SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 3, No. 4, pp. 701–722





# HJB-POD-Based Feedback Design for the Optimal Control of Evolution Problems*

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**Abstract.** The numerical realization of closed loop control for distributed parameter systems is still a significant challenge and in fact infeasible unless specific structural techniques are employed. In this paper we propose the combination of model reduction techniques based on proper orthogonal decomposition (POD) with the numerical treatment of the Hamilton–Jacobi–Bellman (HJB) equation for infinite horizon optimal control problems by a modification of an algorithm originated by Gonzales and Rofman and further developed by Falcone and Ferretti. The feasibility of the proposed methodology is demonstrated numerically by means of optimal boundary feedback control for the Burgers equation with noise in the initial condition and in the forcing function.

**Key words.** dynamic programming, Hamilton–Jacobi–Bellman equation, closed loop control, evolution problems, proper orthogonal decomposition, Burgers equation

**AMS subject classifications.** 35Kxx, 49Lxx, 65Kxx

**DOI.** 10.1137/030600485

## 1. Introduction.

In many applications the discretization of optimal control problems for time dependent partial differential equations, e.g., for the unsteady Navier–Stokes equations, require the solution of nonlinear systems with a large number of degrees of freedom. In particular, to compute closed loop controls in state feedback form we have to solve the Hamilton–Jacobi–Bellman (HJB) equation, which has been numerically infeasible for parabolic differential equations on a standard workstation equipment until today, if classical approximations like finite elements or finite differences are used. In this work model reduction is applied to reduce the number of unknowns significantly. The obtained low-dimensional models should guarantee a reasonable performance of the controlled plant while being computationally tractable. Proper orthogonal decomposition (POD) provides a method for deriving appropriate low-order models. It can be thought of as a Galerkin approximation in the spatial variable, built from functions corresponding to the solution of the physical system at prespecified time instances. These are called the snapshots. Due to possible linear dependence or almost linear dependence a singular value decomposition of the snapshots is carried out and the leading generalized eigenfunctions are chosen as a basis, referred to as the POD basis. Once a low-order model of the dynamical system is available, feedback synthesis based on approximate solutions to the stationary HJB equation becomes feasible.

We demonstrate the feasibility of the proposed approach by means of an optimal boundary control problem for the Burgers equation. Open loop optimal control problems for the

*Received by the editors July 18, 2003; accepted for publication (in revised form) by P. Holmes June 17, 2004; published electronically December 27, 2004.
http://www.siam.org/journals/siads/3-4/60048.html

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Burgers equation was studied by several authors; see, for instance, [7, 14, 17, 24]. Much less attention has been paid to the important problem of closed loop control. We mention the work by Byrnes, Gilliam, and Shubov [6], where a fixed feedback-operator is used and analyzed, and Burns and Kang [?] where the feedback synthesis is based on Riccati operators for the linearized equations. In [12] instantaneous control was applied to construct a feedback law which matches a desired state, but at considerable control costs. In [15] the authors utilized model reduction with POD to construct a suboptimal feedback synthesis, and an optimal output feedback reduced-order control law was designed by POD discretization in [16].

The analysis and use of proper orthogonal decomposition for reduction purposes has a long-lasting history (see [13] and the references given there). Its use in optimal control, while rather recent, has already created a wide range of literature of which we can only mention a few works. In [19] optimal open loop POD-based control of flow around a rotating cylinder is investigated. Control of turbulent flow utilizing POD with the aim of drag reduction is considered in [20], for example. POD-based control of thin film growth in a chemical vapor deposition reactor is investigated in [3]. In [2] the dynamical system is linearized, which allows the use of Riccati synthesis for feedback controller construction, which can favorably be combined with POD-based model reduction. In [9] and [1] the issue of unmodeled dynamics is addressed, i.e., the fact the snapshots for the POD-approximation are typically taken from dynamics which may be different from the controlled dynamics.

The paper is organized in the following manner: In section 2 we review the dynamic programming principle and the HJB equation. Section 3 is devoted to the reduced-order approach based on POD for an abstract optimal control problem. The numerical strategy for the feedback synthesis is explained in section 4. In section 5 we illustrate the efficiency of the proposed method by considering an optimal boundary control problem for the viscous Burgers equation. Conclusions are drawn in the last section. Some facts about the discretization to the HJB equation that we employ are proven in the appendix.

## 2. Review of the dynamic programming principle.

In this section we recall the dynamic programming principle and its infinitesimal version, the Hamilton–Jacobi–Bellman equation. This leads to the design of a feedback synthesis by utilizing the so-called value function. For more details we refer the reader to, e.g., [4, Chapter I], [10].

For $k, n \in \mathbb{N}$ let $U = \mathbb{R}^k$ denote the control space and let $U_{\mathsf{ad}} \subsetneq U$ be a closed, bounded, and convex set. Furthermore, $y_\circ \in \mathbb{R}^n$ is a given initial condition. For a measurable control function $u : [0, \infty) \to U$ the state $y : [0, \infty) \to \mathbb{R}^n$ is governed by the initial value problem

$$\dot{y}(t) = F(y(t), u(t)) \qquad \text{for } t > 0, \tag{2.1a}$$

$$y(0) = y_\circ. \tag{2.1b}$$

To ensure the existence of a unique solution to (2.1) we make use of the following assumption.

Assumption 1. *The (nonlinear) mapping $F : \mathbb{R}^n \times U \to \mathbb{R}^n$ is given in such a way that for every choice of initial condition $y_\circ \in \mathbb{R}^n$ and measurable control function $u$ there exists a unique state $y = y(t)$ to the state equation (2.1).*

At times we write $y(t) = y(t; y_\circ, u)$ or $y = y(y_\circ, u)$ to emphasize the dependence of the state $y$ on $y_\circ$ and $u$. Associated with (2.1) is the cost functional

$$J(y,u) = \int_0^\infty L(y(t),u(t))e^{-\lambda t}\,\mathrm{d}t, \tag{2.2}$$

where $L : \mathbb{R}^n \times \mathbb{R}^k \to [0,\infty)$ is a continuous function and $\lambda > 0$ represents a discount rate.

