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Optimal Control of Obstacle Problems by H^1-Obstacles*

Kazufumi Ito[1] and Karl Kunisch[2]

[1]Department of Mathematics, North Carolina State University,
Raleigh, NC 27695-8205, USA
ito@math.ncsu.edu

[2]Institut für Mathematik und wissenschaftliches Rechnen,
Karl-Franzens-Universität Graz, A-8010 Graz, Austria
karl.kunisch@uni-graz.at

Abstract. Optimal control of variational inequalities with the controls given by the obstacles is considered. Existence optimal solutions are proved for obstacles with H^1 regularity and first-order optimality conditions are derived which, under additional assumptions, are also sufficient. A numerical algorithm is proposed and its practical feasibility is investigated.

Key Words. Control of variational inequalities, Optimality system, Moreau–Yosida approximation.

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1. Introduction

In this paper we consider a control problem where the state satisfies a variational inequality of obstacle-type and the control variable is the obstacle itself. While the techniques that we employ can be applied to a wider class of variational inequalities, we describe them in detail for second-order elliptic obstacle problems: find $y \in K$ such that

$$\langle Ay - f, \varphi - y \rangle \geq 0 \qquad \text{for all} \quad \varphi \in K, \tag{1.1}$$

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where

$$K = \{\varphi \in H_0^1(\Omega),\ y \le \psi\},$$

with

$$f \in H, \qquad \psi \in U_{\mathrm{ad}} \qquad \text{with} \quad \psi \ge 0 \text{ on } \partial\Omega.$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\,\Omega$ and

$$V = H_0^1(\Omega), \qquad X = H^1(\Omega), \qquad H = L^2(\Omega),$$

endowed with the norms $|y|_V = |\nabla y|_{L^2}$ and $|y|_X = (|y|_{L^2}^2 + |\nabla y|_{L^2}^2)^{1/2}$, respectively. Further, $(\cdot,\cdot)$ denotes the inner product on H, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between V and V^* as well as X and X^*, depending on the context, and $A \in \mathcal{L}(V, V^*)$ is defined by

$$\langle A\,\upsilon, w\rangle = \sigma(\upsilon, w) \qquad \text{for all} \quad \upsilon, w \in V,$$

where $\sigma\,(\cdot,\cdot)$ is a bounded bilinear coercive form on $X \times X$, i.e. there exist $M > 0$ and $\omega > 0$ such that

$$\sigma(\upsilon, w) \le M\,|\upsilon|_X |w|_X \qquad \text{for all} \quad \upsilon, w \in X$$

and

$$\sigma(\upsilon,\upsilon) \ge \omega\,|\upsilon|_V^2 \qquad \text{for all} \quad \upsilon \in V. \tag{1.2}$$

Moreover, σ is assumed to satisfy

$$\sigma(\upsilon, \upsilon^+) \ge 0 \qquad \text{for all} \quad \upsilon \in X,$$

where $\upsilon^+(x) = \max(\upsilon(x), 0)$ for a.e. $x \in \Omega$. For fixed $\bar\lambda \in L^2(\Omega)$ satisfying $\bar\lambda \ge 0$ a.e. on Ω the closed convex subset of admissible obstacles U_{ad} of X is defined by

$$\begin{aligned} U_{\mathrm{ad}} = \{\psi \in X\colon\ &\psi(x) \ge 0 \text{ on } \partial\Omega \text{ and} \\ &-\ \sigma(\psi,\upsilon) + (f,\upsilon) \le (\bar\lambda, \upsilon) \text{ for all } \upsilon \in V \text{ with } \upsilon \ge 0\}. \end{aligned} \tag{1.3}$$

Note that $U_{\mathrm{ad}} \ne \emptyset$ since the solution $\tilde\psi \in H_0^1(\Omega)$ to $\Delta\tilde\psi = \bar\lambda - f$ is in U_{ad}. For $\psi \in H^2(\Omega)$, with $\psi \ge 0$ on $\partial\Omega$, we have that $\psi \in U_{\mathrm{ad}}$ if $\Delta\psi \le \bar\lambda - f$ a.e. in Ω. For $\psi \in X$, with $\Delta\psi$ not in $L^2(\Omega)$, the geometrical interpretation of the second inequality in the definition of U_{ad} requires that ψ is convex downward at jumps in the derivative of ψ. From Theorem 2.1 of this paper and also the results in [IK2], it follows that for $\psi \in U_{\mathrm{ad}}$ the variational inequality (1.1) can be equivalently expressed as

$$Ay + \lambda = f, \qquad \lambda = \max(0, \lambda + y - \psi), \tag{1.4}$$

where λ has the property that $\lambda \le \bar\lambda$.

The focus of the paper is directed to the following optimization problem which involves the obstacle as the control variable:

$$\begin{cases} \min J(y,\psi) = g(y) + \dfrac{\alpha}{2}\displaystyle\int_\Omega (|\psi|^2 + |\nabla\psi|^2)\,dx \\ \text{subject to (1.1) and} \quad \psi \in U_{\mathrm{ad}}, \end{cases} \tag{P}$$

where $\alpha > 0$ and g is a C^1 functional on V that is bounded from below. We note that (P) can also be considered as a bilevel optimization problem with the outer level defined by J and the inner optimization given by

$$\min \tfrac{1}{2}\sigma(y, y) - (f, y) \qquad \text{over} \quad y \in K,$$

for which (1.1) is the optimality condition. The main contribution of this work is the development of an optimality system for (P). Such systems are frequently the basis for numerical realizations and sensitivity analysis related to optimal control problems. We also propose an algorithm for solving the optimality system numerically.

Let us consider for a moment the time discretized Black–Scholes model for American options, see, e.g. [S]:

$$-\frac{1}{\Delta t}\,(v - \hat{v}) + \frac{\sigma^2}{2} S^2\, v_{SS} + r S\, v_S - r\, v \le 0 \perp v(S) \ge \psi(S)$$

for a.e. $S \in (0, \infty)$, where ψ is the reward function. Here $S \ge 0$ denotes the price, v is the value of the share, $r > 0$ is the interest rate, and $\sigma > 0$ is the volatility of the market. Several control problems can be formulated but a control problem discussed in this paper is formulated as: given the measured value function $\bar{v}$ find a reward function $\psi \in H^1$ that minimizes

$$\int_0^\infty (|v - \bar{v}|^2\, dS + \alpha \int_0^\infty |\psi_S|^2\, dS,$$

for some $\alpha > 0$. Recall that $\psi(S) = \max(0, K - S)$ for the put and $\psi(S) = \max(0, S - K)$ for the call option, where $K > 0$ is the strike price. Thus the regularity of the involved obstacles is just that allowed by U_{ad}. However, differently from (1.1), the Black–Scholes model is set on an infinite domain and the differential operator has weakly singular coefficients, thus additional work will be necessary to apply the approach we present here.

