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TIME OPTIMAL CONTROL OF THE HEAT EQUATION WITH POINTWISE CONTROL CONSTRAINTS

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Abstract. Time optimal control problems for an internally controlled heat equation with pointwise control constraints are studied. By Pontryagin's maximum principle and properties of nontrivial solutions of the heat equation, we derive a bang-bang property for time optimal control. Using the bang-bang property and establishing certain connections between time and norm optimal control problems for the heat equation, necessary and sufficient conditions for the optimal time and the optimal control are obtained.

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1. Introduction

We can distinguish two distinct versions of time optimal control problems [17]: (i) to reach the target set at a fixed time while delaying the activation of the control as long as possible, and (ii) to reach the target in the shortest time while controlling over the complete timespan. In this paper, we shall consider the above two versions of time optimal control problems for an internally controlled heat equation with pointwise control constraints. Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 1$, with a sufficiently smooth boundary $\partial\Omega$, if $N \geq 2$, and set $C_0(\Omega) = \{y \in C(\overline{\Omega}) : y = 0 \text{ on } \partial\Omega\}$. Further let ω be an open subset of Ω. We formulate the time optimal control problems considered in this paper as follows.

For the first version let $T > 0$ be fixed, and consider the controlled heat equation

$$\begin{cases} y_t - \Delta y = \chi_{(\tau,T)\times\omega} u & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,x) = y_1(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $\chi_{(\tau,T)\times\omega}$ is the characteristic function of the set $(\tau,T)\times\omega$, with $0\le\tau<T$, and $y_1\in C_0(\Omega)$ is a given function. Further u is a control function taken from the set of functions:

$$\mathcal{U}_1\equiv\{v:(0,T)\times\Omega\to\mathbb{R}\text{ measurable};|v(t,x)|\le M_1\text{ for almost all }(t,x)\in(0,T)\times\Omega\},$$

where M_1 is a positive constant. It is well-known that for each $u\in L^\infty((0,T)\times\Omega)$, equation (1.1) has a unique solution denoted by $y(t,x;y_1,\chi_{(\tau,T)\times\omega}u)\in C([0,T];C_0(\Omega))$. In what follows, we write $Q_T,\Sigma_T,Q^\omega_{\tau,T}$ and Q^ω_T for the product sets $(0,T)\times\Omega$, $(0,T)\times\partial\Omega$, $(\tau,T)\times\omega$ and $(0,T)\times\omega$ respectively. The Lebesgue measure of a set D in $\mathbb{R}^d(d\ge1)$ is expressed by $|D|_{\mathbb{R}^d}$. The dual space $C_0(\Omega)$ is denoted by $(C_0(\Omega))^*$. Let $\operatorname{sgn}(r)$ be the sign function, *i.e.*, $\operatorname{sgn}(r)=1$ if $r>0$, $\operatorname{sgn}(r)=-1$ if $r<0$ and $\operatorname{sgn}(r)\in[-1,1]$ if $r=0$. We shall omit the variables t and x for functions of (t,x) and omit the variable x for functions of x, if there is no risk of causing confusion.

Now we are prepared to state the first version of the time optimal control problems under consideration:

$$\sup\ \{\tau:\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)}\le1,\ \tau\in[0,T),\ u\in\mathcal{U}_1\}.\tag{P_1}$$

Without loss of generality we assume that

$$\|y(T,\cdot;y_1,0)\|_{C_0(\Omega)}>1.\tag{1.2}$$

We call

$$\tau^*\equiv\sup\{\tau:\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)}\le1,\ \tau\in[0,T),\ u\in\mathcal{U}_1\}$$

the optimal time for problem (P_1) and $u_1^*\in\mathcal{U}_1$ the associated time-optimal control (or optimal control for simplicity) with corresponding state $y(t,x;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*)$, solution of (1.1), satisfying $\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*)\|_{C_0(\Omega)}\le 1$. We call a control $u\in\mathcal{U}_1$ an admissible control for problem (P_1), if there exists some $\tau\in[0,T)$ such that $\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)}\le1$.

The value of the control in $Q_T\setminus Q^\omega_{\tau,T}$ has no effect on the control system (1.1) and therefore we consistently assign the control to have the value 0 in $Q_T\setminus Q^\omega_{\tau,T}$. *In this paper, we shall prove that the time-optimal control u_1^* for problem (P_1) satisfies the bang-bang property, namely, $|u_1^*(t,x)|=M_1$ for almost all $(t,x)\in Q^\omega_{\tau^*,T}$. Moreover, we shall give necessary and sufficient conditions for τ^* and u_1^* to be the optimal time and the time-optimal control for (P_1).*

For the second version of time optimal control problems studied in this paper we consider the following controlled heat equation

$$\begin{cases}y_t-\Delta y=\chi_\omega u & \text{in } Q_T,\\ y=0 & \text{on } \Sigma_T,\\ y(0,x)=y_2(x) & \text{in } \Omega,\end{cases}\tag{1.3}$$

where $y_2\in C_0(\Omega)$ is a given function, χ_ω is the characteristic function of the set ω and u is a control function taken from

$$\mathcal{U}_2\equiv\{v:(0,+\infty)\times\Omega\to\mathbb{R}\text{ measurable};|v(t,x)|\le M_2\text{ for almost }(t,x)\in(0,\infty)\times\Omega\}.$$

Here M_2 is a positive constant. For each $u\in L^\infty(Q_T)$, we denote the unique solution of (1.3) by $y(t,x;y_2,u)$. The second time optimal control problem under consideration is given by:

$$\inf\ \{T:\|y(T,\cdot;y_2,u)\|_{C_0(\Omega)}\le1,\ T\in(0,\infty),\ u\in\mathcal{U}_2\}.\tag{P_2}$$

Without loss of generality we assume that $\|y_2(\cdot)\|_{C_0(\Omega)}>1$. We call

$$T^*\equiv\inf\{T:\|y(T,\cdot;y_2,u)\|_{C_0(\Omega)}\le1,\ T\in(0,\infty),\ u\in\mathcal{U}_2\}$$

the optimal (minimal) time for problem (P_2) and $u_2^* \in \mathcal{U}_2$ the associated time-optimal control (or optimal control for simplicity) with corresponding state $y(t,x;y_2,u_2^*)$, solution of (1.3), satisfying $\|y(T^*,\cdot;y_2,u_2^*)\|_{C_0(\Omega)} \le 1$. We call a control $u \in \mathcal{U}_2$ an admissible control for problem (P_2), if there exists some $T > 0$ such that $\|y(T,\cdot;y_2,u)\|_{C_0(\Omega)} \le 1$.

The value of the control in $((0,+\infty)\times\Omega)\setminus Q_T^\omega$ has no effect on the control system (1.3) and therefore we consistently assign the control to have the value 0 in $((0,+\infty)\times\Omega)\setminus Q_T^\omega$. *We shall give necessary and sufficient conditions for T^* and u_2^* to be the optimal time and the time-optimal control for* (P_2).

For time optimal control problems, one of the main interests is the bang-bang property of optimal controls. The bang-bang property of optimal controls for time optimal control problems governed by linear evolution equation was first established in [7]. Since then, many results on the bang-bang property of time optimal controls governed by linear and semilinear parabolic differential equations were obtained [3, 4, 20], where the control constraint is in integral form. Certainly pointwise constraints are of interest as well. In [2, 18], Pontryagin's maximum principle was considered, for time optimal control problems governed by semilinear parabolic equations with pointwise constraints in space and time. But the bang-bang property of optimal controls was not established. In [19], the "bang-bang" property of time optimal boundary controls for the heat equation with pointwise control constraints and an arbitrary reachable target set was proved, under an assumption on the bound to which the controls were subjected. In [8], bang-bang property of optimal controls was established for the time optimal control problem governed by the linear parabolic equation, with pointwise control constraint. The target set was a point in the state space and the control acted globally onto the equation. In [12], the bang-bang property of optimal controls was derived for the time optimal control problem governed by the linear Fitzhugh–Nagumo equation with pointwise control constraint, under appropriate assumptions on the initial value of the adjoint equation in Pontryagin's maximum principle. Moreover, in that work the authors pointed out that the time optimal control u_2^* for (P_2) satisfies the bang-bang property, namely, $|u_2(t,x)| = M_2$ for almost all $(t,x) \in Q_{T^*}^\omega$. The above-mentioned works are concerned with the second version of the time optimal control problem. In [17], the authors proved that one-dimensional heat equation with boundary control was exactly null-controllable with control restricted to an arbitrary subset of $[0,T]$ with positive measure. This result implies the bang-bang property of time optimal control for the first version of time optimal control problems. To the best of our knowledge, the bang-bang property of time optimal controls for the first version of time optimal control problems, acting locally onto parabolic equations with pointwise control constraint, was not studied so far. One of the main contributions in this paper is that *the bang-bang property of time optimal control for* (P_1) *is strongly related to the following property for nontrivial solution of the heat equation* (*see e.g. Thm. 4.7.12 in* [8]): if $p \in C^\infty([0,T)\times\overline{\Omega})$ is a nonzero solution to

$$\begin{cases} p_t + \Delta p = 0 & \text{in } Q_T, \\ p = 0 & \text{on } \Sigma_T, \end{cases}$$

then $p(t,x) \neq 0$ a.e. in Q_T.

The other main contribution of this paper are necessary and sufficient conditions for optimal times and optimal controls for (P_1) and (P_2). Time optimal control problems for differential equations were first studied for ordinary differential equations, see *e.g.* [5]. Then such problems were investigated in the context of partial differential equations, see for instance [2,3,13,18,21]. In these works, necessary conditions for time optimal control were given. To the best of our knowledge, for time optimal control problems governed by parabolic equations, there are very few results on sufficient conditions for optimal time and optimal controls, see however [8, 9, 22]. In [8,9] the control acted globally onto the equation and the target set was a point. Moreover, the initial value of the state equation or the target point satisfied some special properties. In [22] the authors obtained necessary and sufficient conditions for the optimal time associated controls for the heat equation, by establishing connections between time and norm optimal control problems. The above-mentioned contributions are concerned with the second version of the time optimal control problem. The idea of our paper utilizes the approach from [22]. More precisely, *for* (P_1) *and* (P_2), *we introduce the norm optimal control problems*

$$\text{Min } \{\|u\|_{L^\infty(Q_T)} : u \in L^\infty(Q_T) \text{ satisfying } \|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)} \le 1\} \qquad (P^\tau_{nm})$$

and

$$\text{Min}\ \{\|u\|_{L^\infty(Q_T)} : u \in L^\infty(Q_T)\ \text{ satisfying }\ \|y(T,\cdot;y_2,u)\|_{C_0(\Omega)} \leq 1\}, \qquad (P_{nmT})$$

and define $N^*_\infty(\tau) = \text{Min}(P^\tau_{nm})$ *and* $\hat{N}^*_\infty(T) = \text{Min}(P_{nmT})$*. By establishing the connections between* (P_1) *and* $(P^{\tau^*}_{nm})$*,* (P_2) *and* (P_{nmT^*})*, respectively, as well as strict monotonicity of* $N^*_\infty(\tau)$ *and* $\hat{N}^*_\infty(T)$*, necessary and sufficient conditions for optimal time and optimal control of* (P_i) *are obtained.* However, there are some main differences between [22] and our paper: (i) the time optimal control problem in [22] is of the second version, while we consider two versions of time optimal control problems. (ii) The methods for the study of the connections between time and norm optimal control problems are different. In [22], the analysis builds on the study of the optimal time T^* as a function of control bound M, while we start by studying the relation of M_i $(i = 1, 2)$ and the minimum of the corresponding norm optimal control problem of (P_i). (iii) In our paper, the control constraint is in pointwise form and the target set is a closed ball in $C_0(\Omega)$, while in [22], the control constraint is in integral form and the target set is 0.

The rest of this paper is organized as follows. In Section 2, we prove that the time optimal control of (P_1) satisfies a bang-bang property. In Section 3, some preliminary results about norm optimal control problems are given, then connections between (P_1) and its corresponding norm optimal control problem are established. In Sections 4 and 5, necessary and sufficient conditions for the optimal time and the optimal control for $(P_i)(i = 1, 2)$ respectively are given. In Appendix A we gather some relevant technical results.

2. Bang-bang property for (P_1)

In this section, we shall present the bang-bang property of the optimal control for problem (P_1) and its proof. To this end, we define the distance function $\overline{d}$ on $\mathcal{U}_1$ by

$$\overline{d}(u,v) = |\{(t,x) \in Q_T : u(t,x) \neq v(t,x)\}|_{\mathbb{R}^{N+1}},\ \ \forall\, u,v \in \mathcal{U}_1.$$

Similarly as for Proposition 3.10 of Chapter 4 in [13], we have that $(\mathcal{U}_1, \overline{d})$ is a complete metric space. We now prove the bang-bang property for (P_1).

Theorem 2.1. *Assume that* τ^* *is the optimal time and let* u^*_1 *be an optimal control for problem* (P_1)*. Then*

$$|u^*_1(t,x)| = M_1\ \ \textit{for almost all}\ \ (t,x) \in Q^\omega_{\tau^*,T}.$$

Proof. The proof is split into five steps.