The optimal control problem is expressed as

$$\min J(y,u) \quad \text{s.t.} \quad y \text{ solves (2.1) and } u \in \mathcal{U}_{\mathsf{ad}}. \tag{2.3}$$

Here, $\mathcal{U}_{\mathsf{ad}}$ denotes the set of all measurable functions from $[0,\infty)$ to $U_{\mathsf{ad}}$. For $u \in \mathcal{U}_{\mathsf{ad}}$ and $y_\circ \in \mathbb{R}^n$ we introduce the reduced cost by

$$\hat{J}(y_\circ,u) = J(y(y_\circ,u),u). \tag{2.4}$$

This gives rise to the value function $v : \mathbb{R}^n \to [0,\infty)$, which is defined by

$$v(y_\circ) = \inf_{u\in\mathcal{U}_{\mathsf{ad}}} \hat{J}(y_\circ,u).$$

It satisfies the *dynamic programming principle*

$$v(y_\circ) = \inf_{u\in\mathcal{U}_{\mathsf{ad}}} \left\{ \int_0^T L(y(t;y_\circ,u),u(t))e^{-\lambda t}\,\mathrm{d}t + v(y(T;y_\circ,u))e^{-\lambda T} \right\} \tag{DPP}$$

for all $y_\circ \in \mathbb{R}^n$ and $T > 0$.

*Remark* 2.1.

(a) (DPP) holds under general conditions on the data. For example, the existence of optimal control has not been assumed.

(b) When $L$ and, consequently, $v$ are bounded, then $w \equiv v$ holds for every function $w = w(y_\circ)$ satisfying (DPP) for all $y_\circ \in \mathbb{R}^n$ and $T > 0$. ■

Suppose that the value function $v$ is differentiable. Dividing both sides in (DPP) by $T$ and letting $T$ tend to zero, we arrive after a short calculation at the infinitesimal version of the dynamic programming principle, the *HJB equation*:

$$\lambda v(y_\circ) + \sup_{u\in U_{\mathsf{ad}}} \left\{ -\nabla v(y_\circ)F(y_\circ,u) - L(y_\circ,u) \right\} = 0. \tag{HJB}$$

If $v$ is only continuous, then (HJB) has to be interpreted in terms of viscosity solutions. The solution to the HJB equation is utilized for the *synthesis procedure*. Due to the Bellman optimality principle, the function

$$h(t) = v(y^*(t))e^{-\lambda t} + \int_0^t L(y^*(s),u^*(s))e^{-\lambda s}\,\mathrm{d}s$$

is constant for $t > 0$ if and only if $(y^*(y_\circ,u^*),u^*)$ is an optimal trajectory and control pair for the initial condition $y_\circ$. Under the hypothesis that $v$ is differentiable, we conclude that $h' \equiv 0$. In particular, we find

$$\lambda v(y^*(t)) - \nabla v(y^*(t))F(y^*(t),u^*(t)) - L(y^*(t),u^*(t)) = 0 \tag{2.5}$$

for almost all $t > 0$. Utilizing (2.5), it can be shown that under appropriate conditions the control $u^* = u^*(t)$ is optimal if and only if

$$u^*(t) = S(y^*(t)) \quad \text{for almost all } t > 0$$

for any choice $S$ such that

$$S(y_\circ) \in \operatorname*{argmax}_{u \in U_{\mathsf{ad}}} \{ -\nabla v(y_\circ) F(y_\circ, u) - L(y_\circ, u) \}, \tag{2.6}$$

i.e., if and only if

$$\sup_{u \in U_{\mathsf{ad}}} \{ -\nabla v(y^*(t)) F(y^*(t), u) - L(y^*(t), u) \} \\ = -\nabla v(y^*(t)) F(y^*(t), u^*(t)) - L(y^*(t), u^*(t)) \quad \text{for almost all } t > 0.$$

If $v$ was known then determining $S$ would be a finite dimensional mathematical programming problem at every $y_\circ \in \mathbb{R}^n$. $S$ is called the *optimal feedback map*. Assuming that $S$ is known results in the closed loop system

$$\begin{aligned} \dot{y}(t) &= F(y(t), S(y(t))) \quad \text{for } t > 0, \\ y(0) &= y_\circ. \end{aligned} \tag{2.7}$$

Its solution $y^*$ and the optimal control $u^*$ are related by

$$u^*(t) = S(y^*(t)), \quad t > 0. \tag{2.8}$$

We refer to the literature for analogous results if $v$ is only continuous.

For the numerical realization we next discretize (2.1) and (HJB). For the grid size $h > 0$, set

$$t_j = jh \quad \text{for } j = 0, 1, \ldots,$$

and consider the discrete time system

$$\begin{aligned} y_{j+1} &= y_j + hF(y_j, u_j) \quad \text{for } j \geq 0, \\ y_0 &= y_\circ \end{aligned} \tag{2.9}$$

and the associated cost

$$J_h(y_\circ, u_h) = \frac{h}{2} \left( L(y_\circ, u_0) + \sum_{j=1}^{\infty} \beta^j \big( L(y_j, u_{j-1}) + L(y_j, u_j) \big) \right) \tag{2.10}$$

for $u_j \in U_{\mathsf{ad}}$, which arises by applying the trapezoidal rule to (2.2) with the assumption that the controls are constant on the subintervals $[t_{j-1}, t_j]$. Here we set $\beta = e^{-\lambda h}$, and $y_h = \{y_\circ, y_1, \ldots\}$ denotes the solution to (2.9) where $u_h = \{u_0, u_1, \ldots\}$ . The approximate minimal value function $v_h : \mathbb{R}^n \to [0, \infty)$ is given by

$$v_h(y_\circ) = \inf_{u_h \in \mathcal{U}^h_{\mathsf{ad}}} J_h(y_\circ, u_h), \tag{2.11}$$

where $\mathcal{U}^h_{\mathsf{ad}} = \{u_h : u_h = \{u_0, u_1, \dots\}$ with $u_i \in U_{\mathsf{ad}}\}$. In the appendix it is verified that $v_h$ is the unique solution to the discrete HJB equation

$$(\mathrm{HJB}_h) \quad v_h(y_\circ) + \sup_{u \in U_{\mathsf{ad}}} \left\{ -\frac{h}{2}\big(L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)\big) - \beta v(y_\circ + hF(y_\circ, u)) \right\} = 0.$$

