), (2.29) and (2.23), we obtain

$$\begin{cases} \partial_t \vec{z} - \Delta \vec{z} + (\vec{z}\cdot\nabla)\vec{z} = \chi_\omega \vec{v} & \text{in } \Omega \times (0,T_0), \\ \vec{z} = \vec{0} & \text{on } \partial\Omega \times (0,T_0), \\ \vec{z}(\cdot,0) = \vec{y}_0 & \text{in } \Omega, \\ \vec{z}(\cdot,T^*) = \vec{0} & \text{in } \Omega. \end{cases}$$

This completes the proof. □

Now we give the proof of Theorem 1.1.

Proof. By a contradiction argument, there would exist a positive constant $\varepsilon_0 < \rho_0$ and a measurable subset $E^* \subset (0,T^*)$ with $|E^*| > 0$ such that

$$\|\vec{u}^*(\cdot,t)\|_{L^q(\Omega)} \leq \rho_0 - \varepsilon_0, \quad \forall\, t \in E^*. \tag{2.30}$$

Take $\delta_0 \in (0,|E^*|/2)$ and denote $E^*_{\delta_0} = \{t \in (0,T^*) : t+\delta_0 \in E^*\}$. Then we have $|E^*_{\delta_0}| > 0$. Indeed, on one hand by the definition of $E^*_{\delta_0}$, if $t \in E^*_{\delta_0}$, then $t+\delta_0 \in E^* \cap (\delta_0,T^*)$. On the other hand, if $t \in E^* \cap (\delta_0,T^*)$, then $t-\delta_0 \in E^*_{\delta_0}$. Hence

$$|E^*_{\delta_0}| = |E^* \cap (\delta_0,T^*)| \geq |E^*| - \delta_0 > 2^{-1}|E^*|.$$

Denote $\vec{y}^*(x,t) = \vec{y}(x,t;\vec{u}^*)$ and $\vec{z}^*_{\delta_0}(x,t) = \vec{y}^*(x,t+\delta_0)$. Then we get that

$$\begin{cases} (\vec{z}^*_{\delta_0})_t - \Delta \vec{z}^*_{\delta_0} + (\vec{z}^*_{\delta_0}\cdot\nabla)\vec{z}^*_{\delta_0} = \chi_\omega \vec{u}^*(\cdot,t+\delta_0) & \text{in } \Omega \times (0,T^*-\delta_0), \\ \vec{z}^*_{\delta_0} = \vec{0} & \text{on } \partial\Omega \times (0,T^*-\delta_0), \\ \vec{z}^*_{\delta_0}(\cdot,0) = \vec{y}^*(\cdot,\delta_0) & \text{in } \Omega, \\ \vec{z}^*_{\delta_0}(\cdot,T^*-\delta_0) = \vec{0} & \text{in } \Omega. \end{cases} \tag{2.31}$$

We claim that there exists a real number $\delta_1 \in (0,\delta_0)$, such that as $\delta \in (\delta_1,\delta_0)$, there exists a couple $(\vec{h}_\delta,\vec{u}_\delta) \in \dot{W}_2^{2,1}(Q_{T^*-\delta_0}) \times L^\infty(0,T^*-\delta_0;L^q(\Omega))$ satisfying

$$\begin{cases} (\vec{h}_\delta)_t - \Delta \vec{h}_\delta + ((\vec{h}_\delta + \vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}_\delta + (\vec{h}_\delta\cdot\nabla)\vec{z}^*_{\delta_0} = \chi_\omega \chi_{E^*_{\delta_0}} \vec{u}_\delta & \text{in } \Omega \times (0,T^*-\delta_0), \\ \vec{h}_\delta = \vec{0} & \text{on } \partial\Omega \times (0,T^*-\delta_0), \\ \vec{h}_\delta(\cdot,0) = \vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0) & \text{in } \Omega, \\ \vec{h}_\delta(\cdot,T^*-\delta_0) = \vec{0} & \text{in } \Omega \end{cases} \tag{2.32}$$

and

$$\|\vec{u}_\delta\|_{L^\infty(0,T^*-\delta_0;L^q(\Omega))} \le c_1\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{L^2(\Omega)}, \tag{2.33}$$

where $c_1>0$ is a constant independent of δ and to be determined later.

We shall use the Kakutani's fixed point theorem to prove (2.32) and (2.33). To this end we set $\tilde{q}=\min\{q,10/3\}$ and define

$$\mathcal{K}_{\delta_0}=\{\vec{\xi}\in L^2(0,T^*-\delta_0;L^2(\Omega)):\ \|\vec{\xi}\|_{\dot{W}_2^{2,1}(Q_{T^*-\delta_0})}+\|\vec{\xi}\|_{L^\infty(0,T^*-\delta_0;W_0^{1,\tilde{q}}(\Omega))}\le 1\}.$$

Let $\delta\in(0,\delta_0)$ be a constant which is fixed later. For any $\vec{\xi}\in\mathcal{K}_{\delta_0}$ consider the linear control system

$$\begin{cases} \vec{h}_t-\Delta\vec{h}+((\vec{\xi}+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}+(\vec{h}\cdot\nabla)\vec{z}^*_{\delta_0}=\chi_\omega\chi_{E^*_{\delta_0}}\vec{u} & \text{in } \Omega\times(0,T^*-\delta_0),\\ \vec{h}=\vec{0} & \text{on } \partial\Omega\times(0,T^*-\delta_0),\\ \vec{h}(\cdot,0)=\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0) & \text{in } \Omega. \end{cases} \tag{2.34}$$

Its adjoint system is

$$\begin{cases} \vec{\psi}_t+\Delta\vec{\psi}-(\nabla\vec{z}^*_{\delta_0})\vec{\psi}+(\mathrm{div}\vec{z}^*_{\delta_0}+\mathrm{div}\vec{\xi})\vec{\psi}+((\vec{\xi}+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{\psi}=\vec{0} & \text{in } \Omega\times(0,T^*-\delta_0),\\ \vec{\psi}=\vec{0} & \text{on } \partial\Omega\times(0,T^*-\delta_0),\\ \vec{\psi}(\cdot,T^*-\delta_0)\in L^2(\Omega), \end{cases}$$

and hence by Theorem 1.2, and the equivalence of observability and controllability, we deduce that there exist a positive constant $c_1\equiv C_1(\Omega,\omega,E^*_{\delta_0},T^*,\delta_0)$ and a control $\vec{u}\in L^\infty(0,T^*-\delta_0;L^q(\Omega))$ such that

$$\vec{h}(T^*-\delta_0;\vec{u})=\vec{0}, \tag{2.35}$$

and

$$\|\vec{u}\|_{L^\infty(0,T^*-\delta;L^q(\Omega))}\le c_1\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{L^2(\Omega)}. \tag{2.36}$$

Now, we define the multivalued map $\Phi_\delta:\mathcal{K}_{\delta_0}\to L^2(0,T^*-\delta_0;L^2(\Omega))$ by

$$\Phi_\delta(\vec{\xi})=\{\vec{h}:\ \text{there exists a control } \vec{u} \text{ such that (2.34), (2.35) and (2.36) hold}\},\quad \text{for } \vec{\xi}\in\mathcal{K}_{\delta_0}.$$

From the above arguments it follows that $\Phi_\delta(\vec{\xi})\neq\emptyset$ for each $\vec{\xi}\in\mathcal{K}_{\delta_0}$.

Next we check in three steps the conditions of Kakutani's fixed point theorem.

Step 1. It is straightforward to check that

$$\mathcal{K}_{\delta_0} \textit{ is a convex, compact set in } L^2(0,T^*-\delta_0;L^2(\Omega)) \textit{ and}$$
$$\Phi_\delta(\vec{\xi}) \textit{ is a convex set in } L^2(0,T^*-\delta_0;L^2(\Omega)) \textit{ for each } \vec{\xi}\in\mathcal{K}_{\delta_0}.$$

Step 2. $\Phi_\delta(\mathcal{K}_{\delta_0})\subset\mathcal{K}_{\delta_0}$.

To achieve this goal, we use that for every $\vec{\xi} \in \mathcal{K}_{\delta_0}$, there exists a control $\vec{u} \in L^\infty(0, T^* - \delta_0; L^q(\Omega))$ satisfying

$$\|\vec{u}\|_{L^\infty(0,T^*-\delta_0;L^q(\Omega))} \leq c_1 \|\vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0)\|_{L^2(\Omega)} \tag{2.37}$$

such that the associated state $\vec{h} = \vec{h}(x,t)$ satisfies

$$\begin{cases} \vec{h}_t - \Delta\vec{h} + ((\vec{\xi} + \vec{z}^*_{\delta_0}) \cdot \nabla)\vec{h} + (\vec{h} \cdot \nabla)\vec{z}^*_{\delta_0} = \chi_\omega \chi_{E^*_{\delta_0}} \vec{u} & \text{in } \Omega \times (0, T^* - \delta_0), \\ \vec{h} = \vec{0} & \text{on } \partial\Omega \times (0, T^* - \delta_0), \\ \vec{h}(\cdot, 0) = \vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0) & \text{in } \Omega, \\ \vec{h}(\cdot, T^* - \delta_0) = \vec{0} & \text{in } \Omega. \end{cases} \tag{2.38}$$