Step 1. *Introduction of a penalty functional* $J_\varepsilon : (\mathcal{U}_1; \overline{d}) \to [0, +\infty)$*.*
For any ε with $0 < \varepsilon < T - \tau^*$, we define the penalty functional $J_\varepsilon : (\mathcal{U}_1; \overline{d}) \to [0, +\infty)$ by:

$$J_\varepsilon(u) = d_W(y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u)),\quad \forall\, u \in \mathcal{U}_1, \tag{2.1}$$

where $W = \{w \in C_0(\Omega) : \|w\|_{C_0(\Omega)} \leq 1\}$ and

$$d_W(y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u)) \equiv \inf_{w\in W} \|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u) - w\|_{C_0(\Omega)}.$$

Due to the embedding theorem and L^p-theory for parabolic equations (see *e.g.* Thm. 1.4.1 in [23] and Thm. 1.14 of Chap. 1 in [11]), we have

$$\begin{aligned}
|J_\varepsilon(u) - J_\varepsilon(v)| &\leq \|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u) - y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}v)\|_{C_0(\Omega)}\\
&\leq \|y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u) - y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}v)\|_{C(\overline{Q}_T)}\\
&\leq C\|y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u) - y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}v)\|_{W^{2,1}_{2(N+2)}(Q_T)}\\
&\leq C\|u - v\|_{L^{2(N+2)}(Q_T)}.
\end{aligned} \tag{2.2}$$

Here and throughout the proof of this theorem, C denotes a generic positive constant independent of ε. Moreover, the set

$$\{y : D^\alpha D_t^r y \in L^{2(N+2)}(Q_T),\ \text{for any } \alpha \text{ and } r \text{ such that } |\alpha| + 2r \leq 2\},$$

endowed with the norm

$$\|y\|_{W^{2,1}_{2(N+2)}(Q_T)} = \left(\int_0^T \int_\Omega \sum_{|\alpha|+2r\leq 2} |D^\alpha D_t^r y|^{2(N+2)} \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2(N+2)}}$$

is denoted by $W^{2,1}_{2(N+2)}(Q_T)$.

Due to (2.2), we can easily check that J_ε is continuous on $(\mathcal{U}_1; \overline{d})$ and it is obvious that $J_\varepsilon(u) > 0$ for each $u \in \mathcal{U}_1$. Due to similar arguments as in (2.2) we have that

$$\begin{aligned} J_\varepsilon(u_1^*) &= d_W(y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_1^*)) \equiv \delta(\varepsilon) \\ &\leq \|y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_1^*) - y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*,T}} u_1^*)\|_{C_0(\Omega)} \\ &\leq C\|\chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_1^* - \chi_{Q^\omega_{\tau^*,T}} u_1^*\|_{L^{2(N+2)}(Q_T)} \to 0 \quad \text{ as } \varepsilon \to 0. \end{aligned}$$

Step 2. *Application of Ekeland's variational principle.*

Due to Ekeland's variational principle (see *e.g.* Cor. 2.2 of Chap. 4 in [13]), we see that there exists a $u_\varepsilon \in \mathcal{U}_1$ such that

$$\overline{d}(u_\varepsilon, u_1^*) \leq [\delta(\varepsilon)]^{\frac{1}{2}} \tag{2.3}$$

and

$$-[\delta(\varepsilon)]^{\frac{1}{2}} \overline{d}(u_\varepsilon, u) \leq J_\varepsilon(u) - J_\varepsilon(u_\varepsilon), \ \ \forall\, u \in \mathcal{U}_1. \tag{2.4}$$

Step 3. *Derivation of the necessary conditions for* $(u_\varepsilon, y(\cdot, \cdot; y_1, \chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_\varepsilon))$.

Let $v \in \mathcal{U}_1$. Then due to Lemma A.1 in Appendix A, we have that for any $\rho \in (0, 1)$, there exists a measurable set $E_\rho \subset Q_T$ such that $|E_\rho|_{\mathbb{R}^{N+1}} = \rho|Q_T|_{\mathbb{R}^{N+1}}$, and the function

$$u_\rho^\varepsilon(t, x) \equiv \begin{cases} u_\varepsilon(t, x),\ (t, x) \in Q_T \setminus E_\rho, \\ v(t, x), \quad (t, x) \in E_\rho \end{cases} \tag{2.5}$$

satisfies $u_\rho^\varepsilon \in \mathcal{U}_1$. Moreover,

$$\|y(\cdot, \cdot; y_1, \chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_\rho^\varepsilon) - y(\cdot, \cdot; y_1, \chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_\varepsilon) - \rho z_\varepsilon\|_{C(\overline{Q}_T)} = o(\rho), \tag{2.6}$$

where z_ε is the unique solution to the following equation

$$\begin{cases} (z_\varepsilon)_t - \Delta z_\varepsilon = \chi_{Q^\omega_{\tau^*+\varepsilon,T}}(v - u_\varepsilon) & \text{in } \ Q_T, \\ z_\varepsilon = 0 & \text{on } \ \Sigma_T, \\ z_\varepsilon(0, x) = 0 & \text{in } \ \Omega. \end{cases} \tag{2.7}$$

From (2.4) and (2.5) it follows that

$$-[\delta(\varepsilon)]^{\frac{1}{2}} \rho |Q_T|_{\mathbb{R}^{N+1}} = -[\delta(\varepsilon)]^{\frac{1}{2}} \overline{d}(u_\varepsilon, u_\rho^\varepsilon) \leq J_\varepsilon(u_\rho^\varepsilon) - J_\varepsilon(u_\varepsilon),$$

which, together with (2.1) and (2.6), implies

$$\begin{aligned} -\,[\delta(\varepsilon)]^{\frac{1}{2}} |Q_T|_{\mathbb{R}^{N+1}} \ &\leq \ \frac{J_\varepsilon(u_\rho^\varepsilon) - J_\varepsilon(u_\varepsilon)}{\rho} \\ &= \ \frac{d_W(y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_\rho^\varepsilon) - d_W(y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*+\varepsilon,T}} u_\varepsilon))}{\rho} \\ &\to \langle \zeta_\varepsilon, z_\varepsilon(T, \cdot) \rangle_{(C_0(\Omega))^*, C_0(\Omega)} \ \text{ as } \ \rho \to 0^+, \end{aligned} \tag{2.8}$$

see *e.g.* Proposition 3.11 of Chapter 4 in [13]. Here $\zeta_\varepsilon \in \partial d_W(y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon))$, which denotes the subdifferential of $d_W(\cdot)$ at $y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon)$. Since τ^* is the optimal time for (P_1), we have $y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon) \notin W$ and

$$\|\zeta_\varepsilon\|_{(C_0(\Omega))^*} = 1, \quad \forall \ \varepsilon > 0. \tag{2.9}$$

Step 4. *Pass to the limit for $\varepsilon \to 0$ in (2.7) and (2.8).*
Due to (2.3) and since $u_\varepsilon, u_1^* \in \mathcal{U}_1$, one can easily show that

$$u_\varepsilon \to u_1^* \ \text{ strongly in } \ L^{2(N+2)}(Q_T) \ \text{ as } \ \varepsilon \to 0. \tag{2.10}$$

Hence, by making use of (1.1), we get

$$\begin{aligned}\|y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon) - y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*)\|_{C(\overline{Q}_T)} &\le \|y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon) - y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*)\|_{W^{2,1}_{2(N+2)}(Q_T)} \\ &\le C\|\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon - \chi_{Q^\omega_{\tau^*,T}}u_1^*\|_{L^{2(N+2)}(Q_T)} \to 0 \ \text{ as } \ \varepsilon \to 0.\end{aligned} \tag{2.11}$$

Due to (2.10), (2.7) and similar arguments as in (2.11), we see that

$$\|z_\varepsilon - z\|_{C(\overline{Q}_T)} \le C\|\chi_{Q^\omega_{\tau^*+\varepsilon,T}}(v-u_\varepsilon) - \chi_{Q^\omega_{\tau^*,T}}(v-u_1^*)\|_{L^{2(N+2)}(Q_T)} \to 0 \ \text{ as } \ \varepsilon \to 0, \tag{2.12}$$

where z is the unique solution to the following equation

$$\begin{cases} z_t - \Delta z = \chi_{Q^\omega_{\tau^*,T}}(v-u_1^*) & \text{in } \ Q_T, \\ z = 0 & \text{on } \ \Sigma_T, \\ z(0,x) = 0 & \text{in } \ \Omega. \end{cases} \tag{2.13}$$

Moreover, due to (2.8) and (2.9), we get that

$$\begin{aligned}\langle \zeta_\varepsilon, z(T,\cdot)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} &= \langle \zeta_\varepsilon, z(T,\cdot) - z_\varepsilon(T,\cdot)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} + \langle \zeta_\varepsilon, z_\varepsilon(T,\cdot)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\ &\ge -\|z_\varepsilon(T,\cdot) - z(T,\cdot)\|_{C_0(\Omega)} - [\delta(\varepsilon)]^{\frac{1}{2}}|Q_T|_{\mathbb{R}^{N+1}}.\end{aligned} \tag{2.14}$$

Applying (2.9) again, we can assume, without loss of generality, that

$$\zeta_\varepsilon \to \zeta_0 \ \text{ weakly star in } \ (C_0(\Omega))^*. \tag{2.15}$$

It follows easily from (2.14), (2.15) and (2.12) that

$$\langle \zeta_0, z(T,\cdot)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \ge 0. \tag{2.16}$$

Step 5. *The bang-bang property for (P_1).*
Now we claim that

$$\zeta_0 \ne 0. \tag{2.17}$$

Indeed, due to (2.9), (2.11) and making use of the definition of the subdifferential ∂d_W, we obtain

$$\begin{aligned}&\langle \zeta_\varepsilon, y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*) - w\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\ =\ &\langle \zeta_\varepsilon, y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*) - y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\ &+\langle \zeta_\varepsilon, y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon) - w\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\ \ge\ &-\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*) - y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon)\|_{C_0(\Omega)} + d_W(y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon)) \\ \ge\ &-\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*,T}}u_1^*) - y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*+\varepsilon,T}}u_\varepsilon)\|_{C_0(\Omega)} \\ \to\ &0, \qquad \forall\ w \in W.\end{aligned}$$

Then due to Lemma A.4 in Appendix A, (2.9), (2.15) and the fact that W is of finite codimensional in $C_0(\Omega)$, inequality (2.17) follows.

Next, let $\psi \in L^1(0,T;W_0^{1,1}(\Omega))$ be the unique solution to the following equation (Lem. A.2 in Appendix A):

$$\begin{cases} \psi_t + \Delta\psi = 0 & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_T, \\ \psi(T,\cdot) = -\zeta_0 & \text{in } \Omega. \end{cases} \tag{2.18}$$

Then on one hand, due to Lemma A.3 in Appendix A, (2.17) and the smoothing effect of the heat equation, we have

$$\psi(t,x) \neq 0, \quad \text{a.e. } (t,x) \in Q_T. \tag{2.19}$$

On the other hand, it follows from (2.16), (2.18), (2.13) and Lemma A.2 that

$$\begin{aligned} 0 \geq \langle -\zeta_0, z(T,\cdot)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} &= \langle \psi(T,\cdot), z(T,\cdot)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\ &= \int_{Q_T} \chi_{Q^{\omega}_{\tau^*,T}}(v-u_1^*)\psi \,\mathrm{d}x\,\mathrm{d}t, \quad \forall\, v \in \mathcal{U}_1. \end{aligned} \tag{2.20}$$

Finally, we denote

$$F(t,x) = \chi_{Q^{\omega}_{\tau^*,T}}(t,x)(M_1 - u_1^*(t,x))\psi(t,x), \text{ for } (t,x) \in Q_T, \tag{2.21}$$

for which we have $F \in L^1(Q_T)$. Therefore there exists a measurable set $A \subset Q_T$, with $|A|_{\mathbb{R}^{N+1}} = |Q_T|_{\mathbb{R}^{N+1}}$, such that any point in A is a Lebesgue point of F, *i.e.*,

$$\lim_{r\to 0^+} |B((\tilde{t},\tilde{x}),r)|^{-1}_{\mathbb{R}^{N+1}} \int_{B((\tilde{t},\tilde{x}),r)} |F(t,x) - F(\tilde{t},\tilde{x})| \,\mathrm{d}x\,\mathrm{d}t = 0, \;\; \forall\, (\tilde{t},\tilde{x}) \in A, \tag{2.22}$$

where $B((\tilde{t},\tilde{x}),r)$ denotes a closed ball with center at $(\tilde{t},\tilde{x})$ and of radius r. Now for any fixed $(\tilde{t},\tilde{x}) \in A$, we define for sufficiently small positive constant r

$$u_r(t,x) = \begin{cases} u_1^*(t,x), & \text{if } (t,x) \in B((\tilde{t},\tilde{x}),r)^c \cap Q_T, \\ M_1, & \text{if } (t,x) \in B((\tilde{t},\tilde{x}),r) \cap Q_T. \end{cases}$$

Here $B((\tilde{t},\tilde{x}),r)^c$ denotes the complement to $B((\tilde{t},\tilde{x}),r)$. From (2.20) with $v = u_r$ it follows that

$$\int_{B((\tilde{t},\tilde{x}),r)\cap Q_T} F(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_{B((\tilde{t},\tilde{x}),r)\cap Q_T} \chi_{Q^{\omega}_{\tau^*,T}}(t,x)(M_1 - u_1^*(t,x))\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq 0. \tag{2.23}$$

Dividing (2.23) by $|B((\tilde{t},\tilde{x}),r)|_{\mathbb{R}^{N+1}}$, we obtain by (2.22) that

$$M_1 \cdot \chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x) \leq u_1^*(t,x)\cdot \chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x), \quad \forall\, (t,x) \in A. \tag{2.24}$$

Since $|A|_{\mathbb{R}^{N+1}} = |Q_T|_{\mathbb{R}^{N+1}}$ this implies that

$$M_1 \cdot \chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x) \leq u_1^*(t,x)\cdot \chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x), \quad \text{a.e. } (t,x) \in Q_T. \tag{2.25}$$

Similarly we obtain

$$-M_1 \cdot \chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x) \leq u_1^*(t,x)\cdot \chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x), \quad \text{a.e. } (t,x) \in Q_T. \tag{2.26}$$

Moreover, since

$$\max_{|a|\leq M_1} (\chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x)\cdot a) = \max_{a\in\{-M_1,M_1\}} (\chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x)\cdot a),$$

we deduce from (2.25) and (2.26) that

$$\chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x)\cdot u_1^*(t,x) = \max_{|a|\leq M_1} (\chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x)\cdot a) = M_1|\chi_{Q^{\omega}_{\tau^*,T}}\psi(t,x)|.$$

This together with (2.19) completes the proof. □

Based on Theorem 2.1, we can easily obtain the following corollary.