Turning to the synthesis problem we define

$$S_h(y_\circ) \in \operatorname*{argmax}_{u \in U_{\mathsf{ad}}} \left\{ -\frac{h}{2}\big(L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)\big) - \beta v(y_\circ + hF(y_\circ, u)) \right\}.$$

Sufficient conditions, given in the appendix, guarantee that $u_j^* = S_h(y_j^*)$ gives an optimal feedback control, i.e.,

$$v_h(y_\circ) = J_h(y_\circ, u^*)$$

and

$$\begin{aligned} y^*_{j+1} &= y^*_j + hF(y^*_j, S_h(y^*_h)) \quad \text{for } j \ge 0, \\ y^*_0 &= y_\circ. \end{aligned} \tag{2.12}$$

Solving ($\mathrm{HJB}_h$) is still a significant challenge and infeasible for high-dimensional discretizations of distributed parametric systems. For this reason we turn to a model reduction technique in the following section which will allow us to reduce the dimension of the state space $y$ in $\mathbb{R}^n$. The discretization of the value function $v_h$ will be discussed in section 4. We do not address dimension issues concerning the control space $U$. Certainly, if it is infinite dimensional, it must be discretized for numerical purposes.

## 3. POD Galerkin approximations for optimal control problems governed by evolution problems.

In this section we propose a reduced-order approach for optimal control problems governed by evolution problems. It is based on POD, which is a method of deriving basis functions containing characteristics of the investigated evolution process. The optimal control problem for an abstract evolution problem and the POD method are introduced in sections 3.1 and 3.2, respectively, and in section 3.3 the reduced-order modeling for the optimal control problem is addressed.

### 3.1. The optimal control problem for an abstract dynamical system.

Let $V$ and $H$ be real separable Hilbert spaces, and suppose that $V$ is dense in $H$ with compact embedding. By $\langle \cdot\,,\cdot\rangle_H$ we denote the inner product in $H$. The inner product in $V$ is given by a symmetric bounded, coercive, bilinear form $a : V \times V \to \mathbb{R}$:

$$\langle \varphi, \psi\rangle_V = a(\varphi, \psi) \quad \text{for all } \varphi, \psi \in V \tag{3.1}$$

with associated norm given by $\|\cdot\|_V = \sqrt{a(\cdot\,,\cdot)}$. We associate with $a$ the linear operator $A$,

$$\langle A\varphi, \psi\rangle_{V',V} = a(\varphi, \psi) \quad \text{for all } \varphi, \psi \in V,$$

where $\langle \cdot\,,\cdot\rangle_{V',V}$ denotes the duality pairing between $V$ and its dual. Then $A$ is an isomorphism from $V$ onto $V'$. For $0 < T \le \infty$ we denote by $L^2(0, T; V)$ the space of equivalence classes of measurable abstract functions $\varphi : (0,T) \to V$, which are square integrable, i.e., $\int_0^T \|\varphi(t)\|_V^2 \, \mathrm{d}t < \infty$. When $t$ is fixed, the expression $\varphi(t)$ stands for the function $\varphi(t,\cdot)$ considered as a function in $\Omega$ only. The space $W(0,T)$ is defined as

$$W(0,T) = \{\varphi \in L^2(0,T;V) : \varphi_t \in L^2(0,T;V')\},$$

which is a Hilbert space endowed with the common inner product (see, for example, [8, p. 473]), and we set $W_{loc}(0,\infty) = \bigcap_{T>0} W(0,T)$. Let $N : V \to V'$ be a nonlinear continuous operator map. Further, let $U$ be a Hilbert space and $U_{\mathsf{ad}} \subset U$ a closed and convex subset, and set $\mathcal{U} = L^2(0,\infty;U)$ and let $\mathcal{U}_{\mathsf{ad}}$ be the subset of $\mathcal{U}$ containing all functions $u : [0,\infty) \to U_{\mathsf{ad}}$. For $y_\circ \in H$ and $u \in \mathcal{U}_{\mathsf{ad}}$ we consider the nonlinear evolution problem on $[0,\infty)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle y(t), \varphi\rangle_H + a(y(t),\varphi) + \langle N(y(t)),\varphi\rangle_{V',V} = \langle B(u(t)),\varphi\rangle_{V',V} \tag{3.2a}$$

for all $\varphi \in V$ and

$$y(0) = y_\circ \quad \text{in } H, \tag{3.2b}$$

where $B : U \to V'$ is a continuous linear operator. We make use of the following assumption.

Assumption 2. *For every $u \in \mathcal{U}_{\mathsf{ad}}$ and $y_\circ \in H$ there exists a unique solution $y$ of (3.2) in $W_{loc}(0,\infty)$.*

This assumption is satisfied for many practical situations, including the controlled viscous Burgers and two-dimensional incompressible Navier–Stokes equations.

Next we introduce the cost functional

$$\mathcal{J}(y,u) = \int_0^\infty e^{-\lambda t} \tilde{L}(y(t),u(t))\,\mathrm{d}t,$$

where $\tilde{L} : V \times U \to \mathbb{R}$. The optimal control problem is given by

$$\min \mathcal{J}(y,u) \quad \text{such that} \quad (y,u) \in W_{loc}(0,\infty;V) \times \mathcal{U}_{\mathsf{ad}} \text{ solves (3.2)}. \tag{P}$$

Its approximation is considered next.