Multiplying the first equation of (2.38) by $2\vec{h}$ and integrating it over Ω, we have that

$$\begin{aligned} & \frac{d}{dt}\|\vec{h}(\cdot,t)\|^2_{L^2(\Omega)} + 2\|\nabla\vec{h}(\cdot,t)\|^2_{L^2(\Omega)} \\ \leq\ & 2\|\vec{\xi} + \vec{z}^*_{\delta_0}\|_{L^\infty(\Omega)}\|\nabla\vec{h}\|_{L^2(\Omega)}\|\vec{h}\|_{L^2(\Omega)} \\ & +2\|\nabla\vec{z}^*_{\delta_0}\|_{L^q(\Omega)}\|\vec{h}\|_{L^2(\Omega)}\|\vec{h}\|_{L^{\frac{2q}{q-2}}(\Omega)} + 2\|\vec{u}\|_{L^2(\Omega)}\|\vec{h}\|_{L^2(\Omega)} \\ \leq\ & \|\nabla\vec{h}(\cdot,t)\|^2_{L^2(\Omega)} + C\|\vec{h}(\cdot,t)\|^2_{L^2(\Omega)} + C\|\vec{u}(\cdot,t)\|^2_{L^2(\Omega)}, \quad \forall\, t \in [0, T^* - \delta_0]. \end{aligned}$$

Here and throughout Step 2, C denotes a generic positive constant independent of δ. Integrating the latter inequality over $(0,t)$, by Gronwall's inequality and (2.37), we obtain

$$\|\vec{h}\|^2_{C([0,T^*-\delta_0];L^2(\Omega))} + \int_0^{T^*-\delta_0} \|\nabla\vec{h}\|^2_{L^2(\Omega)}\, dt \leq C\|\vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0)\|^2_{L^2(\Omega)}. \tag{2.39}$$

From (2.39) it follows that

$$\begin{aligned} & \|((\vec{\xi} + \vec{z}^*_{\delta_0}) \cdot \nabla)\vec{h} + (\vec{h} \cdot \nabla)\vec{z}^*_{\delta_0}\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))} \\ \leq\ & C\|\nabla\vec{h}\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))} + 2\int_0^{T^*-\delta_0} \|\nabla\vec{z}^*_{\delta_0}\|^2_{L^q(\Omega)}\|\vec{h}\|^2_{L^{\frac{2q}{q-2}}(\Omega)}\, dt \\ \leq\ & C\|\nabla\vec{h}\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))} \leq C\|\vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0)\|^2_{L^2(\Omega)}, \end{aligned}$$

which, combined with (2.38) and (2.37), implies that

$$\|\vec{h}\|^2_{C([0,T^*-\delta_0];H^1_0(\Omega))} + \|\vec{h}\|^2_{\dot{W}^{2,1}_2(Q_{T^*-\delta_0})} \leq C\|\vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0)\|^2_{H^1_0(\Omega)}. \tag{2.40}$$

Recalling that $\tilde{q} = \min\{q, 10/3\}$, we claim that

$$\|((\vec{\xi} + \vec{z}^*_{\delta_0}) \cdot \nabla)\vec{h} + (\vec{h} \cdot \nabla)\vec{z}^*_{\delta_0}\|_{L^{\tilde{q}}(0,T^*-\delta_0;L^{\tilde{q}}(\Omega))} \leq C\|\vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0)\|_{H^1_0(\Omega)}. \tag{2.41}$$

Indeed, if $\tilde{q} = \frac{10}{3}$ then

$$\int_0^{T^*-\delta_0}\int_\Omega |((\vec{\xi} + \vec{z}^*_{\delta_0}) \cdot \nabla)\vec{h}|^{10/3}\, dx\, dt \leq C\int_0^{T^*-\delta_0} \|\nabla\vec{h}(\cdot,t)\|^{10/3}_{L^{10/3}(\Omega)}\, dt,$$

which, combined with the interpolation inequality that $\|\nabla\vec{h}\|_{L^{10/3}(\Omega)} \le C\|\nabla\vec{h}\|^{2/5}_{L^2(\Omega)}\|\nabla\vec{h}\|^{3/5}_{L^6(\Omega)}$, implies

$$\int_0^{T^*-\delta_0}\int_\Omega |((\vec{\xi}+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}|^{10/3}\,dx\,dt \le C\int_0^{T^*-\delta_0}\|\nabla\vec{h}\|^{4/3}_{L^2(\Omega)}\|\vec{h}\|^2_{H^2(\Omega)}\,dt.$$

This together with (2.40) implies

$$\|((\vec{\xi}+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}\|_{L^{10/3}(0,T^*-\delta_0;L^{10/3}(\Omega))} \le C\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{H^1_0(\Omega)}. \tag{2.42}$$

On the other hand, if $\tilde{q}=q$, then for any $\vec{\varphi}\in L^{q/(q-1)}(\Omega)$, by Hölder's inequality, we have that

$$\begin{aligned}\left|\int_\Omega(\vec{h}\cdot\nabla)\vec{z}^*_{\delta_0}\cdot\vec{\varphi}\,dx\right| &\le \|\nabla\vec{z}^*_{\delta_0}\|_{C(\overline{\Omega})}\|\vec{h}\|_{L^q(\Omega)}\|\vec{\varphi}\|_{L^{q/(q-1)}(\Omega)}\\ &\le C\|\vec{z}^*_{\delta_0}\|_{W^{2,q}(\Omega)}\|\nabla\vec{h}\|_{L^2(\Omega)}\|\vec{\varphi}\|_{L^{q/(q-1)}(\Omega)},\end{aligned}$$

where we used that $q\le 6$ as $d=3$. From the latter and (2.40) it follows that

$$\int_0^{T^*-\delta_0}\|(\vec{h}\cdot\nabla)\vec{z}^*_{\delta_0}\|^q_{L^q(\Omega)}\,dt \le C\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|^q_{H^1_0(\Omega)},$$

which, combined with (2.42), indicates (2.41).

Now we rewrite (2.38) as

$$\begin{cases}\vec{h}_t-\Delta\vec{h}=\vec{f} & \text{in } \Omega\times(0,T^*-\delta_0),\\ \vec{h}=\vec{0} & \text{on } \partial\Omega\times(0,T^*-\delta_0),\\ \vec{h}(\cdot,0)=\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0) & \text{in } \Omega,\\ \vec{h}(\cdot,T^*-\delta_0)=\vec{0} & \text{in } \Omega,\end{cases}$$

where $\vec{f}=\chi_\omega\chi_{E^*_{\delta_0}}\vec{u}-((\vec{\xi}+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}-(\vec{h}\cdot\nabla)\vec{z}^*_{\delta_0}$. It follows from (2.37) and (2.41) that

$$\|\vec{f}\|_{L^{\tilde{q}}(0,T^*-\delta_0;L^{\tilde{q}}(\Omega))} \le C\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{H^1_0(\Omega)}. \tag{2.43}$$

It is obvious that $\vec{h}=\vec{h}_1+\vec{h}_2$, where $\vec{h}_1$ and $\vec{h}_2$ satisfy

$$\begin{cases}(\vec{h}_1)_t-\Delta\vec{h}_1=\vec{f} & \text{in } \Omega\times(0,T^*-\delta_0),\\ \vec{h}_1=\vec{0} & \text{on } \partial\Omega\times(0,T^*-\delta_0),\\ \vec{h}_1(\cdot,0)=\vec{0} & \text{in } \Omega\end{cases} \tag{2.44}$$

and

$$\begin{cases}(\vec{h}_2)_t-\Delta\vec{h}_2=\vec{0} & \text{in } \Omega\times(0,T^*-\delta_0),\\ \vec{h}_2=\vec{0} & \text{on } \partial\Omega\times(0,T^*-\delta_0),\\ \vec{h}_2(\cdot,0)=\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0) & \text{in } \Omega,\end{cases} \tag{2.45}$$

respectively. By (2.43), (2.44) and the same arguments as for (2.4), we obtain

$$\|\vec{h}_1\|_{\dot{W}^{2,1}_{\tilde{q}}(Q_{T^*-\delta_0})} \leq C\|\vec{f}\|_{L^{\tilde{q}}(0,T^*-\delta_0;L^{\tilde{q}}(\Omega))} \leq C\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{H^1_0(\Omega)}. \tag{2.46}$$

By Remark 8.8 in [3] and Hille-Yosida Theorem, since Ω is a bounded, convex subset of $\mathbb{R}^d$ with smooth boundary, Δ is the infinitesimal generator of a C_0 semigroup of contractions on $W^{1,q}_0(\Omega)$. Considering the fact that $\vec{y}^* \in C([0,T^*];W^{1,q}_0(\Omega))$, from (2.45) we get

$$\|\vec{h}_2\|_{\dot{W}^{2,1}_2(Q_{T^*-\delta_0})} + \|\vec{h}_2\|_{C([0,T^*-\delta_0];W^{1,q}_0(\Omega))} \leq C\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{W^{1,q}_0(\Omega)}.$$

This together with (2.46) implies that

$$\|\vec{h}\|_{\dot{W}^{2,1}_2(Q_{T^*-\delta_0})} + \|\vec{h}\|_{C([0,T^*-\delta_0];W^{1,\tilde{q}}_0(\Omega))} \leq C\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{W^{1,q}_0(\Omega)}.$$

From the latter inequality we obtain that there exists a constant $\delta_1 \in (0,\delta_0)$ such that

$$\|\vec{h}\|_{C([0,T^*-\delta_0];W^{1,\tilde{q}}_0(\Omega))} + \|\vec{h}\|_{\dot{W}^{2,1}_2(Q_{T^*-\delta_0})} \leq 1, \qquad \forall\, \delta \in (\delta_1,\delta_0), \tag{2.47}$$

and thus

$$\Phi_\delta(\mathcal{K}_{\delta_0}) \subset \mathcal{K}_{\delta_0}, \quad \forall\, \delta \in (\delta_1,\delta_0).$$

Step 3. The mapping Φ_δ is upper semicontinuous in $L^2(0,T^*-\delta_0;L^2(\Omega))$, i.e., if $\vec{\xi}_n \in \mathcal{K}_{\delta_0} \to \vec{\xi}$ strongly in $L^2(0,T^*-\delta_0;L^2(\Omega))$ and $\vec{h}_n \in \Phi_\delta(\vec{\xi}_n) \to \vec{h}$ strongly in $L^2(0,T^*-\delta_0;L^2(\Omega))$, then $\vec{h} \in \Phi_\delta(\vec{\xi})$.