Corollary 2.2. *The time optimal control for problem* (P_1) *is unique.*

Proof. Without loss of generality we assume that u_1 and u_2 are optimal controls for problem (P_1). It is obvious that $\frac{u_1+u_2}{2}$ is also an optimal control for problem (P_1). Then due to Theorem 2.1, we have $|u_1(t,x)| = |u_2(t,x)| = |\frac{u_1+u_2}{2}(t,x)| = M_1$ for almost all $(t,x) \in Q^{\omega}_{\tau^*,T}$. Consequently

$$\left|\frac{u_1-u_2}{2}(t,x)\right|^2 = \frac{1}{2}(|u_1(t,x)|^2 + |u_2(t,x)|^2) - \left|\frac{u_1+u_2}{2}(t,x)\right|^2 = 0, \quad \text{a.e. } (t,x) \in Q^{\omega}_{\tau^*,T},$$

and hence $u_1 = u_2$. □

3. The norm optimal control problem corresponding to (P_1)

For fixed $\tau \in [0,T)$ we consider the following norm optimal control problem:

$$\text{Min } \{\|u\|_{L^\infty(Q_T)} : u \in L^\infty(Q_T) \text{ satisfying } \|y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau,T}}u)\|_{C_0(\Omega)} \le 1\}. \qquad (P^{\tau}_{nm})$$

Again u is assigned the value 0 in $Q_T \setminus Q^{\omega}_{\tau,T}$.

We shall show how to construct a solution to (P^{τ}_{nm}). To this end we first study an auxiliary problem:

$$\text{Min } J_\tau(\mu) \text{ over all } \mu \in (C_0(\Omega))^*, \qquad (P^{\tau}_{au})$$

where the functional $J_\tau : (C_0(\Omega))^* \to \mathbb{R}$ is defined by

$$J_\tau(\mu) = \frac{1}{2}\left(\int_{Q^{\omega}_{\tau,T}} |\varphi_\mu| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + \|\mu\|_{(C_0(\Omega))^*} + \langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)},$$

and φ_μ is the unique solution to the equation:

$$\begin{cases} (\varphi_\mu)_t + \Delta\varphi_\mu = 0 & \text{in } Q_T, \\ \varphi_\mu = 0 & \text{on } \Sigma_T, \\ \varphi_\mu(T,\cdot) = \mu & \text{in } \Omega. \end{cases} \qquad (3.1)$$

For (P^{τ}_{au}), we have:

Lemma 3.1. *Problem* (P^{τ}_{au}) *has at least one minimizer. Moreover, its minimizer is not zero.*

Proof. The proof is split into three steps.

Step 1. *The following property holds:*

$$J_\tau(\mu) \to \infty \quad \textit{as} \quad \|\mu\|_{(C_0(\Omega))^*} \to \infty. \qquad (3.2)$$

In fact, we shall show

$$\underline{\lim}_{\|\mu\|_{(C_0(\Omega))^*}\to\infty} \frac{J_\tau(\mu)}{\|\mu\|_{(C_0(\Omega))^*}} \ge 1. \qquad (3.3)$$

It is obvious that (3.3) implies (3.2). In order to prove (3.3), let $\{\mu_n\}_{n=1}^\infty$ be a sequence of initial data for (3.1) with $\|\mu_n\|_{(C_0(\Omega))^*} \to \infty$. We set $\tilde{\mu}_n = \|\mu_n\|^{-1}_{(C_0(\Omega))^*}\mu_n$. Then $\|\tilde{\mu}_n\|_{(C_0(\Omega))^*} = 1$ and

$$\frac{J_\tau(\mu_n)}{\|\mu_n\|_{(C_0(\Omega))^*}} = \frac{1}{2}\|\mu_n\|_{(C_0(\Omega))^*}\left(\int_{Q^{\omega}_{\tau,T}} |\varphi_{\tilde{\mu}_n}| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + 1 + \langle \tilde{\mu}_n, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)}. \qquad (3.4)$$

The following two cases may occur:

Case 1. $\underline{\lim}_{n\to\infty}\int_{Q^{\omega}_{\tau,T}}|\varphi_{\tilde{\mu}_n}|\,\mathrm{d}x\,\mathrm{d}t > 0$. In this case, we obtain

$$\frac{J_\tau(\mu_n)}{\|\mu_n\|_{(C_0(\Omega))^*}} \to \infty,$$

which implies (3.3).

Case 2. $\underline{\lim}_{n\to\infty}\int_{Q^{\omega}_{\tau,T}}|\varphi_{\tilde{\mu}_n}|\,\mathrm{d}x\,\mathrm{d}t = 0$. In this case, since $\|\tilde{\mu}_n\|_{(C_0(\Omega))^*} = 1$, due to Lemma A.2, we deduce that there exist a subsequence, still indexed by n, and $\tilde{\mu}$, such that

$$\tilde{\mu}_n \to \tilde{\mu} \quad \text{weakly star in } \ (C_0(\Omega))^* \tag{3.5}$$

and

$$\varphi_{\tilde{\mu}_n} \to \varphi_{\tilde{\mu}} \quad \text{weakly in } \ L^\delta(0,T;W_0^{1,\delta}(\Omega)) \ \text{ for some } \ \delta > 1. \tag{3.6}$$

Due to (3.6) and the fact that $\underline{\lim}_{n\to\infty}\int_{Q^{\omega}_{\tau,T}}|\varphi_{\tilde{\mu}_n}|\,\mathrm{d}x\,\mathrm{d}t = 0$, we deduce that $\varphi_{\tilde{\mu}} = 0$ a.e. in $Q^{\omega}_{\tau,T}$, which, together with Lemma A.3 and the smoothing effect of the heat equation, implies that $\tilde{\mu} = 0$. It follows from (3.4) and (3.5) that

$$\underline{\lim}_{n\to\infty}\frac{J_\tau(\mu_n)}{\|\mu_n\|_{(C_0(\Omega))^*}} \geq 1.$$

This implies (3.3).

Step 2. *We prove the existence of a solution to problem* (P^{τ}_{au}).

Due to Lemma A.2, we see that the functional $J_\tau : (C_0(\Omega))^* \to \mathbb{R}$ is continuous. This together with (3.2) implies that $\inf_{\mu\in(C_0(\Omega))^*} J_\tau(\mu)$ exists. Let

$$d = \inf_{\mu\in(C_0(\Omega))^*} J_\tau(\mu). \tag{3.7}$$

Then there exists a sequence $\{\mu_n\}_{n=1}^{\infty} \subset (C_0(\Omega))^*$ such that

$$d = \lim_{n\to\infty} J_\tau(\mu_n). \tag{3.8}$$

It follows from (3.2) and (3.8) that there exists a positive constant C independent of n such that $\|\mu_n\|_{(C_0(\Omega))^*} \leq C$. Due to Lemma A.2, we deduce that there exist a subsequence, still indexed by n, and $\hat{\mu}$, such that

$$\mu_n \to \hat{\mu} \quad \text{weakly star in } \ (C_0(\Omega))^* \tag{3.9}$$

and

$$\varphi_{\mu_n} \to \varphi_{\hat{\mu}} \quad \text{weakly in } \ L^\delta(0,T;W_0^{1,\delta}(\Omega)) \ \text{ for some } \ \delta > 1, \tag{3.10}$$

where φ_{μ_n} and $\varphi_{\hat{\mu}}$ are the solutions of (3.1) with initial values μ_n and $\hat{\mu}$ respectively. Hence, we obtain by (3.8)–(3.10) that

$$d = \underline{\lim}_{n\to\infty} J_\tau(\mu_n) \geq J_\tau(\hat{\mu}),$$

which, combined with (3.7), implies that $\hat{\mu}$ is a solution of (P^{τ}_{au}).

Step 3. *Let μ_τ^* be a solution to (P_{au}^τ). Then $\mu_\tau^* \neq 0$.*
By contradiction, if $\mu_\tau^* = 0$, then

$$J_\tau(0) \leq J_\tau(\lambda\mu), \quad \forall\, \lambda \in \mathbb{R} \text{ and } \mu \in (C_0(\Omega))^*,$$

which implies

$$0 \leq \frac{\lambda^2}{2}\left(\int_{Q^\omega_{\tau,T}} |\varphi_\mu| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + |\lambda| \|\mu\|_{(C_0(\Omega))^*} + \lambda\langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)}.$$

Here φ_μ is the solution to (3.1) with initial value μ. After some simple calculations, we obtain

$$|\langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)}| \leq \|\mu\|_{(C_0(\Omega))^*}, \quad \forall\, \mu \in (C_0(\Omega))^*,$$

which shows that $\|y(T,\cdot;y_1,0)\|_{C_0(\Omega)} \leq 1$, and provides a contradiction to (1.2). □

With the help of problem (P_{au}^τ), we have

Lemma 3.2. *Let μ_τ^* be a solution to (P_{au}^τ). Then*

$$u_\tau^*(t,x) = \left(\int_{Q^\omega_{\tau,T}} |\varphi_{\mu_\tau^*}| \,\mathrm{d}x\,\mathrm{d}t\right) \chi_{Q^\omega_{\tau,T}}(t,x)\mathrm{sgn}(\varphi_{\mu_\tau^*}(t,x)) \quad \text{a.e. } (t,x) \in Q_T, \tag{3.11}$$

is a solution to (P_{nm}^τ), where $\varphi_{\mu_\tau^*}$ is the solution of (3.1) with initial value μ_τ^*.

Proof. This will be proven in two steps.

Step 1. *u_τ^* in (3.11) is admissible for (P_{nm}^τ).*
Due to Lemma 3.1, we know that $\mu_\tau^* \neq 0$. Combined with Lemma A.3 and the smoothing effect of the heat equation this shows that $\varphi_{\mu_\tau^*}(t,x) \neq 0$ a.e. in Q_T. Hence $\int_{Q^\omega_{\tau,T}} |\varphi_{\mu_\tau^*}| \,\mathrm{d}x\,\mathrm{d}t \neq 0$. Since μ_τ^* is a solution to (P_{au}^τ),

$$J(\mu_\tau^*) \leq J(\mu_\tau^* + \lambda\mu), \quad \forall\, \lambda \in \mathbb{R},\ \ \mu \in (C_0(\Omega))^*,$$

and consequently

$$\begin{aligned}
&\frac{1}{2}\left(\int_{Q^\omega_{\tau,T}} |\varphi_{\mu_\tau^*}| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + \|\mu_\tau^*\|_{(C_0(\Omega))^*} + \langle \mu_\tau^*, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\
&\leq \frac{1}{2}\left(\int_{Q^\omega_{\tau,T}} |\varphi_{\mu_\tau^*} + \lambda\varphi_\mu| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + \|\mu_\tau^* + \lambda\mu\|_{(C_0(\Omega))^*} + \langle \mu_\tau^* + \lambda\mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\
&\leq \frac{1}{2}\left(\int_{Q^\omega_{\tau,T}} |\varphi_{\mu_\tau^*} + \lambda\varphi_\mu| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + \|\mu_\tau^*\|_{(C_0(\Omega))^*} + |\lambda|\,\|\mu\|_{(C_0(\Omega))^*} + \langle \mu_\tau^* + \lambda\mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)},
\end{aligned} \tag{3.12}$$

where φ_μ is the solution to (3.1) with initial value μ. Due to (3.12), we get that for any $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ and $\mu \in (C_0(\Omega))^*$,

$$\begin{aligned}
&\frac{1}{2}\int_{Q^\omega_{\tau,T}} (|\varphi_{\mu_\tau^*} + \lambda\varphi_\mu| + |\varphi_{\mu_\tau^*}|) \,\mathrm{d}x\,\mathrm{d}t \cdot \int_{Q^\omega_{\tau,T}} \frac{|\varphi_{\mu_\tau^*} + \lambda\varphi_\mu| - |\varphi_{\mu_\tau^*}|}{|\lambda|} \,\mathrm{d}x\,\mathrm{d}t \\
&+\|\mu\|_{(C_0(\Omega))^*} + \frac{\lambda}{|\lambda|}\langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \geq 0.
\end{aligned}$$

Passing to the limits for $\lambda \to 0^+$ and $\lambda \to 0^-$ in the above inequality we obtain

$$\int_{Q^\omega_{\tau,T}} |\varphi_{\mu^*_\tau}| \,\mathrm{d}x\,\mathrm{d}t \cdot \int_{Q^\omega_{\tau,T}} \frac{\varphi_{\mu^*_\tau}}{|\varphi_{\mu^*_\tau}|}\varphi_\mu \,\mathrm{d}x\,\mathrm{d}t + \|\mu\|_{(C_0(\Omega))^*} + \langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \geq 0$$

and

$$\int_{Q^\omega_{\tau,T}} |\varphi_{\mu^*_\tau}| \,\mathrm{d}x\,\mathrm{d}t \cdot \int_{Q^\omega_{\tau,T}} \frac{\varphi_{\mu^*_\tau}}{|\varphi_{\mu^*_\tau}|}\varphi_\mu \,\mathrm{d}x\,\mathrm{d}t - \|\mu\|_{(C_0(\Omega))^*} + \langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \leq 0.$$

These two inequalities together with (3.11) imply

$$\left| \int_{Q_T} \chi_{Q^\omega_{\tau,T}} u^*_\tau \cdot \varphi_\mu \,\mathrm{d}x\,\mathrm{d}t + \langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \right| \leq \|\mu\|_{(C_0(\Omega))^*}, \quad \forall\, \mu \in (C_0(\Omega))^*. \tag{3.13}$$

Furthermore it follows from (1.1) with u replaced by u^*_τ and 0 respectively, and (3.1) that

$$\int_{Q_T} \chi_{Q^\omega_{\tau,T}} u^*_\tau \cdot \varphi_\mu \,\mathrm{d}x\,\mathrm{d}t = \langle \mu, y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}} u^*_\tau)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} - \int_\Omega y_1(x)\cdot\varphi_\mu(0,x)\,\mathrm{d}x$$

and

$$\langle \mu, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} - \int_\Omega y_1(x)\cdot\varphi_\mu(0,x)\,\mathrm{d}x = 0.$$

The above two equalities combined with (3.13) imply

$$|\langle \mu, y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}} u^*_\tau)\rangle_{(C_0(\Omega))^*,C_0(\Omega)}| \leq \|\mu\|_{(C_0(\Omega))^*}, \quad \forall\, \mu \in (C_0(\Omega))^*.$$

Hence $\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}} u^*_\tau)\|_{C_0(\Omega)} \leq 1$. This completes the proof of Step 1.

