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A Semi-smooth Newton Method for Regularized State-constrained Optimal Control of the Navier-Stokes Equations

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Abstract

In this paper, we study semi-smooth Newton methods for the numerical solution of regularized pointwise state-constrained optimal control problems governed by the Navier-Stokes equations. After deriving an appropriate optimality system for the original problem, a class of Moreau-Yosida regularized problems is introduced and the convergence of their solutions to the original optimal one is proved. For each regularized problem a semi-smooth Newton method is applied and its local superlinear convergence verified. Finally, selected numerical results illustrate the behavior of the method and a comparison between the *max-min* and the Fischer-Burmeister as complementarity functionals is carried out.

Mathematics Subject Classifications: 35Q35, 49J20, 65J15, 65K10.

Keywords: Optimal control, Navier-Stokes equations, state constraints, semi-smooth Newton methods.

1. Introduction

In this article, we investigate semi-smooth Newton methods for the numerical solution of the following state-constrained optimal control problem:

$$\begin{cases} \min\ J(y,u) = \frac{1}{2}\int\limits_{\Omega} |y - z_d|^2\, dx + \frac{\alpha}{2}\int\limits_{\tilde{\Omega}} |u|^2\, dx \\ \text{subject to} \\ -\nu\Delta y + (y\cdot\nabla)y + \nabla p = \mathcal{B}u & \text{in } \Omega \\ \operatorname{div} y = 0 & \text{in } \Omega \\ y \in C, \end{cases} \tag{$\mathcal{P}$}$$

where $\alpha > 0$ and C is the closed convex set defined by $C := \{v \in \mathbf{C}(\bar{\Omega}) : v|_\Gamma = g$ and $y_a(x) \le v(x) \le y_b(x)$, for all $x \in \bar{\Omega}_S\}$, with Ω_S a subdomain of Ω. The constraint set realizes pointwise constraints on each component of the velocity vector field, which is motivated by the necessity of diminishing recirculations by limiting the upward vertical velocity or the backward horizontal velocity in some sectors of the domain.

The application of semi-smooth Newton methods to state-constrained linear-quadratic optimal control problems was studied in [1], [3], [13]. In [1], due to the lack of regularity of the Lagrange multiplier associated to the state constraint, the authors apply a semi-smooth Newton method, or equivalently the primal-dual active set strategy, to a discretized version of the original problem. The same approach is adopted in [3], where the authors investigate the efficiency of the method compared to interior point algorithms. Since the discretized version of the problem hides the difficult structure of the multiplier associated to the state constraint, numerical difficulties, primarily on the boundary between active and inactive sets, are encountered. In [13], the authors introduce a family of regularized infinite-dimensional problems to overcome the lack of regularity of the Lagrange multiplier and ensure the applicability of semi-smooth Newton methods. Convergence of the regularized solutions and of the semi-smooth Newton method for each regularized system is proved.

In the context of optimal control of the Navier-Stokes equations, semi-smooth Newton methods were investigated, in presence of control constraints, in [9], [11], [20]. In the boundary control case, the phenomenon of lack of regularity of the multiplier is also present. The approach adopted in [11] consists also in the introduction of a class of regularized problems to cope with the difficulties related to the lack of regularity. Thereafter, the convergence of the regularized solutions and the semi-smooth Newton method is verified.

In this paper, we consider distributed optimal control of the Navier-Stokes equations in the presence of pointwise state constraints of box type. Utilizing the methodology of [11], [13], a family of regularized problems is introduced and the convergence of the regularized solutions towards the original one is proved. For each regularized problem a semi-smooth Newton algorithm is employed and its convergence verified. In the last part of the paper, detailed numerical examples are exhibit. The behavior of the *max-min* and the Fischer-Burmeister functionals in the context of semi-smooth Newton methods is numerically compared.

The outline of the paper is as follows. In Sect. 2, the optimal control problem is stated and the optimality system, which constitutes the starting point of our method, is obtained. In Sect. 3, a family of regularized problems is introduced and the convergence of the regularized solutions towards the original one is verified. Local superlinear convergence of a semi-smooth Newton method for each regularized problem is proved in Sect. 4. Finally, in Sect. 5, detailed numerical examples are given and the behavior of the *max-min* and Fischer-Burmeister complementarity functionals in the context of semi-smooth Newton methods is numerically compared.

2. Problem Statement and Optimality System

Let us firstly introduce some notation to be used. We consider an open bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ of class C^2. On this domain we consider the family of Sobolev spaces $\mathbf{H}^m(\Omega) := H^m(\Omega) \times H^m(\Omega)$. For these spaces a norm is introduced via $\|u\|_{\mathbf{H}^m} = \left(\sum_{[j]\leq m} \|D^j u\|_{\mathbf{L}^2}^2\right)^{1/2}$ and a scalar product is defined in the following way:

$$(u, v)_{\mathbf{H}^m} = \sum_{[j]\leq m} (D^j u, D^j v)_{\mathbf{L}^2}.$$

For the $\mathbf{L}^2$-inner product and norm no subindices are used. The closure of $\mathcal{D}(\Omega)$ in the $\mathbf{H}^m(\Omega)$ norm is denoted by $\mathbf{H}_0^m(\Omega)$ and it can be proved that if Ω is smooth enough, $\mathbf{H}_0^1(\Omega) = \{v \in \mathbf{H}^1(\Omega) : v|_{\partial\Omega} = 0\}$. For this space the Poincaré inequality holds, i.e.,

$$\|u\| \leq \kappa \, \|\nabla u\| \,, \qquad \text{for all } u \in \mathbf{H}_0^1(\Omega),$$

where κ is a constant dependent on Ω.

We also introduce the closed subspaces $V = \{v \in \mathbf{H}_0^1(\Omega) : \text{div } v = 0\}$ and $\mathbf{H}_0^{1/2} = \{v \in \mathbf{H}^{1/2}(\Gamma) : \int_\Gamma v \cdot \vec{n} \, d\Gamma = 0\}$ of $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$, respectively, which constitute themselves Hilbert spaces endowed with the induced scalar product. Additionally, we define a trilinear form $c : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ by $c(u, v, w) = ((u \cdot \nabla)v, w)$ and the associated constant $\mathcal{N} := \sup_{u,v,w \in V} \frac{|c(u,v,w)|}{\|u\|_V \|v\|_V \|w\|_V}$.

The space of continuous functions on $\bar{\Omega}$, vanishing at the boundary, is denoted by $\mathbf{C}_0(\Omega)$. It is well known that the dual space $(\mathbf{C}_0(\Omega))'$ can be associated with the space of regular Borel measures $\mathbf{M}(\Omega)$ endowed with the norm

$$\|\mu\|_{\mathbf{M}(\Omega)} = |\mu|(\Omega),$$

where $|\mu|(\Omega)$ is the total variation of μ (cf. [17, p. 40]). The duality product is then given by

$$\langle \mu, v \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} = \int_\Omega v \, d\mu.$$

The aim of this research is to find a solution $(y^*, u^*) \in \mathbf{C}(\bar{\Omega}) \times \mathbf{L}^2(\Omega)$ of the following optimal control problem:

$$\begin{cases} \min \; J(y,u) = \frac{1}{2} \int_\Omega |y - z_d|^2 \, dx + \frac{\alpha}{2} \int_{\tilde{\Omega}} |u|^2 \, dx \\ \text{subject to} \\ -\nu \Delta y + (y \cdot \nabla) y + \nabla p = \mathcal{B} u & \text{in } \Omega \\ \text{div } y = 0 & \text{in } \Omega \\ y \in C, \end{cases} \tag{$\mathcal{P}$}$$

where $C := \{v \in \mathbf{C}(\bar{\Omega}) : v|_\Gamma = g \text{ and } y_a(x) \leq v(x) \leq y_b(x), \text{ for all } x \in \bar{\Omega}_S\}$, with $y_a, \, y_b \in \mathbf{L}^\infty(\bar{\Omega}_S), \alpha > 0, \; \nu$ is the viscosity coefficient of the fluid, $g \in \mathbf{H}_0^{1/2} \cap \mathbf{H}^{3/2}(\Gamma)$ and $\mathcal{B} \in \mathcal{L}(\mathbf{L}^2(\tilde{\Omega}), \mathbf{L}^2(\Omega))$ stands for the extension by 0 operator, where $\tilde{\Omega}$ is a subdomain of Ω. Throughout we assume the existence of at least one feasible pair (y, u) to $(\mathcal{P})$. Under this assumption, existence of an optimal solution can be verified in a standard manner (see [10, theorem 3.2]).

Remark 2.1: *If $\tilde{\Omega} = \Omega$ and there exists at least one $y \in C \cap \mathbf{H}^2(\Omega)$ with div $y = 0$, then there exists a feasible pair for $(\mathcal{P})$.*

Since there exists a function $\hat{y} \in \mathbf{H}^2(\Omega)$ such that $\hat{y}|_\Gamma = g$ and div $\hat{y} = 0$ (cf. [18, p. 117]), the state equations can be rewritten as

$$\begin{cases} -\nu\Delta w + (w \cdot \nabla)\hat{y} + (\hat{y} \cdot \nabla)w + (w \cdot \nabla)w + \nabla p = F + \mathcal{B}u & \text{in } \Omega \\ \text{div } w = 0 & \text{in } \Omega\,, \\ w = 0 & \text{on } \Gamma \end{cases} \tag{1}$$

where $w := y - \hat{y}$ and $F := \nu\Delta\hat{y} - (\hat{y} \cdot \nabla)\hat{y}$. Using again the fact that $\hat{y} \in \mathbf{H}^2(\Omega)$, it can be verified (see [10]) that w also belongs to $\mathcal{W} := (\mathbf{H}^2(\Omega) \cap V)$, which is embedded in $\mathbf{C}_0(\Omega)$.

