Differentiability properties of the L^1-tracking functional and application to the Robin inverse Problem *

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Abstract

We investigate an optimization problem (OP) in a non-standard form: The cost functional $\mathcal{F}$ measures the L^1 distance between the solution u_φ of the direct Robin problem and a function $f \in L^1(M)$. After proving positivity, monotonicity and control properties of the state u_φ with respect to φ, we prove the existence of an optimal control ψ to the problem (OP) and establish Newton differentiability of the functional $\mathcal{F}$.
As application to this optimization problem the inverse problem of determining a Robin parameter φ_{inv} by measuring the data f on M is considered. In that case f is assumed to be the trace on M of $u_{\varphi_{inv}}$. In spite of the fact that we work with the L^1-norm we prove differentiability of the cost functional $\mathcal{F}$ by using complex analysis techniques. The proof is strongly related to positivity and monotonicity of the derivative of the state with respect to φ. An identifiability result is also proved for the set of admissible parameters Φ_{ad} consisting of positive functions in L^∞.

Introduction

We consider a simply connected bounded Jordan domain Ω in $\mathbb{R}^2$ with $\mathcal{C}^{1,\beta}$ boundary $\partial\Omega$, $\beta \in]0,1[$. Let Γ_1 and Γ_2 two nonempty connected disjoint open subsets of $\partial\Omega$, satisfying $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ and let M be a nonempty connected open set of Γ_1 such that $\partial M \cap \partial\Gamma_1 = \emptyset$.

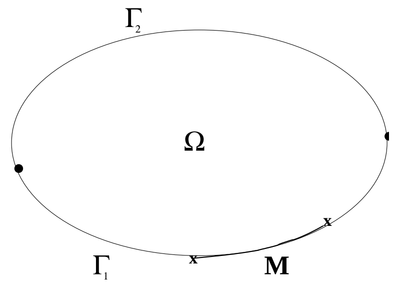

Figure 1: The domain and its boundary.

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Let $c, c' > 0$, and $\mathcal{K}$ be a nonempty connected subset of Γ_2 with $\partial\mathcal{K} \cap \partial\Gamma_2 = \emptyset$. We denote by $\mathcal{D}$ the set:

$$\mathcal{D} = \{\varphi \in \Phi_{ad} \text{ such that } \varphi \leq c' \text{ and } \varphi \geq c\chi_{\mathcal{K}}\},$$

where Φ_{ad} is the set of admissible parameters:

$$\Phi_{ad} = \{\varphi \in L^\infty(\Gamma_2), \quad \varphi \not\equiv 0 \text{ and } \varphi \geq 0 \text{ a. e. on } \Gamma_2\}. \tag{1}$$

Let $\Phi \in \mathcal{C}_0^0(\Gamma_1)$ with $\Phi \geq 0$, $\Phi \not\equiv 0$, and $f \in L^1(M)$. We study the following optimization problem:

$$(OP)\begin{cases} min \int_M |u_\varphi - f|, \\ \text{subject to } \varphi \in \mathcal{D}, \end{cases}$$

where u_φ is the solution of the following problem:

$$(RP)\begin{cases} \Delta u &= 0 \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} &= \Phi \quad \text{on} \quad \Gamma_1, \\ \frac{\partial u}{\partial n} + \varphi u &= 0 \quad \text{on} \quad \Gamma_2. \end{cases}$$

We define the functional $\mathcal{F}$ by:

$$\begin{aligned} \mathcal{F}: \quad \Phi_{ad} \subset L^\infty(\Gamma_2) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \mathcal{F}(\varphi) = \int_M |u_\varphi - f|. \end{aligned}$$

We first prove positivity and monotonicity properties of the solution u_φ with respect to the parameter φ. Such results were established in [6] by using the Hopf maximum principle in case the admissible parameters is set given by:

$$\mathcal{A} = \{\varphi \in \mathcal{C}_0^0(\overline{\Gamma_2}), \quad \varphi \geq 0 \text{ and } \varphi \not\equiv 0\}.$$

If the parameters φ are only nonnegative in the L^∞ sense, as required by Φ_{ad}, then the Hopf maximum principle is not applicable. The main idea in order to prove these two properties is to return to the set $\mathcal{A}$ by density.
Existence of a global minimum ψ to the problem (OP), continuity property of the minimum ψ with respect to the data f and differentiability of the state u_φ with respect to φ are proved in the second part of this work where the positivity and monotonicity properties of the state u_φ with respect to $\varphi \in \Phi_{ad}$ are used. Our next aim is to investigate differentiability properties of the function $\mathcal{F}$: For a given parameter $\varphi \in \Phi_{ad}$, the function $u_\varphi - f$ can be positive in some part of M and negative in another, resulting in a possible lack of differentiability of $\mathcal{F}$. In that case Newton differentiability of the functional $\mathcal{F}$ is proved and a generalized derivative G of $\mathcal{F}$ is established in the third part.
In the last part of this work we study the particular case where the function f is the trace on M of a solution $u_{\varphi_{inv}}$ of (RP):

$$f = u_{\varphi_{inv}|M},$$

which is the case for the following Robin inverse problem:

$$(IP)\begin{cases} \text{Given a prescribed flux } \Phi \text{ and measurements } f \text{ on } M, \\ \text{recover the function } \varphi_{inv} \in \Phi_{ad} \text{ such that } u_{\varphi_{inv}|M} = f. \end{cases}$$

We start this part by establishing an identifiability result allowing for discontinuous parameters φ. This improves an earlier result form [5] where the parameters were required to be continuous. This result implies that φ_{inv} is the global minimum of $\mathcal{F}$ in Φ_{ad}. After proving positivity, monotonicity and an a-priori bound of u^1_φ, the derivative of the state with respect to φ, we establish by complex analysis techniques that the set:

$$S_\varphi \;=\; \{x \in M \text{ such that } u_\varphi(x) \;=\; f(x)\}\,,$$

is a finite for every $\varphi \in \Phi_{ad}$ with $\varphi \neq \varphi_{inv}$. This property, which is strongly related to the fact that the data f are the trace on Γ_1 of the harmonic function $u_{\varphi_{inv}}$, is the main result in order to prove the differentiability of the functional $\mathcal{F}$.

Many authors were interested in the Robin inverse problem. Uniqueness, stability and identification process in the case of a thin plate domains was studied in [10], [14]. In a 2D domain, local, monotonic and logarithmic stability results for the Robin inverse problem were established in [5], [3], [9]. In [2], the stability of the Kohn-Vogelius method with respect to the $H^{\frac{1}{2}}$ data perturbations and the numerical implementation of this method were established. Another method based on complex analysis and analytic functions theory was investigated in [6]. The robustness of this method in the case of $W^{n,2}$, with $n \geq 2$, data perturbations was studied. In the context of the present paper, we can allow for L^1 data perturbations.

The motivation for the L^1-cost functional stems from the theory of robust statistics. In fact, in the case of noisy data f with outliers, the choice of a quadratic cost functional will unnecessarily exaggerate the error in the data in a neighborhood of the outlier and as a consequence have a negative effect on the reconstruction of φ. Special attention was given to the construction of functionals which reduce such effects, see e.g. [13], with a prominent example being L^1-type cost functionals. More recently these ideas were also introduced to area of image processing, [15]. The advantage of the robustness of such functionals against the influence of outliers must be weighed against their lack of differentiability. In the context of L^1-functionals one can rely on optimization algorithms which are specifically designed for such functionals. Alternatively one can employ Fenchel duality theory and characterize the (pre-) dual problem, which is a bilaterally constrained problems. This approach was successfully carried out in [12] for deconvolution problems. The numerical treatment of bilaterally constrained problems in turn has received a significant amount of attention in the past. - While the L^1-cost is preferable over the L^2-cost in the case of aberrant data, the problem of determing φ remains to be illposed and for a numerical realization regularization will be required. In view of an improved cost functional we expect it to be less significant, however.