Since $\vec{h}_n \in \Phi_\delta(\vec{\xi}_n)$, there exists $\vec{u}_n \in L^\infty(0,T^*-\delta_0;L^q(\Omega))$ satisfying

$$\|\vec{u}_n\|_{L^\infty(0,T^*-\delta_0;L^q(\Omega))} \leq c_1\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{L^2(\Omega)}, \quad \forall\, n \geq 1, \tag{2.48}$$

$$\begin{cases} (\vec{h}_n)_t - \Delta\vec{h}_n + ((\vec{\xi}_n + \vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}_n & \\ \qquad\qquad +(\vec{h}_n\cdot\nabla)\vec{z}^*_{\delta_0} = \chi_\omega\chi_{E^*_{\delta_0}}\vec{u}_n & \text{in } \Omega\times(0,T^*-\delta_0), \\ \vec{h}_n = \vec{0} & \text{on } \partial\Omega\times(0,T^*-\delta_0), \\ \vec{h}_n(\cdot,0) = \vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0) & \text{in } \Omega, \\ \vec{h}_n(\cdot,T^*-\delta_0) = \vec{0} & \text{in } \Omega. \end{cases} \tag{2.49}$$

From (2.48), $\{\vec{\xi}_n\}_{n\geq 1} \subset \mathcal{K}_{\delta_0}$ and $\{\vec{h}_n\}_{n\geq 1} \subset \mathcal{K}_{\delta_0}$, it follows that there exist a subsequence of $\{n\}_{n\geq 1}$, still denoted in the same manner, and $\vec{u} \in L^\infty(0,T^*-\delta_0;L^q(\Omega))$ such that

$$\begin{aligned} \vec{\xi}_n \to \vec{\xi},\ \vec{h}_n \to \vec{h} \quad & \text{weakly in } \dot{W}^{2,1}_2(Q_{T^*-\delta_0}),\ \text{weakly star in } L^\infty(0,T;W^{1,\tilde{q}}_0(\Omega)), \\ & \text{strongly in } L^2(0,T^*-\delta_0;H^1_0(\Omega)) \cap C([0,T^*-\delta_0];L^2(\Omega)), \\ & \text{a.e. in } \Omega\times(0,T^*-\delta_0), \end{aligned} \tag{2.50}$$

$$\vec{u}_n \to \vec{u} \ \text{ weakly star in } L^\infty(0,T^*-\delta_0;L^q(\Omega)) \tag{2.51}$$

and

$$\|\vec{u}\|_{L^\infty(0,T^*-\delta_0;L^q(\Omega))} \leq c_1\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{L^2(\Omega)}. \tag{2.52}$$

Next we claim that

$$((\vec{\xi}_n+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}_n \to ((\vec{\xi}+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h} \text{ strongly in } L^2(0,T^*-\delta_0;L^2(\Omega)) \tag{2.53}$$

and

$$(\vec{h}_n\cdot\nabla)\vec{z}^*_{\delta_0} \to (\vec{h}\cdot\nabla)\vec{z}^*_{\delta_0} \text{ strongly in } L^2(0,T^*-\delta_0;L^2(\Omega)). \tag{2.54}$$

Indeed, from (2.50) and Lebesgue's dominated convergence theorem it follows that

$$\begin{aligned}
& \|((\vec{\xi}_n+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}_n-((\vec{\xi}+\vec{z}^*_{\delta_0})\cdot\nabla)\vec{h}\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))} \\
\leq\ & 2\|((\vec{\xi}_n+\vec{z}^*_{\delta_0})\cdot\nabla)(\vec{h}_n-\vec{h})\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))}+2\|((\vec{\xi}_n-\vec{\xi})\cdot\nabla)\vec{h}\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))} \\
\leq\ & C\|\vec{h}_n-\vec{h}\|^2_{L^2(0,T^*-\delta_0;H^1_0(\Omega))}+2\|((\vec{\xi}_n-\vec{\xi})\cdot\nabla)\vec{h}\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))} \to 0
\end{aligned}$$

and

$$\|((\vec{h}_n-\vec{h})\cdot\nabla)\vec{z}^*_{\delta_0}\|^2_{L^2(0,T^*-\delta_0;L^2(\Omega))} \to 0.$$

Passing to the limit for $n\to+\infty$ in (2.49) we obtain from (2.50)-(2.54) that $\vec{h}\in\Phi_\delta(\vec{\xi})$.

By Step 1 - Step 3 and Kakutani's fixed point theorem there exists a $\vec{h}_\delta\in\mathcal{K}_{\delta_0}$ such that $\vec{h}_\delta\in\Phi_\delta(\vec{h}_\delta)$. Thus (2.32) and (2.33) follow.

Using (2.31) and (2.32), we have

$$\begin{cases}
(\vec{h}_\delta+\vec{z}^*_{\delta_0})_t-\Delta(\vec{h}_\delta+\vec{z}^*_{\delta_0})+((\vec{h}_\delta+\vec{z}^*_{\delta_0})\cdot\nabla)(\vec{h}_\delta+\vec{z}^*_{\delta_0}) & \\
\qquad\qquad =\chi_\omega[\vec{u}^*(\cdot,t+\delta_0)+\chi_{E^*_{\delta_0}}\vec{u}_\delta(\cdot,t)] & \text{in } \Omega\times(0,T^*-\delta_0),\\
(\vec{h}_\delta+\vec{z}^*_{\delta_0})=\vec{0} & \text{on } \partial\Omega\times(0,T^*-\delta_0),\\
(\vec{h}_\delta+\vec{z}^*_{\delta_0})(\cdot,0)=\vec{y}^*(\cdot,\delta) & \text{in } \Omega,\\
(\vec{h}_\delta+\vec{z}^*_{\delta_0})(\cdot,T^*-\delta_0)=\vec{0} & \text{in } \Omega.
\end{cases} \tag{2.55}$$

Setting

$$\vec{u}^*_\delta(\cdot,t)\equiv\vec{u}^*(\cdot,t+\delta_0)+\chi_{E^*_{\delta_0}}\vec{u}_\delta(\cdot,t)=\begin{cases}\vec{u}^*(\cdot,t+\delta_0)+\vec{u}_\delta(\cdot,t) & \text{if } t\in E^*_{\delta_0},\\ \vec{u}^*(\cdot,t+\delta_0) & \text{if } t\in(0,T^*-\delta_0)\setminus E^*_{\delta_0},\end{cases} \tag{2.56}$$

by (2.30), (2.33) and (2.56), we obtain that

$$\|\vec{u}^*_\delta(\cdot,t)\|_{L^q(\Omega)}\leq\rho_0-\varepsilon_0+c_1\|\vec{y}^*(\cdot,\delta)-\vec{y}^*(\cdot,\delta_0)\|_{L^2(\Omega)}, \quad \text{a.e. } t\in E^*_{\delta_0}, \tag{2.57}$$

and

$$\|\vec{u}^*_\delta(\cdot,t)\|_{L^q(\Omega)}=\|\vec{u}^*(\cdot,t+\delta_0)\|_{L^q(\Omega)}\leq\rho_0, \quad \text{a.e. } t\in(0,T^*-\delta_0)\setminus E^*_{\delta_0}. \tag{2.58}$$

Now we choose $\delta \in (\delta_1, \delta_0)$, with δ_1 determined in (2.47) such that $\|\vec{y}^*(\cdot,\delta) - \vec{y}^*(\cdot,\delta_0)\|_{L^2(\Omega)} \le \varepsilon_0 c_1^{-1}$. Then from (2.56), (2.57) and (2.58) it follows that

$$\|\vec{u}_\delta^*(\cdot,t)\|_{L^q(\Omega)} \le \rho_0, \quad \text{a.e. } t \in (0, T^* - \delta_0).$$

The latter inequality, together with (2.55), (2.56) and Proposition 2.1, implies that if we take

$$\vec{v}_\delta^*(\cdot,t) = \begin{cases} \vec{u}^*(\cdot,t) & t \in (0,\delta], \\ \vec{u}_\delta^*(\cdot,t-\delta) & t \in (\delta, T^* - \delta_0 + \delta), \\ \vec{0} & t \in [T^* - \delta_0 + \delta, +\infty), \end{cases}$$

then $\vec{v}_\delta^* \in \mathcal{U}$ and the equation

$$\begin{cases} \vec{y}_t - \Delta \vec{y} + (\vec{y} \cdot \nabla)\vec{y} = \chi_\omega \vec{v}_\delta^* & \text{in } \Omega \times (0, T^* - \delta_0 + \delta), \\ \vec{y} = \vec{0} & \text{on } \partial\Omega \times (0, T^* - \delta_0 + \delta), \\ \vec{y}(\cdot,0) = \vec{y}_0 & \text{in } \Omega \end{cases}$$

has a unique solution $\vec{y}(\cdot,\cdot;\vec{v}_\delta^*) \in \dot{W}_q^{2,1}(Q_{T^*-\delta_0+\delta})$ satisfying

$$\vec{y}(\cdot, T^* - \delta_0 + \delta; \vec{v}_\delta^*) = \vec{0} \ \text{ in } \ \Omega.$$

