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RECEDING HORIZON CONTROL WITH INCOMPLETE OBSERVATIONS*

KAZUFUMI ITO† AND KARL KUNISCH‡

Abstract. To overcome the difficulties related to the computational requirements for solving the optimality systems for optimal control problems on long time horizons, receding horizon techniques provide an important alternative. Rather than finding the optimal solution, a suboptimal control is obtained which achieves the design objective with significantly less computational effort. Moreover, the obtained control can be interpreted as a state feedback control. In this work we continue our analysis of receding horizon strategies, considering the situation when only partial state observations are available. The receding horizon strategy is combined with a state estimator framework. A linear quadratic Gaussian design based on a linearization procedure is proposed and its asymptotic performance is analyzed for systems with nonlinear dynamics. Numerical examples validate the proposed methodology.

Key words. receding horizon control, incomplete observations, dynamic compensator

AMS subject classifications. 49N35, 93C41

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1. Introduction.

We consider the optimal control problem of minimizing the performance index subject to a nonlinear control system:

$$\min \int_0^{T_\infty} f^0(x(t), u(t))\, dt, \tag{1.1}$$

where

$$\frac{d}{dt}x(t) = f(x(t)) + Bu(t) \quad \text{for} \quad t > 0, \tag{1.2}$$

$$x(0) = x_0, \tag{1.3}$$

for some $T_\infty \in (0, \infty]$ and $x_0 \in R^n$. In order to present our approach without the technical difficulties associated to infinite dimensional control systems, we assume here that the state space is finite dimensional. We refer to $x(\cdot)$ and $u(\cdot)$ as state and control functions with $x(t) \in R^n$ and $u(t) \in R^m$. Under appropriate conditions, (1.1)–(1.3) admit a solution which satisfies the minimum principle

$$\begin{cases} \frac{d}{dt}x(t) = H_p(x(t), u(t), p(t)),\ x(0) = x_0, \\ \frac{d}{dt}p(t) = -H_x(x(t), u(t), p(t)),\ p(T_\infty) = 0, \\ u(t) = \arg\min_{u \in R^m}\ H(x(t), u, p(t)), \end{cases} \tag{1.4}$$

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†Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205 (kito@math.ncsu.edu).
‡Institut für Mathematik, Karl-Franzens-Universität Graz, A-8010 Graz, Austria (karl.kunisch@uni-graz.at). This author was supported by the Fonds zur Förderung der wissenschaftlichen Forschung under SFB 03 “Optimierung und Kontrolle.”

where H is the Hamiltonian defined by $H(x,u,p) = f^0(x,u) + p \cdot (f(x) + Bu)$. The coupled system of two-point boundary value problems with initial condition for the primal equation and terminal condition for the adjoint equation represents a significant challenge for numerical computations in case the dimension n of the state or the time horizon T_∞ are large. It has therefore been the focus of many research efforts. An alternative approach consists in constructing the feedback solution based on Bellman's dynamic programming principle. Again, due to computational costs, this is tractable only for very limited examples.

One of the possibilities to overcome these difficulties is given by time-domain decomposition based on receding horizon formulations [ABQRW, GPM]. Receding horizon techniques have proved to be effective numerically both for optimal control problems governed by ordinary (e.g., [CA, JYH, K, MM, PND, SMR]) and by partial differential equations, e.g., in the form of the instantaneous control technique for problems in fluid mechanics [B, CHK, CTMC, HV].

To briefly explain the strategy let $0 = T_0 < T_1 \cdots < T_\infty$ describe a grid on $[0, T_\infty)$ and let $T \geq \max\{T_{i+1} - T_i : i = 0, \dots\}$. If $T > T_{i+1} - T_i$, we have overlapping domains. The receding horizon optimal control problem involves the successive finite horizon optimal control problems on $[T_i, T_i + T]$,

$$\min \int_{T_i}^{T_i+T} f^0(x(t), u(t))\, dt + G(x(T_i + T)), \tag{1.5}$$

subject to

$$\frac{d}{dt} x(t) = f(x(t)) + Bu(t),\ t \in [T_i, T_i + T], \tag{1.6}$$

$$x(T_i) = x^*(T_i) \text{ if } i \geq 1 \text{ and } x(0) = x_0 \text{ for } i = 0, \tag{1.7}$$

where x^* is the solution to the auxiliary problem on $[T_{i-1}, T_{i-1} + T]$.

The solution on $[0, T_\infty)$ is obtained by concatenation of the solutions on $[T_i, T_{i+1}]$ for $i = 0, \dots$. If the terminal cost G is chosen as a control Liapunov function, then the asymptotic stability and the performance estimate of the receding horizon synthesis are established in [IK1] in case (1.2) is posed in a finite dimensional state space, and in [IK2] for infinite dimensional state spaces. For example, if G is selected as the optimal value function for (1.1)–(1.3) on the infinite horizon, then the optimal pair (x^*, u^*) for the receding horizon control (1.5)–(1.8) is optimal for the original problem (1.1)–(1.3) on the infinite horizon. In general, G plays the role of a look-ahead term in the sense that it stabilizes the receding horizon synthesis and provides a good suboptimal control law for the given performance (1.1). Since it is impractical to assume that the optimal value functional is known we constructed in [IK1] and [IK2] control Liapunov functions for diverse classes of problems which are based on Liapunov or Riccati equations. For the numerical example of section 5 we select G as the solution to an associated Riccati equation.

If $x(T_i)$ is observed, then the receding horizon control technique is a state feedback method since the control on $[T_i, T_{i+1}]$ is determined as a function of the state $x^*(T_i)$. If the optimal pair $(x^*(t), u^*(t)),\ t \in [T_i, T_i + T]$ is shifted by $-T_i$ it satisfies the two-point boundary value problem (1.4) on the interval $[0, T]$ with the terminal condition $p(T) = G_x(x(T))$ and initial condition $x(0) = x^*(T_i)$.

In this paper we address the state estimator problem for the receding horizon technique. We shall also allow for additive noise in the system dynamics as well as in the observation process and we admit uncertainty in the initial condition. The nonlinear time-independent control system with additive unmodeled disturbance is given by

$$\begin{cases} \frac{d}{dt}x(t) = f(x(t)) + Bu(t) + d(t) \quad \text{for} \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{1.8}$$

where $d(t)$ is an unknown disturbance process. The observation process providing partial observations $y(t) \in R^p$ of the state $x(t)$ is assumed to be of the form

$$y(t) = Cx(t) + n(t), \tag{1.9}$$

where $C \in \mathbb{R}^{p\times n}$ and $n(t)$ is a measurement noise process. The output feedback law will utilize the open loop optimal control u^* with associated optimal state x^* on the interval $[T_i, T_{i+1}]$ computed from (1.5) and (1.6). The initial condition for the computation of x^* is taken to be

$$x(0) = x_0 + \eta_0,$$

where η_0 denotes the uncertainty in the initial condition for $i = 0$, and then it is chosen as the state of the estimator, which will be introduced below, at time T_i, if $i \geq 1$. This can be considered as a preprocessing step for the construction of the feedback law, and in view of the fact that receding time horizon T is considered to be small compared to T_∞ it is considered to be comparatively cheap.