Before turning to our contribution we also comment on some related work. There are many references for variational inequalities, for example [BL1]. Control of obstacle problems with the control given by f rather than ψ has been treated by several authors before, we mention [B] and [MP] and the references given there. In that work no emphasis is put on expressing the optimality system by means of equations rather than variational inequalities. This distinction can be of importance for numerical techniques. In [OKZ] and [H] control of obstacle problems are treated where the control is part of the domain, respectively a coefficient in the differential operator, which relates to a problem in lubrication theory. The work most closely related to ours is presented in [BL2] and [BL3]. There, however, H^2-regularity of the obstacles is required and the optimality system is less complete than the one obtained in this paper. Control of obstacle problems with H^1-regular obstacles are considered in [AL] in the special case that the obstacle satisfies the same Dirichlet boundary conditions as the state variable and the forcing function obeys certain sign conditions.

We next briefly outline the approach that is taken and state the main result. For the Lagrangian

$$L(y, \lambda, \psi, p, \eta) = J(y, \psi) + \langle Ay + \lambda - f, p\rangle + (\max(0, \lambda + y - \psi) - \lambda, \eta)$$

a formal application of the Lagrange multiplier method yields

$$\begin{cases} A^*p + \widehat{\mathrm{sgn}}\,\eta + g'(y) = 0, \\ \langle -\alpha\Delta\psi + \alpha\psi - \widehat{\mathrm{sgn}}\,\eta, \quad \chi - \psi\rangle \geq 0 \qquad \text{for all} \quad \chi \in U_{\mathrm{ad}}, \\ p + (\widehat{\mathrm{sgn}} - 1)\eta = 0, \\ \psi \in U_{\mathrm{ad}}, \end{cases} \tag{1.5}$$

together with (1.4), where $\widehat{\mathrm{sgn}}$ denotes the function defined by

$$\widehat{\mathrm{sgn}}\,(x) = \begin{cases} 0 & \text{if} \quad (\lambda + y - \psi)(x) \leq 0, \\ 1 & \text{if} \quad (\lambda + y - \psi)(x) > 0. \end{cases}$$

System (1.5) is only formal since $x \to \max(0, x)$ is not C^1 regular. Define the C^1-approximation

$$\max_c(0, x) = \begin{cases} x, & x \geq \dfrac{1}{2c}, \\ \dfrac{c}{2}\left(x + \dfrac{1}{2c}\right)^2, & |x| \leq \dfrac{1}{2c}, \\ 0, & x \leq -\dfrac{1}{2c}, \end{cases}$$

where $c > 0$. Then $\max_c(0, x) = \int_{-\infty}^{x} \mathrm{sgn}_c(s)\, ds$, where

$$\mathrm{sgn}_c(x) = \begin{cases} 1, & x \geq \dfrac{1}{2c}, \\ c\left(x + \dfrac{1}{2c}\right), & |x| \leq \dfrac{1}{2c}, \\ 0, & x \leq -\dfrac{1}{2c}. \end{cases}$$

The complementarity system (1.4) will be approximated by means of

$$Ay + \max_c(0, \bar{\lambda} + c(y - \psi)) = f, \tag{1.6}$$

where the max-operation was replaced by a generalized Moreau–Yosida-type regularization. As will be seen from Theorem 2.2 below, this choice of regularization guarantees that the regularized solutions y_c satisfy $y_c \leq \psi$ for all $c > 0$ which, in turn, will imply that for the regularized multipliers $\lambda_c = \max_c(0, \bar{\lambda} + c(y - \psi))$ we have $0 \leq \lambda_c \leq \bar{\lambda} + 1/c$. It will be shown that λ_c converges to λ in (1.4), so that in particular the Lagrange multiplier for each $\psi \in U_{\mathrm{ad}}$ satisfies $0 \leq \lambda \leq \bar{\lambda}$.

For any $c \geq 1$ the regularized optimal control problem is introduced as

$$\begin{cases} \min J(y, \psi) \\ \text{subject to} \quad Ay + \max_c(0, \bar{\lambda} + c(y - \psi)) - f = 0 \quad \text{and} \quad \psi \in U_{\mathrm{ad}}. \end{cases} \tag{P_c}$$

It will be shown in Theorems 2.1–2.3 of Section 2 that (P_c) has a solution $(y_c, \psi_c) \in V \times U_{\mathrm{ad}}$ and every weak cluster point as $c \to \infty$ of solutions to (P_c) is a solution to

(P). Note in particular that the solutions to the regularized problems (P_c) are themselves feasible. This can be achieved by use of the shift $\bar{\lambda}$. The choice of regularization used here constitutes a further development of this technique which was already used for obstacle problems in [IK1] and control of obstacle problems [IK2], where the control was given by the forcing f. For $(y, \psi) \in V \times X$ define

$$F(y, \psi) = Ay + \max_c(0, \bar{\lambda} + c(y - \psi)).$$

Then F is Frechet differentiable from $V \times X$ to V^* with

$$F'(y, \psi)(h, \upsilon) = Ah + c(\widetilde{\operatorname{sgn}}_c)(h - \upsilon),$$

where

$$\widetilde{\operatorname{sgn}}_c = \operatorname{sgn}_c(\bar{\lambda} + c(y - \psi)).$$

Moreover, since

$$\langle A\varphi + c\,\widetilde{\operatorname{sgn}}_c\,\varphi, \varphi\rangle \geq \sigma(\varphi, \varphi) \qquad \text{for all} \quad \varphi \in V,$$

it follows from the Lax–Milgram theorem that $F_y(y, \psi)\colon\ V \to V^*$ is surjective. Thus the necessary optimality condition for (P_c) is given by

$$\begin{cases} Ay_c + \max_c(0, \bar{\lambda} + c\,(y_c - \psi_c)) = f, \\ A^* p_c + c\,\widetilde{\operatorname{sgn}}_c\, p_c + g'(y_c) = 0, \\ \langle -\alpha\Delta\psi_c + \alpha\psi_c - c\,\widetilde{\operatorname{sgn}}_c\, p_c, \chi - \psi_c\rangle \geq 0 \qquad \text{for all} \quad \chi \in U_{\text{ad}}, \\ \widetilde{\operatorname{sgn}}_c = \operatorname{sgn}_c(\bar{\lambda} + c(y_c - \psi_c)), \\ \psi_c \in U_{\text{ad}}. \end{cases} \tag{1.7}$$

Under the additional requirement that

$$\sigma(\upsilon, w) = (a\nabla\upsilon, \nabla w) + (\vec{b}\nabla\upsilon, w) + (d\,\upsilon, w), \tag{1.8}$$

where $a(x) \geq \alpha$ with $\alpha > 0$, $d(x) \geq 0$, and a, $\vec{b}$, d are in $L^\infty(\Omega)$, with $|\overrightarrow{b}|_{L^\infty}$ sufficiently small so that σ still satisfies (1.2), the following main result will be proved in Section 3.