This gives a contradiction and completes the proof for $d \in \{2,3\}$. □

We close the section by giving the sketch for the proof of Theorem 1.1 for $d = 1$. In this case, let $\Omega = (0,1)$ and ω be an open and non-empty subset of Ω. For an arbitrarily fixed $\rho_0 > 0$ we define the constraint set of controls

$$\mathcal{U} \equiv \{u : [0,+\infty) \to L^2(0,1) \text{ is measurable}: \ \|u(\cdot,t)\|_{L^2(0,1)} \le \rho_0 \ \text{ for almost all } t > 0\}.$$

We fix $y_0(\cdot) \in L^2(0,1) \setminus \{0\}$ and consider the controlled Burgers equation

$$\begin{cases} y_t - y_{xx} + y y_x = \chi_\omega u & \text{in } (0,1) \times (0,+\infty), \\ y(0,t) = y(1,t) = 0 & \text{in } (0,+\infty), \\ y(\cdot,0) = y_0 & \text{in } (0,1), \end{cases} \tag{2.59}$$

where $u \in \mathcal{U}$. For any $T > 0$, existence and uniqueness of a solution in $y(\cdot,\cdot;u) \in C([0,T]; L^2(0,1))$ to (2.59) can be ensured by standard arguments. The set of admissible controls is defined to be

$$\mathcal{U}_{ad} \equiv \{u \in \mathcal{U} : y(\cdot,T;u) = 0 \ \text{ over } \ (0,1), \text{ for some } \ T > 0\}.$$

Now carry out the proof in three stages.

Stage 1.

Problem (P) has at least one admissible control.

This will be done by four steps as follows.

Step 1. We consider the equation

$$\begin{cases} y_t - y_{xx} + yy_x = 0 & \text{in } (0,1)\times(0,T_0), \\ y(0,t) = y(1,t) = 0 & \text{in } (0,T_0), \\ y(\cdot,0) = y_0, & \text{in } (0,1), \end{cases} \tag{2.60}$$

where $T_0 > 0$ will be determined later. It is well-known that

$$\|y(\cdot,T_0)\|_{L^2(0,1)} \le e^{-\lambda_1 T_0}\|y_0\|_{L^2(0,1)}, \tag{2.61}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Step 2. Let $z = z(x,t)$ be the solution of

$$\begin{cases} z_t - z_{xx} + zz_x = 0 & \text{in } (0,1)\times(0,2), \\ z(0,t) = z(1,t) = 0 & \text{in } (0,2), \\ z(\cdot,0) = y(\cdot,T_0) & \text{in } (0,1). \end{cases} \tag{2.62}$$

We can check that

$$\|z_x(\cdot,2)\|^2_{L^2(0,1)} \le \|y(\cdot,T_0)\|^2_{L^2(0,1)} e^{C\|y(\cdot,T_0)\|^2_{L^2(0,1)}},$$

where C denotes a generic positive constant independent of T_0. From the latter and (2.61) it follows that

$$\|z_x(\cdot,2)\|^2_{L^2(0,1)} \le Ce^{-2\lambda_1 T_0}. \tag{2.63}$$

Step 3. By standard arguments for local null controllability, Theorem 1.2 and (2.63), for sufficiently large T_0, there exists a $u \in L^\infty(0,2;L^2(0,1))$ with $\|u(\cdot,t)\|_{L^2(0,1)} \le \rho_0$ a.e. $t \in (0,2)$, such that $w = w(x,t)$ satisfies

$$\begin{cases} w_t - w_{xx} + ww_x = \chi_\omega u & \text{in } (0,1)\times(0,2), \\ w(0,t) = w(1,t) = 0 & \text{in } (0,2), \\ w(\cdot,0) = z(\cdot,2) & \text{in } (0,1), \\ w(\cdot,2) = 0 & \text{in } (0,1). \end{cases} \tag{2.64}$$

Step 4. By (2.60), (2.62) and (2.64), we see that

$$\bar{u}(\cdot,t) = \begin{cases} 0, & \text{in } (0,T_0+2), \\ u(\cdot,t), & \text{in } [T_0+2,T_0+4) \\ 0, & \text{in } [T_0+4,+\infty). \end{cases}$$

is an admissible control for the problem (P).

Stage 2. Existence of solution for (P) can be obtained by standard arguments.

Stage 3. The bang-bang property for (P) is obtained by the same arguments as for the case $d = 2,3$, only that in this case $\mathcal{K}_{\delta_0}$ is replaced with

$$\mathcal{K}_{\delta_0} = \{\xi \in L^2(0,T^*-\delta_0;L^2(\Omega)) : \ \|\xi\|_{\dot{W}^{2,1}_2(Q_{T^*-\delta_0})} \le 1\}.$$

3 Proof of the observability estimate

In this section, for the sake of simplicity, we only give the detailed proof of Theorem 1.2 for $m = 1$, i.e., for any solution φ to the equation:

$$\begin{cases} \partial_t \varphi - \Delta\varphi + a\varphi + \vec{b}\cdot\nabla\varphi = 0 & \text{in } \hat{\Omega}\times(0,T),\\ \varphi = 0 & \text{on } \partial\hat{\Omega}\times(0,T),\\ \varphi(\cdot,0) = \varphi_0 \in L^2(\hat{\Omega}), \end{cases} \tag{3.1}$$

the following estimate holds:

$$\|\varphi(\cdot,T)\|_{L^2(\hat{\Omega})} \le e^{C(\hat{\Omega},\hat{\omega},d,\hat{q},E)} e^{C(\hat{\Omega},\hat{\omega},d,\hat{q})[1+(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)(T+1)+\|a\|_\infty^{4/(2-\hat{p})}]} \int_{\hat{\omega}\times E} |\varphi(x,t)|\,dx\,dt. \tag{3.2}$$

As mentioned before, (3.2) is proved by using similar arguments as in [17]. We therefore only sketch the proof below and point out the differences.

Lemma 3.1. *There exists a positive constant $C_0 = C_0(\hat{\Omega}, d, \hat{q})$ such that for any $t \in (0,T]$,*

$$\|\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} \le e^{C_0(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)t}\|\varphi_0\|^2_{L^2(\hat{\Omega})} \tag{3.3}$$

and

$$\|\nabla\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} \le t^{-1} e^{C_0(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)t}\|\varphi_0\|^2_{L^2(\hat{\Omega})}. \tag{3.4}$$

Proof. Multiplying the first equation of (3.1) by 2φ we obtain after some calculations that

$$\frac{d}{dt}\|\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} + \|\nabla\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} \le C(\hat{\Omega},d,\hat{q})(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)\|\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})}.$$

Integrating the latter inequality over $(0,t)$, we have that

$$\begin{aligned} &\|\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} + \int_0^t \|\nabla\varphi(\cdot,s)\|^2_{L^2(\hat{\Omega})}\,ds \\ \le\ & C(\hat{\Omega},d,\hat{q})(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)\int_0^t \|\varphi(\cdot,s)\|^2_{L^2(\hat{\Omega})}\,ds + \|\varphi_0\|^2_{L^2(\hat{\Omega})} \quad \forall\, t\in[0,T]. \end{aligned} \tag{3.5}$$

This, together with Gronwall's inequality, implies

$$\|\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} \le e^{C(\hat{\Omega},d,\hat{q})(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)t}\|\varphi_0\|^2_{L^2(\hat{\Omega})}. \tag{3.6}$$

Moreover, it follows from (3.5) and (3.6) that

$$\int_0^t \|\nabla\varphi(\cdot,s)\|^2_{L^2(\hat{\Omega})}\,ds \le \|\varphi_0\|^2_{L^2(\hat{\Omega})} e^{C(\hat{\Omega},d,\hat{q})(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)t}, \qquad \forall\, t\in[0,T]. \tag{3.7}$$

Multiplying the first equation of (3.1) by $-2t\Delta\varphi$, we have that

$$t\partial_t\|\nabla\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} \le C(\hat{\Omega},d,\hat{q})(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)\|\nabla\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})}t.$$

Integrating the latter over $(0,t)$, we obtain by (3.7) and Gronwall's inequality that

$$t\|\nabla\varphi(\cdot,t)\|^2_{L^2(\hat{\Omega})} \le e^{2C(\hat{\Omega},d,\hat{q})(\|a\|^2_\infty+\|\vec{b}\|^2_\infty)t}\|\varphi_0\|^2_{L^2(\hat{\Omega})}. \tag{3.8}$$

From (3.6) and (3.8) the inequalities (3.3) and (3.4) follow.

□

Let $x_0 \in \hat{\Omega}$ and denote by $B_R \equiv B(x_0,R)$ the open ball with center x_0 and radius R.