Once x^* is computed on $[T_i, T_i + T]$, the output feedback law is chosen to be of the form

$$u(t) = u^*(t) - B^T\Pi(w(t) - x^*(t)), \quad t \in [T_i, T_{i+1}), \tag{1.10}$$

where w denotes the state of the compensator. The construction of the feedback gain $B^T\Pi$ will be specified below. Suggested from linear quadratic Gaussian (LQG) design, the compensator is based on (1.10) together with the state estimator dynamics of the form

$$\frac{dw}{dt} = f'(\bar{x})(w(t) - x^*(t)) + f(x^*(t)) + Bu(t) + \Sigma C^T(y(t) - Cw(t)), \tag{1.11}$$

$$w(T_i) = w(T_i^-) \text{ for } i = 1, 2, \dots, \text{ and } w(0) = x_0 + \eta_0, \tag{1.12}$$

for the state estimator $w(t)$, where $w(T_i^-)$ denotes the value of the estimator on $[T_{i-1}, T_{i-1} + T]$ at T_i and $f'(\bar{x})$ is the Jacobian of f at a reference state $\bar{x}$. This reference state $\bar{x}$ is selected on the basis of the optimal pair (x^*, u^*) on $[T_i, T_i + T]$, for example,

$$\bar{x} = \frac{1}{T}\int_{T_i}^{T_i+T} x^*(t)\,dt \quad \text{or} \quad \bar{x} = x^*(T_i + T). \tag{1.13}$$

The feedback synthesis (1.10)–(1.12) performs tracking of (1.8) to the optimal pair (x^*, u^*) on $[T_i, T_i+T]$ under the uncertainty of the initial condition, observation noise and an additive disturbance in the system dynamics. As described in section 2, our tracking method is based on the linearization of the nonlinearity f at the reference state $\bar{x}$. The control gain $B^T\Pi$ and the filter gain ΣC^T will be constructed by two LQG Riccati equations, which we specify in section 2.

For our stability analysis of the output feedback synthesis (1.11)–(1.13) it is not necessary that (x^*, u^*) is assumed to be the optimal pair for the receding horizon control on $[T_i, T_i + T]$. It is only required that x^* is a stable trajectory of (1.2) corresponding to a given control u^*. However, if G is selected as a control Liapunov function, then the global stability and performance estimate of the receding control are established in [IK1]. Thus, we assume (x^*, u^*) is the optimal pair for the receding horizon control on each horizon $[T_i, T_i+T]$ and the corresponding (1.11)–(1.13) defines our proposed output feedback synthesis for (1.1)–(1.3).

The stability analysis and performance estimates for the proposed procedure are given in section 2. In section 3 the asymptotic behavior of the overall closed-loop system (1.8), with feedback control given by (1.10) and (1.11), is discussed. Our analysis uses the Liapunov stability arguments. In section 4 we briefly address a modification of the proposed method based on an H^∞-synthesis. Numerical examples that illustrate the feasibility of combining the receding horizon strategy with state estimator dynamics are given in section 5, by means of stabilization and tracking problems for the Burgers equation.

The controlled Burgers equation is an infinite dimensional system which, after discretization, is of the form considered in (1.2). In fact, while our analysis is carried out for finite dimensional systems, the proposed concepts can be extended to infinite dimensional systems. The details, however, require further research.

2. LQG design. In this section we describe the construction of the feedback and filters gains for (1.10) and (1.11). Subsequently we establish the stability and the performance estimate for the compensator dynamics (1.10), (1.11) based on the LQG design on a single time horizon $[0, T]$. The iterative procedure on the sequence of time horizons $[T_i, T_{i+1}]$ will be considered in section 3. The Jacobian of f at $x \in \mathbb{R}^n$ will be denoted by $A(x) = f'(x)$ and in particular we set

$$A = f'(\bar{x}).$$

We assume that

$$(A, B, C) \text{ is stabilizable and detectable.} \tag{2.1}$$

Consider the controlled linear equation, which results from linearizing f at $x^*(t)$ and subsequently replacing $x^*(t)$ by $\bar{x}$:

$$\begin{cases} \frac{d}{dt}(\hat{x}(t) - x^*(t)) = A(\hat{x}(t) - x^*(t)) + B(u(t) - u^*(t)) + d(t), \\ \hat{x}(0) = x_0 + \eta_0, \end{cases} \tag{2.2}$$

where $d(t)$ is the disturbance process. We shall further comment on the choice of the linearized equation in Remark 2.2. Let $Q \in \mathbb{R}^{n\times n}$ denote a positive definite matrix and consider the tracking problem to the pair (x^*, u^*):

$$\begin{cases} \min \frac{1}{2}\int_0^\infty ((\hat{x}(t) - x^*(t))^T Q(\hat{x}(t) - x^*(t)) + |u(t) - u^*(t)|^2) dt \\ \text{subject to (2.2) over } u \in L^2(0, \infty; \mathbb{R}^m). \end{cases} \tag{2.3}$$

Throughout this paper we use $|x|^2 = (x, x)$ to denote the square of the Euclidean norm of a vector x and $\|A\|$ for the subordinate matrix norm. Here (x, y) stands for the dot product of vectors x, y which will at times also be denotes by $x^T y$. Let $V\colon \mathbb{R}^n \to \mathbb{R}^+$ denote the value functional associated to (2.3). It can be shown that

$$V(x) = \frac{1}{2}(x - x^*)^T \Pi (x - x^*) + v^T (x - x^*) + c,$$

where c is a constant, and the symmetric positive definite matrix $\Pi \in R^{n\times n}$ and $v \in R^n$ satisfy

$$\left\{ \begin{array}{l} A^T \Pi + \Pi\, A - \Pi\, B\, B^T\, \Pi + Q = 0, \\ (A - B\, B^T\, \Pi)^T v + \Pi\, d = 0, \end{array} \right. \tag{2.4}$$

where we assume that the disturbance is constant in time. The optimal feedback loop control for (2.3) is given by

$$u = u^*(t) - B^T(\Pi(\hat{x}(t) - x^*(t)) + v). \tag{2.5}$$

Since we consider unknown, unmodeled disturbance $d = d(t)$ we do not include the feedforward input v in the feedback form (1.10).