Theorem 1.1. *Assume that $\{g'(y_c)\}_{c\geq 1}$ is bounded in $L^q(\Omega)$ with $q > \min(n/2, 1)$. Then every sequence of solutions $\{(y_c, \psi_c)\}_{c>0}$ contains a weakly in $V \times X$ convergent subsequence with indices $\{c_n\}_{n=1}^\infty$, $\lim c_n = \infty$, and weak limit (y, ψ). Moreover, there exists an associated $p \in V \cap L^\infty(\Omega)$ and $\mu \in C(\Omega)^* \cap H^{-1}(\Omega)$ such that*

$$p_{c_n} \to p \text{ weakly in } V \text{ and weakly star in } L^\infty(\Omega),$$
$$\mu_{c_n} := c\,\widetilde{sgn}_c\, p_{c_n} \to \mu \text{ weakly star in } C(\Omega)^* \text{ and weakly in } V^*.$$

Moreover, we have

$$\begin{cases} Ay + \lambda = f, \qquad \lambda = \max(0, \lambda + y - \psi), \\ A^* p + \mu + g'(y) = 0 \quad \textit{in } V^*, \\ \langle -\alpha\Delta\psi + \alpha\psi - \mu, \chi - \psi \rangle \geq 0 \qquad \textit{for all} \quad \chi \in U_{\mathrm{ad}}, \\ \mu(y - \psi) = 0 \quad \textit{a.e. in } \Omega, \\ p\lambda = 0 \quad \textit{a.e. in } \Omega, \\ \sigma(p, p) + (g'(y), p) \leq 0, \\ \langle \mu, p\,\varphi \rangle_{V^*,V} \geq 0 \qquad \textit{for all} \quad \varphi \in W^{1,\bar{q}}(\Omega), \\ \textit{with} \quad \varphi \geq 0, \quad \textit{where} \quad \bar{q} > n, \\ \psi \in U_{\mathrm{ad}}. \end{cases} \tag{1.9}$$

It will follow from the proof that if $y \to (g'(y), \upsilon)$ is continuous from V to $\mathbb{R}$ for every $\upsilon \in L^\infty(\Omega)$ (e.g. $g(y) = |y - y_d|^2_{L^2}$, $y_d \in L^2(\Omega)$ given), then the second equation in (1.9) also holds in $C(\Omega)^*$. Further, if the problem was cast in finite-dimensional spaces or if μ was an a.e. defined function, then in place of the last inequality in (1.9) we have $\operatorname{sgn} \mu = \operatorname{sgn} p$.

Note that identifying μ by $\widetilde{\operatorname{sgn}}\,\eta$ in (1.5), we find

$$p + \eta = \mu,$$

and the fourth and fifth equations in (1.7) follow from (1.5), since $\widetilde{\operatorname{sgn}} = 0$ for $y < \psi$ and $\widetilde{\operatorname{sgn}} = 1$ if $\lambda > 0$. The last two equations are, however, not contained in (1.5).

Remark 1.1. If in the definition of U_{ad} the space X is replaced by V, then the definition of J can be replaced by $J(y, \psi) = g(y) + (\alpha/2) \int_\Omega |\nabla\psi|^2$ and the result of this paper remains correct if, in the optimality conditions, the term $-\Delta\psi + \psi$ is replaced by $-\Delta\psi$.

2. Existence and Asymptotic Behavior for $c \to \infty$

In this section we establish the existence and asymptotic behavior of solutions to (1.6) and (P_c). Let M denote the embedding constant of V into H, i.e. $|\varphi|_H \leq M|\varphi|_V$, for all $\varphi \in V$.

Theorem 2.1.

(a) *For each $c > 0$ and $\psi \in U_{\mathrm{ad}}$ there exists a unique solution y_c of* (1.6) *with $y_c \leq \psi$ and* $0 \leq \lambda_c = \max_c(0, \bar{\lambda} + c\,(y_c - \psi)) \leq \bar{\lambda} + 1/2c$. *Moreover,*

$$|y_c|_V \leq \frac{M}{\omega}(|\lambda_c|_H + |f|_H) \leq \frac{M}{\omega}\left(|\bar{\lambda}|_H + \frac{|\Omega|^{1/2}}{2c} + |f|_H\right). \tag{2.1}$$

(b) *For every $\psi \in U_{\mathrm{ad}}$ we have $y_c \to y$ strongly in V and $\lambda_c \to \lambda$ weakly in H as $c \to \infty$. Moreover, (y, λ) is the unique solution to* (1.4), $0 \leq \lambda \leq \bar{\lambda}$ *and*

$$|y|_V \leq \frac{M}{\omega}(|\bar{\lambda}|_H + |f|_H). \tag{2.2}$$

(c) *For arbitrary* ψ *and* $\hat{\psi} \in U_{\mathrm{ad}}$ *let* y_c *and* $\hat{y}_c$ *denote the corresponding solutions to* (1.6). *Then*

$$\omega|y_c - \hat{y}_c|_V^2 \le 2\left(|\bar{\lambda}|_H + \frac{|\Omega|^{1/2}}{2c}\right)|\psi - \hat{\psi}|_H. \tag{2.3}$$

Similarly, if y *and* $\hat{y}$ *are the solution to* (1.1) *with* ψ *and* $\hat{\psi}$, *respectively, then*

$$\omega|y - \hat{y}|_V^2 \le 2|\bar{\lambda}|_H\,|\psi - \hat{\psi}|_H. \tag{2.4}$$

Proof. (a) Recall that $\bar{\lambda} \ge 0$ is fixed in $L^2(\Omega)$. Since

$$y \in V \to Ay + \max{}_c(0, \bar{\lambda} + c\,(y - \psi)) \in V^*$$

is an everywhere defined monotone, coercive, and hemicontinous mapping, there exists a unique solution $y_c \in V$ to (1.6). For any $\varphi \in V$ we find

$$\sigma(y_c - \psi, \varphi) + \sigma(\psi, \varphi) + (\lambda_c, \varphi) - (f, \varphi) = 0, \tag{2.5}$$

where $\lambda_c = \max_c(0, \bar{\lambda} + c(y_c - \psi))$. We now utilize (2.5) with $\varphi = \max(0, y_c - \psi)$. Since $\max_c(0, x) \ge \max(0, x) \ge x$ it follows that

$$(\lambda_c, \varphi) \ge (\bar{\lambda}, \varphi) + c|\varphi|_H^2.$$

Moreover, by the assumptions on σ, we have

$$\sigma(y_c - \psi, \max(0, y_c - \psi)) \ge 0.$$

Using these inequalities in (2.5), we find

$$c\,|\varphi|_H^2 + (\bar{\lambda}, \varphi) + \sigma(\psi, \varphi) \le (f, \varphi).$$