Lemma 3.2. *Let $R_0 > 0$ and $\lambda > 0$. Introduce for $t \in [0,T]$ and $x_0 \in \hat{\Omega}$,*

$$G_\lambda(x,t) = \frac{1}{(T-t+\lambda)^{d/2}} e^{-\frac{|x-x_0|^2}{4(T-t+\lambda)}}.$$

Define for $u \in W^{1,2}(0,T;L^2(\hat{\Omega}\cap B_{R_0})) \cap L^2(0,T;H^2(\hat{\Omega}\cap B_{R_0})\cap H^1_0(\hat{\Omega}\cap B_{R_0}))$, $t\in(0,T]$ and $\varepsilon > 0$,

$$N^\varepsilon_\lambda(t) = \frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |\nabla u(x,t)|^2 G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx + \varepsilon}.$$

The following two properties hold:

i)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx + \int_{\hat{\Omega}\cap B_{R_0}} |\nabla u(x,t)|^2 G_\lambda(x,t)\,dx \\ =& \int_{\hat{\Omega}\cap B_{R_0}} u(x,t)(\partial_t - \Delta)u(x,t) G_\lambda(x,t)\,dx.\end{aligned} \tag{3.9}$$

ii) When $\hat{\Omega}\cap B_{R_0}$ is star-shaped with respect to x_0, i.e., $\nu_{\tilde{x}_0}\cdot(\tilde{x}_0 - x_0) \ge 0$ for a.e. $\tilde{x}_0 \in \partial\hat{\Omega}\cap B_{R_0}$,

$$\frac{d}{dt}N^\varepsilon_\lambda(t) \le \frac{1}{T-t+\lambda}N^\varepsilon_\lambda(t) + \frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |\partial_t u - \Delta u|^2 G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx + \varepsilon}. \tag{3.10}$$

Proof. Equality (3.9) follows from direct computations. The proof of (3.10) is the same as that in [15]. □

Lemma 3.3. *Let $R > 0$ and $\delta \in (0,1]$. Then there are constants $C_1 = C_1(\delta,R) > 0, C_2 = C_2(\hat{\Omega},\delta,R,d,\hat{q}) > 0, C_3 = C_3(\hat{\Omega},\delta,R,d,\hat{q}) > 0$ and $C_4 = C_4(\delta,R) > 0$, such that for any $\varphi_0 \in L^2(\hat{\Omega})$ with $\varphi_0 \ne 0$ and $\tilde{\varepsilon} \in (0,\|\varphi_0\|^2_{L^2(\hat{\Omega})})$, the quantity*

$$h_0 = \frac{C_1}{ln\left[(1+C_3)e^{2+\frac{2C_1}{T}} e^{(C_2+2C_0)(\|a\|^2_\infty+\|\vec{b}\|^2_\infty)T} \frac{\|\varphi_0\|^2_{L^2(\hat{\Omega})}}{\int_{\hat{\Omega}\cap B_R}|\varphi(x,T)|^2\,dx+\tilde{\varepsilon}} e^{\|a\|_\infty^{\frac{4}{2-\hat{p}}}+\|a\|^2_\infty+\|\vec{b}\|^2_\infty}\right]} \tag{3.11}$$

satisfies the following two properties:

i)

$$0 < \left[1 + \frac{2C_1}{T} + (C_0 + C_2)(\|a\|_\infty^2 + \|\vec{b}\|_\infty^2)T + \|a\|_\infty^{\frac{4}{2-\hat{p}}} + \|a\|_\infty^2 + \|\vec{b}\|_\infty^2\right] h_0 < C_1. \quad (3.12)$$

ii) For any $t \in [T - h_0, T]$, it holds

$$e^{C_0(\|a\|_\infty^2 + \|\vec{b}\|_\infty^2)T} \int_{\hat{\Omega}} |\varphi_0|^2 \, dx \le e^{1+\frac{C_4}{h_0}} \left(\int_{\hat{\Omega} \cap B_{(1+\delta)R}} |\varphi(x,t)|^2 \, dx + \tilde{\varepsilon} \right). \quad (3.13)$$

Proof. Inequality (3.12) follows from (3.11), (3.3) and the fact that $\tilde{\varepsilon} \in (0, \|\varphi_0\|_{L^2(\hat{\Omega})}^2)$. To verify (3.13) let $h > 0, \rho(x) = |x - x_0|^2$ and $\chi \in C_0^\infty(B_{(1+\delta)R})$ be such that $0 \le \chi \le 1$, $\chi = 1$ on $\{x : |x - x_0| \le (1 + 3\delta/4)R\}$. Multiplying the first equation of (3.1) by $2e^{-\frac{\rho}{h}}\chi^2\varphi$ and integrating over $\hat{\Omega} \cap B_{(1+\delta)R}$, we get

$$\begin{aligned} & \frac{d}{dt} \int_{\hat{\Omega} \cap B_{(1+\delta)R}} e^{-\frac{\rho}{h}} \chi^2 \varphi^2 \, dx + 2 \int_{\hat{\Omega} \cap B_{(1+\delta)R}} e^{-\frac{\rho}{h}} |\chi \nabla \varphi|^2 \, dx \\ \le & \int_{\hat{\Omega} \cap B_{(1+\delta)R}} e^{-\frac{\rho}{2h}} |\chi \nabla \varphi| \left(\frac{4}{h} |x - x_0| e^{-\frac{\rho}{2h}} \chi |\varphi| + 4 |\nabla \chi| e^{-\frac{\rho}{2h}} |\varphi| \right) dx \\ & -2 \int_{\hat{\Omega} \cap B_{(1+\delta)R}} a e^{-\frac{\rho}{h}} \chi^2 \varphi^2 \, dx - 2 \int_{\hat{\Omega} \cap B_{(1+\delta)R}} (\vec{b} \cdot \nabla \varphi) e^{-\frac{\rho}{h}} \chi^2 \varphi \, dx. \end{aligned} \quad (3.14)$$

Considering the following estimate:

$$\begin{aligned} -2 \int_{\hat{\Omega} \cap B_{(1+\delta)R}} a e^{-\frac{\rho}{h}} \chi^2 \varphi^2 \, dx &\le \begin{cases} 2\|a\|_{L^d(\hat{\Omega} \cap B_{(1+\delta)R})} \|e^{-\frac{\rho}{h}} (\chi\varphi)^2\|_{L^{\frac{d}{d-1}}(\hat{\Omega} \cap B_{(1+\delta)R})} & \text{for } d \ge 2, \\ 2\|a\|_{L^1(\hat{\Omega} \cap B_{(1+\delta)R})} \|e^{-\frac{\rho}{h}} (\chi\varphi)^2\|_{L^\infty(\hat{\Omega} \cap B_{(1+\delta)R})} & \text{for } d = 1 \end{cases} \\ \le \; & C(\hat{\Omega}, \delta, R, d, \hat{q}) \|a\|_\infty \|\nabla [e^{-\frac{\rho}{h}} (\chi\varphi)^2]\|_{L^1(\hat{\Omega} \cap B_{(1+\delta)R})}, \end{aligned}$$

we obtain that

$$\begin{aligned} -2 \int_{\hat{\Omega} \cap B_{(1+\delta)R}} a e^{-\frac{\rho}{h}} \chi^2 \varphi^2 \, dx \le \; & C(\hat{\Omega}, \delta, R, d, \hat{q}) \|a\|_\infty \left[\frac{(1+\delta)R}{h} + \|a\|_\infty \right] \int_{\hat{\Omega} \cap B_{(1+\delta)R}} e^{-\frac{\rho}{h}} \chi^2 \varphi^2 \, dx \\ & + C(\hat{\Omega}, \delta, R, d, \hat{q}) \|\nabla \chi\|_{L^\infty(B_{(1+\delta)R})}^2 e^{-\frac{[(1+3\delta/4)R]^2}{h}} \int_{\hat{\Omega} \cap B_{(1+\delta)R}} \varphi^2 \, dx \\ & + \int_{\hat{\Omega} \cap B_{(1+\delta)R}} e^{-\frac{\rho}{h}} \chi^2 |\nabla \varphi|^2 \, dx. \end{aligned}$$

This together with (3.14) and (3.3) implies

$$\begin{aligned} & \frac{d}{dt} \left(e^{-\left\{ \frac{16(1+\delta)^2 R^2}{h^2} + C(\hat{\Omega},\delta,R,d,\hat{q}) \|a\|_\infty [\frac{(1+\delta)R}{h} + \|a\|_\infty] + 2\|\vec{b}\|_\infty^2 \right\} t} \int_{\hat{\Omega} \cap B_{(1+\delta)R}} e^{-\frac{\rho}{h}} \chi^2 \varphi^2 \, dx \right) \\ \le \; & C(\hat{\Omega}, \delta, R, d, \hat{q}) \|\nabla \chi\|_{L^\infty(B_{(1+\delta)R})}^2 e^{-\frac{[(1+3\delta/4)R]^2}{h}} e^{C_0(\|a\|_\infty^2 + \|\vec{b}\|_\infty^2)t} \|\varphi_0\|_{L^2(\hat{\Omega})}^2 \\ & \cdot e^{-\left\{ \frac{16(1+\delta)^2 R^2}{h^2} + C(\hat{\Omega},\delta,R,d,\hat{q}) \|a\|_\infty [\frac{(1+\delta)R}{h} + \|a\|_\infty] + 2\|\vec{b}\|_\infty^2 \right\} t}. \end{aligned} \quad (3.15)$$