Let us now introduce formally the control-to-state operator

$$\begin{aligned} \varphi : \mathbf{L}^2(\tilde{\Omega}) &\to \mathcal{W} \times (L^2_0(\Omega) \cap H^1(\Omega)) \\ u &\mapsto (G(u), H(u)) = (w(u), p(u)), \end{aligned}$$

where (w, p, u) satisfy (1). The derivative of φ at u^* in direction v, denoted by $(w', p') := (G'(u^*)v, H'(u^*)v)$, is given by the unique solution of the system

$$\begin{cases} \begin{aligned} &-\nu\Delta w' + (w' \cdot \nabla)\hat{y} + (\hat{y} \cdot \nabla)w' + (w' \cdot \nabla)w \\ &\quad +(w \cdot \nabla)w' + \nabla p' = \mathcal{B}v \end{aligned} & \text{in } \Omega \\ \text{div } w' = 0 & \text{in } \Omega\,, \\ w' = 0 & \text{on } \Gamma \end{cases} \tag{2}$$

which is equivalent, using the definition of w, to the system

$$\begin{cases} -\nu\Delta y' + (y' \cdot \nabla)y + (y \cdot \nabla)y' + \nabla p' = \mathcal{B}v & \text{in } \Omega \\ \text{div } y' = 0 & \text{in } \Omega\,. \\ y' = 0 & \text{on } \Gamma \end{cases} \tag{3}$$

Problem $(\mathcal{P})$ may therefore be rewritten in reduced form as

$$\begin{cases} \min\ J(u) = \frac{1}{2} \int\limits_\Omega |G(u) + \hat{y} - z_d|^2\, dx + \frac{\alpha}{2} \int\limits_{\tilde{\Omega}} |u|^2\, dx \\ \text{subject to:} \quad G(u) \in \hat{C}, \end{cases} \tag{4}$$

where $\hat{C}$ is the closed convex set given by

$$\hat{C} := \{v \in \mathbf{C}_0(\Omega) : w_a(x) \leq v(x) \leq w_b(x), \text{ for all } x \in \bar{\Omega}_S\},$$

with $w_a(x) := y_a(x) - \hat{y}(x)$ and $w_b(x) := y_b(x) - \hat{y}(x)$.

In the following theorem existence of Lagrange multipliers for the optimal control problem is proved and an appropriate optimality system is obtained. The result relies on a Slater type condition which is stated next and hereafter assumed.

Assumption 2.2: Let $(y^*, u^*) \in C \cap \mathbf{H}^2(\Omega) \times \mathbf{L}^2(\tilde{\Omega})$ be an optimal solution for $(\mathcal{P})$. There exists a pair $(\bar{y}, \bar{u}) \in \mathcal{W} \times \mathbf{L}^2(\tilde{\Omega})$ solution to

$$\begin{aligned} -\nu\Delta\bar{y} + (\bar{y}\cdot\nabla)y^* + (y^*\cdot\nabla)\bar{y} + \nabla\bar{p} &= \mathcal{B}(\bar{u} - u^*) && \text{in } \Omega \\ \operatorname{div}\bar{y} &= 0 && \text{in } \Omega \\ \bar{y} &= 0 && \text{on } \Gamma \end{aligned} \tag{5}$$

such that $y^* + \bar{y} \in \operatorname{int} C$.

Theorem 2.3: *Let* $(y^*, u^*) \in C \cap \mathbf{H}^2(\Omega) \times \mathbf{L}^2(\tilde{\Omega})$ *be an optimal solution for* $(\mathcal{P})$ *with* $\nu > \mathcal{M}(y^*)$, *where* $\mathcal{M}(y) := \sup_{v\in V} \frac{|c(v,y,v)|}{\|v\|_V^2}$. *Then there exist multipliers* $\lambda \in H \cap \mathbf{W}_0^{1,s}(\Omega)$ *with* $s \in [1, 2[$ *and* $\mu \in \mathbf{M}(\Omega)$ *such that the optimal solution of* $(\mathcal{P})$ *is characterized by the following optimality system:*

$$\begin{aligned} -\nu\Delta y^* + (y^*\cdot\nabla)y^* + \nabla p &= \mathcal{B}u^* && \text{in } \Omega \\ div\, y^* &= 0 && \text{in } \Omega \end{aligned} \tag{6}$$

$$\begin{aligned} &-\nu\int_\Omega \lambda\, \Delta w\, dx + \int_\Omega (y^*\cdot\nabla)w\, \lambda\, dx + \int_\Omega (w\cdot\nabla)y^*\, \lambda\, dx \\ &= \int_\Omega (z_d - y^*)w\, dx - \langle \mu, w\rangle_{\mathbf{M}(\Omega),\mathbf{C}_0(\Omega)}, \quad \text{for all } w \in \mathcal{W}, \end{aligned} \tag{7}$$

$$\alpha u^* = \mathcal{B}^\star \lambda, \tag{8}$$

$$y^* \in C, \tag{9}$$

$$\langle \mu, \bar{y} - y^*\rangle_{\mathbf{M}(\Omega),\mathbf{C}_0(\Omega)} \le 0, \quad \text{for all } \bar{y} \in C, \tag{10}$$

where $\mathcal{B}^\star$ *stands for the adjoint operator of* $\mathcal{B}$.

Proof: We begin by studying the properties of the control-to-state mapping. Let us consider the operator $\psi : \mathcal{W} \times (L_0^2(\Omega) \cap H^1(\Omega)) \times \mathbf{L}^2(\tilde{\Omega}) \to \mathbf{L}^2(\Omega)$ defined by

$$\psi(w, p, u) = -\nu\Delta w + (w\cdot\nabla)\hat{y} + (\hat{y}\cdot\nabla)w + (w\cdot\nabla)w + \nabla p - F - \mathcal{B}u.$$

Since (y^*, u^*) is an optimal solution for $(\mathcal{P})$, the triple (w^*, p^*, u^*), with $w^* := y^* - \hat{y}$, satisfies the state equation $\psi(w^*, p^*, u^*) = 0$. It can be verified that ψ is of class C^∞ (see [7, pp. 5–6]). Its partial derivative with respect to (w, p) at (w^*, p^*) in direction (δ_w, δ_p) is given by

$$\begin{aligned} \psi_{(w,p)}(w^*, p^*, u^*)(\delta_w, \delta_p) = &-\nu\Delta\delta_w + (\delta_w\cdot\nabla)\hat{y} + (\hat{y}\cdot\nabla)\delta_w + (\delta_w\cdot\nabla)w^* \\ &+ (w^*\cdot\nabla)\delta_w + \nabla\delta_p. \end{aligned}$$

Since $\nu > \mathcal{M}(y^*)$, the operator $\psi_{(w,p)}(w^*, p^*, u^*)$ is invertible. Utilizing the implicit function theorem, there exists an open neighborhood U of u^* and a control-to-state operator

$$\varphi : U \to \mathcal{W} \times (L_0^2(\Omega) \cap H^1(\Omega))$$
$$u \mapsto (G(u), H(u)) = (w(u), p(u))$$

of class C^∞. The control-to-state mapping is therefore well defined.

Utilizing the general Lagrange multipliers existence theorem stated in [6, p. 1001], with $K = U$, we can assure existence of a real number $\theta \geq 0$ and a measure $\mu \in \mathbf{M}(\Omega)$ such that

$$(\theta J'(u^*) + G'(u^*)^\star \mu, u - u^*) \geq 0, \quad \text{for all } u \in K \tag{11}$$

$$\langle \mu, \bar{w} - w^* \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} \leq 0, \quad \text{for all } \bar{w} \in \hat{C}. \tag{12}$$

In our particular case, since U is open and Assumption 2.2 holds, we may choose $\theta = 1$ and rewrite Eq. (11) as

$$J'(u^*) + G'(u^*)^\star \mu = 0 \quad \text{in } \mathbf{L}^2(\tilde{\Omega}), \tag{13}$$

where $G'(u^*)^\star$ denotes the adjoint operator of $G'(u^*)$. Also, using the form of $\hat{C}$, inequality (12) can be expressed as

$$\langle \mu, \bar{y} - y^* \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} \leq 0, \quad \text{for all } \bar{y} \in C. \tag{14}$$

The derivative of the cost functional in direction $v \in \mathbf{L}^2(\tilde{\Omega})$ is given by

$$(J'(u^*), v) = (y^* - z_d, y') + \alpha(u^*, v),$$

where $y' \in \mathcal{W}$ is the unique solution to the system (3). Therefore, we get that

$$\begin{aligned}(J'(u^*) + G'(u^*)^\star \mu, v) &= (y^* - z_d, y') + (\alpha u^*, v) + \langle \mu, G'(u^*)v \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} \\ &= \langle y^* - z_d + \mu, y' \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} + (\alpha u^*, v),\end{aligned}$$

which, by defining the adjoint state $\lambda \in H \cap \mathbf{W}_0^{1,s}(\Omega)$ as the unique solution (cf. [10, p. 11]) of

$$\begin{aligned}&-\nu \int_\Omega \lambda \, \Delta w \, dx + \int_\Omega (y \cdot \nabla) w \, \lambda \, dx + \int_\Omega (w \cdot \nabla) y \, \lambda \, dx \\ &\quad = \int_\Omega (z_d - y^*) w \, dx - \langle \mu, w \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)}, \quad \text{for all } w \in \mathcal{W},\end{aligned}$$

yields

$$(J'(u^*) + G'(u^*)^\star \mu, v) = \nu(\lambda, \Delta y') - c(y^*, y', \lambda) - c(y', y^*, \lambda) + (\alpha u^*, v).$$

Finally, taking the inner product of (3) with λ and using (13), we obtain that

$$\alpha u^* = \mathcal{B}^\star \lambda. \qquad \square$$

Let us hereafter consider $\Omega_S = \Omega$. Defining the active sets by

$$\mathcal{A}^a = \{x \in \Omega : y^*(x) = y_a(x)\} \quad \text{and} \quad \mathcal{A}^b = \{x \in \Omega : y^*(x) = y_b(x)\},$$

the inactive set by

$$\mathcal{I} = \Omega \backslash (\mathcal{A}^a \cup \mathcal{A}^b)$$

and assuming extra regularity of the state constraint multiplier, for example $\mu \in \mathbf{L}^2(\Omega)$, Eqs. (9)–(10) would be equivalent to the complementarity system:

$$\begin{cases} y_a(x) \leq y(x) \leq y_b(x) \\ \mu|_{\mathcal{A}^b} \geq 0 \\ \mu|_{\mathcal{A}^a} \leq 0 \\ \mu|_{\mathcal{I}} = 0, \end{cases} \tag{15}$$

which can also be written, utilizing the *max* and *min* functions, as the following operator equation:

$$\mu = \max(0, \mu + y - y_b) + \min(0, \mu + y - y_a). \tag{16}$$

3. Regularized Problems

In general, the reformulation (16) of Eqs. (9)–(10) is not possible due to the lack of regularity of the multiplier μ. In this section, following [13], we introduce a family of Moreau-Yosida regularized problems which approximate the original one and allows us to overcome the difficulties resulting from the fact that the Lagrange multiplier associated to the inequality constraints is a measure.