Since $\psi \in U_{\mathrm{ad}}$, this implies that $\varphi = 0$ and hence $y_c \le \psi$. Note that

$$0 \le \max{}_c(0, x) \le \max\left(x, \frac{1}{2c}\right) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{2.6}$$

Hence we have $0 \le \lambda_c \le \bar{\lambda} + 1/2c$ a.e. on Ω, for every $c > 0$. From

$$\sigma(y_c, y_c) = -(\lambda_c, y_c) + (f, y_c)$$

we conclude that

$$\omega|y_c|_V^2 \le |y_c|(|\lambda_c| + |f|) \le M|y_c|_V\left(|\bar{\lambda}| + \frac{|\Omega|^{1/2}}{2c} + |f|\right)$$

and hence

$$|y_c|_V \le \frac{M}{\omega}\left(|\bar{\lambda}|_H + \frac{|\Omega|^{1/2}}{2c} + |f|_H\right). \tag{2.7}$$

(b) From (2.7) it follows that $\{y_c\}_{c\geq 1}$ is bounded in V, and $0 \leq \lambda_c \leq \bar\lambda + 1/2c$ implies that $\{\lambda_c\}_{c\geq 1}$ is bounded in H. Hence there exists $y \in V$ and $\lambda \in H$ such that for a subsequence $\{c_n\}$ of $\{c\}$,

$$(y_{c_n}, \lambda_{c_n}) \text{ converges weakly in } V \times H \text{ to some } (y, \lambda) \in V \times H,$$

where $y \leq \psi$ and $0 \leq \lambda \leq \bar\lambda$ a.e. in Ω.

Taking the limit in (1.6) we find for $n \to \infty$ that

$$Ay + \lambda = f \quad \text{in } V^*. \tag{2.8}$$

Since for $n \to \infty$ by the definition of λ_c and (2.6)

$$0 \leq \left(\lambda_{c_n}, \frac{\bar\lambda}{c_n} + y_{c_n} - \psi\right) + \frac{|\Omega|}{4c_n^3} \to (\lambda, y - \psi) \leq 0,$$

we find that $(\lambda, y - \psi) = 0$, and thus

$$\lambda = \max(0, \lambda + (y - \psi)).$$

Combined with (2.8), this implies that (y, λ) is the unique solution to (1.4). Therefore the whole family $\{y_c, \lambda_c\}_{c\geq 1}$ converges weakly in $V \times H$ to the unique solution (y, λ) of (1.4). Finally, since H embeds compactly into V^*, we have that $\lambda_c \to \lambda$ strongly in V^*. By the Lax–Milgram theorem $A \in \mathcal{L}(V, V^*)$ has a bounded inverse and therefore $y_c \to y$ strongly in V as $c \to \infty$. Taking the limit in (2.1) we obtain (2.2).

(c) From (1.6) we have

$$\sigma(y_c - \hat y_c, y_c - \hat y_c) + (\max\nolimits_c(0, \bar\lambda + c(y_c - \psi)) - \max\nolimits_c(0, \bar\lambda + c(\hat y_c - \psi)), y_c - \hat y_c) \\ = 0.$$

Expressing

$$y_c - y_{\hat c} = \frac{1}{c}(\bar\lambda + c\,(y_c - \psi) - (\bar\lambda + c(\hat y_c - \hat\psi))) + \psi - \hat\psi,$$

and using the monotonicity of $x \to \max_c(0, x)$ we obtain

$$\omega\,|y_c - y_{\hat c}|_V^2 \leq -(\lambda_c - \lambda_{\hat c}, \psi - \hat\psi) \leq 2\left(|\bar\lambda|_H + \frac{|\Omega|^{1/2}}{2c}\right)|\psi - \hat\psi|_H$$

and (2.3) follows. Taking the limit $c \to \infty$ implies (2.4). □

Theorem 2.2. *There exists a solution $(y^*, \psi^*) \in V \times U_{\rm ad}$ to (P). Similarly, for every $c > 0$, there exists a solution $(y_c, \psi_c) \in V \times X$ to (P_c).*

Proof. Let $\{(y_n, \psi_n)\}_{n=1}^\infty$ be a minimizing sequence for (P). Due to the properties of the cost-functional J and by (2.2), the sequence $\{(y_n, \psi_n)\}_{n=1}^\infty$ is bounded in $V \times X$.

Hence there exits a subsequence, denoted by the same symbol and $(y^*, \psi^*) \in V \times U_{\mathrm{ad}}$, satisfying $y^* \leq \psi^*$ such that $\psi_n \to \psi^*$ weakly in X and $y_n \to y^*$ weakly in V. By Theorem 2.1(c), moreover, it follows that $y_n \to y^*$ strongly in V and that y^* is the solution to (1.1) with $\psi = \psi^*$. Due to the weak lower semicontinuity of norms and continuity of $g\colon V \to \mathbb{R}$ it follows that

$$J(y^*, \psi^*) \leq \liminf_{n\to\infty} J(y_n, \psi_n),$$

and consequently (y^*, ψ^*) is a solution to (P). The existence of a solution to (P_c), $c > 0$, can be argued analogously. □

Theorem 2.3. *There exists a weak cluster point in $V \times X$ of the family of solutions $\{(y_c, \psi_c)\}$ to (P_c) as $c \to \infty$, and every such weak cluster point is a solution to (P). Moreover, every weak cluster point is a strong cluster point in $V \times X$.*

Proof. Let (y^*, ψ^*) be a solution to (P) and let y_c^* denote the solution to (1.6) with $\psi = \psi^* \in U_{\mathrm{ad}}$. Then

$$J(y_c, \psi_c) \leq J(y_c^*, \psi^*), \tag{2.9}$$

where (y_c, ψ_c) is any solution to (P_c), $c > 0$. The properties of J together with (2.1) imply that $\{(y_c, \psi_c)\}_{c\geq 1}$ is bounded in $V \times X$. Hence there exists a sequence $\{c_n\}$, with $\lim c_n = \infty$, $y \in V$, and $\psi \in U_{\mathrm{ad}}$ such that $y_{c_n} \to y$ weakly in V, $\psi_{c_n} \to \psi$ weakly in X and strongly in H. By Theorem 2.1(b),(c) we have $y_{c_n} \to y = y(\psi)$ strongly in V, where $y(\psi)$ is the solution to (1.1). Using Theorem 2.1(b), once again we find $y_c^* \to y^*$ strongly in V. Taking the limit in (2.9), we obtain $J(y(\psi), \psi) \leq J(y^*, \psi^*)$ and thus $(y, \psi) = (y(\psi), \psi)$ is a solution of (P). It remains to argue that $\psi_{c_n} \to \psi$ strongly in X. This follows from the fact that

$$g(y_{c_n}) + \frac{\alpha}{2}|\psi_{c_n}|_X^2 \leq g(y_{c_n}^*) + \frac{\alpha}{2}|\psi^*|_X^2,$$

which implies that

$$\limsup_{n\to\infty} |\psi_{c_n}|_X^2 \leq |\psi^*|_X^2 \leq \liminf_{n\to\infty} |\psi_{c_n}|_X^2.$$