Set $c_1 = 17(1+\delta)^2, c_2 = (1+3\delta/4)^2, c_3 = (1+\delta/2)^2$ and $C_1 = \frac{(c_2-c_3)(c_3-1)R^2}{c_1}$. Integrating (3.15) on (t,T), we have after some calculations that there exists a positive constant $C_2 = C_2(\hat{\Omega}, \delta, R, d, \hat{q})$ such that

$$\int_{\hat{\Omega}\cap B_R} \varphi^2(x,T)\,dx \leq e^{\frac{(c_2-c_3+1)R^2}{h}} e^{C_2(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)T} \int_{\hat{\Omega}\cap B_{(1+\delta)R}} \varphi^2(x,t)\,dx + C_2 e^{(C_2+C_0)(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)T} \|\nabla\chi\|^2_{L^\infty(B_{(1+\delta)R})} \frac{c_2-c_3}{c_1} h e^{-\frac{(c_3-1)R^2}{h}} \|\varphi_0\|^2_{L^2(\hat{\Omega})}, \tag{3.16}$$

whenever $0 < T - \frac{c_2-c_3}{c_1}h \leq t \leq T$. Now we set $C_3 = C_2\|\nabla\chi\|^2_{L^\infty(B_{(1+\delta)R})} C_1$ and choose $h = \frac{c_1}{c_2-c_3}h_0$. From (3.12) it follows that for any $0 < T - \frac{c_2-c_3}{c_1}h \leq t \leq T$,

$$C_2 e^{(C_2+C_0)(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)T} \|\nabla\chi\|^2_{L^\infty(B_{(1+\delta)R})} \frac{c_2-c_3}{c_1} h e^{-\frac{(c_3-1)R^2}{h}} \|\varphi_0\|^2_{L^2(\hat{\Omega})} \leq \frac{1}{e}\int_{\hat{\Omega}\cap B_R} |\varphi(x,T)|^2\,dx + \frac{\tilde{\varepsilon}}{e}.$$

This, together with (3.16), (3.11) and (3.12), implies

$$(e-1)e^{C_0(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)T}\|\varphi_0\|^2_{L^2(\hat{\Omega})} \leq e^{1+\frac{C_4}{h_0}} \left(\int_{\hat{\Omega}\cap B_{(1+\delta)R}} \varphi^2(x,t)\,dx + \tilde{\varepsilon} \right)$$

for $C_4 = 3C_1 + \frac{(c_2-c_3+1)(c_2-c_3)R^2}{c_1}$, and (3.13) follows. □

Lemma 3.4. *Let $0 < r < R$. Suppose that $B_r \subset \hat{\Omega}$ and $\hat{\Omega} \cap B_{(1+2\delta)R}$ is star-shaped with respect to x_0 for some $\delta \in (0,1]$. Then there exists a constant $\beta = \beta(\hat{\Omega}, \delta, R, r, d, \hat{q}) > 0$, such that for any $\tilde{\varepsilon} \in (0, \|\varphi_0\|^2_{L^2(\hat{\Omega})})$,*

$$\begin{aligned} &\int_{\hat{\Omega}\cap B_R} |\varphi(x,T)|^2 dx + \tilde{\varepsilon} \\ \leq\ & \left[(1+C_3) e^{2+\frac{2C_1}{T}} e^{(C_2+2C_0)(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)T} \|\varphi_0\|^2_{L^2(\hat{\Omega})} e^{\|a\|_\infty^{\frac{4}{2-\hat{p}}} + \|a\|_\infty^2 + \|\vec{b}\|_\infty^2} \right]^\beta \\ & \left[2\left(\int_{B_r} |\varphi(x,T)|^2\,dx + \tilde{\varepsilon} \right) \right]^{1-\beta}. \end{aligned}$$

Proof. Let $0 < r < R$ and $R_0 = (1+2\delta)R$. Let $\chi \in C_0^\infty(B_{R_0}), 0 \leq \chi \leq 1, \chi = 1$ on $\{x : |x-x_0| \leq (1+3\delta/2)R\}$. We will apply Lemma 3.2 with $u = \chi\varphi$. It is obvious that

$$(\partial_t - \Delta)u = -au - \vec{b}\cdot\nabla u + g$$

with $g = -2\nabla\chi\nabla\varphi - \Delta\chi\varphi + \vec{b}(\varphi\nabla\chi)$. We shall divide the proof into the following three steps.

Step 1. Noticing that g is supported on $\{x : (1+3\delta/2)R \le |x-x_0| \le R_0\}$, and recalling the fact that $\chi = 1$ on $\{x : |x-x_0| \le (1+\delta)R\}$, we have

$$\begin{aligned}
&\frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} u(x,t)g(x,t)G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx + \tilde{\varepsilon}} \\
\le\ & \frac{C(\delta,R)\displaystyle\int_{\hat{\Omega}\cap\{x:(1+3\delta/2)R\le|x-x_0|\le R_0\}} [|\varphi|\cdot|\nabla\varphi| + \varphi^2(\|\vec{b}\|_\infty+1)]\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{(1+\delta)R}} |\varphi(x,t)|^2\,dx + \tilde{\varepsilon}e^{\frac{(1+\delta)^2R^2}{4(T-t+\lambda)}}(T-t+\lambda)^{\frac{d}{2}}} e^{-\frac{C_5}{T-t+\lambda}}
\end{aligned} \tag{3.17}$$

with $C_5 = -\frac{(1+\delta)^2R^2}{4} + \frac{(1+3\delta/2)^2R^2}{4} > 0$. Since $e^{\frac{(1+\delta)^2R^2}{4(T-t+\lambda)}}(T-t+\lambda)^{\frac{d}{2}} \ge C(\delta,R,d) > 0$, we have from (3.17) and Lemma 3.1 that

$$\frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} ugG_\lambda\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |u|^2G_\lambda\,dx + \tilde{\varepsilon}} \le \frac{C(\delta,R,d)(1+\|\vec{b}\|_\infty + t^{-\frac{1}{2}})e^{C_0(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)t}\|\varphi_0\|^2_{L^2(\hat{\Omega})}}{\displaystyle\int_{\hat{\Omega}\cap B_{(1+\delta)R}} |\varphi(x,t)|^2\,dx + \tilde{\varepsilon}} e^{-\frac{C_5}{T-t+\lambda}}. \tag{3.18}$$

Similarly, we obtain

$$\begin{aligned}
&\int_t^T \frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |g(x,s)|^2G_\lambda(x,s)\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |u(x,s)|^2G_\lambda(x,s)\,dx + \tilde{\varepsilon}}\,ds \\
\le\ & \int_t^T \frac{C(\delta,R,d)(1+\|\vec{b}\|^2_\infty + s^{-1})e^{C_0(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)s}\|\varphi_0\|^2_{L^2(\hat{\Omega})}}{\displaystyle\int_{\hat{\Omega}\cap B_{(1+\delta)R}} |\varphi(x,s)|^2\,dx + \tilde{\varepsilon}} e^{-\frac{C_5}{T-s+\lambda}}\,ds.
\end{aligned} \tag{3.19}$$

By (3.12) we have that $h_0 < C_1$ and $h_0 \in (0, T/2)$. Now, for any $t \in [T-\varepsilon, T)$, with $\varepsilon \in (0, h_0]$ to be determined later, we get by (3.13), (3.18) and (3.19) that

$$\begin{aligned}
&2\frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} ugG_\lambda\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2G_\lambda\,dx + \tilde{\varepsilon}} + 3\int_t^T \frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |g|^2G_\lambda(x,s)\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |u|^2G_\lambda(x,s)\,dx + \tilde{\varepsilon}}\,ds \\
\le\ & C(\delta,R,d)\left(1+\|\vec{b}\|^2_\infty + T^{-\frac{1}{2}}\right)e^{\frac{C_4+C_5}{h_0}}e^{-\frac{C_5}{\varepsilon+\lambda}} \triangleq Q_{h_0,\varepsilon,\lambda}.
\end{aligned} \tag{3.20}$$

Step 2. In this step we obtain a bound for $\lambda N_\lambda^{\tilde\varepsilon}(T)$. Firstly, by (3.10), we have

$$\begin{aligned}\frac{d}{dt}N_\lambda^{\tilde\varepsilon}(t) \;&\le\; \frac{1}{T-t+\lambda}N_\lambda^{\tilde\varepsilon}(t)+3\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|au|^2G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u|^2G_\lambda(x,t)\,dx+\tilde\varepsilon}\\ &\quad +3\|\vec b\|_\infty^2\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|\nabla u|^2G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u|^2G_\lambda(x,t)\,dx+\tilde\varepsilon}+3\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|g|^2G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u|^2G_\lambda(x,t)\,dx+\tilde\varepsilon}.\end{aligned}\tag{3.21}$$

Now we deal with the second term on the right hand side of (3.21). Recalling (1.4), we see

$$\begin{aligned}\int_{\hat\Omega\cap B_{R_0}}|au|^2G_\lambda\,dx \;&\le\; \|a^2\|_{L^{\frac{d}{\hat p}}(\hat\Omega\cap B_{R_0})}\|u^2G_\lambda\|_{L^{\frac{d}{d-\hat p}}(\hat\Omega\cap B_{R_0})}\\ &\le\; \|a\|^2_{L^{\frac{2d}{\hat p}}(\hat\Omega\cap B_{R_0})}\|(u^2G_\lambda)^{\frac{1}{\hat p}}\|^{\hat p}_{L^{\frac{d\hat p}{d-\hat p}}(\hat\Omega\cap B_{R_0})}\\ &\le\; C(\hat\Omega,\delta,R,d,\hat q)\|a\|^2_{L^{\hat q}(\hat\Omega\cap B_{R_0})}\left\|\nabla\left((u^2G_\lambda)^{\frac{1}{\hat p}}\right)\right\|^{\hat p}_{L^{\hat p}(\hat\Omega\cap B_{R_0})}\qquad\text{for } d\ge 2,\end{aligned}$$