We consider the following family of penalized optimal control problems:

$$\begin{cases} \min\ J_\gamma(y,u) = J(y,u) + \frac{1}{2\gamma} \int\limits_{\mathcal{A}_\gamma^b} |\bar{\mu} + \gamma(y - y_b)|^2\, dx \\ \qquad\qquad + \frac{1}{2\gamma} \int\limits_{\mathcal{A}_\gamma^a} |\bar{\mu} + \gamma(y - y_a)|^2\, dx \\ \text{subject to} \\ -\nu\Delta y + (y \cdot \nabla) y + \nabla p = \mathcal{B}u \quad \text{in } \Omega \\ \operatorname{div} y = 0 \quad \text{in } \Omega \\ y = g \quad \text{on } \Gamma, \end{cases} \tag{$\mathcal{P}_\gamma$}$$

where $\gamma > 0$ is the regularization parameter, $\bar{\mu} \in \mathbf{L}^2(\Omega)$ and the regularized active and inactive sets are defined by

$$\mathcal{A}_\gamma^a = \{x \in \Omega : \bar{\mu} + \gamma(y_\gamma - y_a) \leq 0 \text{ a.e.}\}, \quad \mathcal{A}_\gamma^b = \{x \in \Omega : \bar{\mu} + \gamma(y_\gamma - y_b) \geq 0 \text{ a.e.}\}$$

and

$$\mathcal{I}_\gamma = \Omega \backslash (\mathcal{A}_\gamma^a \cup \mathcal{A}_\gamma^b).$$

The functional $J_\gamma(y, u)$ can equivalently be written as

$$J_\gamma(y, u) = J(y, u) + \frac{1}{2\gamma} \|\max(0, \bar{\mu} + \gamma(y - y_b))\|^2 + \frac{1}{2\gamma} \left\|\min(0, \bar{\mu} + \gamma(y - y_a))\right\|^2.$$

Special choices of $\bar{\mu} \in \mathbf{L}^2(\Omega)$ are of particular interest in the context of augmented Lagrangian methods (cf. [13, p. 13]). In [13], the authors consider a linear state-constrained optimal control problem and compare an augmented Lagrangian update of $\bar{\mu}$ and the case $\bar{\mu} \equiv 0$. The second approach, combined with a continuation strategy with respect to γ, turned out to be numerically more efficient. In the sequel we concentrate on the case $\bar{\mu} \equiv 0$.

Existence of an optimal solution for $(\mathcal{P}_\gamma)$ can be argued as for the unconstrained or control constrained cases (cf. [9, p. 663]). In the following theorem, convergence of the regularized solutions, as $\gamma \to \infty$, is studied.

Theorem 3.1: *Let $\nu > \mathcal{M}(y^*)$ hold for all solutions of $(\mathcal{P})$. The sequence $\{(y_\gamma, p_\gamma, u_\gamma)\}_{\gamma>0}$ of solutions to $(\mathcal{P}_\gamma)$ contains a subsequence, which converges strongly in $\mathcal{W} \times H^1(\Omega) \times \mathbf{L}^2(\Omega)$ to an optimal solution (y^*, p^*, u^*).*

Proof: Let $(y^*, u^*) \in \mathcal{W} \times U$ be a solution to $(\mathcal{P})$. From the properties of the regularized cost functional we know that

$$J_\gamma(y_\gamma, u_\gamma) \leq J_\gamma(y^*, u^*) = J(y^*, u^*). \tag{17}$$

Consequently, since $\alpha > 0$, the sequence $\{u_\gamma\}_{\gamma>0}$ is uniformly bounded in $\mathbf{L}^2(\tilde{\Omega})$, which implies that $\{y_\gamma\}_{\gamma>0}$ is uniformly bounded in $\mathcal{W}$. Therefore, there exists a subsequence $(y_\gamma, u_\gamma) \subset \mathcal{W} \times \mathbf{L}^2(\tilde{\Omega})$ such that $y_\gamma \rightharpoonup \hat{y}$ in $\mathcal{W}$ and $u_\gamma \rightharpoonup \hat{u}$ in $\mathbf{L}^2(\tilde{\Omega})$.

Additionally, from Eq. (17) the following terms:

$$\frac{1}{\gamma} \left\|\max(0, \gamma(y_\gamma - y_b))\right\|^2 \quad \text{and} \quad \frac{1}{\gamma} \left\|\min(0, \gamma(y_\gamma - y_a))\right\|^2 \tag{18}$$

are uniformly bounded with respect to γ. Hence,

$$\lim_{\gamma\to\infty} \left\|\max(0, y_\gamma - y_b)\right\| = 0 \quad \text{and} \quad \lim_{\gamma\to\infty} \left\|\min(0, y_\gamma - y_a)\right\| = 0.$$

Applying Fatou's Lemma to the previous terms we get that $\hat{y} \leq y_b$, $y_a \leq \hat{y}$ and, consequently, $\hat{y} \in C$. Considering additionally that

$$J(\hat{y}, \hat{u}) \leq \liminf J(y_\gamma, u_\gamma) \leq \liminf J_\gamma(y_\gamma, u_\gamma) \leq J(y^*, u^*), \tag{19}$$

we get that $(\hat{y}, \hat{u})$ is solution of $(\mathcal{P})$. Subsequently, we denote the pair $(\hat{y}, \hat{u})$ by (y^*, u^*).

To verify strong convergence, let us first note that due to (17) and (19)

$$\lim_{\gamma\to\infty} \left\|y_\gamma - z_d\right\|^2 + \alpha \left\|u_\gamma\right\|^2 = \left\|y^* - z_d\right\|^2 + \alpha \left\|u^*\right\|^2$$

and, hence, $u_\gamma \to u^*$ strongly in $\mathbf{L}^2(\tilde{\Omega})$. From the state equations it can be verified that the difference $y_\gamma - y^*$ satisfies the equation

$$\nu(\nabla(y_\gamma - y^*), \nabla v) + c(y_\gamma, y_\gamma, v) - c(y^*, y^*, v) = (u_\gamma - u^*, v), \qquad \text{for all } v \in V, \tag{20}$$

which, considering that

$$\begin{aligned} & c(y_\gamma, y_\gamma, y_\gamma - y^*) - c(y^*, y^*, y_\gamma - y^*) \\ &\quad = c(y_\gamma, y_\gamma, y_\gamma - y^*) + c(y_\gamma - y^*, y^*, y_\gamma - y^*) - c(y_\gamma, y^*, y_\gamma - y^*) \\ &\quad = c(y_\gamma, y_\gamma - y^*, y_\gamma - y^*) + c(y_\gamma - y^*, y^*, y_\gamma - y^*) \\ &\quad \geq -|c(y_\gamma - y^*, y_\gamma - y^*, y^*)| \geq -\mathcal{M}(y^*) \left\| y_\gamma - y^* \right\|_V^2 \end{aligned}$$

yields the following estimate:

$$(\nu - \mathcal{M}(y^*)) \left\| y_\gamma - y^* \right\|_{\mathbf{V}} \leq \left\| u_\gamma - u^* \right\|. \tag{21}$$

Since the nonlinear term is twice Frechét differentiable, it also follows that

$$\left\| (y_\gamma \cdot \nabla) y_\gamma - (y^* \cdot \nabla) y^* \right\| \leq \bar{C} \left\| y_\gamma - y^* \right\|_V. \tag{22}$$

Utilizing (21)–(22) and applying Stokes extra regularity results (cf. [18, p. 25]) to the difference Eqs. (20), we thus obtain

$$\left\| y_\gamma - y^* \right\|_{\mathcal{W}} + \left\| p_\gamma - p^* \right\|_{H^1} \leq C \left\| u_\gamma - u^* \right\| \tag{23}$$

and, consequently, $y_\gamma \to y^*$ strongly in $\mathcal{W}$ and $p_\gamma \to p^*$ strongly in $H^1(\Omega)$ □

From the definition of $\mathcal{M}(\cdot)$ it follows that

$$\begin{aligned} \mathcal{M}(y_\gamma) &= \sup_{v \in V} \frac{|c(v, y_\gamma, v)|}{\|v\|_V^2} \leq \sup_{v \in V} \frac{|c(v, y_\gamma - y^*, v)| + |c(v, y^*, v)|}{\|v\|_V^2} \\ &\leq \mathcal{N} \left\| y_\gamma - y^* \right\|_V + \mathcal{M}(y^*). \end{aligned}$$