**3.2. The POD method.** Throughout we assume that Assumption 2 holds and we denote by $y$ the unique solution to (3.2). For given $n \in \mathbb{N}$ let

$$0 = t_1 < t_2 < \cdots < t_n < \infty \tag{3.3}$$

denote a grid in the interval $[0,\infty)$ and set $\delta t_j = t_j - t_{j-1}$, $j = 1,\dots,n$. Suppose that the *snapshots* $y_j = y(t_j)$ of (3.2) at the given time instances $t_j$, $j = 0,\dots,n$, are known. We set

$$\mathcal{V} = \operatorname{span}\{y_0,\dots,y_n\}.$$

Notice that $\mathcal{V} \subset V$ by construction. Throughout the remainder of this section we let $X$ denote either the space $V$ or the space $H$.

Let $\{\psi_i\}_{i=1}^d$ denote an orthonormal basis for $\mathcal{V}$ with $d = \dim \mathcal{V}$. Then each member of the ensemble $\mathcal{V}$ can be expressed as

$$y_j = \sum_{i=1}^{d} \langle y_j, \psi_i \rangle_X \psi_i \quad \text{for } j = 0, \ldots, n. \tag{3.4}$$

The method of POD consists in choosing an orthonormal basis such that for every $\ell \in \{1, \ldots, d\}$ the mean square error between the elements $y_j$, $0 \le j \le n$, and the corresponding $\ell$th partial sum of (3.4) is minimized on average:

$$\begin{aligned} &\min \mathfrak{J}(\psi_1, \ldots, \psi_\ell) = \sum_{j=0}^{n} \alpha_j \left\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle_X \psi_i \right\|_X^2 \\ &\text{subject to } \langle \psi_i, \psi_j \rangle_X = \delta_{ij} \quad \text{for } 1 \le i \le \ell, 1 \le j \le i. \end{aligned} \tag{3.5}$$

Here $\{\alpha_j\}_{j=0}^n$ are positive weights, which for our purposes are chosen to be

$$\alpha_0 = \frac{\delta t_1}{2}, \quad \alpha_j = \frac{\delta t_j + \delta t_{j+1}}{2} \text{ for } j = 1, \ldots, n-1, \quad \alpha_n = \frac{\delta t_n}{2}.$$

A solution $\{\psi_i\}_{i=1}^\ell$ to (3.5) is called *POD basis of rank* $\ell$. The subspace spanned by the first $\ell$ POD basis functions is denoted by $V^\ell$, i.e.,

$$V^\ell = \text{span } \{\psi_1, \ldots, \psi_\ell\}. \tag{3.6}$$

The solution of (3.5) is characterized by the necessary optimality condition, which can be written as an eigenvalue problem. For that purpose we endow $\mathbb{R}^{n+1}$ with the weighted inner product

$$\langle v, w \rangle_{\mathbb{R}^{n+1}} = \sum_{j=0}^{n} \alpha_j v_j w_j \tag{3.7}$$

for $v = (v_0, \ldots, v_n)^T, w = (w_0, \ldots, w_n)^T \in \mathbb{R}^{n+1}$, and the induced norm. Let us introduce the bounded linear operator $\mathcal{Y}_n : \mathbb{R}^{n+1} \to X$ by

$$\mathcal{Y}_n v = \sum_{j=0}^{n} \alpha_j v_j y_j \quad \text{for } v \in \mathbb{R}^{n+1}. \tag{3.8}$$

Then the adjoint $\mathcal{Y}_n^* : X \to \mathbb{R}^{n+1}$ is given by

$$\mathcal{Y}_n^* z = \left( \langle z, y_0 \rangle_X, \ldots, \langle z, y_n \rangle_X \right)^T \quad \text{for } z \in X. \tag{3.9}$$

It follows that $\mathcal{R}_n = \mathcal{Y}_n \mathcal{Y}_n^* \in \mathcal{L}(X)$ and $\mathcal{K}_n = \mathcal{Y}_n^* \mathcal{Y}_n \in \mathbb{R}^{(n+1)\times(n+1)}$ are given by

$$\begin{aligned} \mathcal{R}_n z &= \sum_{j=0}^{n} \alpha_j \langle z, y_j \rangle_X y_j \quad \text{for } z \in X, \\ \left(\mathcal{K}_n\right)_{ij} &= \alpha_j \left\langle y_j, y_i \right\rangle_X, \end{aligned} \tag{3.10}$$

respectively. Here, $\mathcal{L}(X)$ denotes the Banach space of all bounded linear operators from $X$ into itself. The matrix $\mathcal{K}_n$ is often called a *correlation matrix*.

Using a Lagrangian framework, we can derive the following optimality conditions for the optimization problem (3.5):

$$\mathcal{R}_n\psi = \lambda\psi \tag{3.11}$$

(see, e.g., [13, pp. 88–91] and [23, section 2]). Note that $\mathcal{R}_n$ is a bounded, self-adjoint, and nonnegative operator. Moreover, since the image of $\mathcal{R}_n$ is finite dimensional, $\mathcal{R}_n$ is also compact. By Hilbert–Schmidt theory (see, e.g., [21, p. 203]) there exist an orthonormal basis $\{\psi_i\}_{i=1}^\infty$ for $X$ and a sequence $\{\lambda_i\}_{i=1}^\infty$ of nonnegative real numbers so that

$$\mathcal{R}_n\psi_i = \lambda_i\psi_i, \quad \lambda_1 \geq \cdots \geq \lambda_d > 0, \quad \text{and } \lambda_i = 0 \quad \text{for} \quad i > d. \tag{3.12}$$

Moreover, $\mathcal{V} = \text{span } \{\psi_i\}_{i=1}^d$. Note that $\{\lambda_i\}_{i=0}^\infty$ as well as $\{\psi_i\}_{i=0}^\infty$ depend on $n$. Contents permitting the notation of this dependence is dropped.