Thus $\lim_{n\to\infty} |\psi_{c_n}|_X = |\psi|_X$ which together with weak convergence implies strong convergence of ψ_{c_n} to ψ. □

3. Proof of Theorem 1.1

Let $\{(y_c, \psi_c)\}$ be a sequence of solutions to (P_c) that converges weakly, and hence by Theorem 2.3 also strongly in $V \times U_{\mathrm{ad}}$ to a solution (y, ψ) of (P) as $c \to \infty$. The first equation in (1.9) follows from Theorem 2.3. We henceforth denote the jth equality or inequality in (1.7), respectively (1.9), by (1.7)(j), respectively (1.9)(j). Taking the inner product of (1.7)(ii) by p_c we find

$$\sigma(p_c, p_c) + c(\widetilde{\text{sgn}}_c\, p_c, p_c) = -(g'(y_c), p_c),$$

where $\widetilde{\text{sgn}}_c = \text{sgn}_c(\bar{\lambda} + c(y_c - \psi), 0) \geq 0$. Since σ is coercive on V, there exists a constant M_1 independent of $c \geq 1$ such that

$$|p_c|_V^2 + c(\widetilde{\text{sgn}}_c p_c, p_c) \leq M_1. \tag{3.1}$$

Here we also use the fact that

$$|(g'(y_c), y_c)| \leq |g'(y_c)|_{L^{2n/(n+2)}} |y_c|_{L^{2n/(n-2)}} \leq K|g'(y_c)|_{L^q} |y_c|_V,$$

where K is the embedding constant of V into $L^{2n/(n-2)}$ and L^q into $L^{2n/(n+2)}$. By assumption the term $\{|g'(y_c)|_{L^q}\}$ is bounded.

By Stampacchia's weak maximum principle [T] and the regularity requirements on $a, \vec{b}, d$ in (1.8), there exists M_2 independent of $c \geq 1$ such that $|p_c|_{L^\infty} \leq M_2 |g'(y_c)|_{L^q}$. Again boundedness of $\{|g'(y_c)|_{L^q}\}_{c \geq 1}$ implies the existence of a constant M_3 independent of $c \geq 1$ such that

$$|p_c|_{L^\infty} \leq M_3. \tag{3.2}$$

Next we show that $\mu_c = c\, \widetilde{\text{sgn}}_c\, p_c$ is bounded in $L^1(\Omega)$ uniformly with respect to $c \geq 1$. For $\varepsilon > 0$ define

$$\rho_\varepsilon(x) = \begin{cases} 1, & x \geq \varepsilon, \\ -\dfrac{1}{2\varepsilon^3} x^3 + \dfrac{3}{2\varepsilon} x, & |x| < \varepsilon, \\ -1, & x \leq -\varepsilon, \end{cases}$$

and note that $0 \leq \rho'_\varepsilon(x) \leq 3/2\varepsilon$ on $\mathbb{R}$. Taking the inner product of (1.7)(ii) with $\rho_\varepsilon(p_c)$, we obtain

$$\begin{aligned} &(a\, \rho'_\varepsilon(p_c) \nabla p_c, \nabla p_c) + (\vec{b} \nabla p_c, \rho'_\varepsilon(p_c) p_c) + (d\, p_c, \rho_\varepsilon(p_c)) + (c\, \widetilde{\text{sgn}}_c\, p_c, \rho_\varepsilon(p_c)) \\ &\quad = -(g'(y_c),\ \rho_\varepsilon(p_c)). \end{aligned}$$

Due to (3.1), there exists M_4 independent of $c \geq 1$, such that

$$|(\vec{b}\, \nabla p_c, \rho'_\varepsilon(p_c) p_c)| \leq M_4.$$

Since $\rho_\varepsilon \geq 0$ we have

$$(c\, \widetilde{\text{sgn}}_c\, p_c, \rho_\varepsilon(p_c)) \leq |g'(y_c)|_{L^1} + M_4.$$

Moreover, $0 \leq p_c \rho_\varepsilon(p_c) \to |p_c|$ a.e. in Ω as $\varepsilon \to 0$. Thus by Lebesgue's bounded convergence theorem

$$|\mu_c|_{L^1} \leq |g'(y_c)|_{L^1} + M_4. \tag{3.3}$$

Hence $\{\mu_c\}_{c \geq 1}$ is bounded in $L^1(\Omega)$ and consequently it is also bounded in $C(\Omega)^*$. Hence by a corollary to Alaoglu's theorem and the fact that $C(\Omega)$ is separable there exists a weakly star convergent subsequence with limit $\mu \in C^*(\Omega)$. By (1.7)(ii) the family $\{\mu_c\}_{c\geq 1}$ is also bounded in V^* and thus there exists a further subsequence which converges weakly to μ in V^*. Moreover, $\{p_c\}_{c\geq 1}$ is bounded in $V \cap L^\infty(\Omega)$. Consequently there exist a subsequence $\{c_n\}_{n=1}^\infty$ and $\mu \in C(\Omega)^* \cap V^*$ and $p \in V \cap L^\infty(\Omega)$ such that

$$\begin{cases} \mu_{c_n} \to \mu \quad \text{weakly in } V^* \qquad \text{and weakly star in } C(\Omega)^*, \\ p_{c_n} \to p \quad \text{weakly in } V \qquad \text{and weakly star in } L^\infty(\Omega). \end{cases} \tag{3.4}$$

These are the convergence claims above (1.9). Passing to the limit in (1.7)(ii) and (iii), we obtain (1.9)(ii) and (iii). By Theorem 2.1 we may assume that possibly after choosing yet another subsequence we have that $\lambda_{c_n} = \max_{c_n}(0, \bar\lambda + c_n(y_{c_n} - \psi_{c_n}))$ is weakly convergent in H with limit λ. A short calculation shows that on set $S = \{x \in \Omega\colon\ 0 < \widetilde{\operatorname{sgn}}_c < 1\}$ we have $\lambda_c < (1/2c)\widetilde{\operatorname{sgn}}_c$. From (3.1) and (3.3), we find (deleting the subscript n)

$$\begin{aligned} \int_\Omega |p_c\lambda_c| &\leq \int_S |p_c\lambda_c| + \int_{\Omega\setminus S} |p_c\lambda_c| \leq \frac{1}{2c}\int_S |p_c|\,\widetilde{\operatorname{sgn}}_c + \left(\int_{\Omega\setminus S} |p_c|^2\right)^{1/2} |\lambda_c|_H \\ &\leq \frac{1}{2c^2}|\mu_c|_{L^1} + \left(\int_{\Omega\setminus S} \widetilde{\operatorname{sgn}}_c\, p_c\, p_c\right)^{1/2} |\lambda_c|_H \\ &\leq \frac{1}{2c^2}|\mu_c|_{L^1} + \left(\frac{M_1}{c}\right)^{1/2} |\lambda_c|_H \to 0 \qquad \text{for} \quad c \to \infty. \end{aligned}$$