Step 2. u^*_τ *is optimal for* (P^τ_{nm}).

Taking u from the admissible control set of (P^τ_{nm}), we have by (1.1) and (3.1) with $\mu = \mu^*_\tau$ that

$$\int_{Q_T} \chi_{Q^\omega_{\tau,T}} u \cdot \varphi_{\mu^*_\tau} \,\mathrm{d}x\,\mathrm{d}t - \langle \mu^*_\tau, y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}} u)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} = -\int_\Omega y_1(x)\cdot\varphi_{\mu^*_\tau}(0,x)\,\mathrm{d}x. \tag{3.14}$$

Moreover, since μ^*_τ is a solution to (P^τ_{au}), we have

$$J_\tau(\mu^*_\tau) \leq J_\tau(\mu^*_\tau + \lambda\mu^*_\tau), \quad \forall\, \lambda \in \mathbb{R}.$$

Due to the previous inequality we obtain after some calculations

$$\left(\int_{Q^\omega_{\tau,T}} |\varphi_{\mu^*_\tau}| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + \|\mu^*_\tau\|_{(C_0(\Omega))^*} = -\langle \mu^*_\tau, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)}. \tag{3.15}$$

Noticing that

$$\langle \mu^*_\tau, y(T,\cdot;y_1,0)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} = \int_\Omega y_1(x)\cdot\varphi_{\mu^*_\tau}(0,x)\,\mathrm{d}x,$$

we obtain together with (3.15) and (3.14) that

$$\begin{aligned}\left(\int_{Q^\omega_{\tau,T}} |\varphi_{\mu^*_\tau}| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + \|\mu^*_\tau\|_{(C_0(\Omega))^*} &= \int_{Q_T} \chi_{Q^\omega_{\tau,T}} u \cdot \varphi_{\mu^*_\tau} \,\mathrm{d}x\,\mathrm{d}t - \langle \mu^*_\tau, y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}} u)\rangle_{(C_0(\Omega))^*,C_0(\Omega)} \\ &\leq \|u\|_{L^\infty(Q_T)} \int_{Q^\omega_{\tau,T}} |\varphi_{\mu^*_\tau}| \,\mathrm{d}x\,\mathrm{d}t + \|\mu^*_\tau\|_{(C_0(\Omega))^*}.\end{aligned}$$

Recalling that $\int_{Q^{\omega}_{\tau,T}}|\varphi_{\mu^*_\tau}|\,\mathrm{d}x\,\mathrm{d}t \neq 0$ we have $\int_{Q^{\omega}_{\tau,T}}|\varphi_{\mu^*_\tau}|\,\mathrm{d}x\,\mathrm{d}t \leq \|u\|_{L^\infty(Q_T)}$. This combined with (3.11) and Step 1, completes the proof. $\square$

Remark 3.3. The idea of construction of a solution to problem (P^{τ}_{nm}) by introducing (P^{τ}_{au}) originates from [14,15], where approximate controllability to $u_1 \in L^2(\Omega)$ of the heat equation in $L^2(\Omega)$

$$\begin{cases} u_t - \Delta u = \chi_\omega g & \text{in } \ Q_T, \\ u = 0 & \text{on } \ \Sigma_T, \\ u(0,x) = 0 & \text{in } \ \Omega, \end{cases}$$

is formulated as follows: for any $\varepsilon > 0$, find $g \in L^2(Q_T)$ such that

$$\|u(T,\cdot;0,g) - u_1\|_{L^2(\Omega)} \leq \varepsilon. \tag{3.16}$$

The control g^* satisfying (3.16) with minimum $L^2(Q_T)$-norm can be constructed in the following manner: consider the minimization problem

$$\operatorname{Min} J(\psi_0) \ \text{ over all } \ \psi_0 \in L^2(\Omega), \tag{P_ε}$$

where the functional $J : L^2(\Omega) \to \mathbb{R}$ is defined by

$$J(\psi_0) = \frac{1}{2}\int_0^T \|\psi\|^2_{L^2(\omega)}\,\mathrm{d}t + \varepsilon\|\psi_0\|_{L^2(\Omega)} - \langle u_1, \psi_0\rangle_{L^2(\Omega),L^2(\Omega)},$$

and ψ is the solution to

$$\begin{cases} \psi_t + \Delta\psi = 0 & \text{in } \ Q_T, \\ \psi = 0 & \text{on } \ \Sigma_T, \\ \psi(T,x) = \psi_0(x) & \text{in } \ \Omega. \end{cases} \tag{3.17}$$

We denote the solution of (P_ε) by ψ^*_0. Then $g^* = \chi_\omega\psi^*$, with ψ^* the solution to (3.17) corresponding to initial value ψ^*_0, gives the solution to the approximate controllability problem with minimum $L^2(Q_T)$-norm. Later on, suitable variants of this functional are used to build different types of controls in dealing with approximate controllability, finite-approximate controllability, null controllability and time optimal control problem of partial differential equations [6,16,22,24].

From now on, we denote

$$N^*_\infty(\tau) = \operatorname{Min}(P^{\tau}_{nm}).$$

With the above preparations, we establish the connections between (P_1) and ($P^{\tau^*}_{nm}$).

Lemma 3.4. *Let τ^* be the optimal time for (P_1) and let u^*_1 be the optimal control of (P_1). Then $N^*_\infty(\tau^*) = M_1$.*

Proof. Since u^*_1 is the optimal control of (P_1), it is an admissible control for ($P^{\tau^*}_{nm}$), and hence $N^*_\infty(\tau^*) \leq M_1$. It suffices to show that equality holds. To seek a contradiction, we assume that

$$N^*_\infty(\tau^*) < M_1. \tag{3.18}$$

By the definition of $N^*_\infty(\tau^*)$, we deduce that there exists a sequence $\{u_n\}_{n=1}^\infty$ from the admissible control set of ($P^{\tau^*}_{nm}$) satisfying

$$\lim_{n\to\infty}\|u_n\|_{L^\infty(Q_T)} = N^*_\infty(\tau^*) \ \text{ and } \ \|y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau^*,T}}u_n)\|_{C_0(\Omega)} \leq 1. \tag{3.19}$$

From the equality in (3.19) and (3.18) it follows that for some integer $n_0 > 0$

$$\|u_n\|_{L^\infty(Q_T)} \leq M_1 \ \text{ for all } \ n \geq n_0. \tag{3.20}$$

Due to (3.20) and the inequality in (3.19) we see that u_n is an optimal control for problem (P_1), if $n \geq n_0$. Combined with Theorem 2.1 and (3.18) this implies that

$$\|u_n\|_{L^\infty(Q_T)} = M_1 > N^*_\infty(\tau^*) \quad \forall\, n \geq n_0,$$

which contradicts with the equality in (3.19). $\square$

Based on Corollary 2.2 and Lemma 3.4, we have

Lemma 3.5. *Let τ^* be the optimal time for (P_1). Then ($P_{nm}^{\tau^*}$) has a unique solution and this solution is the optimal control for (P_1).*

Proof. Assume that u_1 and u_2 are solutions to ($P_{nm}^{\tau^*}$). Then on the one hand,

$$\|y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau^*,T}}u_1)\|_{C_0(\Omega)} \le 1 \text{ and } \|y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau^*,T}}u_2)\|_{C_0(\Omega)} \le 1. \tag{3.21}$$

On the other hand, due to Lemma 3.4, we have

$$\|u_1\|_{L^\infty(Q_T)} = \|u_2\|_{L^\infty(Q_T)} = N_\infty^*(\tau^*) = M_1,$$

which, combined with (3.21), implies that u_1 and u_2 are optimal controls for problem (P_1). Hence, due to Corollary 2.2, we deduce that $u_1 = u_2$ a.e. in Q_T. ☐

Finally, due to Lemmas 3.5 and 3.2, we get

Corollary 3.6. *Let τ^* be the optimal time for (P_1). Then*

$$u^*_{\tau^*}(t,x) = \int_{Q^{\omega}_{\tau^*,T}} |\varphi_{\mu^*_{\tau^*}}|\,\mathrm{d}x\,\mathrm{d}t \cdot \chi_{Q^{\omega}_{\tau^*,T}}(t,x)\mathrm{sgn}(\varphi_{\mu^*_{\tau^*}}(t,x)) \;\; a.e. \;\; (t,x) \in Q_T$$

is the unique solution to ($P_{nm}^{\tau^*}$) and (P_1), where $\varphi_{\mu^*_{\tau^*}}$ is the solution to

$$\begin{cases} (\varphi_{\mu^*_{\tau^*}})_t + \Delta\varphi_{\mu^*_{\tau^*}} = 0 & in \;\; Q_T, \\ \varphi_{\mu^*_{\tau^*}} = 0 & on \;\; \Sigma_T, \\ \varphi_{\mu^*_{\tau^*}}(T,\cdot) = \mu^*_{\tau^*} & in \;\; \Omega, \end{cases}$$

and $\mu^*_{\tau^*}$ is a minimizer of ($P_{au}^{\tau^*}$).

4. Necessary and sufficient optimality conditions for (P_1)

In this section necessary and sufficient optimality conditions for the optimal time τ^* and the optimal control u_1^* of (P_1) are obtained. The main result is given first.

Theorem 4.1. *$\tilde{\tau}^*$ and $\tilde{u}^*$ (with $\tilde{u}^*(t,x) = 0$ a.e. in $Q_T \setminus Q^{\omega}_{\tilde{\tau}^*,T}$) are the optimal time and the optimal control for (P_1) if and only if $\mu^*_{\tilde{\tau}^*}$ is a minimizer of ($P_{au}^{\tilde{\tau}^*}$) with the property that*

$$\varphi_{\mu^*_{\tilde{\tau}^*}}(t,x)\cdot\tilde{u}^*(t,x) = \max_{|a|\le M_1}(\varphi_{\mu^*_{\tilde{\tau}^*}}(t,x)\cdot a) \;\; for\ almost\ all \;\; (t,x) \in Q^{\omega}_{\tilde{\tau}^*,T}, \tag{4.1}$$

$$M_1 = \int_{Q^{\omega}_{\tilde{\tau}^*,T}} |\varphi_{\mu^*_{\tilde{\tau}^*}}|\,\mathrm{d}x\,\mathrm{d}t, \tag{4.2}$$

$$\begin{cases} (\varphi_{\mu^*_{\tilde{\tau}^*}})_t + \Delta\varphi_{\mu^*_{\tilde{\tau}^*}} = 0 & in \;\; Q_T, \\ \varphi_{\mu^*_{\tilde{\tau}^*}}(t,x) = 0 & on \;\; \Sigma_T, \\ \varphi_{\mu^*_{\tilde{\tau}^*}}(T,\cdot) = \mu^*_{\tilde{\tau}^*} & in \;\; \Omega. \end{cases} \tag{4.3}$$

To prove the above theorem, we need the following lemma.

Lemma 4.2. *The function* $N_\infty^*(\cdot) : [0, T) \to (0, +\infty)$ *is strictly increasing, continuous and* $\lim_{\tau\to T^-} N_\infty^*(\tau) = +\infty$.

Proof. The proof is split into four steps.