*Remark* 3.1. Setting

$$v_i = \frac{1}{\sqrt{\lambda_i}}\, \mathcal{Y}_n^*\psi_i \quad \text{for } i = 1, \dots, d,$$

we find $\mathcal{K}_n v_i = \lambda_i v_i$ and $\langle v_i, v_j\rangle_{\mathbb{R}^{n+1}} = \delta_{ij}$ for $1 \leq i, j \leq d$. Thus, $\{v_i\}_{i=1}^d$ is an orthonormal basis of eigenvectors of $\mathcal{K}_n$ for the image of $\mathcal{K}_n$. Conversely, if $\{v_i\}_{i=1}^d$ is a given orthonormal basis for the image of $\mathcal{K}_n$, then it follows that the first $d$ eigenfunctions of $\mathcal{R}_n$ can be determined by

$$\psi_i = \frac{1}{\sqrt{\lambda_i}}\, \mathcal{Y}_n v_i \quad \text{for } i = 1, \dots, d.$$

Hence, we can determine the POD basis by solving either the eigenvalue problem for $\mathcal{R}_n$ or the one for $\mathcal{K}_n$. ■

The sequence $\{\psi_i\}_{i=1}^\ell$ solves the optimization problem (3.5). This fact as well as the error formula below were proved in [13, section 3], for example. Let $\lambda_1 \geq \cdots \geq \lambda_d > 0$ denote the positive eigenvalues of $\mathcal{R}_n$ with the associated eigenvectors $\psi_1, \dots, \psi_d \in X$. Then, $\{\psi_i^n\}_{i=1}^\ell$ is a POD basis of rank $\ell \leq d$, and we have the error formula

$$\mathfrak{J}(\psi_1, \dots, \psi_\ell) = \sum_{j=0}^n \alpha_j \left\| y_j - \sum_{i=1}^\ell \langle y_j, \psi_i\rangle_X \psi_i \right\|_X^2 = \sum_{i=\ell+1}^d \lambda_i. \tag{3.13}$$

**3.3. Reduced-order control.** The reduced-order approach to optimal control problems such as (P) is based on approximating the nonlinear dynamics by a Galerkin technique utilizing basis functions that contain characteristics of the controlled dynamic.

To compute a POD solution of (P) we make the ansatz

$$y^\ell(t, x) = \sum_{i=1}^\ell \mathrm{w}_i(t)\psi_i(x). \tag{3.14}$$

We introduce the mass and stiffness matrices by

$$\mathrm{M} = ((\mathrm{m}_{ij})) \in \mathbb{R}^{\ell\times\ell} \text{ with } \mathrm{m}_{ij} = \langle \psi_j, \psi_i \rangle_H,$$
$$\mathrm{S} = ((\mathrm{s}_{ij})) \in \mathbb{R}^{\ell\times\ell} \text{ with } \mathrm{s}_{ij} = a(\psi_j, \psi_i),$$

the nonlinear function $\mathrm{N} : \mathbb{R}^\ell \to \mathbb{R}^\ell$ by

$$(\mathrm{w}_1,\dots,\mathrm{w}_\ell) \mapsto \mathrm{N}(\mathrm{w}_1,\dots,\mathrm{w}_\ell) = (\mathrm{n}_i) \in \mathbb{R}^\ell \text{ with } \mathrm{n}_i = \left\langle N\left(\sum_{j=1}^{\ell} \mathrm{w}_i\psi_j\right), \psi_i \right\rangle_{V',V},$$

and the mapping of the control input $\mathrm{b} : U \to \mathbb{R}^\ell$ by

$$u \mapsto \mathrm{b}(u) = \big(\mathrm{b}(u)_i\big) \in \mathbb{R}^\ell \text{ with } \mathrm{b}(u)_i = \langle Bu, \psi_i \rangle_H.$$

The modal coefficients of the initial condition $y^\ell(0) \in \mathbb{R}^\ell$ are determined by $\mathrm{w}_i(0) = (\mathrm{w}_\circ)_i = \langle y_\circ, \psi_i \rangle_X$, $1 \le i \le \ell$, and the solution vector of the reduced dynamical system is denoted by $\mathrm{w}^\ell(t) \in \mathbb{R}^\ell$. Then the Galerkin approximation of the optimal control problem (P) is given by

$$(\mathrm{P}^\ell) \qquad \begin{cases} \min J^\ell(\mathrm{w}^\ell, u) \\ \text{s.t. } u \in \mathcal{U}_{\mathsf{ad}} \text{ and } \begin{cases} \dot{\mathrm{w}}^\ell(t) = F(\mathrm{w}^\ell(t), u(t)) & \text{for } t > 0, \\ \mathrm{w}^\ell(0) = \mathrm{w}_\circ, \end{cases} \end{cases}$$

where the cost functional is defined as

$$J^\ell(\mathrm{w}^\ell, u) = \int_0^\infty \tilde{L}(y^\ell(t), u(t))e^{-\lambda t}\,\mathrm{d}t$$

with $\mathrm{w}^\ell$ and $y^\ell$ related by (3.14) and the nonlinear mapping $F : \mathbb{R}^\ell \times U \to \mathbb{R}^\ell$ given by

$$F(\mathrm{w}^\ell, u) = \mathrm{M}^{-1}\left(-\mathrm{S}\mathrm{w}^\ell - \mathrm{N}(\mathrm{w}^\ell) + \mathrm{b}(u)\right).$$

Of course, it is tacitly assumed that the dynamical system in $(\mathrm{P}^\ell)$ admits a unique solution for every $u \in \mathcal{U}_{\mathsf{ad}}$. Let us mention that in case of $X = H$ the mass matrix M is just the identity matrix. On the other hand, S is the identity matrix for $X = V$.

The value function $v^\ell$, defined for initial states $\mathrm{w}_\circ \in \mathbb{R}^\ell$, is

$$v^\ell(\mathrm{w}_\circ) = \inf_{u\in\mathcal{U}_{\mathsf{ad}}} \hat{J}^\ell(\mathrm{w}_\circ, u),$$

where $\hat{J}^\ell(\mathrm{w}_\circ, u) = J^\ell(\mathrm{w}^\ell, u)$ and $\mathrm{w}^\ell$ solves the dynamical system in $(\mathrm{P}^\ell)$ with control input $u$ and initial condition $\mathrm{w}_\circ$.