Turning to the estimator we use the (Kalman) filter gain ΣC^T based on the linear system (A, C), where $\Sigma \in \mathbb{R}^{n\times n}$ is the positive definite solution to

$$A\,\Sigma + \Sigma\, A^T - \Sigma\, C^T\, C\,\Sigma + R = 0, \tag{2.6}$$

with $R \in R^{n\times n}$ a positive definite matrix. This results in the following equations for the compensator

$$\frac{dw}{dt} = A(\bar{x})(w(t) - x^*(t)) + f(x^*(t)) + Bu(t) + \Sigma C^T(y(t) - Cw(t)), \tag{2.7}$$

$$w(0) = x_0 + \eta_0 \tag{2.8}$$

and the associated feedback law

$$u(t) = u^*(t) - B^T \Pi(w(t) - x^*(t)). \tag{2.9}$$

Note that (2.1) guarantees the existence of Π and Σ with the specified properties. Moreover the spectra of $A - B\, B^T\, \Pi$ and $A - \Sigma C^T C$ are strictly contained in the left half of the complex plane. We henceforth assume that

$$d \in L^1_{loc}(0, \infty; R^n), \text{ and } n \in L^1_{loc}(0, \infty; R^p).$$

We further assume the existence of solutions x and w to (1.8) and (2.7), (2.8) on $[0, T]$ for every $u \in L^1(0, T; R^m)$, where y is given by (1.9).

Proposition 2.1. *Let $W(x) = \frac{1}{2}x^T \Pi x$. For $t \in [0, T]$ we have*

$$\begin{aligned} \frac{d}{dt}W(x(t) - x^*(t)) = {} & -\frac{1}{2}(|B^T \Pi(x(t) - x^*(t))|^2 + (Q(x(t) - x^*(t)), x(t) - x^*(t))) \\ & + (d(t) + r(x(t), x^*(t)), \Pi(x(t) - x^*(t))) + (B^T \Pi(x(t) - w(t)), B^T \Pi(x(t) - x^*(t))), \end{aligned}$$

where

$$r(x, x^*) = f(x) - f(x^*) - A(\bar{x})(x - x^*),$$

x is the solution to (1.8) *with u given in* (2.9), *and x^* is the solution to* (1.6) *with the optimal open loop control u^*.*

Proof. Using (1.6) and (1.8) we have

$$\frac{d}{dt}(x(t)-x^*(t)) = A(x(t)-x^*(t)) - BB^T\Pi(x(t)-x^*(t))$$
$$+BB^T\Pi(x(t)-w(t)) + d(t) + r(x(t),x^*(t)).$$

Thus,

$$\frac{d}{dt}W(x(t)-x^*(t)) = ((A-BB^T\Pi)(x(t)-x^*(t)) + BB^T\Pi(x(t)-w(t))$$
$$+d(t)+r(x(t),x^*(t)),\Pi(x(t)-x^*(t))$$

and the proposition follows from the fact that

$$((A-BB^T\Pi)x,\Pi x) = -\frac{1}{2}\,(|B^T\Pi x|^2 + (Qx,x)). \qquad \square$$

PROPOSITION 2.2. *Let $\tilde W(z) = \frac{1}{2}\,z^T\Sigma^{-1}z$. For $t\in[0,T]$ we have*

$$\frac{d}{dt}\tilde W(x(t)-w(t)) = -\frac{1}{2}(|C(x(t)-w(t))|^2 + (R\Sigma^{-1}(x(t)-w(t)),\Sigma^{-1}(x(t)-w(t)))$$
$$+(r(x(t),x^*(t))+d(t),\Sigma^{-1}(x(t)-w(t))) - (n(t),C(x(t)-w(t)),$$

where x and w are the solutions to (1.8) *and* (2.7) *for some $u\in L^2(0,T;R^m)$.*

Proof. From (1.8) and (2.7) we have

$$\frac{d}{dt}(x(t)-w(t)) = A(x(t)-w(t)) + r(x(t),x^*(t)) + d(t) - \Sigma\, C^T(C(x(t)-w(t))+n(t)),$$

and hence

$$\frac{d}{dt}\tilde W(x(t)-w(t)) = ((A-\Sigma C^TC)(x(t)-w(t))$$
$$+r(x(t),x^*) + d(t) - \Sigma\, C^Tn(t),\Sigma^{-1}(x(t)-w(t))).$$

Thus the proposition follows from the fact that

$$((A-\Sigma C^TC)z,\Sigma^{-1}z) = -\frac{1}{2}\,(|Cz|^2 + (R\Sigma^{-1}z,\Sigma^{-1}z)). \qquad \square$$

To quantify the performance of the compensator we set

$$E(t) = (x(t)-x^*(t),\Pi(x(t)-x^*(t)))^{1/2} + (x(t)-w(t),\Sigma^{-1}(x(t)-w(t)))^{1/2}.$$

Note that $E(t) = (W(x(t)-x^*(t)) + \tilde W(x(t)-w(t)))^{1/2}$. We further introduce positive constants $\alpha_1,\ \alpha_2$ and $\beta_1,\ \beta_2$ such that

$$\alpha_1\, I \le \Pi \le \alpha_2\, I, \qquad \beta_1\, I \le \Sigma^{-1} \le \beta_2\, I, \tag{2.10}$$

where I is the identity matrix on R^n. We shall require the following assumptions: there exists $\omega > 0$ such that

$$\begin{aligned}\frac{1}{2}(|B^T\Pi x|^2 + (Qx,x) + |Cz|^2 + (R\Sigma^{-1}z,\Sigma^{-1}z))\\ -(B^T\Pi\Sigma\,\Sigma^{-1}z, B^T\Pi x) \geq \omega\,(W(x)+\tilde W(z)) \text{ for all } x,z\in R^n,\end{aligned} \tag{2.11}$$

and

$$E(0) \leq \frac{\delta}{2} \text{ and } |x^*(t)-\bar x| \leq \delta \text{ on } [0,T] \text{ for some } \delta>0. \tag{2.12}$$

Turning to the effect of the nonlinearity we observe that

$$r(x,x^*) = f(x) - f(x^*) - A(x^*)(x-x^*) + (A(x^*)-A(\bar x))(x-x^*).$$

We assume that there exist a constant L such that

$$|r(x,x^*)| \leq L\,(|x^*-\bar x| + |x-x^*|)\,|x-x^*| \text{ for all } x,x^*\in\mathcal{S}, \tag{2.13}$$

where $\mathcal{S} = \{x\in R^n : |x-\bar x| \leq \delta(1+\frac{1}{\sqrt{\alpha_1}})\}$. We shall further assume that

$$\tilde\omega = \omega - L\delta\,\frac{\sqrt{\alpha_1}+1}{\alpha_1}\,\frac{\sqrt{\alpha_2}+\sqrt{\alpha_2+\beta_2}}{2} > 0. \tag{2.14}$$

This requirement is trivially satisfied if f is linear. Let us define

$$\rho(t) = \sqrt{\alpha_2+\beta_2}\,|d(t)| + \frac{1}{\sqrt{\beta_1}}\,\|C\||n(t)|.$$

We require the following smallness condition on the noise processes:

$$\int_0^t e^{-\tilde\omega(t-s)}\rho(s)\,ds < \frac{\delta}{2} \qquad \text{for all } t\in[0,T]. \tag{2.15}$$

THEOREM 2.1. *If* (2.12) *and the stability conditions* (2.11), (2.13), (2.14) *and the smallness condition on the noise processes* (2.15) *are satisfied, then*

$$E(t) \leq e^{-\tilde\omega t}\,E(0) + \int_0^t e^{-\tilde\omega(t-s)}\rho(s)\,ds \tag{2.16}$$

holds for all $t\in[0,T]$.