Step 1. $N_\infty^*(\cdot) : [0, T) \to (0, +\infty)$ *is strictly increasing.*
Let $0 \le \tau_1 < \tau_2 < T$. We shall show that

$$N_\infty^*(\tau_1) < N_\infty^*(\tau_2). \tag{4.4}$$

Due to Lemmas 3.2 and 3.1, we see that for $i = 1, 2$,

$$u_{\tau_i}^*(t, x) = \int_{Q_{\tau_i,T}^\omega} |\varphi_{\mu_{\tau_i}^*}| \,\mathrm{d}x\,\mathrm{d}t \cdot \chi_{Q_{\tau_i,T}^\omega}(t, x)\mathrm{sgn}(\varphi_{\mu_{\tau_i}^*}(t, x)) \quad \text{a.e. } (t, x) \in Q_T \tag{4.5}$$

is a solution to $(P_{nm}^{\tau_i})$, where $\varphi_{\mu_{\tau_i}^*}$ is the solution of (3.1) with initial value $\mu_{\tau_i}^*$ with $\mu_{\tau_i}^*$ a solution to $(P_{au}^{\tau_i})$ and

$$\varphi_{\mu_{\tau_i}^*}(t, x) \neq 0 \quad \text{for a.e. } (t, x) \in Q_T.$$

Consider the equation

$$\begin{cases} z_t - \Delta z = \chi_\omega u & \text{in} \quad (\tau_1, \tau_2) \times \Omega, \\ z = 0 & \text{on} \quad (\tau_1, \tau_2) \times \partial\Omega, \\ z(\tau_1, x) = \delta y(\tau_1, x; y_1, 0) & \text{in} \quad \Omega, \end{cases} \tag{4.6}$$

where $\delta \in (0, 1)$ will be determined later. Due to Theorem 3.1 in [10], we have that there exists a control $u_\delta \in L^\infty((\tau_1, \tau_2) \times \Omega)$, such that the solution $z_\delta(\cdot, \cdot; u_\delta)$ of (4.6) corresponding to u_δ satisfies

$$z_\delta(\tau_2, \cdot; u_\delta) = 0 \tag{4.7}$$

and

$$\|u_\delta\|_{L^\infty((\tau_1,\tau_2)\times\Omega)} \le c_0\, \delta \|y(\tau_1, \cdot; y_1, 0)\|_{L^2(\Omega)}. \tag{4.8}$$

Here c_0 is a positive constant independent of δ. Now take $\delta \in (0, 1)$ such that

$$c_0\, \delta \|y(\tau_1, \cdot; y_1, 0)\|_{L^2(\Omega)} < \|u_{\tau_2}^*\|_{L^\infty(Q_T)}.$$

This combined with (4.8) gives

$$\|u_\delta\|_{L^\infty((\tau_1,\tau_2)\times\Omega)} < \|u_{\tau_2}^*\|_{L^\infty(Q_T)}. \tag{4.9}$$

Let

$$\tilde{u}_\delta = \begin{cases} u_\delta & \text{in} \quad (\tau_1, \tau_2) \times \Omega, \\ 0 & \text{in} \quad [\tau_2, T) \times \Omega. \end{cases} \tag{4.10}$$

Then due to (4.10), (4.9), (4.6) and (4.7), we can easily check that

$$\|\tilde{u}_\delta\|_{L^\infty((\tau_1,T)\times\Omega)} < \|u_{\tau_2}^*\|_{L^\infty(Q_T)}, \tag{4.11}$$

and the solution to

$$\begin{cases} (\tilde{y}_\delta)_t - \Delta \tilde{y}_\delta = \chi_\omega \tilde{u}_\delta & \text{in} \quad (\tau_1, T) \times \Omega, \\ \tilde{y}_\delta = 0 & \text{on} \quad (\tau_1, T) \times \partial\Omega, \\ \tilde{y}_\delta(\tau_1, x) = \delta y(\tau_1, x; y_1, 0) & \text{in} \quad \Omega \end{cases} \tag{4.12}$$

satisfies

$$\tilde{y}_\delta(t, \cdot) = 0, \quad \forall\, t \in [\tau_2, T]. \tag{4.13}$$

Next we consider the following equation

$$\begin{cases} (\hat{y}_\delta)_t - \Delta \hat{y}_\delta = (1-\delta)\chi_{Q^\omega_{\tau_2,T}} u^*_{\tau_2} & \text{in} \quad (\tau_1, T)\times\Omega, \\ \hat{y}_\delta = 0 & \text{on} \quad (\tau_1,T)\times\partial\Omega, \\ \hat{y}_\delta(\tau_1, x) = (1-\delta) y(\tau_1, x; y_1, 0) & \text{in} \quad \Omega, \end{cases} \tag{4.14}$$

and we deduce that

$$\hat{y}_\delta(t,\cdot) = (1-\delta) y(t,\cdot; y_1, \chi_{Q^\omega_{\tau_2,T}} u^*_{\tau_2}), \quad \forall\, t \in [\tau_1, T]. \tag{4.15}$$

It follows from (4.12), (4.14), (4.13) and (4.15) that

$$\begin{cases} (\tilde{y}_\delta + \hat{y}_\delta)_t - \Delta(\tilde{y}_\delta + \hat{y}_\delta) = \chi_\omega \tilde{u}_\delta + (1-\delta)\chi_{Q^\omega_{\tau_2,T}} u^*_{\tau_2} & \text{in} \quad (\tau_1,T)\times\Omega, \\ \tilde{y}_\delta + \hat{y}_\delta = 0 & \text{on} \quad (\tau_1,T)\times\partial\Omega, \\ (\tilde{y}_\delta + \hat{y}_\delta)(\tau_1, x) = y(\tau_1, x; y_1, 0) & \text{in} \quad \Omega \end{cases} \tag{4.16}$$

and

$$\|\tilde{y}_\delta(T,\cdot) + \hat{y}_\delta(T,\cdot)\|_{C_0(\Omega)} \le 1 - \delta < 1. \tag{4.17}$$

Hence, due to (4.16), (4.17), (4.10) and (4.11), we see that the function

$$\hat{u}_\delta = \begin{cases} 0 & \text{in} \quad (0,\tau_1]\times\Omega, \\ \chi_\omega u_\delta & \text{in} \quad (\tau_1,\tau_2)\times\Omega, \\ (1-\delta)\chi_\omega u^*_{\tau_2} & \text{in} \quad [\tau_2, T)\times\Omega \end{cases}$$

is an admissible control for $(P^{\tau_1}_{nm})$ and $\|\hat{u}_\delta\|_{L^\infty(Q_T)} < \|u^*_{\tau_2}\|_{L^\infty(Q_T)}$. These facts together with (4.5) provide (4.4).

Step 2. $N^*_\infty(\cdot) : [0, T) \to (0, +\infty)$ *is left continuous.*
Let $\tau_0 \in (0, T)$ be fixed. Due to Step 1, we infer that

$$\lim_{\tau\uparrow\tau_0} N^*_\infty(\tau)$$

exists. We shall prove

$$\lim_{\tau\uparrow\tau_0} N^*_\infty(\tau) = N^*_\infty(\tau_0). \tag{4.18}$$

By contradiction, we assume that there did exist a sequence $\{\tau_n\}_{n=1}^\infty$ with $\tau_n \uparrow \tau_0$ such that

$$\lim_{\tau_n\uparrow\tau_0} N^*_\infty(\tau_n) = N^*_\infty(\tau_0) - \varepsilon \quad \text{for some positive constant } \varepsilon.$$

We denote by $u^*_{\tau_n}$ a solution to $(P^{\tau_n}_{nm})$. Then

$$\|u^*_{\tau_n}\|_{L^\infty(Q_T)} = N^*_\infty(\tau_n) \uparrow N^*_\infty(\tau_0) - \varepsilon.$$

Hence there exist a subsequence, still indexed by n, and $\tilde{u} \in L^\infty(Q_T)$, such that

$$u^*_{\tau_n} \to \tilde{u} \ \text{ weakly star in } \ L^\infty(Q_T) \tag{4.19}$$

and

$$\|\tilde{u}\|_{L^\infty(Q_T)} \le N^*_\infty(\tau_0) - \varepsilon < N^*_\infty(\tau_0). \tag{4.20}$$

Due to (4.19) we have

$$\chi_{Q^\omega_{\tau_n,T}} u^*_{\tau_n} \to \chi_{Q^\omega_{\tau_0,T}} \tilde{u} \ \text{ weakly star in } \ L^\infty(Q_T).$$

This, combined with L^p- theory for parabolic equation (see *e.g.* Thm. 1.14 of Chap. 1 in [11]) shows that for a subsequence, still indexed by n,

$$y(\cdot,\cdot;y_1,\chi_{Q^{\omega}_{\tau_n,T}}u^*_{\tau_n})-y(\cdot,\cdot;y_1,\chi_{Q^{\omega}_{\tau_0,T}}\tilde{u})\to 0 \;\text{ weakly in }\; W^{2,1}_{2(N+2)}(Q_T).$$

Due to embedding Theorem (see *e.g.* Thm. 1.4.1 in [23]), we deduce that $W^{2,1}_{2(N+2)}(Q_T)$ is compactly embedded in $C(\overline{Q}_T)$, and hence for a subsequence, still indexed by n,

$$\|y(\cdot,\cdot;y_1,\chi_{Q^{\omega}_{\tau_n,T}}u^*_{\tau_n})-y(\cdot,\cdot;y_1,\chi_{Q^{\omega}_{\tau_0,T}}\tilde{u})\|_{C(\overline{Q}_T)}\to 0,$$

which implies that

$$\|y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau_n,T}}u^*_{\tau_n})-y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau_0,T}}\tilde{u})\|_{C_0(\Omega)}\to 0.$$

Together with the fact that $\|y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau_n,T}}u^*_{\tau_n})\|_{C_0(\Omega)}\le 1$ this shows that

$$\|y(T,\cdot;y_1,\chi_{Q^{\omega}_{\tau_0,T}}\tilde{u})\|_{C_0(\Omega)}\le 1. \tag{4.21}$$

Hence $N^*_\infty(\tau_0)\le\|\chi_{Q^{\omega}_{\tau_0,T}}\tilde{u}\|_{L^\infty(Q_T)}\le\|\tilde{u}\|_{L^\infty(Q_T)}$. This contradicts (4.20) and (4.18) follows.

Step 3. $N^*_\infty(\cdot):[0,T)\to(0,+\infty)$ *is right continuous.*
Let $\tau_0\in[0,T)$ be fixed. Due to Step 1, we infer that

$$\lim_{\tau\downarrow\tau_0}N^*_\infty(\tau)$$

exists. We shall prove that

$$\lim_{n\to\infty}N^*_\infty(\tau_n)=N^*_\infty(\tau_0),\quad\text{where }\;\tau_n=\tau_0+\frac{1}{2n^{2(N+2)}}\cdot \tag{4.22}$$

Let $u^*_{\tau_0}$ be a solution of $(P^{\tau_0}_{nm})$ and let y^δ_n and y_δ be the solutions to

$$\begin{cases}(y^\delta_n)_t-\Delta y^\delta_n=\chi_{Q^{\omega}_{\tau_n,T}}(1+n^{-1})(1-\delta)u^*_{\tau_0} & \text{in}\quad Q_T,\\ y^\delta_n=0 & \text{on}\quad \Sigma_T,\\ y^\delta_n(0,x)=(1-\delta)y_1(x) & \text{in}\quad \Omega\end{cases} \tag{4.23}$$

and

$$\begin{cases}(y_\delta)_t-\Delta y_\delta=\chi_{Q^{\omega}_{\tau_0,T}}(1-\delta)u^*_{\tau_0} & \text{in}\quad Q_T,\\ y_\delta=0 & \text{on}\quad \Sigma_T,\\ y_\delta(0,x)=(1-\delta)y_1(x) & \text{in}\quad \Omega\end{cases} \tag{4.24}$$

respectively, where $\delta=\delta(n)\in(0,1)$ will be determined later. It follows from (4.23), (4.24), L^p- theory for parabolic equation (see *e.g.* Thm. 1.14 of Chap. 1 in [11]) and embedding theorem (see *e.g.* Thm. 1.4.1 in [23]) that

$$\begin{aligned}\|y^\delta_n-y_\delta\|_{C(\overline{Q}_T)}\le C\|y^\delta_n-y_\delta\|_{W^{2,1}_{2(N+2)}(Q_T)}&\le C\|\chi_{Q^{\omega}_{\tau_n,T}}(1+n^{-1})(1-\delta)u^*_{\tau_0}-\chi_{Q^{\omega}_{\tau_0,T}}(1-\delta)u^*_{\tau_0}\|_{L^{2(N+2)}(Q_T)}\\ &=C\Bigg(\int_{\tau_0}^{\tau_n}\int_\Omega|\chi_\omega(1-\delta)u^*_{\tau_0}|^{2(N+2)}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_{\tau_n}^{T}\int_\Omega|\chi_\omega n^{-1}(1-\delta)u^*_{\tau_0}|^{2(N+2)}\,\mathrm{d}x\,\mathrm{d}t\Bigg)^{\frac{1}{2(N+2)}}\\ &\le C\|u^*_{\tau_0}\|_{L^\infty(Q_T)}(1-\delta)\big[(\tau_n-\tau_0)^{\frac{1}{2(N+2)}}+n^{-1}\big]\\ &\le C(1-\delta)n^{-1}.\end{aligned}$$

Here and throughout the proof of this step, C denotes different positive constants independent of n and δ. The previous inequality, together with the fact that $y_\delta(t,x) = (1-\delta)y(t,x;y_1,\chi_{Q^\omega_{\tau_0,T}}u^*_{\tau_0})$, implies that

$$\begin{aligned}\|y^\delta_n(T,\cdot)\|_{C_0(\Omega)} &\leq (1-\delta)\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau_0,T}}u^*_{\tau_0})\|_{C_0(\Omega)} + C(1-\delta)n^{-1}\\ &\leq (1-\delta)[1+Cn^{-1}].\end{aligned} \tag{4.25}$$