and

$$\begin{aligned}\int_{\hat\Omega\cap B_{R_0}}|au|^2G_\lambda\,dx \;&\le\; \|a\|^2_{L^2(\hat\Omega\cap B_{R_0})}\|u^2G_\lambda\|_{L^\infty(\hat\Omega\cap B_{R_0})}\\ &\le\; C(\hat\Omega,\delta,R,d,\hat q)\|a\|^2_{L^{\hat q}(\hat\Omega\cap B_{R_0})}\left\|\nabla\left(u^2G_\lambda\right)\right\|_{L^1(\hat\Omega\cap B_{R_0})}\qquad\text{for } d=1.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\hat\Omega\cap B_{R_0}}|au|^2G_\lambda\,dx\\ \le\;& C(\hat\Omega,\delta,R,d,\hat q)\|a\|_\infty^2\int_{\hat\Omega\cap B_{R_0}}(|u|^{2-\hat p}|\nabla u|^{\hat p}G_\lambda+u^2(G_\lambda)^{1-\hat p}|\nabla G_\lambda|^{\hat p})\,dx\\ \le\;& \int_{\hat\Omega\cap B_{R_0}}|\nabla u|^2G_\lambda\,dx+C(\hat\Omega,\delta,R,d,\hat q)\left[\|a\|_\infty^{\frac{4}{2-\hat p}}+\frac{\|a\|_\infty^2}{(T-t+\lambda)^{\hat p}}\right]\int_{\hat\Omega\cap B_{R_0}}|u|^2G_\lambda\,dx,\end{aligned}$$

which, combined with (3.21), implies

$$\begin{aligned}&\frac{d}{dt}[(T-t+\lambda)e^{-3(1+\|\vec b\|_\infty^2)t}N_\lambda^{\tilde\varepsilon}(t)]\\ \le\;& C(\hat\Omega,\delta,R,d,\hat q)e^{-3(1+\|\vec b\|_\infty^2)t}\left[(T-t+\lambda)\|a\|_\infty^{\frac{4}{2-\hat p}}+\frac{\|a\|_\infty^2}{(T-t+\lambda)^{\hat p-1}}\right]\\ &+3(T-t+\lambda)e^{-3(1+\|\vec b\|_\infty^2)t}\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|g|^2G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u|^2G_\lambda(x,t)\,dx+\tilde\varepsilon}.\end{aligned}$$

Integrating the latter inequality from t to T, we obtain after some calculations that

$$
\begin{aligned}
& e^{-3(1+\|\vec{b}\|_\infty^2)\varepsilon} \frac{\lambda}{\varepsilon+\lambda} N_\lambda^{\tilde{\varepsilon}}(T) \left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon} \right) \\
\leq & \left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon} \right) \\
& \left\{ N_\lambda^{\tilde{\varepsilon}}(t) + C(\hat{\Omega}, \delta, R, d, \hat{q}) \left[\varepsilon \|a\|_\infty^{\frac{4}{2-\hat{p}}} + \|a\|_\infty^2 (\varepsilon+\lambda)^{1-\hat{p}} \right] \right. \\
& \left. +3 \int_t^T \frac{\int_{\hat{\Omega}\cap B_{R_0}} |g|^2 G_\lambda(x,s) \, dx}{\int_{\hat{\Omega}\cap B_{R_0}} |u|^2 G_\lambda(x,s) \, dx + \tilde{\varepsilon}} \, ds \right\}, \quad \forall\, 0 < T - \varepsilon \leq t < T.
\end{aligned} \tag{3.22}
$$

Secondly, by (3.9), we have that

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon} \right) + N_\lambda^{\tilde{\varepsilon}}(t) \left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon} \right) \\
= & -\int_{\hat{\Omega}\cap B_{R_0}} a u^2 G_\lambda \, dx - \int_{\hat{\Omega}\cap B_{R_0}} u (\vec{b} \cdot \nabla u) G_\lambda \, dx \\
& + \frac{\int_{\hat{\Omega}\cap B_{R_0}} u g G_\lambda \, dx}{\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon}} \left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon} \right).
\end{aligned} \tag{3.23}
$$

Since

$$
-\int_{\hat{\Omega}\cap B_{R_0}} a u^2 G_\lambda \, dx \leq \begin{cases} \|a\|_{L^d(\hat{\Omega}\cap B_{R_0})} \|u^2 G_\lambda\|_{L^{\frac{d}{d-1}}(\hat{\Omega}\cap B_{R_0})} & \text{for } d \geq 2, \\ \|a\|_{L^1(\hat{\Omega}\cap B_{R_0})} \|u^2 G_\lambda\|_{L^\infty(\hat{\Omega}\cap B_{R_0})} & \text{for } d = 1, \end{cases}
$$

we have that

$$
\begin{aligned}
& -\int_{\hat{\Omega}\cap B_{R_0}} a u^2 G_\lambda \, dx \\
\leq & \; C(\hat{\Omega}, \delta, R, d, \hat{q}) \|a\|_{L^{\hat{q}}(\hat{\Omega}\cap B_{R_0})} \|\nabla(u^2 G_\lambda)\|_{L^1(\hat{\Omega}\cap B_{R_0})} \\
\leq & \; \frac{1}{4} N_\lambda^{\tilde{\varepsilon}}(t) \left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon} \right) \\
& + C(\hat{\Omega}, \delta, R, d, \hat{q}) \left(\|a\|_\infty^2 + \frac{\|a\|_\infty}{T - t + \lambda} \right) \left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,t)|^2 G_\lambda \, dx + \tilde{\varepsilon} \right).
\end{aligned} \tag{3.24}
$$

From (3.23), (3.24) and (3.22) it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\hat\Omega\cap B_{R_0}}|u(x,t)|^2G_\lambda\,dx+\tilde\varepsilon\right)+e^{-3(1+\|\vec b\|_\infty^2)\varepsilon}\frac{\lambda}{\varepsilon+\lambda}N_\lambda^{\tilde\varepsilon}(T)\left(\int_{\hat\Omega\cap B_{R_0}}|u(x,t)|^2G_\lambda\,dx+\tilde\varepsilon\right)\\
\le\ &\left[C(\hat\Omega,\delta,R,d,\hat q)\left(\|a\|_\infty^2+\frac{\|a\|_\infty}{T-t+\lambda}+\varepsilon\|a\|_\infty^{\frac{4}{2-\hat p}}+\|a\|_\infty^2(\varepsilon+\lambda)^{1-\hat p}\right)+2\|b\|_\infty^2\right]\\
&\cdot\left(\int_{\hat\Omega\cap B_{R_0}}|u(x,t)|^2G_\lambda\,dx+\tilde\varepsilon\right) \qquad (3.25)\\
&+\left(2\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}ugG_\lambda\,dx}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u(x,t)|^2G_\lambda\,dx+\tilde\varepsilon}+3\int_t^T\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|g|^2G_\lambda(x,s)\,dx}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u|^2G_\lambda(x,s)\,dx+\tilde\varepsilon}\,ds\right)\\
&\cdot\left(\int_{\hat\Omega\cap B_{R_0}}|u(x,t)|^2G_\lambda\,dx+\tilde\varepsilon\right),\quad \forall\,0<T-\varepsilon\le t<T.
\end{aligned}$$

For $t\in[T-\varepsilon,T-\varepsilon/2]$, $(T-t+\lambda)^{-1}\le 2(\varepsilon+\lambda)^{-1}$, from (3.25) and (3.20) it follows that

$$\begin{aligned}
&\frac{d}{dt}\Bigg[e^{\left(\frac{e^{-3(1+\|\vec b\|_\infty^2)\varepsilon}\lambda}{\varepsilon+\lambda}N_\lambda^{\tilde\varepsilon}(T)-2\|\vec b\|_\infty^2-Q_{h_0,\varepsilon,\lambda}\right)t}\\
&\quad e^{-C(\hat\Omega,\delta,R,d,\hat q)\left(\|a\|_\infty^2+\frac{\|a\|_\infty}{\varepsilon+\lambda}+\varepsilon\|a\|_\infty^{\frac{4}{2-\hat p}}+\|a\|_\infty^2(\varepsilon+\lambda)^{1-\hat p}\right)t}\left(\int_{\hat\Omega\cap B_{R_0}}|u(x,t)|^2G_\lambda\,dx+\tilde\varepsilon\right)\Bigg]\le 0.
\end{aligned}$$

Integrating the latter inequality over $(T-\varepsilon,T-\varepsilon/2)$ and after some calculations, we obtain that

$$\begin{aligned}
&e^{\frac{e^{-3(1+\|\vec b\|_\infty^2)\varepsilon}}{2}\frac{\varepsilon}{\varepsilon+\lambda}\lambda N_\lambda^{\tilde\varepsilon}(T)}\\
\le\ &e^{C(\hat\Omega,\delta,R,d,\hat q)\varepsilon\left[\|a\|_\infty^2+\frac{\|a\|_\infty}{\varepsilon+\lambda}+\varepsilon\|a\|_\infty^{\frac{4}{2-\hat p}}+\|a\|_\infty^2(\varepsilon+\lambda)^{1-\hat p}\right]+\varepsilon\|\vec b\|_\infty^2+\frac{\varepsilon}{2}Q_{h_0,\varepsilon,\lambda}} \qquad (3.26)\\
&\cdot\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u(x,T-\varepsilon)|^2G_\lambda(x,T-\varepsilon)\,dx+\tilde\varepsilon}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u(x,T-\varepsilon/2)|^2G_\lambda(x,T-\varepsilon/2)\,dx+\tilde\varepsilon}.
\end{aligned}$$

Thirdly, we claim that

$$\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u(x,T-\varepsilon)|^2G_\lambda(x,T-\varepsilon)\,dx+\tilde\varepsilon}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u(x,T-\varepsilon/2)|^2G_\lambda(x,T-\varepsilon/2)\,dx+\tilde\varepsilon}\le e^{\frac{(1+\delta)^2R^2}{\varepsilon/2+\lambda}}e^{1+\frac{C_4}{h_0}}e^{\ln\left[\frac{d!(2C_1)^{\frac{d}{2}}}{(1+\delta)^{2d}R^{2d}}+1\right]}. \qquad (3.27)$$