Proof. Due to (2.12) there exists τ such that

$$E(\tau) \leq \delta \text{ on } [0,\tau].$$

Note that $\sqrt{\alpha_1}|x(t)-x^*(t)| \leq E(t)$ and therefore $x(t)\in\mathcal{S}$ for $t\in[0,\tau]$. Set $X(t) = |x(t)-x^*(t)|_\Pi$ and $Y(t) = |x(t)-w(t)|_{\Sigma^{-1}}$. Suppressing the dependence on t we obtain by (2.12), (2.13) that

$$\begin{aligned}&(r(x,x^*),\Pi(x-x^*)) + (r(x,x^*),\Sigma^{-1}(x-w))\\ &\leq L\delta\,\frac{\sqrt{\alpha_1}+1}{\alpha_1}(\sqrt{\alpha_2}X^2 + \sqrt{\beta_2}XY)\\ &\leq L\delta\,\frac{\sqrt{\alpha_1}+1}{\alpha_1}\,\frac{\sqrt{\alpha_2}+\sqrt{\alpha_2+\beta_2}}{2}\,(X^2+Y^2) \text{ for } t\in[0,\tau].\end{aligned}$$

Since

$$E(t)\frac{d}{dt}\,E(t)=\frac{d}{dt}(W(x(t)-x^*(t))+\tilde{W}(x(t)-w(t))),$$

we find by Proposition 2.1, 2.2, with u given in (2.9), and (2.11), (2.13), (2.14) that

$$\begin{aligned}&\frac{d}{dt}(W(x-x^*)+\tilde{W}(x-w))\\&\le -\omega(W(x-x^*)+\tilde{W}(x-w))+(d+r(x,x^*),\Pi(x-x^*)+\Sigma^{-1}(x-w))-(n,C(x-w))\\&\le -\tilde{\omega}(W(x-x^*)+\tilde{W}(x-w))+(d,\Pi(x-x^*))+(d,\Sigma^{-1}(x-w))-(n,C(x-w))\\&\le -\tilde{\omega}(W(x-x^*)+\tilde{W}(x-w))+\rho(t)(W(x-x^*)+\tilde{W}(x-w))^{1/2}.\end{aligned}$$

Consequently

$$\frac{d}{dt}E(t)\le -\tilde{\omega}E(t)+\rho(t).$$

This implies (2.16) on $[0,\tau]$.

By (2.15) and continuity of $t\to\int_0^t e^{-\omega(t-s)}\rho(s)ds$ there exists $\alpha\in(0,1)$ such that

$$\int_0^t e^{-\tilde{\omega}(t-s)}\rho(s)ds\le\frac{\alpha\delta}{2}\ \text{ for all } t\in[0,T],$$

and consequently

$$E(t)<\frac{\delta}{2}(1+\alpha)\ \text{ for all } t\in[0,\tau]. \tag{2.17}$$

A continuation argument implies that (2.16) and (2.17) hold on $[0,T]$. In fact, if (2.17) holds on $[0,T]$, then (2.16) can be continued from $[0,\tau]$ to $[0,T]$. Assuming that (2.17) is not valid on $[0,T]$, let $\tilde{\tau}$ denote the smallest value in $(0,T)$ such that $E(\tilde{\tau})=\frac{\delta}{2}(1+\alpha)$. Then repeating the argument leading to (2.16) it can be shown that there exists $\epsilon>0$ such that (2.16) holds in $[0,\tilde{\tau}+\epsilon]$. This implies $E(\tilde{\tau}+\epsilon)<\frac{\delta}{2}(1+\alpha)$, which is a contradiction. Thus (2.17) holds on $[0,T]$. $\square$

Remark 2.1. Concerning assumption (2.12), we note that the indefinite term $-(B^T\Pi\Sigma\Sigma^{-1},B^T\Pi x)$ can be estimated by

$$-(B^T\Pi\Sigma\Sigma^{-1}z,B^T\Pi x)\ge -\frac{1}{2\alpha}|B^T\Pi\Sigma\Sigma^{-1}z|^2-\frac{\alpha}{2}|B^T\Pi x|^2$$

and the terms on the right-hand side can be combined with (Qx,x) and $(R\Sigma^{-1}z,\Sigma^{-1}z)$, and $\alpha>0$ appropriately chosen, to check for (2.12). However, since Π and Σ depend nonlinearly on Q and R, respectively, we must check the validity of (2.12) for any given system. We have done so numerically for our test examples.

Remark 2.2. The linear equation (2.2) as a basis for the construction of the compensator is well suited for our purposes. Other choices are briefly indicated.

1. Linearizing (1.8) at $\bar{x}$ results in

$$\frac{d}{dt}(\hat{x}(t)-\bar{x})=f(\bar{x})+A(\hat{x}(t)-\bar{x})+B(u(t)-u^*(t))+d(t).$$

The resulting Riccati synthesis is of the form (2.4) with d replaced by $d+f(\bar{x})$. Since d is unknown it cannot be used for the construction of the estimator, and $f(\bar{x})$ remains as a known bias term. It necessitates to modify W to be $W(x) = \frac{1}{2}x^T \Pi x + x^T v$, with the bias v changing from one horizon to the next.

2. It is also possible to employ the time varying linearization

$$\left\{ \begin{array}{l} \frac{d}{dt}(\hat{x}(t) - x^*(t)) = A(t)(\hat{x}(t) - x^*(t)) + B(u - u^*(t)) + \tilde{n}(t), \\ \\ \hat{x} = x_0, \end{array} \right. \tag{2.18}$$

where $A(t) = A(x^*(t))$ and use the corresponding time-varying gains $B^T \Pi(t)$ and $\Sigma(t)C^T$ determined by

$$\frac{d}{dt}\Pi(t) + A(t)^T \Pi(t) + \Pi(t)A(t) - \Pi(t)BB^T \Pi(t) + Q = 0$$

and

$$\frac{d}{dt}\Sigma(t) - A(t)\Sigma(t) - \Sigma(t)A(t)^T + \Sigma(t)BB^T \Sigma(t) - R = 0.$$

One can adapt our analysis and establish an error estimate analogous to (2.16) assuming that $\omega > 0$ and L are independent of $t \in [0, T]$.