1 Positivity and monotonicity properties of u_φ

Let $\mathcal{S} \;=\; \{\varphi \in L^2(\Gamma_2) \;:\; \varphi \not\equiv 0 \text{ and } \varphi \geq 0 \text{ a. e. on } \Gamma_2\}$, be endowed with the L^2 norm and choose $\varphi \in \mathcal{S}$. We denote by L and A the linear and bilinear form defined on $H^1(\Omega)$ by:

$$L(v) \;=\; \int_{\Gamma_1} \Phi v \;;\quad A(u,v) \;=\; \int_\Omega \nabla u\, \nabla v \;+\; \int_{\Gamma_2} \varphi\, u\, v,$$

We refer to [4] for the proof of the following lemmas:

Lemma 1 *The mapping:*

$$\begin{aligned} \eta:\ \ & \mathcal{S} \longrightarrow H^1 \\ & \varphi \longmapsto u_\varphi \end{aligned} \tag{2}$$

is well defined and locally Lipschitzian.

Lemma 2 *The mapping η is a decreasing function on $\mathcal{S}$, i.e. $0 \leq \varphi \leq \psi$ implies: $0 \leq u_\psi \leq u_\varphi$.*

Lemma 3 *For every $\varphi \in \Phi_{ad}$ we have*

$$\inf_{x \in \overline{\Omega}} u_\varphi(x) \;>\; 0.$$

2 Existence of an optimal control to (OP) and differentiability property of the mapping $\varphi \longmapsto u_\varphi$.

We start by proving existence for (OP).

Theorem 1 *The optimization problem (OP) has a global minimum ψ.*

Proof:
Let $\delta = \inf_{\varphi \in \mathcal{D}} \mathcal{F}(\varphi)$ and $\varphi_n \in \mathcal{D}$ a minimizing sequence: $\delta = \lim_{n\to\infty} \mathcal{F}(\varphi_n)$.
We have $0 \leq \varphi_n \leq c'$. Referring to [1], there exist $\psi \in \mathcal{D}$ and a subsequence of φ_n also denoted by φ_n such that:

(3) $$\varphi_n \rightharpoonup \psi \text{ in } L^\infty \text{ weak } * .$$

¿From Lemma 2, the function $u_n = u_{\varphi_n}$ satisfies:

(4) $$u_{c'} \leq u_n \leq u_c \text{ for every } n \in \mathbb{N},$$

where u_c and $u_{c'}$ denote the solutions of the Robin problem (RP) for $\varphi = c\chi_{\mathcal{K}}$ and $\varphi = c'$ respectively.
Using Robin boundary conditions and equation (4), there exist a constant $\beta > 0$ such that:

(5) $$\|\frac{\partial u_n}{\partial n}\|_{L^\infty(\Gamma_2)} \leq \beta.$$

Let $c_n = \frac{\int_{\partial\Omega} u_n}{mes(\partial\Omega)}$ and $w_n = u_n - c_n$. The function w_n satisfies the following problem:

$$\begin{cases} \Delta w_n & = 0 & \text{in} & \Omega, \\ \frac{\partial w_n}{\partial n} & = \Phi & \text{on} & \Gamma_1, \\ \frac{\partial w_n}{\partial n} & = \dfrac{\partial u_n}{\partial n} & \text{on} & \Gamma_2, \\ \int_{\partial\Omega} w_n & = 0. \end{cases}$$

¿From (5), we have that $\{\dfrac{\partial w_n}{\partial n}\}$ is bounded in $L^2(\partial\Omega)$. Due to the shift theorem [8], the function w_n is bounded independently of n in $W^{\frac{3}{2},2}(\Omega)$. Using again Lemma 2, the sequence $\{c_n\}$ is also bounded independently of n. Consequently, there exist a constant $\gamma > 0$ independent of n such that:

$$\|u_n\|_{W^{\frac{3}{2},2}(\Omega)} \leq \gamma.$$

Then, there exist a function $u \in H^1(\Omega)$ and a subsequence of u_n also denoted by u_n such that:

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } H^1(\Omega) \\ u_n \longrightarrow u \text{ strongly in } L^2(\partial\Omega) \end{cases}$$

We shall prove now that $u = u_\psi$. For every $v \in H^1(\Omega)$, we have

$$\int_{\Gamma_2} \varphi_n u_n v = \int_{\Gamma_2} \varphi_n (u_n - u) v + \int_{\Gamma_2} \varphi_n u v.$$

Due to the strong convergence of u_n to u in $L^2(\Gamma_2)$, boundedness of φ_n in the L^∞, and from (3) we get:

$$\lim_{n\longrightarrow\infty} \int_{\Gamma_2} \varphi_n u_n v = \int_{\Gamma_2} \psi u v.$$

In the other hand we have:

$$\int_\Omega \nabla u_n \nabla v + \int_{\Gamma_2} \varphi_n u_n v = \int_{\Gamma_1} \Phi v.$$

Consequently, the function u satisfies:

$$\int_\Omega \nabla u \nabla v + \int_{\Gamma_2} \psi u v = \int_{\Gamma_1} \Phi v.$$

Consequently $u = u_\psi$ and $\delta = \lim_{n\to\infty} \mathcal{F}(\varphi_n) = \mathcal{F}(\psi)$ from the strong convergence in $L^2(\partial\Omega)$ of u_n to u. ■

The following result establishes stability of the solution ψ with respect to the data f.

Theorem 2 *Let $\{f_n\}$ be a sequence of $L^1(M)$ converging strongly to f in $L^1(M)$ and set $\psi_n \in \mathcal{D}$ such that: $\int_M |u_{\psi_n} - f_n| = \min_{\varphi\in\mathcal{D}} \int_M |u_\varphi - f_n|$. Then, there exist a subsequence of ψ_n also denoted by ψ_n and $\psi \in \mathcal{D}$ a global minimum of (OP) such that: $\psi_n \rightharpoonup \psi$ in L^∞ weak $*$.*

Proof:
¿From the boundedness of the sequence $\{\psi_n\}$, there exist $\psi \in \mathcal{D}$ and a subsequence of $\{\psi_n\}$ also denoted by $\{\psi_n\}$ such that:

$$\psi_n \rightharpoonup \psi \text{ in } L^\infty \text{ weak } * . \tag{6}$$

By using the same technique as in the proof of Theorem 1, there exists a subsequence of u_{ψ_n} also denoted by u_{ψ_n} such that:

$$\begin{cases} u_{\psi_n} \rightharpoonup u_\psi \text{ weakly in } H^1(\Omega) \\ u_{\psi_n} \longrightarrow u_\psi \text{ strongly in } L^2(\partial\Omega). \end{cases}$$

We have: $\int_M |u_{\psi_n} - f_n| \le \int_M |u_\varphi - f_n|$ for every $\varphi \in \mathcal{D}$, where $f_n \longrightarrow f$ strongly in $L^1(M)$ and $u_{\psi_n} \longrightarrow u_\psi$ strongly in $L^2(\partial\Omega)$. Consequently:

$$\int_M |u_\psi - f| \le \int_M |u_\varphi - f|,$$

and hence, ψ is a global minimum to the (OP). ■

For $\varphi \in \Phi_{ad}$, denote by u^1_φ the linear mapping:

$$u^1_\varphi : \begin{array}{ll} L^\infty(\Gamma_2) & \longrightarrow H^1(\Omega) \\ h & \longmapsto u^1_\varphi(h), \end{array} \tag{7}$$

where $u^1_\varphi(h)$ is the solution to the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} + \varphi u = -h\, u_\varphi & \text{on } \Gamma_2. \end{cases} \tag{8}$$