Indeed, on the one hand, by the definition of G_λ, (3.3) and (3.13), we have

$$
\begin{aligned}
&\frac{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u(x,T-\varepsilon)|^2G_\lambda(x,T-\varepsilon)\,dx+\tilde\varepsilon}{\displaystyle\int_{\hat\Omega\cap B_{R_0}}|u(x,T-\varepsilon/2)|^2G_\lambda(x,T-\varepsilon/2)\,dx+\tilde\varepsilon}\\
\le\ &\frac{\displaystyle\int_{\hat\Omega}|\varphi(x,T-\varepsilon)|^2\,dx+\tilde\varepsilon(\varepsilon/2+\lambda)^{\frac d2}}{\displaystyle\int_{\hat\Omega\cap B_{(1+\delta)R}}|\varphi(x,T-\varepsilon/2)|^2\,dx\cdot e^{-\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}}+\tilde\varepsilon(\varepsilon/2+\lambda)^{\frac d2}} \qquad (3.28)\\
\le\ &\frac{e^{\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}}e^{1+\frac{C_4}{h_0}}\left(\displaystyle\int_{\hat\Omega\cap B_{(1+\delta)R}}|\varphi(x,T-\varepsilon/2)|^2\,dx+\tilde\varepsilon\right)+\tilde\varepsilon(\varepsilon/2+\lambda)^{\frac d2}e^{\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}}}{\displaystyle\int_{\hat\Omega\cap B_{(1+\delta)R}}|\varphi(x,T-\varepsilon/2)|^2\,dx+\tilde\varepsilon(\varepsilon/2+\lambda)^{\frac d2}e^{\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&e^{\frac{(1+\delta)^2R^2}{\varepsilon/2+\lambda}}e^{1+\frac{C_4}{h_0}}e^{\ln\left[\frac{d!(2C_1)^{\frac d2}}{(1+\delta)^{2d}R^{2d}}+1\right]}\left[\int_{\hat\Omega\cap B_{(1+\delta)R}}|\varphi(x,T-\varepsilon/2)|^2\,dx+\tilde\varepsilon(\varepsilon/2+\lambda)^{\frac d2}e^{\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}}\right]\\
\ge\ &e^{\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}}e^{1+\frac{C_4}{h_0}}\int_{\hat\Omega\cap B_{(1+\delta)R}}|\varphi(x,T-\varepsilon/2)|^2\,dx+\tilde\varepsilon(\varepsilon/2+\lambda)^{\frac d2}e^{\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}} \qquad (3.29)\\
&+e^{\frac{(1+\delta)^2R^2}{4(\varepsilon/2+\lambda)}}e^{1+\frac{C_4}{h_0}}\left(\frac{2C_1}{\varepsilon/2+\lambda}\right)^{d/2}\tilde\varepsilon.
\end{aligned}
$$

Now we choose $\lambda=\mu\varepsilon$ with $\mu\in(0,1)$ to be determined later. Recalling that $0<\varepsilon\le h_0<C_1$ from (3.12), we obtain from (3.29) and (3.28) that (3.27) holds.

Next we set $\varepsilon=\frac{C_5}{2(C_4+C_5)}h_0$. It follows from (3.26) and (3.27) that

$$
\begin{aligned}
\varepsilon\lambda N_\lambda^{\tilde\varepsilon}(T)\ \le\ & e^{3(1+\|\vec b\|_\infty^2)\varepsilon}C(\hat\Omega,\delta,R,d,\hat q)(\varepsilon^2\|a\|_\infty^2+\varepsilon\|a\|_\infty+\varepsilon^3\|a\|_\infty^{\frac{4}{2-\hat p}}+\varepsilon^{3-\hat p}\|a\|_\infty^2)\\
&+e^{3(1+\|\vec b\|_\infty^2)\varepsilon}[4\varepsilon^2\|\vec b\|_\infty^2+2\varepsilon^2Q_{h_0,\varepsilon,\lambda}+4(1+\delta)^2R^2+4\varepsilon+4\varepsilon C_4h_0^{-1}] \qquad (3.30)\\
&+4\varepsilon e^{3(1+\|\vec b\|_\infty^2)\varepsilon}\ln\left[\frac{d!(2C_1)^{\frac d2}}{(1+\delta)^{2d}R^{2d}}+1\right].
\end{aligned}
$$

Recalling (3.12) and the definition of $Q_{h_0,\varepsilon,\lambda}$ in (3.20), we have by (3.30) that

$$\varepsilon\lambda N_\lambda^{\tilde\varepsilon}(T)\le C(\hat\Omega,\delta,R,d,\hat q).$$

From this estimate and the fact that $\varepsilon<h_0<C_1$, we get

$$\frac{16\lambda}{r^2}\left(\frac d4+\lambda N_\lambda^{\tilde\varepsilon}(T)\right)=\mu\left(\frac{4d}{r^2}\varepsilon+\frac{16}{r^2}\varepsilon\lambda N_\lambda^{\tilde\varepsilon}(T)\right)\le\mu(1+C_6) \qquad (3.31)$$

for some positive constant $C_6 = C_6(\hat{\Omega}, \delta, R, r, d, \hat{q})$.

Step 3. Now, by the same arguments as in Lemma 4 of [17] and (3.31), we have

$$\begin{aligned}&\int_{\hat{\Omega}\cap B_{R_0}} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx \\ \le\ & \int_{B_r} |\varphi(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx + \mu(1+C_6)\left(\int_{\hat{\Omega}\cap B_{R_0}} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx + \tilde{\varepsilon}\right).\end{aligned} \tag{3.32}$$

Then we choose $\mu = \frac{1}{2(1+C_6)}$. This, together with (3.32), implies

$$\int_{\hat{\Omega}\cap B_{R_0}} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx + \tilde{\varepsilon} \le 2\left(\int_{B_r} |\varphi(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx + \tilde{\varepsilon}\right).$$

Using the fact that $\lambda = \frac{C_5}{4(C_4+C_5)(1+C_6)} h_0$ and (3.11), we obtain that Lemma 3.4 holds with $\beta = \frac{(1+C_6)(C_4+C_5)R^2}{(1+C_6)(C_4+C_5)R^2 + C_1C_5}$. □

Lemma 3.5. *Let $\tilde{\omega}$ be a non-empty open subset of $\hat{\Omega}$. Then there are $C = C(\hat{\Omega}, \tilde{\omega}, d, \hat{q}) > 0$ and $\tilde{\beta} = \tilde{\beta}(\hat{\Omega}, \tilde{\omega}, d, \hat{q}) \in (0,1)$ such that for any $T > 0$ and $\varphi_0 \in L^2(\hat{\Omega})$,*

$$\int_{\hat{\Omega}} |\varphi(x,T)|^2\,dx \le e^{C[1+T^{-1}+(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)(T+1)+\|a\|_\infty^{4/(2-\hat{p})}]} \|\varphi_0\|_{L^2(\hat{\Omega})}^{2\tilde{\beta}} \|\varphi(\cdot,T)\|_{L^2(\tilde{\omega})}^{2(1-\tilde{\beta})}.$$

Proof. Indeed, by Lemma 3.4 and the same arguments as Lemma 5 in [17], we have

$$\int_{\hat{\Omega}} |\varphi(x,T)|^2\,dx + \tilde{\varepsilon} \le e^{C[1+T^{-1}+(\|a\|_\infty^2+\|\vec{b}\|_\infty^2)(T+1)+\|a\|_\infty^{4/(2-\hat{p})}]} \|\varphi_0\|_{L^2(\hat{\Omega})}^{2\tilde{\beta}} (\|\varphi(\cdot,T)\|_{L^2(\tilde{\omega})}^2 + \tilde{\varepsilon})^{1-\tilde{\beta}}$$

for some constants $C = C(\hat{\Omega}, \tilde{\omega}, d, \hat{q}) > 0$ and $\tilde{\beta} = \tilde{\beta}(\hat{\Omega}, \tilde{\omega}, d, \hat{q}) \in (0,1)$, where $\tilde{\varepsilon} \in (0, \|\varphi_0\|_{L^2(\hat{\Omega})}^2)$. Passing to the limit for $\tilde{\varepsilon} \to 0$ in the above inequality completes the proof of this lemma. □

By the same arguments as Theorem 4 in [17] and Lemma 3.5, we arrive at (3.2).

Remark 3.6. *By Lemma 3.5, if $\varphi(\cdot,T) = 0$ in a non-empty open set of $\hat{\Omega}$, then $\varphi(\cdot,T) = 0$ in $\hat{\Omega}$. This together with Théorème II.1 in [4] implies that $\varphi(\cdot,t) \equiv 0$ in $\hat{\Omega} \times (0,T)$.*

Remark 3.7. *If $m > 1$, for each $\vec{u} \in W^{1,2}(0,T; L^2(\hat{\Omega}\cap B_{R_0})) \cap L^2(0,T; H^2(\hat{\Omega}\cap B_{R_0}) \cap H_0^1(\hat{\Omega}\cap B_{R_0}))$, $t \in (0,T]$ and $\varepsilon > 0$, we define the frequency function as*

$$N_\lambda^\varepsilon(t) = \frac{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |\nabla \vec{u}(x,t)|^2 G_\lambda(x,t)\,dx}{\displaystyle\int_{\hat{\Omega}\cap B_{R_0}} |\vec{u}(x,t)|^2 G_\lambda(x,t)\,dx + \varepsilon}.$$

Then by the same arguments as above the estimate in Theorem 1.2 can be obtained.

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