3. Asymptotic performance of a closed-loop system. We will apply Theorem 2.1 repeatedly on the intervals $[T_i, T_i + T]$. Let us briefly recall the procedure. The open loop solution x^* to (1.5)–(1.7) is computed on $[T_i, T_{i+1}]$, and based on it $\bar{x}$ is determined; see (1.13). To refer to a specific horizon we henceforth use $\bar{x}_{T_i}$ for $\bar{x}$. This determines $A(\bar{x}_{T_i})$ and allows us to compute $\Pi = \Pi(\bar{x}_{T_i})$ and $\Sigma = \Sigma(\bar{x}_{T_i})$ as solutions to the corresponding Riccati equations. The compensator can then be defined on the basis of (1.10)–(1.12). To simplify the following discussion we assume that $T_{i+1} - T_i = T$ for all i. We shall assume that

$$E(0) \le \frac{\delta}{2} \text{ and } |x^*(t) - \bar{x}| \le \delta \text{ on } [T_i, T_{i+1}] \text{ for all } i = 0, \ldots \tag{3.1}$$

and that (2.10) and (2.11) hold uniformly on all horizons $[T_i, T_{i+1}]$. In view of the continuity of $x \to A(x)$, we have continuity of $x \to \Pi(x)$ and $x \to \Sigma(x)$. Due to these properties and the fact that the open loop control x^* is typically guaranteed to be continuous and bounded on $[0, \infty)$ (see, e.g., [IK1]), assumptions (2.10), (2.11), and (3.1) are natural ones. As $\bar{x}_{T_i}$ changes from one horizon to the next, so does $\mathcal{S} = \mathcal{S}_{T_i}$ in (2.13). We assume that (2.13) holds uniformly for all horizons as well and that

$$\int_0^t e^{-\tilde{\omega}(t-s)} \rho(s + T_i)\, ds < \frac{\delta}{2} \text{ for } t \in [0, T] \text{ and all } i = 0, 1 \ldots. \tag{3.2}$$

Finally we require that

$$\int_0^T e^{-\tilde{\omega}(T-t)}\, \rho(T_i + t)\, dt \le \frac{\delta}{2}(1 - e^{-\tilde{\omega}T}) \text{ for } i = 0, 1 \ldots. \tag{3.3}$$

These assumptions imply that

$$E(t) \le e^{-\tilde{\omega}\,(t-T_i)} E(T_i) + \int_0^{t-T_i} e^{-\tilde{\omega}(t-T_i-s)}\, \rho(T_i + s)\, ds \text{ for } t \in [T_i, T_{i+1}] \tag{3.4}$$

for every i, provided that $E(T_i) \le \frac{\delta}{2}$. Note that (3.3), (3.4) imply

$$E(T_{i+1}) \le e^{-\tilde{\omega}\, T}\, E(T_i) + \frac{\delta}{2}(1 - e^{-\tilde{\omega}T}) \le \frac{\delta}{2},$$

and hence by induction $E(T_i) \le \frac{\delta}{2}$ for all i.

THEOREM 3.1. *If* (3.1)–(3.3) *hold and* (2.10), (2.11), *and* (2.13) *are satisfied uniformly, then*

$$E(t) \le e^{-\tilde{\omega}\,(t-T_i)} E(T_i) + \int_0^{t-T_i} e^{-\tilde{\omega}(t-T_i-s)}\, \rho(T_i + s)\, ds$$

for $t \in [T_i, T_{i+1}]$ *and all* i. *In particular this implies*

$$E(t) \le e^{-\tilde{\omega}t} E(0) + \int_0^t e^{-\tilde{\omega}(t-s)} \rho(s)\, ds \text{ for all } t > 0.$$

Note that on each horizon $[T_i, T_{i+1}]$ we must have $|x^*(t) - \bar{x}| \le \delta$ on $[0, T]$. This may necessitate to take T smaller than $T_{i+1} - T_i$ to ensure that $|x^*(t) - \bar{x}| \le \delta$ on $[T_i, T_i + T]$. In this case, we can further partition the interval $[T_i, T_{i+1}]$ into subintervals and use consecutive linearization on each subinterval so that the condition is satisfied. The extreme case of this procedure results in the time-varying synthesis as in Remark 2.2.

Concerning the condition $|x^*(t) - \bar{x}| \le \delta$ we can use an alternative approach motivated by the H^∞ Riccati equation. This will be discussed in the next section.

4. H^∞ Riccati synthesis. In this section we present a modification of the approach proposed in section 1 motivated by H^∞ synthesis. We assume that there exists an attenuation bound $\gamma > 0$ such that

$$A^T\Pi + \Pi\, A - \Pi \left(B\, B^T - \frac{1}{\gamma} I \right) \Pi + \frac{1}{\gamma} I + Q = 0 \tag{4.1}$$

has a positive definite solution Π and

$$A\Sigma + \Sigma A^T - \Sigma \left(C^T C - \frac{1}{\gamma} I \right) \Sigma + \frac{1}{\gamma} I + R = 0 \tag{4.2}$$

has a positive definite solution Σ, where again $A = f'(\bar{x})$. These Riccati equations are similar to those used in the equivalence between H^∞ controllers and linear quadratic zero-some differential games where in our case u and w are the two players [BB]. In the following proposition x and w denote the solutions to (1.8) and (2.7)with u given in (2.9) and x^* is the solution to (1.6) with the optimal open loop control u^*.

PROPOSITION 4.1. *For* $t \in [0, T]$ *we have*

$$\begin{aligned}
&\frac{d}{dt} W(x - x^*) \\
&= -\tfrac{1}{2}(|B^T\Pi(x - x^*)|^2 + (Q(x - x^*), x - x^*) + \tfrac{1}{\gamma}\, |\Pi(x - x^*)|^2 + \tfrac{1}{\gamma}|x - x^*|^2) \\
&\qquad + (r(x, x^*) + d, \Pi(x - x^*)) + (B^T\Pi(x - w), B^T\Pi(x - x^*))
\end{aligned}$$

and

$$\frac{d}{dt}\tilde{W}(x-w)$$
$$= -\frac{1}{2}(|C(x-w)|^2 + (R\Sigma^{-1}(x-w), \Sigma^{-1}(x-w)) + \frac{1}{\gamma}|x-w|^2 + \frac{1}{\gamma}|\Sigma^{-1}(x-w)|^2)$$
$$+(r(x,x^*) + d, \Sigma^{-1}(x-w)) - (n, C(x-w)),$$

where

$$r(x,x^*) = f(x) - f(x^*) - A(\bar{x})(x-x^*),$$

and the dependence of the variables on t is suppressed.

Proof. The proposition follows from the proofs of Propositions 2.1 and 2.2 observing that

$$((A - BB^T\Pi)x, \Pi x) = -\frac{1}{2}\left(|B^T\Pi x|^2 + (Qx,x) + \frac{1}{\gamma}|x|^2 + \frac{1}{\gamma}|\Pi x|^2\right),$$

and

$$((A - \Sigma C^T C)z, \Sigma^{-1}z) = -\frac{1}{2}\left(|Cz|^2 + (R\Sigma^{-1}z, \Sigma^{-1}z) + \frac{1}{\gamma}|z|^2 + \frac{1}{\gamma}|\Sigma^{-1}z|^2\right). \qquad \square$$