Thus $p_c\lambda_c \to 0$ in $L^1(\Omega)$ and hence in $C(\Omega)^*$. Since $p_c \to p$ strongly in H, $p_c\lambda_c \to p\lambda$ weakly star in $C(\Omega)^*$ and thus $p\lambda = 0$ a.e. and (1.9)(v) holds. Note that

$$\mu_c(y_c - \psi_c) = c\operatorname{sgn}_c p_c(y_c - \psi_c) \to 0 \quad \text{strongly in } L^1(\Omega) \text{ as } \quad c \to \infty. \tag{3.5}$$

In fact, with $T = \{x \in \Omega\colon\ \widetilde{\operatorname{sgn}}_c(x) > 0\}$ we have by (3.1)

$$\begin{aligned} \int_\Omega |\mu_c(y_c - \psi)| &= c\int_\Omega |\widetilde{\operatorname{sgn}}_c\, p_c(y_c - \psi_c)| \\ &\leq c(\widetilde{\operatorname{sgn}}_c\, p_c, p_c)^{1/2}\left(\int_T \widetilde{\operatorname{sgn}}_c\, |y_c - \psi_c|^2\right)^{1/2} \\ &\leq (\widetilde{\operatorname{sgn}}_c\, p_c, p_c)^{1/2}\left(\int_T |\bar\lambda + \frac{1}{2c}|^2\right)^2 \to 0 \qquad \text{for} \quad c \to \infty. \end{aligned}$$

For $\varphi \in W_0^{1,p}(\Omega)$, $p > n$, we have $\varphi(y_c - \psi_c) \in V$, $\varphi(y - \psi) \in V$ and

$$\begin{aligned} &(\mu_c(y_c - \psi_c), \varphi) - \langle \mu, \varphi\,(y - \psi)\rangle_{V^*,V} \\ &\quad = (\mu_c, \varphi\,(y_c - \psi_c - (y - \psi))) + \langle \mu_c - \mu, \varphi(y - \psi)\rangle_{V^*,V} \end{aligned}$$

as $c \to \infty$. Hence, for all $\varphi \in W_0^{1,p}$, $p > n$, we have

$$\begin{aligned} 0 &= \lim_{c\to\infty} \langle \mu_c(y_c - \psi_c), \varphi\rangle_{L^1,\,L^\infty} = \langle \mu_c, (y_c - \psi_c)\varphi\rangle_{V^*,V} \\ &= \langle \mu, (y - \psi)\varphi\rangle_{V^*,V} = \langle \mu(y - \psi), \varphi\rangle_{(W_0^{1,p})^*,\,W_0^{1,p}}. \end{aligned}$$

This implies that $\mu(y - \psi) \in L^1(\Omega)$, $\mu(y - \psi) = 0$ a.e. and (1.9)(iv) follows.

Inequality (1.9)(vi) follows from (1.7)(ii). Note that

$$p_c\, c\, \widetilde{\mathrm{sgn}}_c\, p_c\varphi \geq 0 \quad \text{a.e. in } \Omega \text{ for every} \quad \varphi \in W^{1,\bar q}(\Omega)$$
$$\text{with} \quad \varphi \geq 0, \quad \text{where} \quad \bar q > n.$$

Consequently $\langle \mu, p\,\varphi\rangle_{V^*,V} \geq 0$ for all such φ, which is (1.9)(vii).

Remark 3.1 (Sufficiency of Optimality Condition (1.9)). Assume that

$$g(y) = \tfrac{1}{2}|y - y_d|^2_{L^2(\Omega)} \tag{3.6}$$

for some $y_d \in L^2(\Omega)$, that $(y^*, \psi^*, \lambda^*, p^*, \mu^*) \in V \times X \times H \times V \times (L^\infty(\Omega)^* \cap V^*)$ satisfies (1.9), and that

$$\psi^* \leq y_d \qquad \text{on} \quad \mathcal{I} = \{x\colon\ \lambda^*(x) = 0\}. \tag{3.7}$$

Then (y^*, ψ^*) solves (P). In fact, let $(y, \psi, \lambda) \in V \times U_{\mathrm{ad}} \times H$ satisfy (1.4), i.e.

$$Ay + \lambda = f, \qquad \lambda = \max(0, \lambda + y - \psi),$$

with $y \neq y^*$. Then by (1.9)(i)–(v)

$$\begin{aligned} J(y,\psi) - J(y^*,\psi^*) &= g(y) - g(y^*) + \frac{\alpha}{2}\,|\psi|^2_X - \frac{\alpha}{2}\,|\psi^*|^2_X \\ &> (y^* - y_d, y - y^*)_H + \alpha(\psi^*, \psi - \psi^*)_X \\ &= -\langle \mu^*, y - y^*\rangle - \sigma(p^*, y - y^*) + \langle \mu^*, \psi - \psi^*\rangle \\ &= \langle \mu^*, \psi - y\rangle + (\lambda, p^*). \end{aligned}$$

From (1.9) (vii) we deduce that $\langle \mu^*, \min(0, p^*)\rangle \geq 0$.

Hence from (1.9) (ii), (v) and (3.6)

$$\begin{aligned} 0 &\leq \sigma(p^*, \min(0,p^*)) + \langle \mu^*,\ \min(0,p^*)\rangle = (y_d - y^*, \min(0,p^*)) \\ &\leq (y_d - \psi^* + \psi^* - y^*,\ \min(0,p^*)) \leq (y_d - \psi^*,\ \min(0,p^*))_{L^2(\Omega\setminus\mathcal{A})} \leq 0 \end{aligned}$$

and therefore $p^* \geq 0$, and $(\lambda, p^*) \geq 0$. This implies that $(p^*, \psi - y) \geq 0$ and since $\psi - y \in V$ we have $\langle \mu^*, \psi - y\rangle \geq 0$ by (1.9)(vii). Thus we find $J(y,\psi) > J(y^*,\psi^*)$, and hence (y^*, ψ^*) is optimal for (P) and (1.9) is sufficient, if (3.7) holds.

4. Some Numerical Examples

In this section we propose a numerical realization of the optimality system (1.9). It appears that in earlier work algorithmic aspects for optimal control of obstacle problems with control given by the obstacle were not addressed. The algorithm is described on the continuous level. Clearly for implementation purposes an adequate discretization must be utilized.

Algorithm

(i) Set $k = 1$, choose ψ_1, $\gamma \in (0, 1]$ and $\varepsilon > 0$.

(ii) Solve

$$Ay + \lambda = f,$$

$$\lambda = \max(0, \lambda + y - \psi_k)$$

for $(y_k, \lambda_k) \in V \times H$. Set $\mathcal{I}_k = \{x\colon\ y(x) \le \psi_k(x)\}$, $\mathcal{A}_k = \Omega - \mathcal{I}_k$.