Since by Theorem 3.1 $y_\gamma \to y^*$ strongly in $\mathcal{W}$, there exists a sufficiently large $\bar{\gamma}$ such that $\nu > \mathcal{M}(y_\gamma)$, for all $\gamma > \bar{\gamma}$. Introducing the Lagrangian for $(\mathcal{P}_\gamma)$

$$\mathcal{L}_\gamma(y, u, \lambda) = J_\gamma(y, u) + \nu(\nabla y, \nabla \lambda) + c(y, y, \lambda) - (\mathcal{B}u, \lambda),$$

existence of Lagrange multipliers for $\gamma > \bar{\gamma}$ is justified and the solution satisfies the following optimality system in variational sense (cf. [9, 11]):

$$\begin{aligned} -\nu \Delta y_\gamma + (y_\gamma \cdot \nabla) y_\gamma + \nabla p_\gamma &= \mathcal{B}u_\gamma && \text{in } \Omega \\ \operatorname{div} y_\gamma &= 0 && \text{in } \Omega. \\ y_\gamma &= g && \text{on } \Gamma \end{aligned} \tag{24}$$

$$\begin{aligned} -\nu\Delta\lambda_\gamma - (y_\gamma\cdot\nabla)\lambda_\gamma + (\nabla y_\gamma)^T\lambda_\gamma + \nabla q_\gamma &= z_d - y_\gamma - \mu_\gamma && \text{in } \Omega \\ \operatorname{div}\lambda_\gamma &= 0 && \text{in } \Omega\,, \\ \lambda_\gamma &= 0 && \text{on } \Gamma \end{aligned} \tag{25}$$

$$\alpha u_\gamma = \mathcal{B}^\star\lambda_\gamma, \tag{26}$$

$$\mu_\gamma = \max(0, \gamma(y_\gamma - y_b)) + \min(0, \gamma(y_\gamma - y_a)). \tag{27}$$

Next, convergence of the regularized variables $(\lambda_\gamma, \mu_\gamma)$ is verified.

Theorem 3.2: *Let $\nu > \mathcal{M}(y^*)$ hold for all solutions of $(\mathcal{P})$. The sequence $\{(\lambda_\gamma, \mu_\gamma)\}_{\gamma>\check{\gamma}}$, with $\check{\gamma} := \max(1, \bar{\gamma})$, of multipliers associated with $(\mathcal{P}_\gamma)$ contains a subsequence which converges to a pair $(\hat{\lambda}, \hat{\mu})$ in the sense that $\lambda_\gamma \rightharpoonup \hat{\lambda}$ weakly in $\mathbf{L}^2(\Omega)$ and $\mu_\gamma \rightharpoonup^* \hat{\mu}$ weakly* in $\mathbf{M}(\Omega)$. The pair $(\hat{\lambda}, \hat{\mu})$ solves, together with (y^*, p^*, u^*), the optimality system* (6)–(10). *Moreover, $\lambda_\gamma|_{\tilde{\Omega}} \to \hat{\lambda}|_{\tilde{\Omega}}$ strongly in $\mathbf{L}^2(\tilde{\Omega})$.*

Proof: Since for $\gamma \geq 1$,

$$\begin{aligned} J_\gamma(y_\gamma, u_\gamma) &\leq J(y_\gamma, u_\gamma) + \frac{1}{2}\|\max(0, \gamma(y - y_b))\|^2 + \frac{1}{2}\left\|\min(0, \gamma(y - y_a))\right\|^2 \\ &\leq J(y^*, u^*), \end{aligned}$$

the terms

$$\left\|\max(0, \gamma(y_\gamma - y_b))\right\|^2 \quad \text{and} \quad \left\|\min(0, \gamma(y_\gamma - y_a))\right\|^2$$

are uniformly bounded with respect to γ. From (27), the sequence $\{\mu_\gamma\}_{\gamma>\check{\gamma}}$ is uniformly bounded in $\mathbf{L}^2(\Omega)$ and therefore uniformly bounded in $\mathbf{M}(\Omega)$. Taking into account the bijectivity of the adjoint operator (see [10, theorem 4.4]), it follows from (25) that $\{\lambda_\gamma\}_{\gamma>\check{\gamma}}$ is also uniformly bounded in $\mathbf{L}^2(\Omega)$. Therefore, there exists a subsequence, also denoted by $(\lambda_\gamma, \mu_\gamma)$, such that $\lambda_\gamma \rightharpoonup \hat{\lambda}$ weakly in $\mathbf{L}^2(\Omega)$ and $\mu_\gamma \rightharpoonup^* \hat{\mu}$ weakly* in $\mathbf{M}(\Omega)$.

To verify that $\hat{\mu}$ satisfies (10), let us first consider the set $\mathcal{A}_\gamma^a$ with $\bar{\mu} = 0$. From (27), it follows that $\mu_\gamma \leq 0$ a.e. in $\mathcal{A}_\gamma^a$. For $\bar{y} \in C$, then $y_\gamma - \bar{y} \leq 0$ in $\mathcal{A}_\gamma^a$ and $(\mu_\gamma, \bar{y} - y_\gamma)_{\mathcal{A}_\gamma^a} \leq 0$. On the set $\mathcal{A}_\gamma^b$, we obtain that $\mu_\gamma \geq 0$ a.e. and $y_\gamma - \bar{y} \geq 0$, for all $\bar{y} \in C$. Therefore $(\mu_\gamma, \bar{y} - y_\gamma)_{\mathcal{A}_\gamma^b} \leq 0$. Finally, on $\mathcal{I}_\gamma$ we obtain that $\mu_\gamma = 0$ a.e. and therefore $(\mu_\gamma, \bar{y} - y_\gamma)_{\mathcal{I}_\gamma} = 0$, for all $\bar{y} \in C$. Consequently, considering all three cases and since $\Omega = \mathcal{A}_\gamma^b \cup \mathcal{A}_\gamma^b \cup \mathcal{I}_\gamma$, we get that

$$\langle \mu_\gamma, \bar{y} - y_\gamma \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} \leq 0, \qquad \text{for all } \bar{y} \in C. \tag{28}$$

Passing to the limit in (28) yields (10).

Considering (25) in very weak form and passing to the limit, we obtain that $(\hat{\lambda}, \hat{\mu})$ satisfies Eq. (7). Finally, passing to the limit in (26) yields that $\lambda_\gamma|_{\tilde{\Omega}} \to \hat{\lambda}|_{\tilde{\Omega}}$ strongly in $\mathbf{L}^2(\tilde{\Omega})$. □

4. Semi-smooth Newton Method

In this section, we introduce a semi-smooth Newton algorithm for the numerical solution of each regularized problem $(\mathcal{P}_\gamma)$. The well posedness of the algorithm and sufficient conditions for local superlinear convergence of the method are investigated.

4.1. Formulation using max *and* min *functions*

The algorithm is motivated by the regularized version, expressed by Eq. (27), of the original complementarity system. The formulation in terms of the Newton differentiable *max* and *min* functions allows the application of a semi-smooth Newton method to the optimality system of $(\mathcal{P}_\gamma)$. Expressed as an active set strategy, the algorithm determines in each iteration the following active and inactive sets by $\mathcal{A}_n^b = \{x : \gamma(y_{n-1} - y_b) \geq 0\}$, $\mathcal{I}_n = \{x : \gamma(y_{n-1} - y_b) < 0 < \gamma(y_{n-1} - y_a)\}$ and $\mathcal{A}_n^a = \{x : \gamma(y_{n-1} - y_a) \leq 0\}$, and then solves the optimal control problem on the inactive set. The complete algorithm can be stated as follows:

Algorithm 4.1: (1) *Initialization: choose* $(u_0, \ y_0, \ \lambda_0) \in \mathbf{L}^2(\Omega) \times \mathcal{W} \times \mathbf{L}^2(\Omega)$ *and set* $n = 1$.

(2) *Until a stopping criteria is satisfied, set*

$$\mathcal{A}_n^b = \{x : \gamma(y_{n-1} - y_b) \geq 0\} \qquad \mathcal{A}_n^a = \{x : \gamma(y_{n-1} - y_a) \leq 0\}$$

and

$$\mathcal{I}_n = \{x : \gamma(y_{n-1} - y_b) < 0 < \gamma(y_{n-1} - y_a)\}.$$

Find the solution $(y_n, p_n, u_n, \lambda_n, \phi_n, \mu_n)$ *of:*

$$\begin{aligned} -\nu\Delta y_n + (y_{n-1}\cdot\nabla)y_n + (y_n\cdot\nabla)y_{n-1} + \nabla p_n &= \mathcal{B}u_n + (y_{n-1}\cdot\nabla)y_{n-1} \\ div\ y_n &= 0 \\ y_n|_\Gamma &= g \end{aligned} \tag{29}$$

$$\begin{aligned} -\nu\Delta\lambda_n - (y_n\cdot\nabla)\lambda_{n-1} - (y_{n-1}\cdot\nabla)\lambda_n + (\nabla y_{n-1})^T\lambda_n + (\nabla y_n)^T\lambda_{n-1} & \\ +\nabla\phi_n = z_d - y_n - \mu_n - (y_{n-1}\cdot\nabla)\lambda_{n-1} + (\nabla y_{n-1})^T\lambda_{n-1} & \\ div\ \lambda_n = 0 & \\ \lambda_n|_\Gamma = 0. & \end{aligned} \tag{30}$$

$$\alpha u_n = \mathcal{B}^\star\lambda_n \tag{31}$$

$$\mu_n = \begin{cases} \gamma(y_n - y_b) & \text{in } \mathcal{A}_n^b \\ 0 & \text{in } \mathcal{I}_n \\ \gamma(y_n - y_a) & \text{in } \mathcal{A}_n^a \end{cases} \tag{32}$$

and set $n = n + 1$.