Referring to [4] we have the following theorem:

Theorem 3 *The linear mapping u^1_φ is continuous and for every $\varphi,\,\psi\,\in\,\Phi_{ad}$, we have:*

$$\lim_{\|\varphi-\psi\|_{L^\infty(\Gamma_2)}\to 0}\frac{\|u_\psi\,-\,u_\varphi\,-\,u^1_\varphi(\psi\,-\,\varphi)\|_{H^1(\Omega)}}{\|\varphi\,-\,\psi\|_{L^\infty(\Gamma_2)}}\;=\;0.$$

We also have for every φ , $\psi\in\Phi_{ad}$ satisfying $\|\varphi\,-\,\psi\|_{L^\infty(\Gamma_2)}\,<\,Min\left\{\frac{\|\varphi\|_{L^\infty(\Gamma_2)}}{2}\,,\,\frac{c_\varphi}{\alpha^2}\right\}$:

$$\|u^1_\psi\,-\,u^1_\varphi\|_{\mathcal{L}(L^\infty(\Gamma_2)\,,\,H^1(\Omega))}\;\le\;K_\varphi\,\|\psi\,-\,\varphi\|_{L^\infty(\Gamma_2)},$$

where K_φ is a constant independent of ψ and h. In particular, $\varphi\mapsto u_\varphi$ is Frechet differentiable at every φ in the interior of Φ_{ad}.

3 Newton differentiability of the functional $\mathcal{F}$

We first recall the definition of Newton differentiability. Then we prove that $\mathcal{F}$ is Newton differentiable and we calculate a generalized derivative G of $\mathcal{F}$.

Definition 1 *Let X and Z be two Banach spaces, and D an open subset of X.*
A function $F\,:\,D\,\longrightarrow\,Z$ is called Newton differentiable in the open subset $U\,\subset\,D$ if there exist a family of mappings $G:\,U\,\longrightarrow\mathcal{L}(X,Z)$ such that:

$$\lim_{h\longrightarrow 0}\frac{\|F(x\,+\,h)\,-\,F(x)\,-\,G(x\,+\,h)h\|}{\|h\|}\;=\;0,$$

for every $x\,\in\,U$.

We note that the function G is not required to be unique. Moreover, if the function F is Newton differentiable, then Newton's method for the resolution of the equation $F(x)\;=\;0$ converges super-linearly for appropriate choices of the initialization [11], [16].
Let us designate by G, the family of mapping:

$$\begin{array}{rccl} G: & \mathrm{Int}(\Phi^0_{ad}) & \longrightarrow & \mathcal{L}\left(L^\infty(\Gamma_2)\,,\,L^1(\Gamma_2)\right), \\ & \varphi & \longmapsto & G(\varphi), \end{array}$$

where for $h\,\in\,L^\infty(\Gamma_2)$:

$$G(\varphi)(h)(x)\;=\;\left\{\begin{array}{ccc} u^1_\varphi(h)(x) & \text{if} & u_\varphi(x)\,>\,f(x), \\ -\,u^1_\varphi(h)(x) & \text{if} & u_\varphi(x)\,<\,f(x), \\ 0 & \text{if} & u_\varphi(x)\,=\,f(x). \end{array}\right.$$

Theorem 4 *G is a generalized derivative of the map:*

$$\begin{array}{rccl} \zeta: & Int(\Phi_{ad})\,\subset\,L^\infty(\Gamma_2) & \longrightarrow & L^1(\Gamma_2) \\ & \varphi & \longmapsto & |u_\varphi\,-\,f|. \end{array} \qquad (9)$$

Proof:
Some ideas of this proof are inspired from [11], where Newton differentiability of the map:

$$\begin{array}{rcl} max(0\,,\,.)\;\;L^q(\Omega) & \longrightarrow & L^p(\Omega) \\ u & \longmapsto & max(0\,,\,u), \end{array}$$

is studied. Let $h \in L^\infty(\Gamma_2)$ and denoted by:

$$D_{\varphi,h} = |u_{\varphi+h} - f| - |u_\varphi - f| - G(\varphi + h)h.$$

We have:

$$|D_{\varphi,h}(x)| = \begin{cases} |u_{\varphi+h}(x) - u_\varphi(x) - u^1_{\varphi+h}(h)(x)| & \text{if} \quad [u_\varphi(x) - f(x)][u_{\varphi+h}(x) - f(x)] > 0, \\ |u_{\varphi+h}(x) - u_\varphi(x)| & \text{if} \quad u_{\varphi+h}(x) = f(x), \\ |u_{\varphi+h}(x) - u_\varphi(x) - u^1_{\varphi+h}(h)(x)| & \text{if} \quad u_\varphi(x) = f(x), \\ |u_{\varphi+h}(x) + u_\varphi(x) - 2f(x) - u^1_{\varphi+h}(h)(x)| & \text{if} \quad [u_\varphi(x) - f(x)][u_{\varphi+h}(x) - f(x)] < 0. \end{cases}$$

Let us denote:

$$o_1(h) = u_{\varphi+h} - u_\varphi - u^1_{\varphi+h}(h),$$

$$\mathcal{A}_h = \{x \in \Gamma_2 \text{ such that } [u_\varphi(x) - f(x)][u_{\varphi+h}(x) - f(x)] \geq 0 \text{ and } u_{\varphi+h}(x) \neq f(x)\},$$

and:

$$\mathcal{B}^0_h = \{x \in \Gamma_2 \text{ such that } [u_\varphi(x) - f(x)][u_{\varphi+h}(x) - f(x)] \leq 0 \text{ and } u_\varphi(x) \neq f(x)\},$$

and observe that $\Gamma_2 = \mathcal{A}_h \cup \mathcal{B}^0_h$. The proof of theorem is now divided in two steps:

Step1: On $\mathcal{A}_h$ we have $\|D_{\varphi,h}\|_{L^1(\mathcal{A}_h)} \leq \tau \sqrt{mes(\Gamma_2)}\ \|o_1(h)\|_{L^2(\Gamma_2)}$.

By Theorem 3 and the continuity of the trace mapping, we have $\lim\limits_{\|h\|_{L^\infty(\Gamma_2)} \longrightarrow 0} \dfrac{\|o_1(h)\|_{L^2(\Gamma_2)}}{\|h\|_{L^\infty(\Gamma_2)}} = 0,$ and therefore

(10) $$\lim_{\|h\|_{L^\infty(\Gamma_2)} \to 0} \frac{\|D_{\varphi,h}(x)\|_{L^1(\mathcal{A}_h)}}{\|h\|_{L^\infty(\Gamma_2)}} = 0.$$

Step 2: On $\mathcal{B}^0_h$ we first establish the following inequalities:

(11) $$|u_\varphi(x) - f(x)| \leq \left| u^1_\varphi(h)(x) + o(h)(x) \right|,$$

(12) $$|D_{\varphi,h}(x)| \leq 3 \left| u^1_\varphi(h)(x) + o(h)(x) \right| + |u^1_{\varphi+h}(h)(x)|,$$

for some $o(h) \in H^1(\Omega)$ satisfying $\lim\limits_{\|h\|_{L^\infty(\Gamma_2)} \longrightarrow 0} \dfrac{\|o(h)\|_{L^2(\Gamma_2)}}{\|h\|_{L^\infty(\Gamma_2)}} = 0.$

Let $x \in \mathcal{B}^0_h$ and consider two cases:

If $u_{\varphi+h}(x) = f(x)$, then by Theorem 3 and continuity of the trace mapping, there exist $o(h) \in H^1(\Omega)$ with the desired asymptotic behavior such that

$$u_{\varphi+h} = u_\varphi + u^1_\varphi(h) + o(h).$$

We then have

$$|u_{\varphi+h}(x) - f(x)| = \left| D_{\varphi,h}(x) \right| \leq \left| u^1_\varphi(h)(x) + o(h)(x) \right|.$$

If $\Big[u_\varphi(x) - f(x)\Big]\Big[u_{\varphi+h}(x) - f(x)\Big] < 0$, then

$$\Big[u_\varphi(x) - f(x)\Big]\ \Big[u_\varphi(x) - f(x) + u^1_\varphi(h)(x) + o(h)(x)\Big] < 0,$$

and hence

$$|u_\varphi(x) - f(x)| \leq \left| u^1_\varphi(h)(x) + o(h)(x) \right|.$$

Moreover in this case

$$\left| D_{\varphi,h}(x) \right| = \left| 2\Big(u_\varphi(x) - f(x)\Big) - u^1_{\varphi+h}(h)(x) + u^1_\varphi(h)(x) + o(h)(x) \right|,$$

and thus

$$\Big|D_{\varphi,h}(x)\Big| \;\leq\; 3\;\Big|\,u^1_\varphi(h)(x) \;+\; o(h)(x)\;\Big| \;+\; |u^1_{\varphi\,+\,h}(h)(x)|,$$

establishing (11) and (12). For $\varepsilon > 0$, we denote by:

$$\mathcal{B}^\varepsilon_h \;=\; \Big\{x \in \mathcal{B}^0_h \text{ such that } \Big[u_\varphi(x) \,-\, f(x)\Big]\Big[u_{\varphi+h}(x) \,-\, f(x)\Big] \leq 0 \text{ and } |u_\varphi(x) \,-\, f(x)| \;>\; \varepsilon\Big\}.$$

On $\mathcal{B}^\varepsilon_h$ we have $\varepsilon \;\leq\; |u_\varphi(x) \,-\, f(x)| \;\leq\; \Big|\,u^1_\varphi(h)(x) \,+\, o(h)(x)\,\Big|$ and therefore

$$\varepsilon\; mes(\mathcal{B}^\varepsilon_h) \;\leq\; \sqrt{mes(\Gamma_2)}\Big[\tau\,\big\|u^1_\varphi\big\|_{\mathcal{L}(L^\infty(\Gamma_2),H^1(\Omega))}\;\|h\|_{L^\infty(\Gamma_2)} \;+\; \|o(h)\,\|_{L^2(\Gamma_2)}\Big].$$

It follows that, for every fixed $\varepsilon \;>\; 0$, we have:

$$\lim_{\|h\|_{L^\infty(\Gamma_2)\,\to\,0}} \; mes\,(\mathcal{B}^\varepsilon_h) \;=\; 0. \tag{13}$$

Let us define also the set $\mathcal{C}^\varepsilon$ by:

$$\mathcal{C}^\varepsilon \;=\; \{\,x \;\in\; \Gamma_2 \;\text{ such that } 0 \;<\; |u_\varphi(x) \;-\; f(x)| \;\leq\; \varepsilon\Big\} \;\subset\; \{x:\; u_\varphi(x) \;\neq\; f(x)\Big\}.$$

Note that $\mathcal{C}^\varepsilon \;\subset\; \mathcal{C}^{\varepsilon'}$ for $0 \;<\; \varepsilon \;\leq\; \varepsilon'$ and $\bigcap_{\varepsilon>0} \mathcal{C}^\varepsilon \;=\; \emptyset$, and therefore

$$\lim_{\varepsilon\,\to\,0^+} mes(\mathcal{C}^\varepsilon) \;=\; 0. \tag{14}$$

¿From (12), we have:

$$\|D_{\varphi,h}\|_{L^1(\mathcal{B}^\varepsilon_h)} \;\leq\; (\,3\,\big\|\,u^1_\varphi(h) \,+\, \mathrm{o}(h)\,\big\|_{L^2(\mathcal{B}^\varepsilon_h)} \;+\; \|u^1_{\varphi\,+\,h}(h)\|_{L^2(\mathcal{B}^\varepsilon_h)})\,\sqrt{mes(\mathcal{B}^\varepsilon_h)},$$

and hence there exist a constant $c_1 \;>\; 0$ independent of h and ε such that:

$$\frac{\|D_{\varphi,h}\|_{L^1(\mathcal{B}^\varepsilon_h)}}{\|h\|_{L^\infty(\Gamma_2)}} \;\leq\; c_1\,\sqrt{mes(\mathcal{B}^\varepsilon_h)}. \tag{15}$$

Referring to (12) once again we find

$$\|D_{\varphi,h}\|_{L^1(\mathcal{B}^0_h\backslash\mathcal{B}^\varepsilon_h)} \;\leq\; (\,3\,\big\|\,u^1_\varphi(h) \,+\, \mathrm{o}(h)\,\big\|_{L^2(\mathcal{B}^0_h\backslash\mathcal{B}^\varepsilon_h)} \;+\; \|u^1_{\varphi\,+\,h}(h)\|_{L^2(\mathcal{B}^0_h\backslash\mathcal{B}^\varepsilon_h)})\,\sqrt{mes(\mathcal{B}^0_h\backslash\mathcal{B}^\varepsilon_h)}).$$

Since $\mathcal{B}^0_h\backslash\mathcal{B}^\varepsilon_h \;\subset\; \mathcal{C}_\varepsilon$, there exist a constant $c_2 \;>\; 0$ independent of h and ε such that:

$$\frac{\|D_{\varphi,h}\|_{L^1(\mathcal{B}^0_h\backslash\mathcal{B}^\varepsilon_h)}}{\|h\|_{L^\infty(\Gamma_2)}} \;\leq\; c_2\,\sqrt{mes(\mathcal{C}^\varepsilon)}. \tag{16}$$

Let $\eta \;>\; 0$. By (14), (15) and (16), there exists $\overline{\varepsilon} \;>\; 0$ such that:

$$\frac{\|D_{\varphi,h}\|_{L^1(\mathcal{B}^0_h)}}{\|h\|_{L^\infty(\Gamma_2)}} \;\leq\; c_1\,\sqrt{mes(\mathcal{B}^{\overline{\varepsilon}}_h)} \;+\; \eta. \tag{17}$$

We therefore have:

$$\frac{\|D_{\varphi\,,\,h}\|_{L^1(\Gamma_2)}}{\|h\|_{L^\infty(\Gamma_2)}} \;\leq\; \frac{\|D_{\varphi\,,\,h}\|_{L^1(\mathcal{A}_h)}}{\|h\|_{L^\infty(\Gamma_2)}} \;+\; c_1\,\sqrt{mes(\mathcal{B}^{\overline{\varepsilon}}_h)} \;+\; \eta$$

and from equations (10) and (13)

$$\lim_{\|h\|_{L^\infty(\Gamma_2)\to 0}} \frac{\|D_{\varphi\,,\,h}\|_{L^1(\Gamma_2)}}{\|h\|_{L^\infty(\Gamma_2)}} \;\leq\; \eta,$$

Since this inequality is satisfied for every $\eta \;>\; 0$, we have

$$\lim_{\|h\|_{L^\infty(\Gamma_2)\to 0}} \frac{\|D_{\varphi\,,\,h}\|_{L^1(\Gamma_2)}}{\|h\|_{L^\infty(\Gamma_2)}} \;=\; 0,$$

as desired ■

4 Particular case: $f = u_{\varphi_{inv}|M}$ and application to the Robin inverse problem

We study in this part the inverse problem (IP) of determining of the Robin parameter $\varphi_{inv} \in \mathcal{D}$ by measuring the state $u_{\varphi_{inv}}$ on M. In this case the function f is assumed to be the trace on M of a solution $u_{\varphi_{inv}}$ of (RP):

(18) $$f = u_{\varphi_{inv}|M}.$$

First, we establish an identifiability result in Φ_{ad} improving the one established in [5]. Then we prove positivity, monotonicity and a priori bound of the state derivative u^1_φ. By using complex analysis techniques, we prove that the set S_φ is a finite whenever $\varphi \neq \varphi_{inv}$ which is the main idea to prove the differentiability of the cost functional $\mathcal{F}$.