(iii) Solve

$$A^* p = -g'(y) \quad \text{in } \mathcal{I}_k,$$

$$p = 0 \quad \text{on } \partial\mathcal{I}_k$$

for p_k on $\mathcal{I}_k$ and set

$$p_k = 0 \quad \text{on } \mathcal{A}_k.$$

(iv) Set

$$\mu_k = \begin{cases} 0 \quad \text{on } \mathcal{I}_k, \\ -g'(y_k) - A^* p_k \quad \text{on } \mathcal{A}_k. \end{cases}$$

(v) Solve $-\alpha\Delta\psi + \alpha\psi = \mu_k$ for $\hat{\psi}_{k+1}$.

(vi) Set $\psi_{k+1} = \gamma\,\hat{\psi}_{k+1} + (1 - \gamma)\psi_k$.

(vii) Terminate if $|\lambda_k - \max(0,\ \lambda_k + y_k - \psi_{k+1})|_{L^2} < \varepsilon$, otherwise update $k = k + 1$ and go to (ii).

Below we report on numerical results with this algorithm based on a finite-difference approximation involving the three-, respectively five-, point approximation of the Laplacian in dimensions one or two, and g given by (3.6). First we make a series of remarks.

Remark 4.1. (i) The algorithm uses an iteration loop with respect to the variable ψ. As a result one iteration of the algorithm requires the solution of one obstacle problem in step (i), and two elliptic solves, one for p_k on the subset $\mathcal{I}_k$ of Ω in step (ii) and one for $\hat{\psi}_{k+1}$ in step (v). We relate the algorithm to the optimality system (1.9). Clearly (ii)–(v) of the algorithm correspond to (1.9)(i)–(iii). Moreover, $\mu(y - \psi) = 0$, i.e. (1.9)(iv), is realized by algorithm (iv), and $p\lambda = 0$, i.e. (1.9)(v), is realized by algorithm (iii). Finally (1.9)(vi) and (vii) are realized since $\mu_k p_k = 0$ componentwise on each iteration level k and by (iii) of the algorithm.

(ii) The existence of a unique solution to (iii) is guaranteed if $\mathcal{I}_k$ is a sufficiently regular domain (e.g. $\partial\mathcal{I}_k$ Lipschitzian).

(iii) For some examples we restricted ψ to satisfy homogenous Dirichlet boundary conditions, i.e. $\psi \in V$. In this case the potential term $\alpha\,\psi$ in (v) was eliminated.

(iv) Unless specified otherwise we ignored the constraint $-(\nabla\psi,\ \nabla v) + (f, v) \le (\bar{\lambda}, v)$ for all $v \ge 0$ in the definition of U_{ad}. A possible numerical realization can be based on a penalty term of the form $\beta|(\Delta\psi + f - \bar{\lambda})^+|^2_{L^2}$.

(v) Concerning the stopping criterion, note that if $\lambda_k = \max(0, \lambda_k + y_k - \psi_{k+1})$ then $(y_{k+1}, \lambda_{k+1}) = (y_k, \lambda_k)$. This follows from the fact that for given ψ_k (respectively

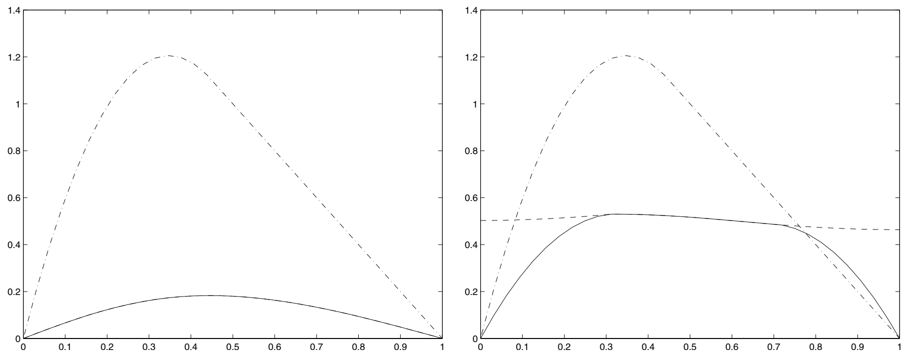

Figure 1. Example 1. y_d: dash-dot, $\psi^* = y^*$: solid line. (Left) $\psi = 0$ on boundary. (Right) No boundary conditions for ψ.

ψ_{k+1}) the solution to the system in (ii) is unique. Unless specified otherwise we stopped the algorithm with $\varepsilon = 10^{-11}$ in the one-dimensional case and with $\varepsilon = 10^{-7}$ in the two-dimensional case.

(vi) For solving the complementarity system (ii) we used the primal–dual active set strategy, see [IK3] and the references therein. We recall that this algorithm typically has finite step convergence for discretized problems.

(vii) Clearly the algorithm requires further investigations, which are not within the scope of this paper. These include a convergence analysis, investigation of the effects of lack of strict complementarity, and proper choices for the relaxation parameter γ. Alternatives for solving (1.9) or the original problem (1.1) are of interest as well.

Example 1. This is a one-dimensional example with $\Omega = (0, 1)$, $\alpha = 0.5$, and $f = 10$. The choice of y_d is based on solving $-\Delta\bar{y} + \bar{\lambda} = f$, $\bar{\lambda} = \max(0, \bar{\lambda} + \bar{y} - \psi)$ with $\bar{\psi}(x) = 1 - x$ and setting $y_d = 2\bar{y}$. In Figure 1 we depict the solutions for the grid size $h = \frac{1}{32}$ for the cases $U_{\mathrm{ad}} = H_0^1(\Omega)$ and $U_{\mathrm{ad}} = \{\psi \in H^1(\Omega)\colon \psi \geq 0 \text{ on } \partial\Omega\}$, respectively. Not surprisingly the results are completely different. By the choice of y_d it can be expected that for the solution y we have $y < y_d$ in Ω, which is confirmed by the numerical results. Further, for the case $U_{\mathrm{ad}} = H_0^1(\Omega)$ one can conjecture that the solution to (1.9) is such that all of Ω constitutes the active set, i.e. that for the solution we have $y = \psi$. Assuming strict complementarity this implies that $p = 0$, since $\lambda p = 0$. Consequently, $\mu = y_d - y > 0$ in Ω which, using (1.9)(iii) with $U_{\mathrm{ad}} = H_0^1(\Omega)$, gives the equation, for ψ,

$$-\alpha\Delta\psi + \alpha\psi = y_d,$$

which finally determines all variables $(y, \lambda, p, \mu, \psi)$ of the solution to (1.9). Precisely this solution is found numerically.