Let us note that the system to be solved in step (2) results from linearization of system (24)–(27) and corresponds to the optimality system of the following linear quadratic optimal control problem:

$$\begin{cases} \min\limits_{\delta_x \in \mathbf{H}^1(\Omega)\times \mathbf{L}^2(\Omega)} \frac{1}{2}\langle \mathcal{L}_\gamma''(x_{n-1},\lambda_{n-1},\xi_{n-1})\delta_x,\delta_x\rangle + \langle \mathcal{L}_\gamma'(x_{n-1},\lambda_{n-1},\xi_{n-1}),\delta_x\rangle \\ \text{subject to} \\ -\nu\Delta\delta_y + (\delta_y\cdot\nabla)y_{n-1} + (y_{n-1}\cdot\nabla)\delta_y + \nabla\delta_p \\ \quad = \mathcal{B}\delta_u + \nu\Delta y_{n-1} - (y_{n-1}\cdot\nabla)y_{n-1} - \nabla p_{n-1} \\ \operatorname{div}\delta_y = 0 \\ \delta_y|_\Gamma = 0, \end{cases} \tag{33}$$

where $x_n = (y_n, u_n)$ and $\delta_x = x_n - x_{n-1}$. Taking $\|y_{n-1} - y_\gamma\|_V$ sufficiently small such that $\nu - \mathcal{M}(y_{n-1}) > \frac{1}{2}(\nu - \mathcal{M}(y_\gamma)) > 0$, existence of Lagrange multipliers $(\lambda_n, \mu_n, \phi_n)$ can be verified. Moreover, system (29)–(32) has a unique solution, equivalent to the solution of (33), if a second-order condition of the type

$$\langle \mathcal{L}_\gamma''(y_{n-1},u_{n-1},\lambda_{n-1})(w,h),(w,h)\rangle \geq \|(w,h)\|^2_{\mathbf{H}^1(\Omega)\times\mathbf{L}^2(\Omega)}, \tag{34}$$

is satisfied for all (w,h) in the *ker* of the linear state equation in (33). This is accomplished if $\|\lambda_{n-1} - \lambda_\gamma\|_V$ is sufficiently small and a second-order sufficient condition for the regularized optimal pair (y_γ, u_γ) holds (cf. [11, p. 19]).

From the quadratic properties of the trilinear from we get, introducing the operators

$$\mathcal{H}: \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega) \to (\mathbf{H}^1(\Omega))' \qquad \tilde{\mathcal{H}}: \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\to(\mathbf{H}^1(\Omega))'$$
$$(v,w)\to(v\cdot\nabla)w \qquad\qquad (v,w)\to(\nabla v)^T w$$

that

$$\begin{aligned} E_1 &:= ((y_n - y_\gamma)\cdot\nabla)(y_n - y_\gamma) = \mathcal{H}(y_n) - \mathcal{H}(y_\gamma) - \mathcal{H}'(y_\gamma)(y_n - y_\gamma) \\ &= \frac{1}{2}\mathcal{H}''(y_\gamma)(y_n - y_\gamma)(y_n - y_\gamma), \\ E_2 &:= ((y_n - y_\gamma)\cdot\nabla)(\lambda_n - \lambda_\gamma) \\ &= \mathcal{H}(y_n,\lambda_n) - \mathcal{H}(y_\gamma,\lambda_\gamma) - \mathcal{H}'(y_\gamma,\lambda_\gamma)(y_n - y_\gamma, \lambda_n - \lambda_\gamma) \\ &= \frac{1}{2}\mathcal{H}''(y_\gamma,\lambda_\gamma)(y_n - y_\gamma,\lambda_n - \lambda_\gamma)(y_n - y_\gamma,\lambda_n - \lambda_\gamma), \\ E_3 &:= (\nabla(y_n - y_\gamma))^T(\lambda_n - \lambda_\gamma) \\ &= \tilde{\mathcal{H}}(y_n,\lambda_n) - \tilde{\mathcal{H}}(y_\gamma,\lambda_\gamma) - \tilde{\mathcal{H}}'(y_\gamma,\lambda_\gamma)(y_n - y_\gamma,\lambda_n - \lambda_\gamma) \\ &= \frac{1}{2}\tilde{\mathcal{H}}''(y_\gamma,\lambda_\gamma)(y_n - y_\gamma,\lambda_n - \lambda_\gamma)(y_n - y_\gamma,\lambda_n - \lambda_\gamma). \end{aligned}$$

In the following theorem a local convergence result for the semi-smooth Newton method is stated. The result is formulated in terms of the constants $\sigma := (\nu - \mathcal{M}(y_\gamma))^{-1}, \theta := \frac{\alpha}{16\sigma^2\kappa^2}$, and it relies on the frequently used hypothesis $\nu > \mathcal{M}(y_\gamma)$

and a smallness condition on the adjoint state λ_γ. From control constrained optimal control, it is known that a sufficient condition for the latter to hold is the closedness of the reached controlled state with respect to the desired one.

Theorem 4.2: *If* $\nu > \mathcal{M}(y_\gamma)$, $\mathcal{N}\,\|\lambda_\gamma\|_V < \theta$ *and* $\|y_0 - y_\gamma\|_{\mathcal{W}}$, $\|\lambda_0 - \lambda_\gamma\|_{\mathbf{V}}$ *are sufficiently small, then the sequence* $\{(y_n, u_n, \lambda_n, \mu_n)\}$ *generated by the algorithm converges superlinearly in* $\mathcal{W} \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ *to* $(y_\gamma, u_\gamma, \lambda_\gamma, \mu_\gamma)$.

Proof: Let us take $\delta > 0$ such that

$$\begin{aligned} \nu - \mathcal{M}(y) &\geq \frac{1}{2}(\nu - \mathcal{M}(y_\gamma)) > 0 \quad \text{and} \\ \theta - \mathcal{N}\,\|\lambda_n\|_V &\geq \frac{1}{2}(\theta - \mathcal{N}\,\|\lambda_\gamma\|_V) > 0 \end{aligned} \tag{35}$$

for all (y, λ) with $\|y - y_\gamma\|_{\mathcal{W}} < \delta$ and $\|\lambda - \lambda_\gamma\|_V < \delta$

Introducing the notation $\delta_u = u_{n+1} - u_\gamma$, $\delta_y = y_{n+1} - y_\gamma$ and similarly for δ_λ, δ_p, δ_μ, and taking into account the systems of equations satisfied by the regularized solution and the iterate $(y_n, p_n, u_n, \lambda_n, \mu_n)$ we obtain the system:

$$\begin{cases} -\nu\Delta\delta_y + (y_n \cdot \nabla)\delta_y + (\delta_y \cdot \nabla)y_n + \nabla\delta_p = \mathcal{B}\delta_u + E_1 \\ \operatorname{div}\,\delta_y = 0 \\ \delta_y|_\Gamma = 0 \\ -\nu\Delta\delta_\lambda - (y_n \cdot \nabla)\delta_\lambda - (\delta_y \cdot \nabla)\lambda_n + (\nabla y_n)^T\delta_\lambda \\ \quad +(\nabla\delta_y)^T\lambda_n + \nabla\delta_\phi = E_3 - E_2 - \delta_y - \delta_\mu \\ \operatorname{div}\,\delta_\lambda = 0 \\ \delta_\lambda|_\Gamma = 0 \\ \alpha\delta_u = \mathcal{B}^\star\delta_\lambda \\ \delta_\mu = \gamma G^k_{\max}\delta_y + \gamma G^k_{\min}\delta_y + R, \end{cases} \tag{36}$$

where

$$\begin{aligned} R = {} & \max(0, \gamma(y_\gamma + (y_k - y_\gamma) - y_b)) - \max(0, \gamma(y_\gamma - y_b)) + \gamma G^k_{\max}(y_k - y_\gamma) \\ & + \min(0, \gamma(y_\gamma + (y_k - y_\gamma) - y_a)) - \min(0, \gamma(y_\gamma - y_a)) + \gamma G^k_{\min}(y_k - y_\gamma), \end{aligned}$$

$$G^k_{\max}\phi = \begin{cases} \phi \text{ on } \mathcal{A}^b_{n+1} \\ 0 \text{ in } \Omega\backslash\mathcal{A}^b_{n+1} \end{cases} \quad \text{and} \quad G^k_{\min}\phi = \begin{cases} \phi \text{ on } \mathcal{A}^a_{n+1} \\ 0 \text{ in } \Omega\backslash\mathcal{A}^a_{n+1} \end{cases}.$$

Due to Newton differentiability of the $max(0, \cdot)$ and $min(0, \cdot)$ functions (cf. [12]) we obtain that

$$\|R\|_{\mathbf{L}^2} = o(\|y_k - y_\gamma\|_{\mathbf{L}^p}), \tag{37}$$

with $p > 2$.