4.1 Identifiability result in the set Φ_{ad}

Theorem 5 *Let φ, $\psi \in \Phi_{ad}$ and assume that $u_{\varphi|M} = u_{\psi|M}$. Then $\varphi = \psi$ a. e. on Γ_2.*

Proof: Let $w = u_\varphi - u_\psi$. Then w satisfies the Cauchy problem:

$$\begin{cases} \Delta w = 0 \text{ in } \Omega, \\ \frac{\partial w}{\partial n} = 0 \text{ on } M, \\ w = 0 \text{ on } M. \end{cases}$$

By using Holmgren's uniqueness theorem we obtain $w = 0$ in Ω, and therefore $u_\varphi = u_\psi$ on Ω. Using the Robin boundary condition we obtain $(\varphi - \psi)\, u_\varphi = 0$ a. e. on Γ_2, and by Lemma 3 we have $\varphi = \psi$ a. e. on Γ_2. ■

Theorem 6 *The function φ_{inv} is the unique global minimum of $\mathcal{F}$. Moreover, if $\{f_n\}$ is a sequence of perturbed data in $L^1(M)$ such that $f_n \longrightarrow f$ strongly in $L^1(M)$, and $\psi_n \in \mathcal{D}$ such that: $\displaystyle\int_M |u_{\psi_n} - f_n| = \min_{\varphi \in \mathcal{D}} \int_M |u_\varphi - f_n|$, then:*

$$\psi_n \rightharpoonup \varphi_{inv} \text{ in } L^\infty \text{ weak } * .$$

Proof: ¿From (18), φ_{inv} is a global minimum for $\mathcal{F}$ satisfying: $\mathcal{F}(\varphi_{inv}) = 0$. Uniqueness of this minimum is obtained by Theorem 5.
Let $v \in L^1(M)$. According to Theorem 2 and uniqueness of the minimum φ_{inv}, the bounded real sequence $k_n = \displaystyle\int_{\Gamma_2} \psi_n v$ has a unique accumulation point $\delta = \displaystyle\int_{\Gamma_2} \varphi_{inv} v$ then k_n converge to δ and consequently $\psi_n \rightharpoonup \varphi_{inv}$ in L^∞ weak $*$. ■

4.2 Positivity, monotonicity and a-priori bound of the state derivative u^1_φ

Theorem 7 *For $h \in L^\infty(\Gamma_2)$ the mapping $\varphi \mapsto u^1_\varphi(h)$ is locally Lipschitzian from Φ_{ad} equipped with the L^2 norm to $H^1(\Omega)$.*

Proof:

Let $h \in L^\infty(\Gamma_2)$ and $\varphi,\ \psi \in \Phi_{ad}$. The function $Z = u^1_\psi(h) - u^1_\varphi(h)$ satisfies:

$$\int_\Omega |\nabla Z|^2 + \int_{\Gamma_2} \varphi\,|Z|^2 = -\int_{\Gamma_2} (\psi - \varphi) u^1_\psi(h) Z + \int_{\Gamma_2} (u_\psi - u_\varphi)\,h\,Z.$$

¿From the coercivity of the bilinear form $A(u,v) = \displaystyle\int_\Omega \nabla u\,\nabla v + \int_{\Gamma_2} \varphi\,u\,v$ (see [4]) and the continuity of the trace mapping, there exists two positive constants c_φ and τ such that:

$$c_\varphi \|Z\|^2_{H^1(\Omega)} \le \|\psi - \varphi\|_{L^2(\Gamma_2)} \|u^1_\psi(h)\|_{L^4(\Gamma_2)} \|Z\|_{L^4(\Gamma_2)} + \tau^2 \|h\|_{L^\infty(\Gamma_2)} \|u_\psi - u_\varphi\|_{H^1(\Omega)} \|Z\|_{H^1(\Omega)}.$$

This implies

$$c_\varphi \|Z\|_{H^1(\Omega)} \le \alpha^2 \|\psi - \varphi\|_{L^2(\Gamma_2)} \|u^1_\psi(h)\|_{H^1(\Omega)} + \tau^2 \|h\|_{L^\infty(\Gamma_2)} \|u_\psi - u_\varphi\|_{H^1(\Omega)}. \tag{19}$$

Where α denotes the norm of trace mapping from $H^1(\Omega)$ into $L^4(\Gamma_2)$. On the other hand we have:

$$\int_\Omega |\nabla u^1_\psi(h)|^2 + \int_{\Gamma_2} \varphi\,|u^1_\psi(h)|^2 = \int_{\Gamma_2} (\varphi - \psi)|u^1_\psi(h)|^2 - \int_{\Gamma_2} u_\psi\,h\,u^1_\psi(h),$$

and consequently

$$\left(c_\varphi - \alpha^2 \|\psi - \varphi\|_{L^2(\Gamma_2)}\right) \|u^1_\psi(h)\|_{H^1(\Omega)} \le \tau^2 \|h\|_{L^\infty(\Gamma_2)} \|u_\psi\|_{H^1(\Omega)}.$$

Equation (19) gives:

$$\|Z\|_{H^1(\Omega)} \le \frac{\tau^2 \|h\|_{L^\infty(\Gamma_2)}}{c_\varphi} \left[\frac{\alpha^2 \|\psi - \varphi\|_{L^2(\Gamma_2)} \left(\|u_\psi - u_\varphi\|_{H^1(\Omega)} + \|u_\varphi\|_{H^1(\Omega)}\right)}{\left(c_\varphi - \alpha^2 \|\psi - \varphi\|_{L^2(\Gamma_2)}\right)} + \|u_\psi - u_\varphi\|_{H^1(\Omega)} \right],$$

for every $\psi \in \Phi_{ad}$ satisfying $\|\psi - \varphi\|_{L^2(\Gamma_2)} < \dfrac{c_\varphi}{\alpha^2}$. By using Lemma 1, the map $\varphi \mapsto u^1_\varphi(h)$ is locally Lipschitzian. ■

Theorem 8 *For $\varphi \in \Phi_{ad}$ and $h_1,\ h_2 \in L^\infty(\Gamma_2)$ with $h_1 \le h_2$, we have:*

$$u^1_\varphi(h_1) \ge u^1_\varphi(h_2).$$

Proof: Due to the linearity of $h \mapsto u^1_\varphi(h)$, it suffices to prove that $u^1_\varphi(h) \le 0$ for every $h \in L^\infty(\Gamma_2)$ with $h \ge 0$.
For every such h let us consider two cases:
First case: $\varphi \in \mathcal{C}^0_0(\Gamma_2)$. Let h_n be a sequence in $\mathcal{C}^0_0(\Gamma_2)$ satisfying:

$$h_n \longrightarrow h \text{ strongly in } L^2(\Gamma_2), \text{ and } h_n \ge 0. \tag{20}$$

For simplicity, we denote $u^1_n = u^1_\varphi(h_n)$. Using the Hopf maximum principle, we have two cases:

- u^1_n **is constant in** $\overline{\Omega}$**:** Due to the Robin boundary conditions, the positivity of φ, h_n and u_φ this constant is necessarily positive.
- u^1_n **is not constant in** $\overline{\Omega}$**:** In this case, we can find $x_n \in \partial\Omega$ such that:

$$M_n \;=\; \sup_{x\in\overline{\Omega}} u_n^1(x) \;=\; u_n^1(x_n), \text{ and } \frac{\partial u_n^1}{\partial n}(x_n) > 0.$$

¿From the positivity of u_φ, we obtain that $x_n \in \Gamma_2$ and $M_n < 0$. On the other hand, for every $\varphi \in \Phi_{ad}$ and $h \in L^\infty(\Gamma_2)$, we have:

$$\int_\Omega |\nabla u_\varphi^1(h)|^2 + \int_{\Gamma_2} \varphi\, |u_\varphi^1(h)|^2 \;=\; -\int_{\Gamma_2} h_n\, u_\varphi\, u_\varphi^1(h).$$

By coercivity of the bilinear form A (see [4]) and continuity of the trace mapping from $H^1(\Omega)$ into $L^2(\Gamma_2)$, we obtain:

(21) $$c_\varphi \|u_\varphi^1\|_{H^1(\Omega)} \;\leq\; \tau\, \|h\|_{L^2(\Gamma_2)}\, \|u_\varphi\|_{L^\infty(\Gamma_2)}.$$

Then, the linear mapping $h \mapsto u_\varphi^1(h)$ is continuous from $L^\infty(\Gamma_2)$ equipped with the $L^2(\Gamma_2)$ norm into $H^1(\Omega)$ and hence $u_\varphi^1(h) \leq 0$ by (20).

Second case: $\varphi \in \Phi_{ad}$. Let $h \in L^\infty(\Gamma_2)$ and φ_n be a sequence of $\mathcal{C}_0^0(\Gamma_2)$ satisfying:

(22) $$\varphi_n \longrightarrow \varphi \text{ strongly in } L^2(\Gamma_2), \text{ and } \varphi_n \geq 0.$$

¿From the first case, we have $u_{\varphi_n}^1(h) \leq 0$ for every $n \in \mathbb{N}$, and consequently $u_\varphi^1(h) \leq 0$ by Theorem 7. ■

Theorem 9 *The following properties hold:*

- **Monotonicity:** *If $\varphi,\ \psi \in \Phi_{ad}$ satisfy $\psi \leq \varphi$, then $u_\psi^1(1) \leq u_\varphi^1(1)$.*
- **Positivity:** *$u_\varphi^1(1)(x) < 0$ for every $x \in \overline{\Omega}$.*
- **A priori bound:** *For every $\varphi \in \Phi_{ad}$ and $h \in L^\infty(\Gamma_2)$, we have $\left|u_\varphi^1(h)\right| \;\leq\; -\|h\|_{L^\infty(\Gamma_2)} u_\varphi^1(1)$.*

Proof:

Monotonicity: For $\varphi,\ \psi \in \Phi_{ad}$ with $\psi \leq \varphi$, we can find two sequences φ_n and $\psi_n \in \mathcal{C}_0^0(\Gamma_2)$ such that:

$$\varphi_n \longrightarrow \varphi \text{ strongly in } L^2(\Gamma_2),\ \psi_n \longrightarrow \psi \text{ strongly in } L^2(\Gamma_2) \text{ and } 0 \;\leq\; \psi_n \;\leq\; \varphi_n.$$

The function $W_n \;=\; u_{\varphi_n}^1(1) \;-\; u_{\psi_n}^1(1)$ is a solution to:

$$\left\{\begin{array}{rcll} \Delta W_n & = & 0 & \text{in} \quad \Omega, \\ \frac{\partial W_n}{\partial n} & = & 0 & \text{on} \quad \Gamma_1, \\ \frac{\partial W_n}{\partial n} + \varphi_n W_n & = & (\psi_n - \varphi_n) u_{\psi_n}^1 + (u_{\psi_n} - u_{\varphi_n}) & \text{on} \quad \Gamma_2. \end{array}\right.$$

¿From the regularity of parameters φ_n and ψ_n, the function $W_n \in \mathcal{C}(\overline{\Omega})$ and by Lemma 2 and Theorem 8 we obtain:

(23) $$(\psi_n - \varphi_n) u_{\psi_n}^1 + (u_{\psi_n} - u_{\varphi_n}) \geq 0 \text{ on } \Gamma_2.$$

According to the Hopf maximum principle, we consider two cases:

- W_n **is constant in** $\overline{\Omega}$**:** From the Robin boundary conditions and equation (23), this constant is necessarily positive.
- u_n^1 **is not constant in** $\overline{\Omega}$**:** In this case, we can find $y_n \in \partial\Omega$ such that:

$$m_n \;=\; \inf_{x\in\overline{\Omega}} W_n(x) \;=\; W_n(y_n), \text{ and } \frac{\partial W_n}{\partial n}(y_n) < 0.$$

Then, $y_n \in \Gamma_2$ and

$$\frac{\partial W_n}{\partial n}(y_n) \;=\; -\,\varphi_n(y_n)m_n \,+\, (\psi_n \,-\, \varphi_n)(y_n)u^1_{\psi_n}(y_n) \,+\, (u_{\psi_n} \,-\, u_{\varphi_n})(y_n) < 0.$$

¿From (23) and the positivity of φ_n, we have: $m_n > 0$, and by continuity of the map: $\varphi \longmapsto u^1_\varphi(1)$ from Φ_{ad} equipped with the L^2 norm to $H^1(\Omega)$ (see theorem 8), we have $u_\varphi \geq u_\psi$.

Positivity: Let $\varphi \in \Phi_{ad}$ and set $\gamma \;=\; \|\varphi\|_{L^\infty(\Gamma_2)}$. From the monotonicity of the mapping $\varphi \mapsto u^1_\varphi(1)$, we have $u^1_\varphi(1) \leq u^1_\gamma(1)$ and by Theorem 8, we have $M \;=\; \sup_{x\in\overline{\Omega}} u^1_\gamma(1)(x) \leq 0$.

To prove $M < 0$, we use again the Hopf maximum principle and consider two cases:

- $u^1_\gamma(1)$ **is constant in** Ω**:** From the Robin boundary condition and the strict positivity of u_γ (see Lemma 3), we have $u^1_\gamma(1)(x) \;=\; M < 0$.

- $u^1_\gamma(1)$ **not constant in** Ω**:** In this case, there exists $y \in \overline{\Gamma_2}$ such that:

$$u^1_\gamma(1)(y) \;=\; M \text{ and } \frac{\partial u^1_\gamma(1)}{\partial n}(y) > 0. \tag{24}$$

If $M = 0$, then by Robin's boundary conditions, continuity of $\dfrac{\partial u^1_\gamma(1)}{\partial n}$ in a neighborhood of y and positivity of u_γ we have $\dfrac{\partial u^1_\gamma(1)}{\partial n}(y) < 0$. This is in contradiction with (24) and hence $M < 0$.

A priori bound: Since $-\,\|h\|_{L^\infty(\Gamma_2)} \;\leq\; h \;\leq\; \|h\|_{L^\infty(\Gamma_2)}$, Theorem 8, linearity of the map $h \mapsto u^1_\varphi(h)$ and negativity of $u^1_\varphi(1)$, imply that $\left|u^1_\varphi(h)\right| \;\leq\; -\|h\|_{L^\infty(\Gamma_2)}u^1_\varphi(1)$. ■

4.3 Differentiability of the function $\mathcal{F}$

Referring to [4] we have the following lemma:

Lemma 4 *The set* $S_\varphi \;=\; \{x \in M :\; u_\varphi(x) \;=\; f(x)\}$ *is a finite set for every* $\varphi \in \Phi_{ad}$ *such that* $\varphi \neq \varphi_{inv}$.