## 4. Numerical strategy for the closed loop design.

Here we briefly explain the numerical realization of $(\mathrm{HJB}_h)$. While $(\mathrm{HJB}_h)$ is defined on $\mathbb{R}^n$ for practical purposes, we restrict ourselves to a computational domain $\Upsilon_h$ which is a bounded subset of $\mathbb{R}^n$. This is justified if $y+hF(y,u) \in \Upsilon_h$ for all $y \in \Upsilon_h$ and $u \in U_{\mathsf{ad}}$. Here we choose $\Upsilon_h = [a_1, b_1]\times[a_2, b_2]\times\cdots\times[a_\ell, b_\ell]$ with $a_1 \ge a_2 \ge \cdots \ge a_\ell$ and $b_1 \ge b_2 \ge \cdots \ge b_\ell$. Let $\{S_j\}_{j=1}^k$ denote the hypercubes of a rectilinear partition of $\Upsilon_h$ with $N$ vertices $\{y_j\}$. We consider the space $W^k$ of piecewise $\ell$-linear functions $w^k : \Upsilon_h \to \mathbb{R}$ which are continuous on $\Upsilon_h$. We look for a solution $w^k \in W^k$ of

$$(\mathrm{HJB}_h^k) \qquad w^k(y_j) = \inf_{u \in U_{\mathsf{ad}}} \left\{ \frac{h}{2}(L(y_j,u) + \beta L(y_j + hF(y_j,u),u)) + \beta w^k(y_j + hF(y_j,u)) \right\}$$

for every vertex $y_j \in \Upsilon_h$. If $v_h$ is a solution to ($\mathrm{HJB}_h$), then it satisfies ($\mathrm{HJB}_h^k$) and the convex interpolant

$$w^k(y) = \sum_{j=1}^N \lambda_j v(y_j) \text{ for } y = \sum_{j=1}^N \lambda_j y_j$$

belongs to $W^k$. The intervals $[a_j, b_j]$ must be chosen such that they contain the components of the expected controlled trajectories while simultaneously keeping them as small as possible for computational purposes. Since the dynamic model is derived from POD, the magnitude of the components in the solution representation of the trajectory in terms of the POD basis functions are rapidly decreasing. We therefore choose the intervals $[a_j, b_j]$ such that their lengths are rapidly decreasing. Moreover, the mesh sizes decrease as $j$ increases, since we expect the solution to be more sensitive to modes with lower index. The evaluation of the right-hand side of ($\mathrm{HJB}_h^k$) requires us to solve a constrained nonlinear programming problem. For this purpose we used `fmincon` from Matlab. It would certainly be worthwhile to investigate possible speed-up by employing different techniques for this task. To solve ($\mathrm{HJB}_h^k$) on $\Upsilon_h$, a fixed point iteration with a multilevel acceleration strategy was used. Once ($\mathrm{HJB}_h^k$) is solved, the optimal value function and corresponding optimal control at each grid point are available.

1. Multilevel method. The convergence rate of the fixed point iteration depends on $h$. The HJB equation is first solved for $h = 0.2$; the result is taken as initial guess to solve the HJB equation for $h = 0.05$. These two results are utilized to predict a new guess by means of the secant method. Then the HJB equation is solved for $h = 0.0125$. This multilevel method significantly accelerates the fixed point iteration.

2. Parallel computation. In solving the dynamic programming equation, the constrained minimizing problem must be solved at each point of the polyhedron; these computations are independent of each other and can therefore be performed in a fine-grained parallel strategy. In this parallelism, the same set of codes runs simultaneously on different pieces of data on various processors. The technique of *massage passing interface* (MPI) [11] was used in our numerical tests. The mechanism used in MPI to distribute data (or information) is through explicit sending and receiving of data among the processors. The newest MPI standard was released in 1997. We refer to [18].

## 5. Application to the viscous Burgers equation.

In this section we demonstrate the efficiency of the proposed methodology by means of optimal boundary control of the viscous Burgers equation.

### 5.1. The optimal control problem.

Define the domains $\Omega = (0,1) \subset \mathbb{R}$, $Q = (0,\infty) \times \Omega$, and $\Sigma = (0,\infty) \times \partial\Omega$. In the context of section 3.1, we set $H = L^2(\Omega)$, $V = H^1(\Omega)$, and we define

$$a(\varphi, \phi) = \nu \int_\Omega \varphi' \phi' \,\mathrm{d}x \quad \text{for } \varphi, \phi \in V,$$

with $\nu > 0$ and $B \in \mathcal{L}(\mathbb{R}, V')$ by

$$\langle Bu, \phi\rangle_{V',V} = u\,\phi(0).$$

Let $u_a \le u_b$. We set $U_{\mathsf{ad}} = \{u \in \mathbb{R} : u_a \le u \le u_b\}$ and define the set of admissible controls

$$\mathcal{U}_{\mathsf{ad}} = \{u \in L^2_{loc}(0,\infty) : u(t) \in U_{\mathsf{ad}} \text{ for almost all } t \in (0,\infty)\}. \tag{5.1}$$