From the following equation

$$\begin{cases} z_t - \Delta z = \chi_{Q^\omega_{\tau_n,T}} u & \text{in} \quad Q_T,\\ z = 0 & \text{on} \quad \Sigma_T,\\ z(0,x) = \delta y_1(x) & \text{in} \quad \Omega, \end{cases} \tag{4.26}$$

it is obvious that

$$\begin{cases} z_t - \Delta z = 0 & \text{in} \quad (0,\tau_n)\times\Omega,\\ z = 0 & \text{on} \quad (0,\tau_n)\times\partial\Omega,\\ z(0,x) = \delta y_1(x) & \text{in} \quad \Omega. \end{cases} \tag{4.27}$$

Multiplying the first equation of (4.27) by z and integrating on Ω, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t,\cdot)\|^2_{L^2(\Omega)} = -2\|\nabla z(t,\cdot)\|^2_{L^2(\Omega)}, \quad \forall\, t\in(0,\tau_n),$$

which implies that the function $\|z(t,\cdot)\|^2_{L^2(\Omega)}$ is decreasing on $[0,\tau_n]$. Hence

$$\|z(\tau_n,\cdot)\|_{L^2(\Omega)} \leq \|z(0,\cdot)\|_{L^2(\Omega)} = \delta\|y_1\|_{L^2(\Omega)}. \tag{4.28}$$

From Theorem 3.1 in [10] it follows that there exists a function $\tilde{u}^\delta_n \in L^\infty((\tau_n,T)\times\Omega)$ such that the solution of (4.26) corresponding to $u = u^\delta_n \equiv \begin{cases} 0 & \text{in} \ \ (0,\tau_n]\times\Omega,\\ \tilde{u}^\delta_n & \text{in} \ \ (\tau_n,T)\times\Omega, \end{cases}$ denoted by z^δ_n, satisfies

$$z^\delta_n(T,\cdot) = 0 \tag{4.29}$$

and

$$\|\tilde{u}^\delta_n\|_{L^\infty((\tau_n,T)\times\Omega)} \leq \mathrm{e}^{C[1+T-\tau_n+(T-\tau_n)^{-1}]}\|z^\delta_n(\tau_n,\cdot)\|_{L^2(\Omega)} \leq C\|z^\delta_n(\tau_n,\cdot)\|_{L^2(\Omega)}.$$

The previous inequality, combined with (4.23), (4.25), (4.26), (4.28) and (4.29), implies that there exists a positive constant $c_0>1$ independent on n and δ such that

$$(y^\delta_n + z^\delta_n)(\cdot,\cdot) = y(\cdot,\cdot;y_1,\chi_{Q^\omega_{\tau_n,T}}[(1+n^{-1})(1-\delta)u^*_{\tau_0} + u^\delta_n]), \quad \forall\,(t,x)\in Q_T, \tag{4.30}$$

$$\|(y^\delta_n + z^\delta_n)(T,\cdot)\|_{C_0(\Omega)} \leq (1-\delta)(1+c_0 n^{-1}), \tag{4.31}$$

and

$$\|(1+n^{-1})(1-\delta)u^*_{\tau_0} + u^\delta_n\|_{L^\infty(Q_T)} \leq (1+n^{-1})(1-\delta)\|u^*_{\tau_0}\|_{L^\infty(Q_T)} + c_0\delta. \tag{4.32}$$

Now we take $\delta = c_0(c_0+n)^{-1}$. Then it follows from (4.31) and (4.32) that

$$\|(y^\delta_n + z^\delta_n)(T,\cdot)\|_{C_0(\Omega)} \leq 1 \tag{4.33}$$

and

$$\|(1+n^{-1})(1-\delta)u^*_{\tau_0} + u^\delta_n\|_{L^\infty(Q_T)} \leq \|u^*_{\tau_0}\|_{L^\infty(Q_T)} + c_0^2(c_0+n)^{-1}.$$

Hence we deduce from the result of Step 1, (4.30), (4.33) and the previous inequality that

$$N_\infty^*(\tau_0) < N_\infty^*(\tau_n) \le \|\chi_{Q^\omega_{\tau_n,T}}[(1+n^{-1})(1-\delta)u^*_{\tau_0} + u_n^\delta]\|_{L^\infty(Q_T)} \le N_\infty^*(\tau_0) + c_0^2(c_0+n)^{-1},$$

which gives (4.22).

Step 4. $\lim_{\tau\to T^-} N_\infty^*(\tau) = +\infty$.

Assume that this property does not hold. Then there would be a positive constant M_0 and a sequence $\{\tau_n\}_{n=1}^\infty$ with $\tau_n \uparrow T$ such that

$$N_\infty^*(\tau_n) \le M_0. \tag{4.34}$$

Let $u^*_{\tau_n}$ be a solution to problem $(P_{nm}^{\tau_n})$. Then due to (4.34), we deduce

$$\|u^*_{\tau_n}\|_{L^\infty(Q_T)} = N_\infty^*(\tau_n) \le M_0 \tag{4.35}$$

and

$$\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau_n,T}}u^*_{\tau_n})\|_{C_0(\Omega)} \le 1. \tag{4.36}$$

It follows from (4.35) that

$$\chi_{Q^\omega_{\tau_n,T}}u^*_{\tau_n} \to 0 \text{ weakly star in } L^\infty(Q_T),$$

which, combined with (4.36) and the same arguments as (4.21), indicates

$$\|y(T,\cdot;y_1,0)\|_{C_0(\Omega)} \le 1.$$

This contradicts assumption (1.2). □

We turn to the proof of Theorem 4.1.

Proof. The"only if" part can be easily derived by Corollary 3.6.

Concerning the "if" part, it follows from (4.2) and (4.3) that $\mu^*_{\tilde\tau^*} \ne 0$ and $\varphi_{\mu^*_{\tilde\tau^*}} \ne 0$ a.e. in Q_T. Then due to (4.1)–(4.2) and Lemma 3.2, we deduce that

$$\begin{aligned}\tilde u^*(t,x) &= M_1\cdot\chi_{Q^\omega_{\tilde\tau^*,T}}(t,x)\mathrm{sgn}(\varphi_{\mu^*_{\tilde\tau^*}}(t,x))\\ &= \int_{Q^\omega_{\tilde\tau^*,T}} |\varphi_{\mu^*_{\tilde\tau^*}}|\,\mathrm{d}x\,\mathrm{d}t\cdot\chi_{Q^\omega_{\tilde\tau^*,T}}(t,x)\mathrm{sgn}(\varphi_{\mu^*_{\tilde\tau^*}}(t,x)); \text{ a.e. in } Q_T\end{aligned}$$

is a solution of the problem $(P_{nm}^{\tilde\tau^*})$, which implies $M_1 = N_\infty^*(\tilde\tau^*)$. This together with Lemma 3.4 shows that

$$N_\infty^*(\tilde\tau^*) = N_\infty^*(\tau^*) = M_1. \tag{4.37}$$

By Lemma 4.2 and (4.37), we obtain $\tilde\tau^* = \tau^*$. Furthermore, it follows from Lemma 3.5 that $\tilde u^*$ is the optimal control for (P_1). □

Remark 4.3.

(i) Assigning the control to have the value 0 in $Q_T\setminus Q^\omega_{\tau,T}$ is essential to obtain Theorem 4.1. Analogously this will be the case in Theorem 5.5 of the following section.

(ii) In the proof of Theorem 4.1, from the properties of $N_\infty^*(\cdot)$ given in Lemma 4.2, we only use the fact that $N_\infty^*(\cdot):[0,T)\to(0,+\infty)$ is strictly increasing.

(iii) The fact that $M_1 = N_\infty^*(\tau^*) = \int_{Q^\omega_{\tau^*,T}}|\varphi_{\mu^*_{\tau^*}}|\,\mathrm{d}x\,\mathrm{d}t$, Lemma 4.2 and Theorem 4.1 give us some directions to numerically calculate the optimal time τ^* and optimal control u_1^* for (P_1).

5. Necessary and sufficient optimality conditions for (P_2)

In this section necessary and sufficient optimality conditions for optimal time T^* and optimal control u_2^* of (P_2) are given. Let us first recall the following result from [12].

Lemma 5.1. *Let T^* be the optimal time and let u_2^* be an optimal control for problem (P_2). Then $|u_2^*(t,x)| = M_2$ for almost all $(t,x) \in Q_{T^*}^{\omega}$.*

It follows from Lemma 5.1 and the same arguments as Corollary 2.2 that

Corollary 5.2. *The time optimal control for (P_2) is unique.*

Next, let $T > 0$ be fixed. We introduce two minimization problems as in Section 3. The first one is the following norm optimal control problem:

$$\text{Min } \{\|u\|_{L^\infty(Q_T)} : u \in L^\infty(Q_T) \text{ satisfying } \|y(T,\cdot;y_2,u)\|_{C_0(\Omega)} \leq 1\}, \tag{P_{nmT}}$$

where the control u is set to 0 in $Q_T \setminus Q_T^{\omega}$.

The second one is an auxiliary problem:

$$\text{Min } J_T(\mu) \text{ over all } \mu \in (C_0(\Omega))^*, \tag{P_{auT}}$$

where the functional $J_T : (C_0(\Omega))^* \to \mathbb{R}$ is defined by

$$J_T(\mu) = \frac{1}{2}\left(\int_{Q_T^{\omega}} |\varphi_\mu| \,\mathrm{d}x\,\mathrm{d}t\right)^2 + \|\mu\|_{(C_0(\Omega))^*} + \langle \mu, y(T,\cdot;y_2,0)\rangle_{(C_0(\Omega))^*, C_0(\Omega)},$$

and φ_μ is the unique solution to the equation:

$$\begin{cases} (\varphi_\mu)_t + \Delta\varphi_\mu = 0 & \text{in } Q_T, \\ \varphi_\mu = 0 & \text{on } \Sigma_T, \\ \varphi_\mu(T,\cdot) = \mu & \text{in } \Omega. \end{cases} \tag{5.1}$$

Due to Lemma 5.1, Corollary 5.2 and similar arguments as those in Section 3, we have

Lemma 5.3.

(i) *Problem (P_{auT}) has at least a minimizer. Moreover, its minimizer is zero if and only if $\|y(T,\cdot;y_2,0)\|_{C_0(\Omega)} \leq 1$.*

(ii) *If $\|y(T,\cdot;y_2,0)\|_{C_0(\Omega)} > 1$ and μ_T^* is a solution to (P_{auT}), then*

$$u_T^*(t,x) = \int_{Q_T^{\omega}} |\varphi_{\mu_T^*}| \,\mathrm{d}x\,\mathrm{d}t \cdot \chi_\omega(x)\mathrm{sgn}(\varphi_{\mu_T^*}(t,x)) \quad a.e. \ (t,x) \in Q_T,$$

is a solution of (P_{nmT}), where $\varphi_{\mu_T^*}$ is the solution to (5.1) with initial value μ_T^*.

(iii) *Let T^* be the optimal time for problem (P_2). Then $\hat{N}_\infty^*(T^*) = M_2$, where $\hat{N}_\infty^*(T)$ denotes the minimum of (P_{nmT}).*

(iv) *Let T^* be the optimal time for problem (P_2). Then problem (P_{nmT^*}) has a unique solution. This solution, after being extended to be 0 on $[T^*, +\infty) \times \Omega$, is the optimal control for problem (P_2).*

Henceforth for a given solution of (P_{nmT}), we extend it to be 0 on $[T, +\infty) \times \Omega$. From Lemma 5.1 it follows that $\|y(T^*,\cdot;y_2,0)\|_{C_0(\Omega)} > 1$. Then, due to (ii) and (iv) in Lemma 5.3, we obtain the following result.

Corollary 5.4. *Let T^* be the optimal time for problem (P_2). Then*

$$u^*_{T^*}(t,x) = \int_{Q^\omega_{T^*}} |\varphi_{\mu^*_{T^*}}| \,\mathrm{d}x\,\mathrm{d}t \cdot \chi_\omega(x)\mathrm{sgn}(\varphi_{\mu^*_{T^*}}(t,x)) \;\; a.e. \;\; (t,x) \in Q_{T^*}$$

is the unique solution to (P_{nmT^*}) and (P_2), where $\varphi_{\mu^*_{T^*}}$ is the solution to

$$\begin{cases} (\varphi_{\mu^*_{T^*}})_t + \Delta\varphi_{\mu^*_{T^*}} = 0 & in \;\; Q_{T^*}, \\ \varphi_{\mu^*_{T^*}} = 0 & on \;\; \Sigma_{T^*}, \\ \varphi_{\mu^*_{T^*}}(T^*,\cdot) = \mu^*_{T^*} & in \;\; \Omega, \end{cases}$$

and $\mu^*_{T^*}$ is a minimizer of (P_{auT^*}).

The main result of this section is given next.

Theorem 5.5. *$\tilde{T}^*$ and $\tilde{u}^*$ (with $\tilde{u}^* = 0$ on $((0,+\infty)\times\Omega)\setminus Q^\omega_{\tilde{T}^*}$) are the optimal time and the optimal control for (P_2) if and only if*

$$\varphi_{\mu^*_{\tilde{T}^*}}(t,x)\cdot\tilde{u}^*(t,x) = \max_{|a|\le M_2}(\varphi_{\mu^*_{\tilde{T}^*}}(t,x)\cdot a) \;\; \textit{for almost all} \;\; (t,x)\in Q^\omega_{\tilde{T}^*}, \tag{5.2}$$

$$M_2 = \int_{Q^\omega_{\tilde{T}^*}} |\varphi_{\mu^*_{\tilde{T}^*}}| \,\mathrm{d}x\,\mathrm{d}t, \tag{5.3}$$

$$\begin{cases} (\varphi_{\mu^*_{\tilde{T}^*}})_t + \Delta\varphi_{\mu^*_{\tilde{T}^*}} = 0 & in \;\; Q_{\tilde{T}^*}, \\ \varphi_{\mu^*_{\tilde{T}^*}}(t,x) = 0 & on \;\; \Sigma_{\tilde{T}^*}, \\ \varphi_{\mu^*_{\tilde{T}^*}}(\tilde{T}^*,\cdot) = \mu^*_{\tilde{T}^*} & in \;\; \Omega, \end{cases} \tag{5.4}$$

where $\mu^*_{\tilde{T}^*}$ is a minimizer of $(P_{au\tilde{T}^*})$.