Example 2. For this two-dimensional example, $\Omega = (0, 1) \times (0, 1)$, $\alpha = 0.05$, and $f = 30$. Further, y_d is based on solving $-\Delta\bar{y} + \bar{\lambda} = f$, $\bar{\lambda} = \max(0, \bar{\lambda} + \bar{y} - \psi)$

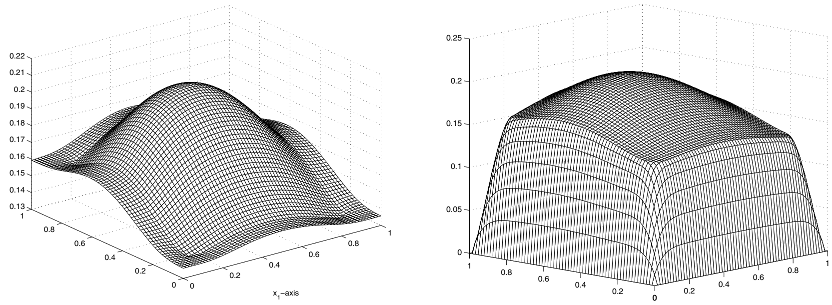

Figure 2. Example 2: Optimal ψ (left) and optimal y with no boundary conditions on ψ.

with $\psi(x_1, x_2) = 2 - 1.5x_1$. Its solution is rotated clockwise by 90° and multiplied by 0.2 to obtain y_d. The numerical results for ψ and y with meshsize $h = \frac{1}{64}$ and $U_{\text{ad}} = \{\psi \in H^1(\Omega)\colon\ \psi \geq 0 \text{ on } \partial\Omega\}$ are depicted in Figure 2. The corresponding data y_d is given on the left of Figure 3. The numerical result for $U_{\text{ad}} = H_0^1(\Omega)$ and otherwise the same problem data is given on the right of Figure 4. This is again an example where, in the case $U_{\text{ad}} = H_0^1(\Omega)$, all of Ω is the active set and $y = \psi$ at the solution. In this case the algorithm required 15 iterations with the stopping criterion set to $\varepsilon = 10^{-7}$ and 26 iterations for $\varepsilon = 10^{-10}$. Next we turn to the case, alluded to in Remark 4.1(iii), where the upper bound on $\Delta\psi + f - \bar{\lambda}$ is realized with a penalty term, with penalty parameter 1. The corresponding results for ψ and y with $\bar{\lambda} = f$ are given in Figure 4. Comparing the results for ψ in Figure 2 to Figure 4 we can observe that the procedure is effective.

Example 3. This is another two-dimensional example with $\Omega = (0, 1) \times (0, 1)$, $\alpha = 0.5$, $f = 30$ except on $(0, 1) \times (0.2344, 0.6250)$ where $f = -5$, and $y_d = 0.5$. In this case for both choices of U_{ad} described in Example 2, only a part of Ω is active. The numerical results for $h = \frac{1}{64}$, with and without homogenous Dirichlet boundary conditions, are given in Figures 5 and 6, respectively.

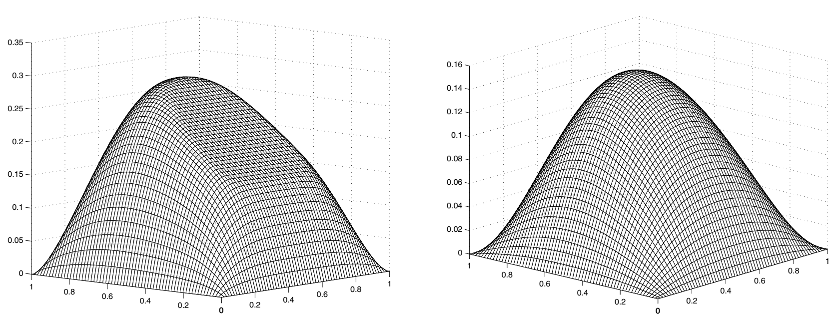

Figure 3. Example 2: desired state y_d (left) and optimal ψ with Dirichlet boundary conditions.

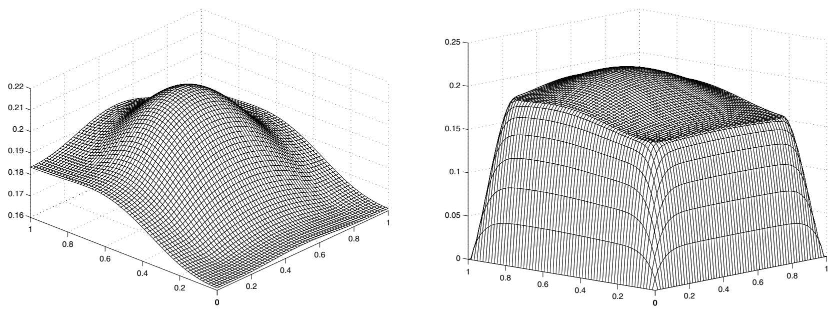

Figure 4. Example 2: optimal ψ (left) and optimal y with no boundary conditions on ψ and upper bound for $(\Delta\psi)^+$.

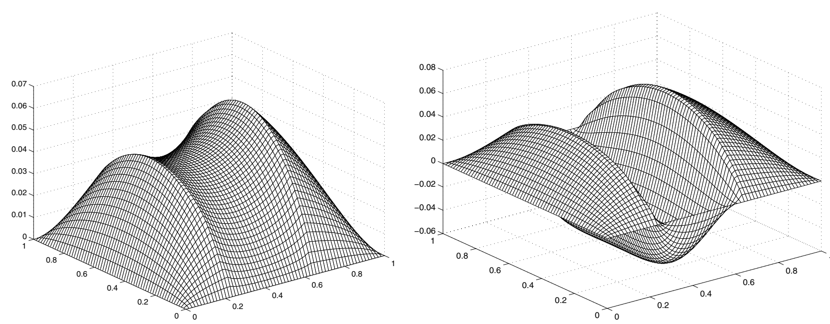

Figure 5. Example 3: optimal ψ (left) and optimal y with Dirichlet conditions on ψ.

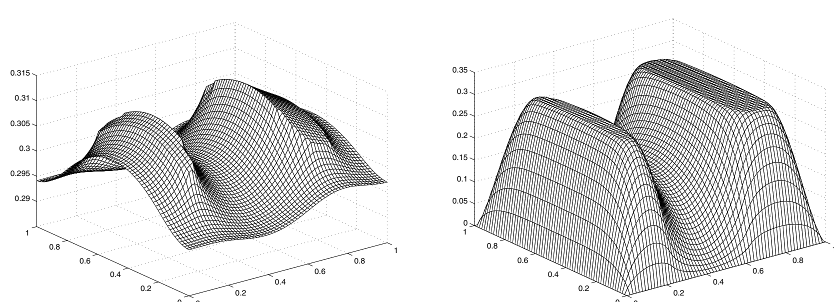

Figure 6. Example 3: optimal ψ (left) and optimal y without boundary conditions on ψ.

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