Theorem 10 *For* $\varphi \;\in \Phi_{ad}$ *we have*

$$\lim_{\substack{\|\psi - \varphi\|_{L^\infty(\Gamma_2)} \;\to 0^+ \\ \psi \in \Phi_{ad}}} \frac{\mathcal{F}(\psi) \,-\, \mathcal{F}(\varphi)}{\|\psi \,-\, \varphi\|_{L^\infty(\Gamma_2)}} \;=\; \int_M u^1_\varphi(\psi \,-\, \varphi)sign(u_\varphi \,-\, f),$$

where $u^1_\varphi(\psi \;-\; \varphi)$ *is the solution of (8) with* $h \;=\; \psi \;-\; \varphi$. *Moreover, the function* $\mathcal{F}$ *is differentiable in* $Int(\Phi_{ad})$.

Proof:

Let us first fix a positive sense and denote by $]b, c[$ the arc M. For $x\;\in]b\,,\,c[$ and $s\;>\;0$ small enough, we denote by $x_s \;=\; x + s$ the unique point x_s of $]x\,,\,c[$ such that $|x \,-\, x_s| \;=\; s$. Moreover $y_s \;=\; x \,-\, s$ denote the unique point of $]b\,,\,x[$ such that $|x \,-\, y_s| \;=\; s$.

By Lemma 4, there exist $x_0,\, x_1,\, x_2,\, \ldots\,,\, x_n \in M$ such that $S_\varphi \;=\; \{\, x_0,\, x_1, \ldots, x_n \,\}$, and

$$|x_0 \,-\, b| \;<\; |x_1 \,-\, b| \;<|x_2 \,-\, b| \;<\; \ldots <\; |x_{n-1} \,-\, b| \;<\; |x_n \,-\, b|.$$

We set

$$\delta = \inf \left\{ |x_0 - b|, \ |x_n - c|, \ \inf_{0 \leq i \leq n-1} |x_i - x_{i+1}| \right\}.$$

For $\varepsilon \in]0\,,\,\frac{\delta}{3}[$ we denote

(25) $$\begin{cases} b_i^\varepsilon = x_{i-1} - \varepsilon, & \text{for } 1 \leq i \leq n+1 \text{ and } b_0^\varepsilon = b, \\ c_i^\varepsilon = x_i + \varepsilon, & \text{for } 0 \leq i \leq n \text{ and } c_{n+1}^\varepsilon = c, \end{cases}$$

and further:

- $\mathcal{B}_0^\varepsilon$ the closed arc limited by b_0^ε and b_1^ε,
- $\mathcal{B}_i^\varepsilon$ the closed arc limited by c_{i-1}^ε and b_{i+1}^ε for $1 \leq i \leq n$,
- $\mathcal{B}_{n+1}^\varepsilon$ the closed arc limited by c_n^ε and c_{n+1}^ε,
- $\mathcal{C}_i^\varepsilon$ the closed arc limited by b_{i+1}^ε and c_i^ε for $0 \leq i \leq n$.

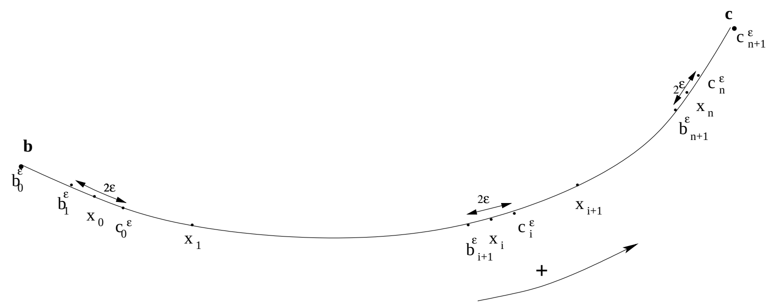

Figure 2: The arcs $\mathcal{B}_\varepsilon^\varepsilon$ and $\mathcal{C}_\varepsilon^\varepsilon$.

We have:

(26) $$\mathcal{F}(\psi) - \mathcal{F}(\varphi) = \sum_{i=0}^{n+1} B_i^\varepsilon + \sum_{i=0}^{n} C_i^\varepsilon,$$

where:

$$B_i^\varepsilon = \int_{\mathcal{B}_i^\varepsilon} |u_\psi - f| - |u_\varphi - f|, \quad \text{and} \quad C_i^\varepsilon = \int_{\mathcal{C}_i^\varepsilon} |u_\psi - f| - |u_\varphi - f|.$$

By the Theorem 3, we have:

(27) $$u_\psi = u_\varphi + u_\varphi^1(\psi - \varphi) + o(\psi - \varphi),$$

where:

(28) $$\lim_{\substack{\|\psi - \varphi\|_{L^\infty(\Gamma_2)} \to 0 \\ \psi \in \Phi_{ad}}} \frac{\|o(\psi - \varphi)\|_{H^1(\Omega)}}{\|\psi - \varphi\|_{L^\infty(\Gamma_2)}}.$$

This implies

$$|C_i^\varepsilon| \leq \int_{\mathcal{C}_i^\varepsilon} \left| u_\varphi^1(\psi - \varphi) + o(\psi - \varphi) \right|.$$

and by using Cauchy Schwartz inequality, we obtain:

$$|C_i^\varepsilon| \leq \sqrt{2\varepsilon}\, \|\psi - \varphi\|_{L^\infty(\Gamma_2)} \left[\frac{\|u_\varphi^1(\psi - \varphi)\|_{L^2(\mathcal{C}_i^\varepsilon)}}{\|\psi - \varphi\|_{L^\infty(\Gamma_2)}} + \frac{\|o(\psi - \varphi)\|_{L^2(\mathcal{C}_i^\varepsilon)}}{\|\psi - \varphi\|_{L^\infty(\Gamma_2)}} \right].$$

By (28), Theorem 3 and the continuity of the trace mapping from $H^1(\Omega)$ into $L^2(M)$, there exist a constant $\gamma > 0$ depending only on φ such that:

$$\sum_{i=0}^{n} |C_i^\varepsilon| \leq \gamma \sqrt{2\varepsilon}\, \|\psi - \varphi\|_{L^\infty(\Gamma_2)}. \tag{29}$$

Let us study the second sum $\sum_{i=0}^{n+1} B_i^\varepsilon$. From (8), the normal derivative of the harmonic function $u_\varphi^1(\psi - \varphi)$ belongs to $L^2(\partial\Omega)$ and hence $u_\varphi^1(\psi - \varphi) \in W^{\frac{3}{2},2}(\Omega)$. Since Ω is a regular domain in $\mathbb{R}^2$, the function $u_\varphi^1(\psi - \varphi)$ belongs to $\mathcal{C}(\overline{\Omega})$ by Sobolev's embedding theorem. Let us denoted by $M = \sup_{x \in \overline{\Omega}} \left|u_\varphi^1(1)(x)\right|$. We have $u_\varphi(x) \neq f(x)$ for all $x \in \mathcal{B}_i^\varepsilon$ and therefore

$$m_i^\varepsilon = \inf_{x \in \mathcal{B}_i^\varepsilon} |u_\varphi(x) - f(x)| > 0.$$

Denote by $h_0^\varepsilon = \dfrac{\inf_{0 \leq i \leq n} m_i^\varepsilon}{M}$. To prove now that $\left| B_i^\varepsilon - \int_{\mathcal{B}_i^\varepsilon} sign\{u_\varphi - f\} u_\varphi^1(\psi - \varphi) \right| \leq \|o(\psi - \varphi)\|_{L^1(\mathcal{B}_i^\varepsilon)}$ we consider two cases:

First case: $u_\varphi(x) - f(x) > 0$ for all $x \in \mathcal{B}_i^\varepsilon$ Let $\psi \in \Phi_{ad}$ be such that $\|\psi - \varphi\|_{L^\infty(\Gamma_2)} \in]0\,,\, h_0^\varepsilon[$. By Theorem 9 with $h = \psi - \varphi$, we obtain: $u_\varphi(x) - f(x) + u_\varphi^1(\psi - \varphi)(x) > 0$, and consequently

$$\left| B_i^\varepsilon - \int_{\mathcal{B}_i^\varepsilon} u_\varphi^1(\psi - \varphi) \right| = \left| \int_{\mathcal{B}_i^\varepsilon} |u_\varphi - f + u_\varphi^1(\psi - \varphi) + o(\psi - \varphi)| - \left[u_\varphi - f + u_\varphi^1(\psi - \varphi)\right] \right|.$$

Since $\Big| |\alpha + \beta| - \alpha \Big| \leq |\beta|$ for all $\alpha > 0$ and $\beta \in \mathbb{R}$ we have

$$\left| B_i^\varepsilon - \int_{\mathcal{B}_i^\varepsilon} u_\varphi^1(\psi - \varphi) \right| \leq \int_{\mathcal{B}_i^\varepsilon} |o(\psi - \varphi)|\,.$$

Second case: $u_\varphi(x) - f(x) < 0$ for all $x \in \mathcal{B}_i^\varepsilon$.
For every $\psi \in \Phi_{ad}$ such that $\|\psi - \varphi\|_{L^\infty(\Gamma_2)} \in]0\,,\, h_0^\varepsilon[$ we have: $u_\varphi(x) - f(x) + u_\varphi^1(\psi - \varphi)(x) < 0$, and thus

$$\left| B_i^\varepsilon + \int_{\mathcal{B}_i^\varepsilon} u_\varphi^1(\psi - \varphi) \right| = \left| \int_{\mathcal{B}_i^\varepsilon} |f - u_\varphi - u_\varphi^1(\psi - \varphi) - o(\psi - \varphi)| - \left[f - u_\varphi - u_\varphi^1(\psi - \varphi)\right] \right|.$$

and

$$\left| B_i^\varepsilon + \int_{\mathcal{B}_i^\varepsilon} u_\varphi^1(\psi - \varphi) \right| \leq \int_{\mathcal{B}_i^\varepsilon} |o(\psi - \varphi)|\,.$$

By (26) and (29) we find

$$\left|\mathcal{F}(\psi) - \mathcal{F}(\varphi) - \sum_{i=0}^{n+1} \int_{\mathcal{B}_i^\varepsilon} sign\,(u_\varphi - f)\, u_\varphi^1(\psi - \varphi)\right| \leq \gamma \sqrt{2\varepsilon}\|\psi - \varphi\|_{L^\infty(\Gamma_2)} + \int_M |o(\psi - \varphi)|\,.$$

This implies

$$\begin{aligned}\left|\mathcal{F}(\psi) - \mathcal{F}(\varphi) - \int_M sign\,(u_\varphi - f)\, u_\varphi^1(\psi - \varphi)\right| \; &\leq \gamma \sqrt{2\varepsilon}\|\psi - \varphi\|_{L^\infty(\Gamma_2)} + \int_M |o(\psi - \varphi)| \\ &+ \left|\sum_{i=0}^{n} \int_{\mathcal{C}_i^\varepsilon} sign\,(u_\varphi - f)\, u_\varphi^1(\psi - \varphi)\right|.\end{aligned}$$

We have $\left|\int_{\mathcal{C}_i^\varepsilon} sign\,(u_\varphi - f)\, u_\varphi^1(\psi - \varphi)\right| \leq \sqrt{2\varepsilon}\left(\|u_\varphi^1(\psi - \varphi)\|_{L^2(\mathcal{C}_i^\varepsilon)}\right)$. Denoting by τ_1 the norm of the trace mapping from $H^1(\Omega)$ to $L^2(M)$, there exist a constant $\eta > 0$ such that:

$$\begin{aligned}\left|\mathcal{F}(\psi) - \mathcal{F}(\varphi) - \int_M sign\,(u_\varphi - f)\, u_\varphi^1(\psi - \varphi)\right| \leq \; &\tau_1 \sqrt{mes(M)}\, \|o(\psi - \varphi)\|_{H^1(\Omega)} \\ &+ (\gamma + \eta)\sqrt{2\varepsilon}\|u_\varphi^1(\psi - \varphi)\|_{H^1(\Omega)}.\end{aligned}$$

By (28) there exist $h_1^\varepsilon > 0$ such that for every $\psi \in \Phi_{ad}$ satisfying $0 < \|\psi - \varphi\|_{L^\infty(\Gamma_2)} \leq h_1^\varepsilon$ we have:

$$\frac{\|o(\psi - \varphi)\|_{H^1(\Omega)}}{\|\psi - \varphi\|_{L^\infty(\Gamma_2)}} \leq \sqrt{2\varepsilon}.$$

Then, for $\psi \in \Phi_{ad}$ such that $\|\psi - \varphi\|_{L^\infty(\Gamma_2)} \in]0\,, inf\{h_0^\varepsilon\,, h_1^\varepsilon\}[$, we find:

$$\begin{aligned}\frac{\left|\mathcal{F}(\psi) - \mathcal{F}(\varphi) - \int_M sign(u_\varphi - f)u_\varphi^1(\psi - \varphi)\right|}{\|\psi - \varphi\|_{L^\infty(\Gamma_2)}} \; &\leq \sqrt{2\varepsilon}[(\gamma + \eta)\|u_\varphi^1\|_{\mathcal{L}(H^1(\Omega), L^\infty(\Gamma_2))}\|\psi - \varphi\|_{L^\infty(\Gamma_2)} \\ &+ \tau_1 \sqrt{mes(M)}],\end{aligned}$$

and

$$\lim_{\substack{\|\psi - \varphi\|_{L^\infty(\Gamma_2)} \to 0 \\ \psi \in \Phi_{ad}}} \left(\frac{\mathcal{F}(\psi) - \mathcal{F}(\varphi)}{\|\psi - \varphi\|_{L^\infty(\Gamma_2)}}\right) = \int_M u_\varphi^1(\psi - \varphi) sign(u_\varphi - f).$$

For $\varphi \in Int(\Phi_{ad})$, and $h \in L^\infty(\Gamma_2)$, we have $D\mathcal{F}_\varphi(h) = \int_M u_\varphi^1(h) sign(u_\varphi - f)$. The Cauchy Schwartz inequality, continuity of the trace mapping and Theorem 3, imply:

$$|D\mathcal{F}_\varphi(h)| \leq \tau_1 \sqrt{mes(M)}\, \|u_\varphi^1\|_{\mathcal{L}(H^1(\Omega), L^\infty(\Gamma_2))}\, \|h\|_{L^\infty}.$$

Finally, $\mathcal{F}$ is differentiable in $Int(\Phi_{ad})$. ■

Conclusion: Existence of an optimal control to the optimization problem (OP) and differentiability properties of the functional $\mathcal{F}$ were established. In the case of no error on the data f which is the case of the Robin inverse problem, the characterization of the set S_φ together with the positivity, monotonicity and a priori bound of the state derivative u_φ^1 with respect to the parameter φ allow to prove the differentiability of the functional $\mathcal{F}$.
The functional $\mathcal{F}$ and its differential expression permit to define a Newton-type method in order to solve numerically the Robin inverse problem. This will be the next step of our work.

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