Since

$$r(x(t),x^*(t)) = \left(\int_0^1 A(x^*(t) + \theta(x(t) - x^*(t)))\,d\theta - A\right)(x(t) - x^*(t)),$$

we can observe that the assumption

$$\left\|\int_0^1 A(x^*(t) + \theta(x(t) - x^*(t)))\,d\theta - A\right\| \le \frac{1}{\sqrt{2}\gamma} \text{ for all } t \in [0,T] \tag{4.3}$$

implies that

$$(r(x(t),x^*(t)), \Pi(x(t) - x^*(t))) \le \frac{1}{2\gamma}\left(|\Pi(x(t) - x^*(t))|^2 + \frac{1}{2}|x(t) - x^*(t)|^2\right), \tag{4.4}$$

and

$$(r(x(t),x^*(t)), \Sigma^{-1}(x(t) - x^*(t))) \le \frac{1}{2\gamma}\left(|\Sigma^{-1}(x(t) - w(t))|^2 + \frac{1}{2}|x(t) - x^*(t)|^2\right) \tag{4.5}$$

Using (2.11) and (4.4), (4.5) we find

$$\frac{d}{dt}(W(x-x^*)+\tilde W(x-w))$$

$$\le -\omega(W(x-x^*)+\tilde W(x-w))+(d+r(x,x^*),\Pi(x-x^*)+\Sigma^{-1}(x-w))-(n,C(x-w))$$

$$-\frac{1}{2\gamma}(|\Pi(x-x^*)|^2+|x-x^*|^2+|x-w|^2+|\Sigma^{-1}(x-w)|^2)$$

$$\le -\omega(W(x-x^*)+\tilde W(x-w))+\rho(t)(W(x-x^*)+\tilde W(x-w))^{1/2}$$

$$+\frac{1}{2\gamma}(|\Pi(x-x^*)|^2+|x-x^*|^2+|\Sigma^{-1}(x-w)|^2)$$

$$-\frac{1}{2\gamma}(|\Pi(x-x^*)|^2+|x-x^*|^2+|x-w|^2+|\Sigma^{-1}(x-w)|^2),$$

where $\rho(t)$ was defined below (2.14). Hence it follows that

$$\frac{d}{dt}(W(x-x^*)+\tilde W(x-w))\le -\omega(W(x-x^*)+\tilde W(x-w))+\rho(t)((W(x-x^*)+\tilde W(x-w)))^{\frac12},$$

and therefore

$$\frac{d}{dt}E(t)\le -\omega E(t)+\rho(t).$$

We can summarize the above developments in the following result.

THEOREM 4.1. *If* (4.3) *and* (4.2) *admit solutions for* $\gamma>0$ *and* (2.10), (2.11), (4.3) *hold on* $[0,T]$, *then*

$$E(t)\le e^{-\omega\, t}E(0)+\int_0^t e^{-\omega(t-s)}\,\rho(s)\,ds \text{ for } t\in[0,T]. \tag{4.6}$$

If the assumptions hold uniformly on all intervals, then (4.6) *holds for all* $t\in[0,\infty)$.

5. Numerical examples. We validate the proposed approach by means of a class of optimal control problems for the Burgers equation

$$\min\ \frac12\int_0^{T_\infty}|y(t)-z(t)|^2_{L^2(\Omega)}dt\ +\ \frac{\sigma}{2}\int_0^{T_\infty}|u(t)|^2_{L^2(\tilde\Omega)}dt \tag{5.1}$$

subject to $u\in L^2(\tilde\Omega)$ and

$$\begin{cases} \frac{d}{dt}\,y(t,x)=\nu\, y_x x(t,x)-y\cdot y_x(t,x)+Bu(t,x)+d(t,x), \quad t>0,\\ y(t,0)=y(t,1)=0, \quad t>0,\\ y(0,x)=\varphi(x), \quad x\in(0,1), \end{cases} \tag{5.2}$$

where $y(t)=y(t,x)$, $x\in\Omega=(0,1)$, and ν and σ are positive constants. Here the initial condition $\varphi\in L^2(\Omega)$, the control domain $\tilde\Omega$ is a subset of Ω, and B is the extension-by-zero-operator from $\tilde\Omega$ to Ω. Finally d represents noise to the system and the observation data are supposed to be of the form

$$\hat y(t)=C_{\hat\Omega}\,y(t)+n(t), \tag{5.3}$$

where C is the restriction operator from Ω to $\hat{\Omega}$ with $\hat{\Omega}$ a subset of Ω, and n stands for observation noise. We further need to specify the operators G for the terminal weight in the receding horizon cost (1.5), the tracking weight Q in (2.3) and the operator R in the Kalman filter equation. We shall set

$$R = r\, I \quad \text{and} \quad Q = q\, I. \tag{5.4}$$

After several tests G was taken to be 0 for stabilization problems ($z = 0$) and as a scalar multiple of $\frac{1}{2}(x, \Pi x)$, where Π is the nonnegative solution of the algebraic Riccati equation (2.4) with $A = -\nu\, \partial_{xx}$ for tracking problems with $z \neq 0$. In this way G does not depend on the specific receding horizon level and can be computed during the initialization phase. Choosing G instead as $G(x) = \alpha\, |x|^2$ resulted in longer computing times and less favorable tracking properties.

The spatial discretization was done by the standard Galerkin scheme applied to (5.2) based on the basis of linear finite elements with meshsize dx. The ordinary differential equations resulting from (5.2) and (2.2), (2.7) were solved with an implicit Euler scheme with stepsize dt, while resolving the nonlinearities by Newton's method. The resulting linear systems were solved by inexact GMRES iterations. Unless specified otherwise we took

$$\nu = .01, \quad dt = .05, \quad dx = .025,$$

and for the receding horizon, $T = .5$. Further, unless quoted otherwise we chose $\Omega = \tilde{\Omega} = \hat{\Omega}$, $q = 10^{-5}$, and $r = 10^3$. For Example 1 we took $\sigma = .0175$ and for Example 2 we calculated with $\sigma = 10^{-3}$. The MATLAB-routine CARE was used to solve the algebraic Riccati equations. Below, $\hat{J}$ denotes the tracking part of the cost in (5.1). Noise was simulated by choosing uniformly distributed random numbers in the interval $[-\delta, \delta]$. It was added to the spatial-temporal grid points for either d or n representing disturbance to the equation and noise in the observation, respectively. Analogously, noise in the initial data φ was simulated by adding random numbers in $[-\delta, \delta]$ to the values of φ at the spatial grid points. The initial condition for (5.2) on receding horizon intervals with $i \geq 2$ was chosen as the state of the estimator at time T_i.