For a control $u \in \mathcal{U}_{\mathsf{ad}}$ we consider the viscous Burgers equation

$$y_t - \nu y_{xx} + y y_x = 0 \quad \text{in } Q, \tag{5.2a}$$
$$\nu y_x(\cdot,0) + \sigma_0 y(\cdot,0) = u \quad \text{in } (0,\infty), \tag{5.2b}$$
$$\nu y_x(\cdot,1) + \sigma_1 y(\cdot,1) = g \quad \text{in } (0,\infty), \tag{5.2c}$$
$$y(0,\cdot) = y_\circ \quad \text{in } \Omega, \tag{5.2d}$$

where $y_\circ \in L^2(\Omega)$ is a given initial condition and $\sigma_0, \sigma_1$, and $g$ are real numbers. Henceforth we consider weak solutions $y \in W_{loc}(0,\infty;V)$ of (5.2) satisfying (5.2d) and

$$\begin{aligned} &\langle y_t(t), \varphi\rangle_{V',V} + \sigma_1 y(t,1)\varphi(1) - \sigma_0 y(t,0)\varphi(0) \\ &a(y,\varphi) + \int_\Omega y(t)y'(t)\varphi\,\mathrm{d}x = g\varphi(1) - \langle Bu, \varphi\rangle_{V',V} \end{aligned} \tag{5.3}$$

for all $\varphi \in H^1(\Omega)$ and $t \in (0,\infty)$ a.e. For the functional analytic treatment of (5.2) we refer to [22, 24], for example. We shall consider the cost functional

$$J(y,u) = \int_0^\infty \left(\frac{1}{2}\int_\Omega |y(t,x) - z(x)|^2\,\mathrm{d}x + \frac{\beta}{2}\,|u(t)|^2\right) e^{-\lambda t}\,\mathrm{d}t,$$

where $z \in L^2(\Omega)$ is a given desired state and $\lambda, \beta > 0$ are positive constants.

The optimal control problem is given by

$$\min J(y,u) \quad \text{such that} \quad (y,u) \in W_{loc}(0,\infty) \times \mathcal{U}_{\mathsf{ad}} \text{ satisfies (5.2)}, \tag{$\tilde{\mathrm{P}}$}$$

as a weak solution. It is straightforward to argue the existence of an optimal control for ($\tilde{\mathrm{P}}$).

**5.2. Reduced-order control.** Suppose that we have computed a POD basis utilizing, e.g., a finite element code for the viscous Burgers equation and determined the basis functions as described in section 3.2. To compute a POD solution of ($\tilde{\mathrm{P}}$) we make the ansatz (3.14) for the state variable. In addition to the matrices and vectors defined in section 3.3 we introduce the tensor

$$\mathrm{T} = (((\mathrm{b}_{ijk}))) \in \mathbb{R}^{\ell\times\ell\times\ell} \text{ with } \mathrm{b}_{ijk} = \int_\Omega \psi_j \psi_k' \psi_i\,\mathrm{d}x,$$

and the vectors for the boundary conditions

$$\mathrm{d} = (\mathrm{d}_i) \in \mathbb{R}^\ell \text{ with } \mathrm{d}_i = \psi_i(0), \quad \mathrm{e} = (\mathrm{e}_i) \in \mathbb{R}^\ell \text{ with } \mathrm{e}_i = \psi_i(1).$$

**Table 1**
*Construction of parallel computations.*

| Node | Portion |
|---|---|
| master | 12/18 |
| slave1 | 3/18 |
| slave2 | 2/18 |
| slave3 | 1/18 |

Then the Galerkin approximation of the optimal control problem $(\tilde{\mathrm{P}})$ is given by

$$(\tilde{\mathrm{P}}^\ell) \qquad \begin{cases} \min J^\ell(\mathrm{w}^\ell, u) \\ \text{s.t. } u \in \mathcal{U}_{\mathsf{ad}} \text{ and } \begin{cases} \dot{\mathrm{w}}^\ell(t) = F(\mathrm{w}^\ell(t), u^\ell(t)) & \text{for } t > 0, \\ \mathrm{w}^\ell(0) = \mathrm{w}_\circ, \end{cases} \end{cases}$$

where the nonlinear mapping $F : \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}^\ell$ is defined by

$$F(\mathrm{w}^\ell, u) = \mathrm{M}^{-1}\left(( - \mathrm{S} - (\mathrm{T} : \mathrm{w}^\ell))\mathrm{w}^\ell + \mathrm{d}\left(\mathrm{d}^\mathsf{T}\sigma_0 \mathrm{w}^\ell - u\right) - \mathrm{e}(\mathrm{e}^\mathsf{T}\sigma_1 \mathrm{w}^\ell - g)\right).$$

The value function $v$, defined for any initial state $\mathrm{w}_\circ \in \mathbb{R}^\ell$, is

$$v(\mathrm{w}_\circ) = \inf_{u \in \mathcal{U}_{\mathsf{ad}}} \hat{J}^\ell(\mathrm{w}_\circ, u),$$

where $\hat{J}^\ell(\mathrm{w}_\circ, u) = J^\ell(\mathrm{w}^\ell, u)$ and $\mathrm{w}^\ell$ solves the dynamical system in $(\tilde{\mathrm{P}}^\ell)$ with initial condition $\mathrm{w}_\circ$ and control input $u$.

**5.3. Numerical experiments.** This subsection is devoted to demonstrate the efficiency of the feedback synthesis proposed in section 4.

In practical implementations, three Matlab sessions are started on three slaves remotely from the master. Then the required data are transferred to the slaves via MPI. On receiving data, each slave can perform computations concurrently. The portion of the computational work to be performed on each slave can be adjusted according to the performance of the slaves. After all computations are done on the slaves, the data will be collected from the slaves. The distribution of the parallel computation is shown in Table 1. The consumed time (in seconds) are displayed in Table 2 for the parallel and serial computations to calculate one iteration of the fixed point scheme. The specific numbers correspond to the example with discontinuous initial data, given below. The last row shows the ratio of the parallel time cost to the serial time cost. With the number of grid points increasing, the ratio is increasing, partly because more time is consumed to transfer required data to and from the slaves.