Multiplying δ_μ by δ_y and considering the variational formulation of the adjoint equations, we get

$$(\delta_\mu, \delta_y) = (E_3 - E_2, \delta_y) - \left\|\delta_y\right\|^2 - \nu(\nabla\delta_\lambda, \nabla\delta_y) + c(y_n, \delta_\lambda, \delta_y) \\ + c(\delta_y, \lambda_n, \delta_y) - c(\delta_y, y_n, \delta_\lambda) - c(\delta_y, \delta_y, \lambda_n),$$

which, utilizing also the variational formulation of the state equations yields

$$(\delta_\mu, \delta_y) = -\alpha \left\|\delta_u\right\|^2 - \left\|\delta_y\right\|^2 - 2c(\delta_y, \delta_y, \lambda_n) + (E_3 - E_2, \delta_y) - (E_1, \delta_\lambda)$$

Consequently,

$$\begin{aligned} &\alpha \left\|\delta_u\right\|^2 + \left\|\delta_y\right\|^2 - 2\mathcal{N} \left\|\lambda_n\right\|_V \left\|\delta_y\right\|_V^2 \\ &\quad \leq \|R\| \left\|\delta_y\right\| + C_1(\left\|y_n - y_\gamma\right\|_V \left\|y_n - y_\gamma\right\|_{\mathbf{W}^{1,4}} \left\|\delta_\lambda\right\| \\ &\quad\quad + \left\|y_n - y_\gamma\right\|_{\mathbf{W}^{1,4}} \left\|\lambda_n - \lambda_\gamma\right\| \left\|\delta_y\right\|_V). \end{aligned} \tag{38}$$

From the state equations increment system (36) we obtain the estimate

$$(\nu - \mathcal{M}(y_n)) \left\|\delta_y\right\|_V \leq \kappa(\left\|\delta_u\right\| + \left\|E_1\right\|), \tag{39}$$

which, considering the smallness condition (35), yields

$$\left\|\delta_u\right\| \geq \frac{1}{2\kappa\sigma} \left\|\delta_y\right\|_V - \left\|E_1\right\|. \tag{40}$$

Taking the square on both sides of (40) we get

$$\frac{\alpha}{2} \left\|\delta_u\right\|^2 \geq \frac{\alpha}{8\kappa^2\sigma^2} \left\|\delta_y\right\|_V^2 - \frac{\alpha}{2\kappa\sigma} \left\|E_1\right\| \left\|\delta_y\right\|_V + \frac{\alpha}{2} \left\|E_1\right\|^2. \tag{41}$$

From (38) and (41), we thus obtain

$$\begin{aligned} &\frac{\alpha}{2} \left\|\delta_u\right\|^2 + \left\|\delta_y\right\|^2 + \left(\frac{\alpha}{8\kappa^2\sigma^2} - 2\mathcal{N} \left\|\lambda_n\right\|_V\right) \left\|\delta_y\right\|_V^2 \\ &\quad \leq \|R\| \left\|\delta_y\right\| + \frac{\alpha}{2\kappa\sigma} \left\|E_1\right\| \left\|\delta_y\right\|_V + C_1\Big(\left\|y_n - y_\gamma\right\|_V \left\|y_n - y_\gamma\right\|_{\mathbf{W}^{1,4}} \left\|\delta_\lambda\right\| \\ &\quad\quad + \left\|y_n - y_\gamma\right\|_{\mathbf{W}^{1,4}} \left\|\lambda_n - \lambda_\gamma\right\| \left\|\delta_y\right\|_V \Big). \end{aligned}$$

Utilizing the smallness condition $\theta - \mathcal{N} \left\|\lambda_n\right\|_V \geq \frac{1}{2}(\theta - \mathcal{N} \left\|\lambda_\gamma\right\|_V) > 0$ we get that

$$\begin{aligned} &\frac{\alpha}{2} \left\|\delta_u\right\|^2 + \left\|\delta_y\right\|^2 + \beta_\gamma \left\|\delta_y\right\|_V^2 \\ &\quad \leq \kappa \|R\| \left\|\delta_y\right\|_V + \frac{\alpha}{2\kappa\sigma} \left\|y_n - y_\gamma\right\|_{\mathbf{W}^{1,4}}^2 \left\|\delta_y\right\|_V \\ &\quad\quad + C_1 \left(\left\|y_n - y_\gamma\right\|_{\mathbf{W}^{1,4}}^2 \left\|\delta_\lambda\right\| + \left\|y_n - y_\gamma\right\|_{\mathbf{W}^{1,4}} \left\|\lambda_n - \lambda_\gamma\right\| \left\|\delta_y\right\|_V\right), \end{aligned}$$

with $\beta_\gamma := \theta - \mathcal{N} \left\|\lambda_\gamma\right\|_V$.

Considering the increment optimality condition $\delta_\lambda = \alpha\delta_u$ and using (40), together with the estimate $\|E_1\| \leq \|y_n - y_\gamma\|^2_{\mathbf{W}^{1,4}}$, yields

$$\begin{aligned}&\frac{\alpha}{2}\|\delta_u\|^2 + \|\delta_y\|^2 + \beta\|\delta_y\|_V^2\\ &\quad\leq C_1\alpha\|y_n - y_\gamma\|^2_{\mathbf{W}^{1,4}}\|\delta_u\| + (2\kappa^2\sigma\|R\| + \alpha\|y_n - y_\gamma\|^2_{\mathbf{W}^{1,4}}\\ &\qquad+2\kappa C_1\sigma\|y_n - y_\gamma\|_{\mathbf{W}^{1,4}}\|\lambda_n - \lambda_\gamma\|)(\|\delta_u\| + \|y_n - y_\gamma\|^2_{\mathbf{W}^{1,4}}).\end{aligned}$$

Taking into account that $\|y_n - y_\gamma\|_{\mathcal{W}} < \delta$, $\|\lambda_n - \lambda_\gamma\|_V < \delta$ and using $2ab \leq a^2 + b^2$ for all $a, b > 0$, we get the existence of a constant $C_2 > 0$ such that

$$\frac{\alpha}{4}\|\delta_u\| + \frac{1}{4}\|\delta_\lambda\| \leq C_2(\|y_n - y_\gamma\|^2_{\mathbf{W}^{1,4}} + \|\lambda_n - \lambda_\gamma\|^2) + o(\|y_n - y_\gamma\|_{\mathbf{L}^p}). \quad (42)$$

Utilizing the state equations increment system again, it can be verified, proceeding as in the proof of Theorem 3.1, that

$$\|\delta_y\|_{\mathcal{W}} \leq \hat{C}\|\delta_u\|.$$

Consequently, there exists a constant $C > 0$ such that

$$\|\delta_y\|_{\mathcal{W}} + \|\delta_u\| + \|\delta_\lambda\| \leq C(\|y_n - y_\gamma\|^2_{\mathcal{W}} + \|\lambda_n - \lambda_\gamma\|^2) + o(\|y_n - y_\gamma\|_{\mathcal{W}}) \quad (43)$$

and, therefore, superlinear convergence of the iterates is verified. □

4.2. Formulation using Fischer-Burmeister function

For the introduction of the Fischer-Burmeister nonlinear complementarity functional for the regularized optimality system (24)–(27), we decompose the multiplier $\mu_\gamma = \mu_b - \mu_a$, with

$$\mu_b := \frac{1}{2}(-\mu_\gamma + |\mu_\gamma|) \quad \text{and} \quad \mu_a := \frac{1}{2}(\mu_\gamma + |\mu_\gamma|)$$

and introduce the auxiliar variable $\tilde{y} := y_\gamma - \frac{1}{\gamma}\mu_\gamma$. With these definitions, Eq. (27) with $\bar{\mu} = 0$ can be rewritten as

$$\mu_\gamma = \max(0, \mu_\gamma + \gamma(\tilde{y} - y_b)) + \min(0, \mu_\gamma + \gamma(\tilde{y} - y_a))$$

or, equivalently, as complementarity system:

$$\begin{cases}\mu_a,\ \mu_b \geq 0,\\ y_a \leq \tilde{y} \leq y_b,\\ (\mu_b, y_b - \tilde{y}) = (\mu_a, \tilde{y} - y_a) = 0.\end{cases} \quad (44)$$

Using Fischer-Burmeister's function, system (44) can be replaced by

$$\Phi_1(\mu_b, \tilde{y}) = \sqrt{\mu_b^2 + \gamma^2(y_b - \tilde{y})^2} - \mu_b - \gamma(y_b - \tilde{y}) = 0, \tag{45}$$

$$\Phi_2(\mu_a, \tilde{y}) = \sqrt{\mu_a^2 + \gamma^2(\tilde{y} - y_a)^2} - \mu_a - \gamma(\tilde{y} - y_a) = 0. \tag{46}$$

By defining the sets $\mathcal{S}^b = \{x : \mu_b(x) = \tilde{y}(x) - y_b(x) = 0\}$, $\mathcal{S}^a = \{x : \mu_a(x) = \tilde{y}(x) - y_a(x) = 0\}$ and $\mathcal{I} = \Omega\backslash(\mathcal{S}^b \cup \mathcal{S}^a)$, it can be verified that the Newton derivatives of (45) and (46), in the directions $(\delta_{\mu_b}, \delta_{\tilde{y}})$ and $(\delta_{\mu_a}, \delta_{\tilde{y}})$, satisfy

$$(d_1, d_2) \in \begin{cases} \{((\tau_1 - 1)\delta_{\mu_b}, -(\tau_2 - 1)\gamma\delta_{\tilde{y}}) : \tau_1^2 + \tau_2^2 \leq 1\} \text{ on } \mathcal{S}^b \\ \{\Phi_1'(\mu_b, \tilde{y})\} \text{ else} \end{cases}$$

and

$$(e_1, e_2) \in \begin{cases} \{((\tau_1 - 1)\delta_{\mu_a}, (\tau_2 - 1)\gamma\delta_{\tilde{y}}) : \tau_1^2 + \tau_2^2 \leq 1\} \text{ on } \mathcal{S}^a \\ \{\Phi_2'(\mu_a, \tilde{y})\} \text{ else,} \end{cases}$$

respectively (cf. [20, p. 831]). Choosing in particular $\tau_1 = \tau_2 = 1/2$ for the derivative candidates, the complete algorithm can be stated through the following steps:

Algorithm 4.3: (1) *Initialization: choose* $(u_0,\ y_0,\ \lambda_0,\ \mu_{b,0},\ \mu_{a,0}) \in \mathbf{L}^2(\Omega) \times \mathcal{W} \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ *and set* $n = 1$.