Example 5.1 (stabilization). *Here we choose*

$$\varphi(x) = \begin{cases} 5\sin 2\pi x & \text{on } (0, \frac{1}{2}], \\ 0 & \text{on } (\frac{1}{2}, 1), \end{cases}$$

and $z = 0$. Further $T^\infty = 5$, $T = 0.5$, $T_i = iT$ so that we have 10 receding horizon intervals. The uncontrolled solution is depicted in Figure 5.1. For Figures 5.2–5.4 we chose $n = 0$ (no observation noise) and simulated disturbances to the system with noise level $\delta = 10$ for d. The result for Figure 5.2 shows an open-loop solution, i.e., first we compute the open-loop optimal control u^* to (5.1)–(5.2), with $d = 0$, and then use the optimal open-loop control u^* against the noisy Burgers equation with $d = 10$. Figure 5.3 is for the full observation case, obtained with the state estimator procedure introduced in section 1, with $\hat{\Omega} = \tilde{\Omega} = \Omega$. For Figure 5.4 we test a partial observation case, i.e., we chose the control domain $\tilde{\Omega} = (0, \frac{1}{2})$ and the observation domain $\hat{\Omega} = (\frac{1}{2}, 1)$. Thus control and observation domain are non-overlapping. It should be noted that the hyperbolic nature of the Burgers equation is such that information moves from smaller to larger x-values. Figures 5.5 and 5.6 show

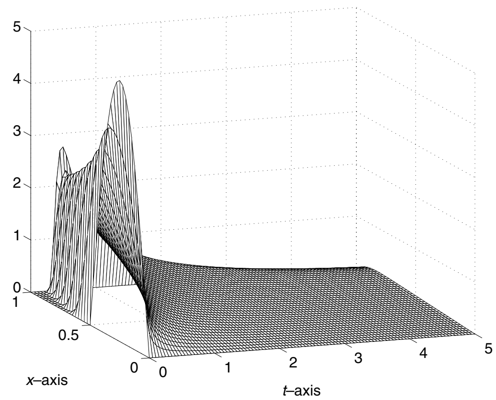

FIG. 5.1. *Uncontrolled solution, no noise.*

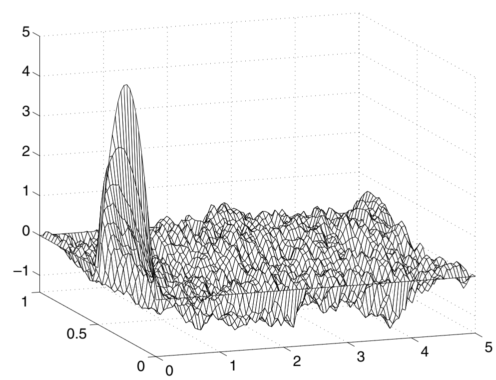

FIG. 5.2. *Open-loop controlled solution.*

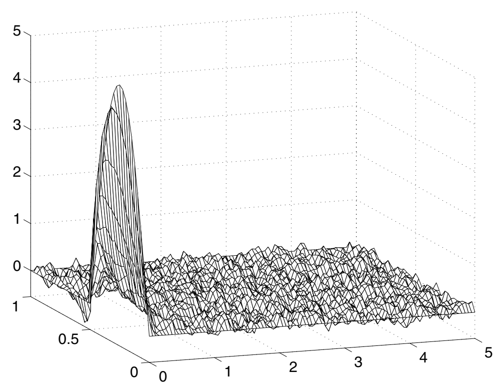

FIG. 5.3. *Estimator closed loop.*

results for the noisy observation case and we take again $\tilde{\Omega} = (0, \frac{1}{2})$ *and* $\hat{\Omega} = (\frac{1}{2}, 1)$, *i.e., the control and observation domains, which are as those used for Figure* 5.4. *Figure* 5.5 *disturbance to the equation was set* $d = 0$, *while now the observation data were taken to be noisy with noise level* $\delta = 10$ *for* n. *For Figure* 5.6 *noise was added to the initial condition* ($\delta = 5$) *and to the observation data* ($\delta = 10$).

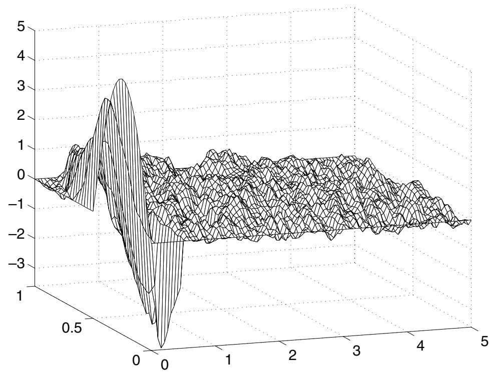

Fig. 5.4. *Partial control and observation domain.*

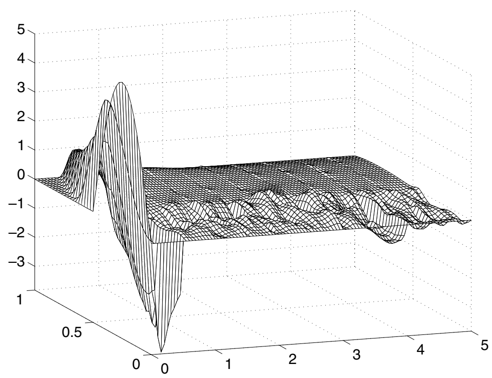

Fig. 5.5. *Noise in observation.*

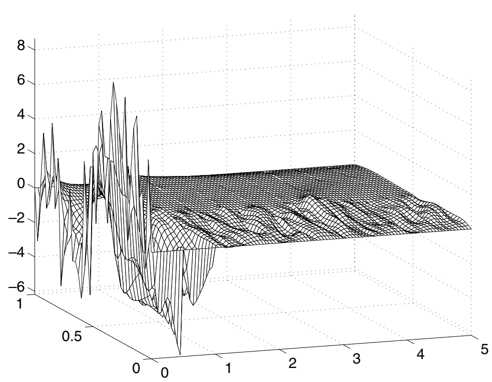

Fig. 5.6. *Noise in observation and initial condition.*

The run-times depend slightly on the random numbers that enter in d *or* n*. Typical times for Figures* 5.3–5.6 *are* 16 *to* 17 *units, whereas the run-time for Figure* 5.2*, which uses the open-loop optimal control on the complete interval* $[0, T^\infty]$*, is* 181 *units. The stabilization value for Figure* 5.2 *is* $\hat{J} = .540$*, whereas it is only* $\hat{J} = .146$ *for Figure* 5.3*. In Table* 1 *we give the stabilization values* $\hat{J}_-$ *for the results of Figures* 5.2–5.6 *on* $[0.5, 5]$*, i.e., for the receding horizon intervals* 2–10*.*

TABLE 1
$\hat{J}_-$.

Fig 5.2	Fig 5.3	Fig 5.4	Fig 5.5	Fig 5.6
.17-.5	.0008-.0009	.070-.114	.056-.065	.1-.5

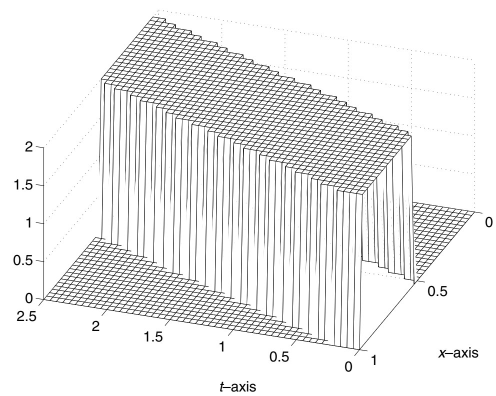

FIG. 5.7. *Desired state.*

We ran all tests with five randomly chosen initializations of the random number generator. In Table 1 we report the range of obtained values for $\hat{J}_-$. We can conclude that the receding horizon strategy combined with the dynamic observer introduced in section 1 attenuates noise significantly better than the open-loop controlled system; compare column 1 in Table 1 to columns 2–4. The computing times for the receding horizon strategy are much shorter than for the open-loop optimal control on the complete time interval. Finally the strategy can cope with partial observations.