To prove the above theorem, we need some properties of the function $\hat{N}^*_\infty(\cdot)$. For this purpose, we define

$$T_0 = \inf\{T: \; \|y(T,\cdot;y_2,0)\|_{C_0(\Omega)} \le 1, \; T>0\}. \tag{5.5}$$

The following properties are satisfied by $\hat{N}^*_\infty$.

Lemma 5.6. *We have $T_0<\infty$ and the function $\hat{N}^*_\infty$ is strictly decreasing and continuous on $(0,T_0]$. Moreover, $\hat{N}^*_\infty(T)=0$ for $T\ge T_0$ and $\lim_{T\to 0^+}\hat{N}^*_\infty(T)=+\infty$.*

Proof. The proof is split into four steps.

Step 1. *$\hat{N}^*_\infty(\cdot):(0,T_0]\to[0,+\infty)$ is strictly decreasing, $T_0<+\infty$ and $\hat{N}^*_\infty(T)=0$ for $T\ge T_0$.*
Let $0<T_1<T_2<T_0$. It is obvious that

$$\|y(T_1,\cdot;y_2,0)\|_{C_0(\Omega)} > 1 \text{ and } \|y(T_2,\cdot;y_2,0)\|_{C_0(\Omega)} > 1. \tag{5.6}$$

We shall show that

$$\hat{N}^*_\infty(T_1) > \hat{N}^*_\infty(T_2). \tag{5.7}$$

Due to (5.6), and (i), (ii) in Lemma 5.3, we see that for $i=1,2$,

$$u^*_{T_i}(t,x) = \int_{Q^\omega_{T_i}} |\varphi_{\mu^*_{T_i}}| \,\mathrm{d}x\,\mathrm{d}t \cdot \chi_\omega(x)\mathrm{sgn}(\varphi_{\mu^*_{T_i}}(t,x)) \quad \text{a.e.} \;\; (t,x)\in Q_{T_i} \tag{5.8}$$

is a solution to (P_{nmT_i}), where $\varphi_{\mu^*_{T_i}}$ is the solution of (5.1) with initial value $\mu^*_{T_i}$, and $\mu^*_{T_i}$ is a solution to (P_{auT_i}). We also have

$$\varphi_{\mu^*_{T_i}}(t,x) \neq 0 \ \text{ for a.e. } \ (t,x) \in Q_{T_i} \quad (\mu^*_{T_i} \neq 0). \tag{5.9}$$

Let y_n be the solution to

$$\begin{cases} (y_n)_t - \Delta y_n = \chi_\omega (1-n^{-1}) u^*_{T_1} & \text{in } \ (0,+\infty)\times\Omega, \\ y_n = 0 & \text{on } \ (0,+\infty)\times\partial\Omega, \\ y_n(0,x) = y_2(x) & \text{in } \ \Omega. \end{cases} \tag{5.10}$$

It is easy to check that

$$\|y_n(T_1,\cdot) - y(T_1,\cdot; y_2, u^*_{T_1})\|_{C_0(\Omega)} \le Cn^{-1},$$

which implies

$$\|y_n(T_1,\cdot)\|_{C_0(\Omega)} \le 1 + Cn^{-1}. \tag{5.11}$$

Here and throughout the proof of this step, C denotes different positive constants independent of n. Set $z_n(t,x) = y_n(t+T_1,x)$, for $(t,x) \in Q_{T_2-T_1}$. Then we have

$$\begin{cases} (z_n)_t - \Delta z_n = 0 & \text{in } \ Q_{T_2-T_1}, \\ z_n = 0 & \text{on } \ \Sigma_{T_2-T_2}, \\ z_n(0,x) = y_n(T_1,x) & \text{in } \ \Omega. \end{cases}$$

Define

$$\tilde z_n(0,x) = \begin{cases} |y_n(T_1,x)| & \text{in } \ \Omega, \\ 0 & \text{in } \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.12}$$

and let $\tilde z_n$ satisfy the heat equation $(\tilde z_n)_t - \Delta \tilde z_n = 0$ for $x \in \mathbb{R}^N, t > 0$. Then

$$\tilde z_n(t,x) = \int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x-s|^2}{4t}} \tilde z_n(0,s)\,\mathrm{d}s, \ \ \forall\,(t,x) \in (0,+\infty)\times\mathbb{R}^N, \tag{5.13}$$

and

$$|z_n(t,x)| \le \tilde z_n(t,x), \quad \forall\,(t,x) \in Q_{T_2-T_1}. \tag{5.14}$$

It follows from (5.14), (5.13) and (5.12) that

$$\begin{aligned} |z_n(T_2-T_1,x)| &\le \int_{\mathbb{R}^N} [4\pi(T_2-T_1)]^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x-s|^2}{4(T_2-T_1)}} \tilde z_n(0,s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}^N} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|s|^2}{2}} \tilde z_n\Big(0, x + \sqrt{2(T_2-T_1)}s\Big)\,\mathrm{d}s \\ &= \int_{\{s\in\mathbb{R}^N : x+\sqrt{2(T_2-T_1)}s \in\Omega\}} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|s|^2}{2}} \Big| y_n\Big(T_1, x + \sqrt{2(T_2-T_1)}s\Big)\Big|\,\mathrm{d}s \\ &\le \|y_n(T_1,\cdot)\|_{C_0(\Omega)} \int_{\{s\in\mathbb{R}^N : x+\sqrt{2(T_2-T_1)}s \in\Omega\}} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|s|^2}{2}}\,\mathrm{d}s, \qquad \forall\, x\in\Omega. \end{aligned} \tag{5.15}$$

It is easy to check that there exists an open, bounded set $\hat\Omega$ (depending on $T_2 - T_1$) in $\mathbb{R}^N$ such that

$$\Big\{ s \in \mathbb{R}^N : x + \sqrt{2(T_2-T_1)}s \in \Omega \Big\} \subset \hat\Omega, \quad \forall\, x \in \Omega.$$

This together with (5.15) implies

$$|z_n(T_2-T_1,x)| \le \|y_n(T_1,\cdot)\|_{C_0(\Omega)} \int_{\hat\Omega} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|s|^2}{2}}\,\mathrm{d}s, \qquad \forall\, x\in\Omega,$$

from which it follows that

$$\|z_n(T_2-T_1,\cdot)\|_{C_0(\Omega)} \leq \|y_n(T_1,\cdot)\|_{C_0(\Omega)} \int_{\hat{\Omega}} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}x.$$

Hence

$$\|y_n(T_2,\cdot)\|_{C_0(\Omega)} \leq \|y_n(T_1,\cdot)\|_{C_0(\Omega)} \int_{\hat{\Omega}} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}x. \tag{5.16}$$

This together with (5.11) leads to

$$\|y_n(T_2,\cdot)\|_{C_0(\Omega)} \leq [1+Cn^{-1}] \int_{\hat{\Omega}} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}x. \tag{5.17}$$

Moreover, we know that $\int_{\mathbb{R}^N} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}x = 1$, which implies $\int_{\hat{\Omega}} (2\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}x < 1$. It follows from (5.17) and the latter that there exists a positive integer n_0 such that

$$\|y_{n_0}(T_2,\cdot)\|_{C_0(\Omega)} \leq 1. \tag{5.18}$$

Hence due to (5.18), (5.10), (5.8) and (5.9), we obtain

$$\hat{N}^*_\infty(T_2) \leq \|\chi_\omega(1-n_0^{-1})u^*_{T_1}\|_{L^\infty(Q_{T_2})} \leq \|(1-n_0^{-1})u^*_{T_1}\|_{L^\infty(Q_{T_1})} < \|u^*_{T_1}\|_{L^\infty(Q_{T_1})} = \hat{N}^*_\infty(T_1),$$

which completes the proof of (5.7).

Furthermore by the same arguments as (5.16), we deduce that there exists a constant $a_0 \in (0,1)$ such that for any integer $n \geq 0$,

$$\|y(n+1,\cdot;y_2,0)\|_{C_0(\Omega)} \leq a_0 \|y(n,\cdot;y_2,0)\|_{C_0(\Omega)},$$

which implies

$$\|y(n,\cdot;y_2,0)\|_{C_0(\Omega)} \leq a_0^n \|y_2(\cdot)\|_{C_0(\Omega)}.$$

Hence there exists a positive integer n_0 satisfying $\|y(n_0,\cdot;y_2,0)\|_{C_0(\Omega)} \leq 1$. This together with definition (5.5) implies $T_0 < +\infty$ and $\hat{N}^*_\infty(T_0) = 0$. Similarly, for $T > T_0$, since

$$\|y(T-T_0,\cdot;y(T_0,\cdot;y_2,0),0)\|_{C_0(\Omega)} \leq \|y(T_0,\cdot;y_2,0)\|_{C_0(\Omega)} \leq 1$$

and $y(T,x;y_2,0) = y(T-T_0,x;y(T_0,\cdot;y_2,0),0), \forall\, x \in \Omega$, we get $\|y(T,\cdot;y_2,0)\|_{C_0(\Omega)} \leq 1$. Hence $\hat{N}^*_\infty(T) = 0$.

Step 2. $\hat{N}^*_\infty(\cdot) : (0,T_0) \to [0,+\infty)$ *is right continuous.*

Let $\tilde{T} \in (0,T_0)$ be fixed. Due to Step 1, we deduce that $\lim_{T\downarrow\tilde{T}} \hat{N}^*_\infty(T)$ exists. We shall prove

$$\lim_{T\downarrow\tilde{T}} \hat{N}^*_\infty(T) = \hat{N}^*_\infty(\tilde{T}). \tag{5.19}$$

By contradiction, we assume there exists a sequence $\{T_n\}_{n=1}^\infty$ with $T_n \downarrow \tilde{T}$ such that

$$\lim_{T_n\downarrow\tilde{T}} \hat{N}^*_\infty(T_n) = \hat{N}^*_\infty(\tilde{T}) - \varepsilon \quad \text{for some positive constant } \varepsilon.$$

We denote by u_n a solution to (P_{nmT_n}). Then

$$\|u_n\|_{L^\infty(Q_{T_n})} = \hat{N}^*_\infty(T_n) \uparrow \hat{N}^*_\infty(\tilde{T}) - \varepsilon.$$

Hence there exist a subsequence, still indexed by n, and $\tilde{u} \in L^\infty((0,+\infty)\times\Omega)$, such that

$$u_n \to \tilde{u} \ \text{ weakly star in } \ L^\infty((0,+\infty)\times\Omega) \tag{5.20}$$

and

$$\|\tilde{u}\|_{L^\infty(Q_{\tilde{T}})} \le \hat{N}^*_\infty(\tilde{T}) - \varepsilon < \hat{N}^*_\infty(\tilde{T}). \tag{5.21}$$

Due to the similar arguments as those of Step 2 in Lemma 4.2, we obtain that

$$\|y(\tilde{T},\cdot;y_2,\tilde{u})\|_{C_0(\Omega)} \le 1.$$

Hence $\hat{N}^*_\infty(\tilde{T}) \le \|\chi_\omega \tilde{u}\|_{L^\infty(Q_{\tilde{T}})} \le \|\tilde{u}\|_{L^\infty(Q_{\tilde{T}})}$. This contradicts (5.21) and therefore, (5.19) follows.