(2) *Until a stopping criteria is satisfied, set*

$$\mathcal{S}_n^b = \left\{x : \mu_{b,n-1} = y_{n-1} - y_b - \frac{1}{\gamma}\mu_{b,n-1} + \frac{1}{\gamma}\mu_{a,n-1} = 0\right\}$$
$$\mathcal{S}_n^a = \left\{x : \mu_{a,n-1} = y_{n-1} - y_a - \frac{1}{\gamma}\mu_{b,n-1} + \frac{1}{\gamma}\mu_{a,n-1} = 0\right\}$$
$$\mathcal{I}_n = \Omega\backslash(\mathcal{S}^b \cup \mathcal{S}^a).$$

Find the solution $(y_n, p_n, u_n, \lambda_n, \phi_n, \mu_{a,n}, \mu_{b,n})$ *of*

$$\begin{aligned} -\nu\Delta y_n + (y_{n-1} \cdot \nabla)y_n + (y_n \cdot \nabla)y_{n-1} + \nabla p_n &= u_n + (y_{n-1} \cdot \nabla)y_{n-1} \\ div\ y_n &= 0 \\ y_n|_\Gamma &= g, \end{aligned} \tag{47}$$

$$\begin{aligned} -\nu\Delta\lambda_n - (y_n \cdot \nabla)\lambda_{n-1} - (y_{n-1} \cdot \nabla)\lambda_n + (\nabla y_{n-1})^T\lambda_n + (\nabla y_n)^T\lambda_{n-1} \\ +\nabla\phi_n = z_d - y_n - \mu_{b,n} + \mu_{a,n} - (y_{n-1} \cdot \nabla)\lambda_{n-1} + (\nabla y_{n-1})^T\lambda_{n-1} \\ div\ \lambda_n = 0 \\ \lambda_n|_\Gamma = 0. \end{aligned} \tag{48}$$

$$\alpha u_n = \lambda_n \tag{49}$$

$$-\mu_{b,n} + \gamma \tilde{y}_n = -\frac{\mu_{b,n-1} \cdot \delta\mu_b - \gamma^2 (y_b - \tilde{y}_{n-1}) \cdot \delta_{\tilde{y}}}{\sqrt{\mu_{b,n-1}^2 + \gamma^2 (y_b - \tilde{y}_{n-1})}} - \sqrt{\mu_{b,n-1}^2 + \gamma^2 (y_b - \tilde{y}_{n-1})} + \gamma y_b \quad \text{on} \quad \mathcal{S}_n^a \cup \mathcal{I} \tag{50}$$

$$-\mu_{b,n} + \gamma \tilde{y}_n = \gamma y_b \quad \text{on } \mathcal{S}_n^b \tag{51}$$

$$-\mu_{a,n} - \gamma \tilde{y}_n = -\frac{\mu_{a,n-1} \cdot \delta\mu_b - \gamma^2 (\tilde{y}_{n-1} - y_a) \cdot \delta_{\tilde{y}}}{\sqrt{\mu_{a,n-1}^2 + \gamma^2 (\tilde{y}_{n-1} - y_a)}} - \sqrt{\mu_{a,n-1}^2 + \gamma^2 (\tilde{y}_{n-1} - y_a)} - \gamma y_a \quad \text{on } \mathcal{S}_n^b \cup \mathcal{I} \tag{52}$$

$$-\mu_{a,n} - \gamma \tilde{y}_n = -\gamma y_a \quad \text{on } \mathcal{S}_n^a \tag{53}$$

$$\tilde{y}_n = y_n - \frac{1}{\gamma}\mu_{b,n} + \frac{1}{\gamma}\mu_{a,n} \tag{54}$$

and set $n = n + 1$.

5. Numerical Results

In this section, we describe some numerical tests, which illustrate the performance of the semi-smooth Newton method applied to a class of state constrained optimal control problems of the stationary Navier-Stokes equations. The *max-min* and Fischer-Burmeister functionals are chosen as examples of Newton differentiable NCP-functions and their behavior is numerically compared.

For the numerical simulations, a forward facing step channel geometry is used. We consider a channel of length 1 and height 0.5. The fluid flows from left to right with inflow boundary condition of parabolic type (with maximum value 1) and outflow "do nothing" condition (cf. [19]). For the discretization of the domain a homogeneous staggered grid with step h is utilized. A first-order upwind finite differences scheme is used for the approximation of the partial differential equations.

The target of the control problem is to drive the fluid to an almost linear behavior given by the Navier-Stokes flow with Reynolds number equal to 1 and, through the presence of pointwise state constraints, avoid recirculations before and/or after the step. In that sense, the $Re = 1$ flow is chosen as desired state z_d. The uncontrolled flow with $Re = 1000$, depicted in Fig. 1, illustrates the main recirculation zones in the channel.

The semi-smooth Newton algorithm is terminated when the norm of the increments reaches the precision tol, whose value is typically set equal to 10^{-5}. The resulting linear systems in each semi-smooth Newton iteration are solved exactly using Matlab's sparse solver.

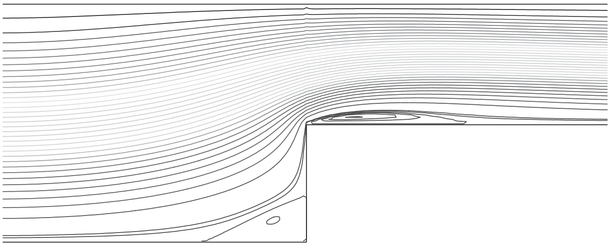

Fig. 1. Streamlines of the uncontrolled state

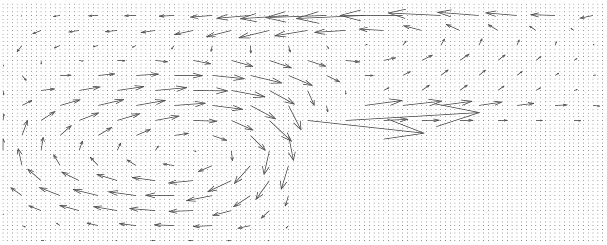

Fig. 2. Example 1: Plot of the control vector; $\|u_\gamma\|_{\mathbf{L}^2} = 0.120096$

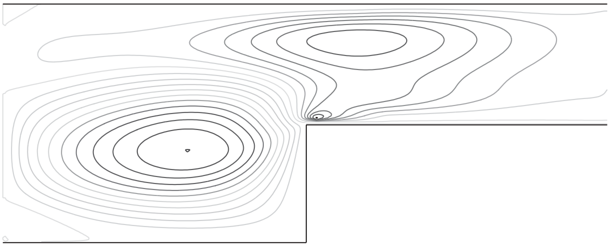

Fig. 3. Example 1: Contour plot of the control

Example 1: In this first example, we choose $Re = 1000$ (see Fig. 1) and set the state constraint $y_1 \geq -10^{-7}$ in Ω in order to reach the objective of avoiding recirculations. The control regularization parameter is set to the value $\alpha = 0.1$. Figures 2 and 3 show the optimal control obtained for the penalization parameter $\gamma = 10^6$. With this penalization parameter, the minimum of the regularized horizontal velocity takes the value $-2.4247 \cdot 10^{-4}$.

The final controlled state is depicted in Fig. 4, where it can be observed that, at the scale of numerical resolution, no recirculations are present. To obtain this solution, the control reaches high values, which occur mainly in the recirculation zones. By numerical evaluation the L^2 and L^∞ control norms reach the values 0.120096 and 7.331976, respectively.

In Table 1, the performance of the penalization approach is numerically tested. The number of iterations, the size of the active set and the values of the cost functional are tabulated for different values of γ. It can be seen that the dependence of the iteration number on the penalization parameter is significantly larger for the Fischer-Burmeister than the *max-min* complementarity functional. Moreover, the total

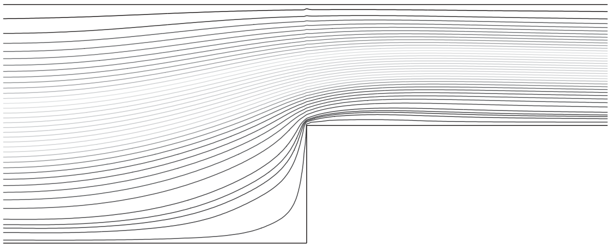

Fig. 4. Example 1: Plot of the streamlines of the final controlled state

Table 1. Example 1, $h = 1/240$, $\varepsilon = 10^{-4}$

γ	10	10^2	10^3	10^4	10^5	10^6
Iter. *max*	5	5	8	8	8	9
$\lvert\mathcal{A}_\gamma^a \cup \mathcal{A}_\gamma^b\rvert$	62	58	40	30	28	28
Iter. F-B	7	7	9	15	37	54
$J(y_\gamma, u_\gamma)$	0.003997	0.004002	0.004080	0.004339	0.004443	0.004458

number of iterations is consistently larger for the Fischer-Burmeister than the *max*-*min* functional. Similar observations were made in [8, 15] for finite dimensional optimization problems.

The data for the performance of the semi-smooth Newton method with the *max*-*min* functional, are reported in Table 2. The size of the active set, the values of the cost functional, the difference between two consecutive iterates of the velocity field, the convergence rate estimate and the residual values of the nonlinear complementarity functional are tabulated for each SSN iteration. Local superlinear convergence of the method can be observed numerically.