EXAMPLE 5.2 (tracking). *Here we set the initial condition as*

$$\varphi(x) = \begin{cases} 1 & \text{on } (0, \frac{1}{2}], \\ 0 & \text{on } (\frac{1}{2}, 1), \end{cases}$$

and the target z as the characteristic function of the set $\{(t,x)\colon 2.5-5x < t < 5-5x,\ x \in (0,1)\}$; see Figure 5.7. Further $T^\infty = 2.5$, $T = 0.5$, $T_i = iT$ so that we have five receding horizon intervals. Note that the uncontrolled solution of the Burgers equation would transport the jump at $x = \frac{1}{2}$ toward increasing x as t increases, while decreasing its height. Also, the desired state z moves into the direction which is opposite to the one of the characteristics of the Burgers equation. Thus this example can be considered as a challenging one. For the results presented here we took the control domain as $\tilde{\Omega} = \Omega$. The noise levels are 5 for disturbance to the equation and 10 for observation noise and disturbance to the initial condition. If noise is not mentioned, then it is set equal to 0. The random number generator is initialized at the same seed, independently for disturbances and observations.

In Figures 5.8 and 5.9 we show the open-loop and the full-observation ($\hat{\Omega} = \Omega$) closed-loop results, each with disturbances to the equation. Figures 5.10 and 5.11 show partial observation results with the observation domain $\hat{\Omega} = (1/4, 1)$ for nonzero disturbance d and nonzero observation noise n, respectively. Figure 5.12 provides the result for the case with the partial observation on $\hat{\Omega} = (1/4, 1)$, $d = 0$, and noise

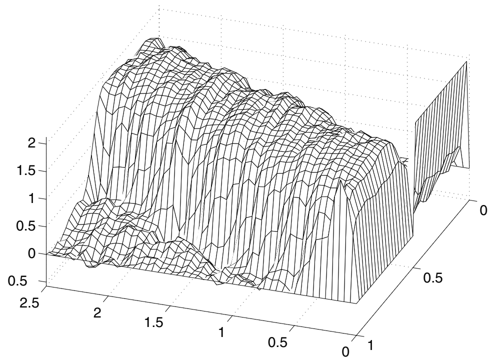

FIG. 5.8. *Open-loop controlled solution.*

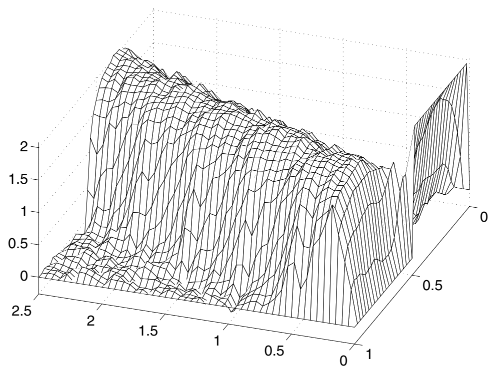

FIG. 5.9. *Estimator closed loop.*

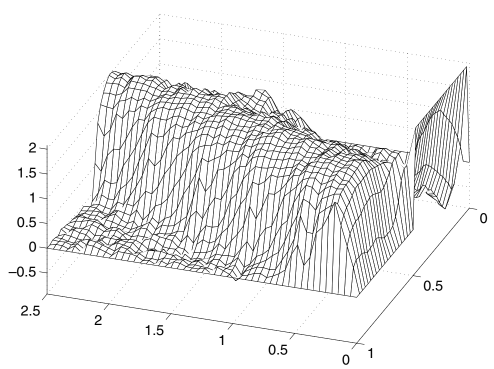

FIG. 5.10. *Estimator closed loop, partial observation.*

entering to the observations as well as the initial condition. The tracking costs for Figures 5.8–5.12 *are given in Table* 2.

For Figure 5.8 *we again computed the open-loop optimal control from* $[0, T^{\infty}]$ *and applied it to the perturbed Burgers equation. The computing time for Figure* 5.8 *is* 56 *units, whereas for the other figures it is* 16.3 *to* 20 *units. Thus the ratio of the computing times for the open-loop control on* $[0, T^{\infty}]$ *and the receding horizon*

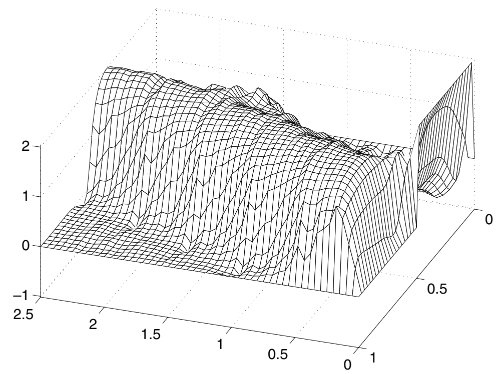

Fig. 5.11. *Estimator, noise in data.*

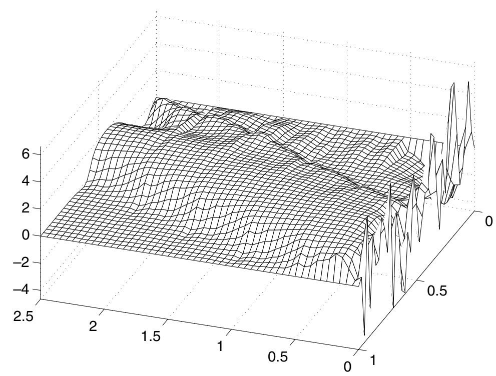

Fig. 5.12. *Noise in data and initial condition.*

Table 2
$\hat{J}$.

Fig 5.8	Fig 5.9	Fig 5.10	Fig 5.11	Fig 5.12
.179	.0816	.0814	.0812	.0887

controls is less favorable than for Example 5.1*. This is because the resulting two-point boundary value problems are less well-conditioned as a consequence of the terminal weight* $G \neq 0$*.*

We also tested with smaller observation domain $\hat{\Omega}$ *and obtained comparable results. Reducing the control domain* $\tilde{\Omega}$ *results in a significant increase of* $\hat{J}$ *due to the challenging nature of the problem. For example, for the settings of Figure* 5.10*, if* $\tilde{\Omega} = (0, .8)$*, then* $\hat{J} = .490$*.*

6. Conclusion. The receding horizon technique is a sequential time-domain decomposition method to reduce the computational requirements for solving the optimality systems on long time horizons. The obtained controls on each receding horizon can be interpreted as a state feedback control. In this paper we developed a compensator design in the context of receding horizon control when only partial state observations are available. The receding horizon synthesis was combined with the state estimator dynamics. An LQG design based on a linearization procedure was used and its asymptotic performance was analyzed for nonlinear systems with disturbances and observation noise. Numerical examples for control problems for the Burgers equation validate the proposed methodology.

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