Step 3. $\hat{N}^*_\infty(\cdot) : (0,T_0] \to [0,+\infty)$ *is left continuous.*

Let $\tilde{T} \in (0,T_0]$ be fixed. Due to Step 1, we deduce that $\lim_{T\uparrow\tilde{T}} \hat{N}^*_\infty(T)$ exists. We shall prove that for any sequence $T_n \uparrow \tilde{T}$,

$$\lim_{n\to\infty} \hat{N}^*_\infty(T_n) = \hat{N}^*_\infty(\tilde{T}). \tag{5.22}$$

Consider the following equation

$$\begin{cases} y_t - \Delta y = \chi_\omega u & \text{in } \ Q_{T_n}, \\ y = 0 & \text{on } \ \Sigma_{T_n}, \\ y(0,x) = \delta y_2(x) & \text{in } \ \Omega, \end{cases} \tag{5.23}$$

where $\delta = \delta(n) \in (0,1)$ will be determined later. Due to Theorem 3.1 in [10] there exists a control $u_n^\delta \in L^\infty(Q_{T_n})$, such that the solution to (5.23) with $u = u_n^\delta$, denoted by y_n^δ, satisfies

$$y_n^\delta(T_n,\cdot) = 0. \tag{5.24}$$

Moreover we have

$$\|u_n^\delta\|_{L^\infty(Q_{T_n})} \le \mathrm{e}^{c_1(1+T_n+T_n^{-1})} \cdot \delta\|y_2\|_{L^2(\Omega)} \le c_2\delta. \tag{5.25}$$

Here c_1 and c_2 are positive constants independent of n and δ. Define

$$y_\delta(t,x) = (1-\delta)y(t,x;y_2,\tilde{u}), \quad \forall\,(t,x) \in Q_{T_n}, \tag{5.26}$$

where $\tilde{u}$ is a solution of problem $(P_{nm\tilde{T}})$. Then it follows from (5.23)–(5.26) that

$$\begin{cases} (y_n^\delta + y_\delta)_t - \Delta(y_n^\delta + y_\delta) = \chi_\omega[u_n^\delta + (1-\delta)\tilde{u}] & \text{in } \ Q_{T_n}, \\ y_n^\delta + y_\delta = 0 & \text{on } \ \Sigma_{T_n}, \\ (y_n^\delta + y_\delta)(0,x) = y_2(x) & \text{in } \ \Omega, \end{cases} \tag{5.27}$$

$$\begin{aligned} \|(y_n^\delta + y_\delta)(T_n,\cdot)\|_{C_0(\Omega)} &= (1-\delta)\|y(T_n,\cdot;y_2,\tilde{u})\|_{C_0(\Omega)} \\ &\le (1-\delta)(\|y(T_n,\cdot;y_2,\tilde{u}) - y(\tilde{T},\cdot;y_2,\tilde{u})\|_{C_0(\Omega)} + 1) \end{aligned} \tag{5.28}$$

and

$$\|u_n^\delta + (1-\delta)\tilde{u}\|_{L^\infty(Q_{T_n})} \le c_2\delta + (1-\delta)\|\tilde{u}\|_{L^\infty(Q_{\tilde{T}})}. \tag{5.29}$$

Choosing

$$\delta = \frac{\|y(T_n,\cdot;y_2,\tilde{u}) - y(\tilde{T},\cdot;y_2,\tilde{u})\|_{C_0(\Omega)}}{1 + \|y(T_n,\cdot;y_2,\tilde{u}) - y(\tilde{T},\cdot;y_2,\tilde{u})\|_{C_0(\Omega)}},$$

we deduce from (5.28) and (5.29) that

$$\|(y_n^\delta + y_\delta)(T_n, \cdot)\|_{C_0(\Omega)} \le 1 \tag{5.30}$$

and

$$\overline{\lim}_{n\to\infty} \|u_n^\delta + (1-\delta)\tilde{u}\|_{L^\infty(Q_{T_n})} \le \|\tilde{u}\|_{L^\infty(Q_{\tilde{T}})}. \tag{5.31}$$

It follows from the result of Step 1, (5.27) and (5.30) that

$$\hat{N}_\infty^*(\tilde{T}) < \hat{N}_\infty^*(T_n) \le \|\chi_\omega[u_n^\delta + (1-\delta)\tilde{u}]\|_{L^\infty(Q_{T_n})} \le \|u_n^\delta + (1-\delta)\tilde{u}\|_{L^\infty(Q_{T_n})},$$

which, combined with (5.31) and the fact that $\|\tilde{u}\|_{L^\infty(Q_{\tilde{T}})} = \hat{N}_\infty^*(\tilde{T})$, implies (5.22).

Step 4. $\lim_{T\to 0^+} \hat{N}_\infty^*(T) = +\infty$.

Assume that this property does not hold. Then there would be a positive constant M_0 and a sequence $\{T_n\}_{n=1}^\infty$ with $T_n \downarrow 0$ such that

$$\hat{N}_\infty^*(T_n) \le M_0. \tag{5.32}$$

Let $u_{T_n}^*$ be a solution to problem (P_{nmT_n}). Then due to (5.32), we deduce

$$\|u_{T_n}^*\|_{L^\infty((0,+\infty)\times\Omega)} = \|u_{T_n}^*\|_{L^\infty(Q_{T_n})} = \hat{N}_\infty^*(T_n) \le M_0 \tag{5.33}$$

and

$$\|y(T_n, \cdot; y_2, u_{T_n}^*)\|_{C_0(\Omega)} \le 1. \tag{5.34}$$

Due to the similar arguments as those of Step 4 in Lemma 4.2, we obtain that $\|y_2(\cdot)\|_{C_0(\Omega)} \le 1$. Thus this contradicts with assumption $\|y_2(\cdot)\|_{C_0(\Omega)} > 1$. □

Now we turn to the proof of Theorem 5.5.

Proof. The "only if" part can be easily derived by Corollary 5.4.

Now we turn to the proof of the "if" part. It follows from (5.3) and (5.4) that $\mu_{\tilde{T}^*}^* \ne 0$ and $\varphi_{\mu_{\tilde{T}^*}^*}(t,x) \ne 0$ a.e. in $Q_{\tilde{T}^*}$. Then due to (5.2)–(5.3) and (i), (ii) in Lemma 5.3, we have

$$\tilde{u}^*(t,x) = M_2 \cdot \chi_\omega(x)\mathrm{sgn}(\varphi_{\mu_{\tilde{T}^*}^*}(t,x)) = \int_{Q_{\tilde{T}^*}^\omega} |\varphi_{\mu_{\tilde{T}^*}^*}|\,\mathrm{d}x\,\mathrm{d}t \cdot \chi_\omega(x)\mathrm{sgn}(\varphi_{\mu_{\tilde{T}^*}^*}(t,x)) \ \text{ a.e. in } \ Q_{\tilde{T}^*}$$

is a solution of the problem $(P_{nm\tilde{T}^*})$, which implies $M_2 = \hat{N}_\infty^*(\tilde{T}^*)$. This together with (iii) in Lemma 5.3 implies

$$\hat{N}_\infty^*(\tilde{T}^*) = \hat{N}_\infty^*(T^*) = M_2. \tag{5.35}$$

Then due to Lemma 5.6 and (5.35), we get that $\tilde{T}^* = T^* \in (0, T_0)$. Furthermore, it follows from (iv) in Lemma 5.3 that $\tilde{u}^*$ is the optimal control for (P_2). □

Remark 5.7.

(i) In the proof of Theorem 5.5, from the properties of $\hat{N}_\infty^*(\cdot)$ given in Lemma 5.6, we only use the fact that $\hat{N}_\infty^*(\cdot) : (0, T_0] \to [0, +\infty)$ is strictly decreasing.

(ii) The fact that $M_2 = \hat{N}_\infty^*(T^*) = \int_{Q_{T^*}^\omega} |\varphi_{\mu_{T^*}^*}|\,\mathrm{d}x\,\mathrm{d}t$, Lemma 5.6 and Theorem 5.5 suggest a method to numerically calculate the optimal time T^* and optimal control u_2^* for (P_2).

Appendix A

We quote from [1, 8, 13] the following results, which are used in the present paper.

Lemma A.1. *Let $\rho \in (0,1)$ be a fixed positive constant. Then for u_1, u_2 in $\mathcal{U}_1$, there exists a measurable subset $E_\rho \subset Q_T$ such that*

$$|E_\rho|_{\mathbb{R}^{N+1}} = \rho |Q_T|_{\mathbb{R}^{N+1}}.$$

Moreover, if we set

$$u_\rho(t,x) = \begin{cases} u_1(t,x), & (t,x) \in Q_T \setminus E_\rho, \\ u_2(t,x), & (t,x) \in E_\rho, \end{cases}$$

then

$$\left\| \frac{y_\rho - y_1}{\rho} - z \right\|_{C(\overline{Q}_T)} \to 0 \ \ as \ \ \rho \to 0,$$

where y_ρ and y_1 are the solutions to (1.1) corresponding to u_ρ and u_1 respectively, and z is the solution to the equation

$$\begin{cases} z_t = \Delta z + \chi_{Q^\omega_{\tau,T}}(u_2 - u_1) & in \ \ Q_T, \\ z = 0 & on \ \ \Sigma_T, \\ z(0,x) = 0 & in \ \ \Omega. \end{cases}$$

Lemma A.2. *Let $\mu \in (C_0(\Omega))^*$. Then the following equation*

$$\begin{cases} y_t - \Delta y = 0 & in \ \ Q_T, \\ y = 0 & on \ \ \Sigma_T, \\ y(0,x) = \mu & in \ \ \Omega \end{cases}$$

has a unique weak solution $y \in L^1(0,T; W_0^{1,1}(\Omega))$. Moreover, there exists a positive constant $\delta > 1$, such that $y \in L^\delta(0,T; W_0^{1,\delta}(\Omega))$ and

$$\|y\|_{L^\delta(0,T;W_0^{1,\delta}(\Omega))} \le C \|\mu\|_{(C_0(\Omega))^*},$$

where C is a positive constant independent of y.

Lemma A.3. *If $p \in C^\infty([0,T) \times \overline{\Omega})$ is a nonzero solution to*

$$\begin{cases} p_t + \Delta p = 0 & in \ \ Q_T, \\ p = 0 & on \ \ \Sigma_T, \end{cases}$$

then $p(t,x) \neq 0$ a.e. in Q_T.

Lemma A.4. *Let X be a Banach space and let S_1 be finite codimensional in X.*

(*i*) *Let $S_2 \subset X$. Then, for any $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, the set*

$$aS_1 - bS_2 \equiv \{as_1 - bs_2 | s_1 \in S_1, s_2 \in S_2\}$$

is finite codimensional in X.

(*ii*) *Let $\{f_n\}_{n\ge1} \subset X^*$ with*

$$\|f_n\|_{X^*} \ge \delta > 0, \ \ f_n \to f \text{ weakly star in } X^*,$$

and

$$\langle f_n, s\rangle_{X^*,X} \ge -\varepsilon_n, \quad \forall\, s \in S_1,\ n \ge 1,$$

where δ is a constant and $\varepsilon_n \to 0$. Then $f \neq 0$.

Lemmas A.1 and A.2 are special cases of Theorems 5.1 and 4.2 in [1], Lemma A.3 is a special case of Theorem 4.7.12 in [8] and Lemma A.4 is taken from [13] (see Prop. 3.4 and Lem. 3.6 of Chap. 4 in [13]).

References

[1] N. Arada and J.P. Raymond, Dirichlet boundary control of semilinear parabolic equations, Part 1: Problems with no state constraints. *Appl. Math. Optim.* **45** (2002) 125–143.

[2] N. Arada and J.P. Raymond, Time optimal problems with Dirichlet boundary controls. *Discrete Contin. Dyn. Syst.* **9** (2003) 1549–1570.

[3] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*. Academic Press, Boston (1993).

[4] V. Barbu, The time optimal control of Navier-Stokes equations. *Syst. Control Lett.* **30** (1997) 93–100.

[5] R.E. Bellman, I. Glicksberg and O.A. Gross, On the "bang-bang" control problem. *Q. Appl. Math.* **14** (1956) 11–18.

[6] C. Fabre, J.P. Puel and E. Zuazua, Approximate controllability of the semilinear heat equation. *Proc. R. Soc. Edinburgh* **125** (1995) 31–61.

[7] H.O. Fattorini, Time optimal control of solutions of operational differential equations. *SIAM J. Control* **2** (1964) 54–59.

[8] H.O. Fattorini, Infinite Dimensional Linear Control Systems: The Time Optimal and Norm Optimal Problems. *North-Holland Math. Stud.* **201** (2005).

[9] H.O. Fattorini, Sufficiency of the maximum principle for time optimality. *Cubo: A. Math. J.* **7** (2005) 27–37.

[10] E. Fernández-Cara and E. Zuazua, Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **17** (2000) 583–616.

[11] A.V. Fursikov, Optimal Control of Distributed Systems, *Theory and Applications*. American Mathematical Society, Providence (2000).

[12] K. Kunisch and L.J. Wang, Time optimal controls of the linear Fitzhugh–Nagumo equation with pointwise control constraints. *J. Math. Anal. Appl.* (2012), `doi: 10.1016/j.jmaa.2012.05.028`.

[13] X.J. Li and J.M. Yong, *Optimal Control Theory for Infinite Dimensional Systems*. Birkhäuser, Boston (1995).

[14] J.L. Lions, Remarques sur la contrôlabilité approchée, in *Jornadas Hispano-Francesas Sobre Control de Sistemas Distribuidos*. University of Málaga, Spain (1991) 77–87.

[15] J.L. Lions, Remarks on approximate controllability. *J. Anal. Math.* **59** (1992) 103–116.

[16] S. Micu and E. Zuazua, An introduction to the controllability of partial differential equations, in *Quelques questions de théorie du contròle*, edited by T. Sari. Collection Travaux en Cours Hermann (2004) 69–157.

[17] V.J. Mizel and T.I. Seidman, An abstract bang-bang principle and time optimal boundary control of the heat equation. *SIAM J. Control Optim.* **35** (1997) 1204–1216.

[18] J.P. Raymond and H. Zidani, Pontryagin's principle for time-optimal problems. *J. Optim. Theory Appl.* **101** (1999) 375–402.

[19] E.J.P.G. Schmidt, The "bang-bang" principle for the time-optimal problem in boundary control of the heat equation. *SIAM J. Control Optim.* **18** (1980) 101–107.

[20] G.S. Wang and L.J. Wang, The bang-bang principle of time optimal controls for the heat equation with internal controls. *Syst. Control Lett.* **56** (2007) 709–713.

[21] L.J.Wang and G.S. Wang, The optimal time control of a phase-field system. *SIAM J. Control Optim.* **42** (2003) 1483–1508.

[22] G.S. Wang and E. Zuazua, On the equivalence of minimal time and minimal norm controls for heat equations. *SIAM J. Control Optim.* **50** (2012) 2938–2958.

[23] Z.Q. Wu, J.X. Yin and C.P. Wang, *Elliptic and Parabolic Equations*. World Scientific Publishing Corporation, New Jersey (2006).

[24] E. Zuazua, Approximate controllability for semilinear heat equations with globally Lipschitz nonlinearities. *Control Cybern.* **28** (1999) 665–683.