Next, we consider the limit case when the tracking type component of the cost functional is dropped, i.e., $J(y, u) = \frac{1}{2}\|u\|^2$. The solution to this problem corresponds to the minimum control norm required to eliminate the fluid recirculations via the state constraints. From Fig. 5, it can be observed that the control action in this case is significantly concentrated in the recirculation zones. The state constraint

Table 2. Example 1, $h = \frac{1}{240}$, $\varepsilon = 10^{-7}$, $\gamma = 10^4$

Iteration	$\lvert \mathcal{A}_n \rvert$	$J(y, u)$	$\|y_n - y_{n-1}\|$	$\frac{\|y_n - y_{n-1}\|}{\|y_{n-1} - y_{n-2}\|}$	NCP
1	0	0.00447153	19.0394	–	74.7580
2	9	0.00424673	4.8617	0.2553	881.0017
3	45	0.00435671	0.5394	0.110967	69.5120
4	33	0.00434033	0.446645	0.827893	5.8360
5	31	0.00433979	0.005584	0.0125	0.4800
6	30	0.00433980	$6.179 \cdot 10^{-4}$	0.110648	0
7	30	0.00433980	$3.550 \cdot 10^{-7}$	$5.745 \cdot 10^{-4}$	0
8	30	0.00433980	$1.361 \cdot 10^{-14}$	$3.835 \cdot 10^{-8}$	0

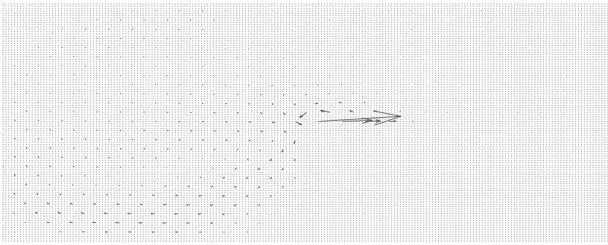

Fig. 5. Example 1: Minimum control norm problem: control; $\|u_\gamma\|_{\mathbf{L}^2} = 0.083488$

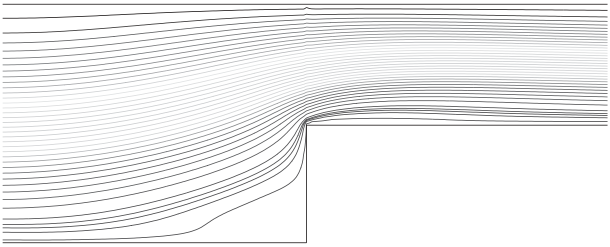

Fig. 6. Example 1: Minimum control norm problem: final state

$y_1 \geq -10^{-7}$ is satisfied on the whole domain and the recirculations are reduced (see Fig. 6). The cost functional takes the optimal value $J(y_\gamma, u_\gamma) = 0.0139407$ and the number of SSN iterations needed to reach the solution is 23.

Example 2: For this example, we chose the state constraint $y_1 \leq 1.75$ in Ω. The remaining parameter values are $Re = 1000$, $\alpha = 0.01$ and $\gamma = 10^4$. It is anticipated that this type of constraint results in a more homogeneous outflow horizontal velocity and that large velocity gradients in the last part of the channel are diminished. Besides that, this type of constraint is imposed in order to obtain a bigger active set, which allows the visualization of the Lagrange multiplier structure. The constraint $y_1 \leq 1.75$ results in a reduction of 7.5% of the maximum value of the horizontal velocity in the uncontrolled flow. It can be seen from Fig. 7 that this constraint also results in a remarkable reduction of the recirculation zones. In fact, in the last part of the channel, the bubble almost disappears.

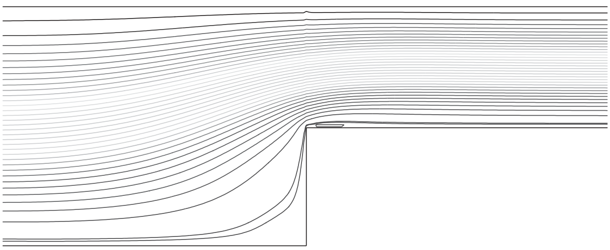

Fig. 7. Example 2: Plot of the streamlines of the final controlled state, $\gamma = 10^4$

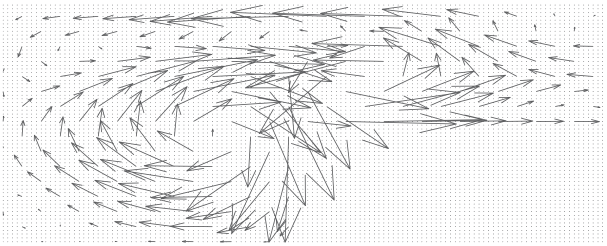

Fig. 8. Example 2: Plot of the control vector; $\|u_\gamma\|_{\mathbf{L}^2} = 0.2803324$

As can be seen in Fig. 8, in this case the optimal control is not concentrated on the recirculations zones, but acts in a more distributed way throughout the channel.

In Table 3, the behavior of the penalized problem is reported. As in Example 1, there is a clear difference in the number of iterations of the semi-smooth Newton method when applied to the *max-min* and the Fischer-Burmeister functionals. The data show clearly a better behavior of the *max-min* function based algorithm.

In order to further improve the efficiency of the latter, we also test a continuation procedure to accelerate the convergence. In this case the solution of the regularized problem with value γ is used as initialization for the next larger γ-value. The data for this approach are given in Table 4.

The fact that the Lagrange multiplier μ^* is only a measure can be observed from the behavior of its approximation μ_γ along the boundary of the active set (see Fig. 9).

Next, we turn to the situation of controls localized on sub-domains $\tilde{\Omega} = \Omega_c$. Our two choices for sub-domains are depicted in Fig. 10. In the first case, when the control domain is located on the recirculation zone, the controlled velocity field satisfies the state constraint on the whole domain. The maximum of the regularized horizontal velocity takes the value 1.7527. As expected, the effect of recirculation diminishing is now smaller than in the case $\tilde{\Omega} = \Omega$. The cost functional in this case takes the

Table 3. Example 2, $h = 1/240$, $\varepsilon = 10^{-4}$

γ	10	10^2	10^3	10^4	10^5	10^6
Iter. *max*	5	7	10	17	24	25
$\lvert\mathcal{A}_\gamma^a \cup \mathcal{A}_\gamma^b\rvert$	2257	2063	1955	1907	1888	1878
Iter. F-B	7	11	17	31	62	84
$J(y_\gamma, u_\gamma)$	0.0023302	0.0026332	0.0026801	0.0026859	0.0026866	0.0026867

Table 4. Example 2, $h = 1/240$, $\varepsilon = 10^{-4}$

γ	10^2	10^4	10^6	
Iter.	7	7	4	$\sum = 18$
$\lvert\mathcal{A}_\gamma^a \cup \mathcal{A}_\gamma^b\rvert$	2063	1907	1879	

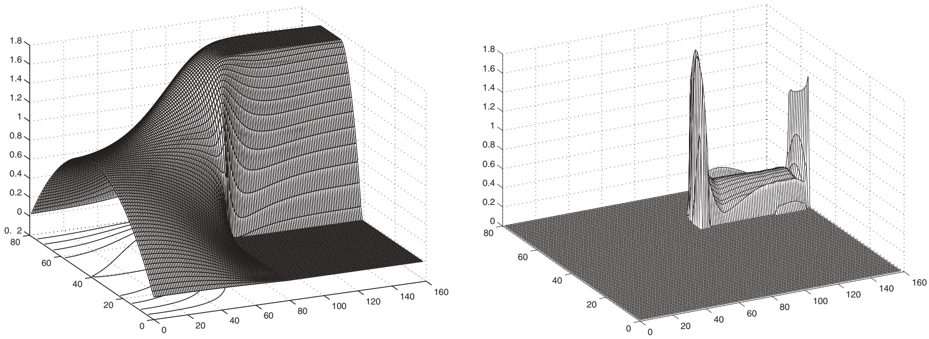

Fig. 9. Example 2: Horizontal velocity y_γ and multiplier μ_γ

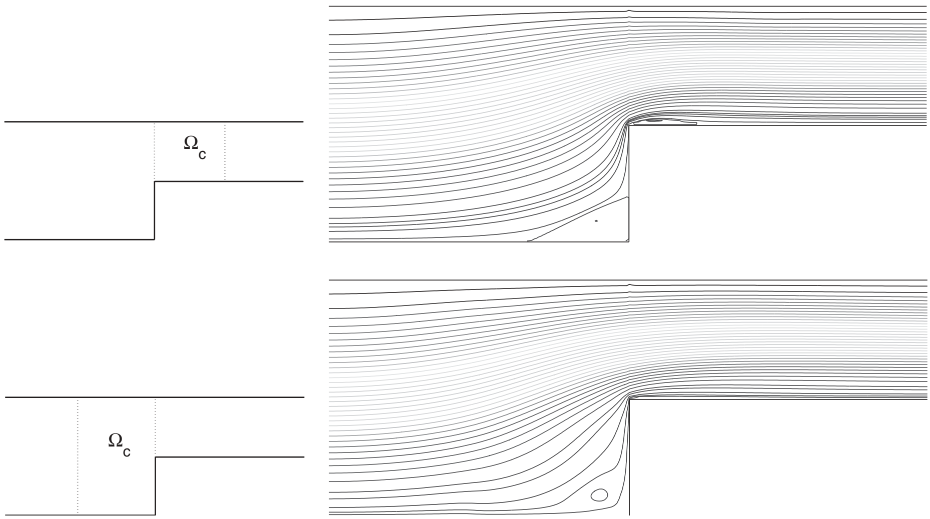

Fig. 10. Example 2: Subdomains of control and correspondent reached states

optimal value $J(y_\gamma, u_\gamma) = 0.004927$ and the number of SSN iterations, with the *max-min* NCP function, is 22.

In the second case with the location of the control sub-domain before the step, again an effective control action throughout the whole fluid is achieved. The state constraint is satisfied on Ω, and the recirculation after the step is eliminated. Compared to the previous sub-domain control case and also to the case $\tilde{\Omega} = \Omega$ (see Fig. 7), the effect of recirculation diminishing is more significant with respect to the vortex after the step, but fails by diminishing the one before the step. The cost functional takes the optimal value $J(y_\gamma, u_\gamma) = 0.004961$ and the number of SSN iterations needed is 17.

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