Two computational tests will be presented, one with continuous initial condition and the other with discontinuous initial condition. For the sake of comparison we also compute open loop solutions. This can be done efficiently by means of SQP techniques applied to $(\tilde{\mathrm{P}})$ [24], where the constraint in the form of the Burgers equation is discretized by a finite element technique. Moreover, the infinite time horizon was replaced by a finite horizon $[0, T]$, with $T$ chosen sufficiently large so that it has little effect on the numerical results. The parameter

**Table 2**

*Comparisons of parallel and serial computations: CPU times in seconds.*

| Grid points | 625 | 5525 | 9945 | 17901 |
|---|---|---|---|---|
| Parallel | 20.94 | 194.43 | 438.86 | 814.90 |
| Serial | 35.09 | 344.60 | 643.37 | 1094.16 |
| | 59.68% | 56.42% | 68.21% | 74.75% |

**Table 3**

*Parameter settings.*

| Symbol | Value | Description |
|---|---|---|
| $\lambda$ | 2.0 | discount rate |
| $\beta$ | 0.05 | weighting coefficient for the control |
| $T$ | 5 | time horizon |
| $z$ | 0 | desired state |

settings are listed in Table 3. Concerning the boundary conditions for the Burgers equation (5.2), we set $\sigma_0 = \sigma_1 = 0, g = 0$. We took 251 equidistant snapshots from the uncontrolled dynamics. For both examples four basis functions are used for the POD approximation. In terms of the ratio $r(\ell) = \sum_1^\ell \lambda_i / \sum_{i=\ell+1}^d \lambda_i$ this means that $r(4) \geq .985$ for the first example below, and $r(4) \geq .9999$ for the second example. Unless specified otherwise, the grid size was chosen to be $24 \times 16 \times 4 \times 4$. We also report on the effect of the choice of this grid. In our numerical tests we frequently replaced the explicit Euler approximation $y_\circ + hF(y_\circ, u)$ of $y(h)$ by a semiimplicit approximation of $y(h)$. This improved the performance without qualitatively changing the results. Finally, let us comment on the choice of snapshots, which were taken from the uncontrolled dynamics for the results to be presented below. We also carried out tests with taking snapshots from the dynamics, controlled by the open loop optimal control, and combination of the former and the latter. There was little effect on the value of the cost $J$ (evaluated for the closed loop optimal control and the associated trajectory). However, the difference between this value for $J$ and the value of the value-function obtained from the HJB equation, which, as we explain below, is used for validation of our procedure, increases. This comes as no surprise for the class of test problems under consideration. In fact, the controlled states converge to the origin rather quickly and hence contain significantly less information than the uncontrolled snapshots resulting in a decrease of the approximation property of the HJB equation. This in turn could possibly be counteracted by taking nonuniformly spaced snapshots, an issue that we do not want to pursue in this work.

**Continuous initial condition.** In this case the continuous initial condition is $y(0) = (1-x)\sin(3\pi(x-0.5))$, and the viscosity coefficient is $\nu = 0.05$. The state evolutions without control and with feedback optimal control are displayed in Figure 1. As expected, the controlled state decreases as time evolves. The feedback and open loop controls are compared in Figure 2. In Table 4, we can see that the cost functional is decreased from 0.01818 to 0.00766 in the feedback design and from 0.01812 to 0.00681 in the open loop design. This minor difference is not unexpected since the feedback design is based on the reduced system obtained by the POD technique, whereas the open loop optimal control is computed by means of an

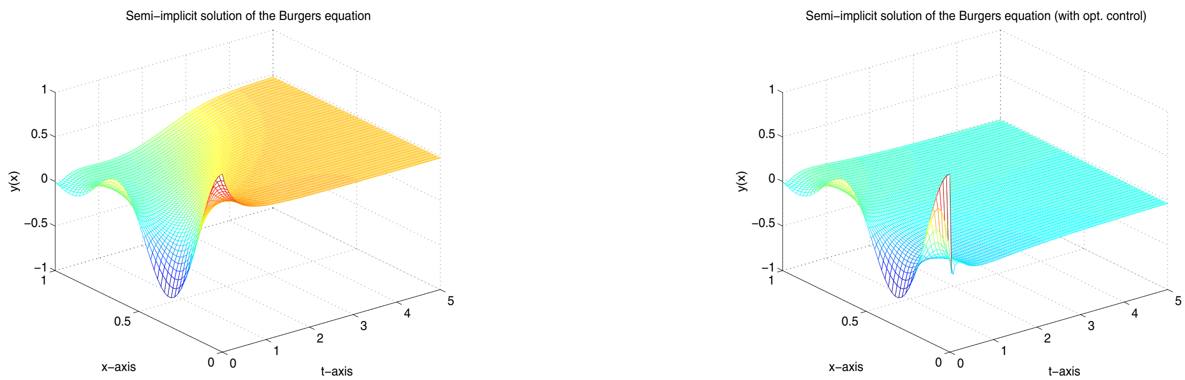


**Figure 1.** *Uncontrolled state (left) and optimal state (right): Continuous initial condition.*

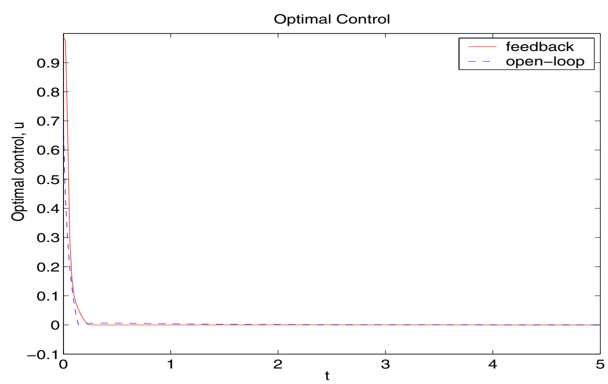


**Figure 2.** *Comparisons of optimal controls from feedback and open-loop design: Continuous initial condition.*

SQP technique for a high-resolution finite element discretization of a continuous system (5.2). A further validation of the numerical results is obtained by comparing the values of the cost function obtained from (i) the open loop control as in the second row of the right column of Table 4 , (ii) inserting the controls and the controlled state into the cost as in the second row of the left column, and (iii) from the numerical approximation to the HJB equation, shown in the first column's last row.

**Discontinuous initial condition.** In this test case, the initial condition is

$$y_\circ(x) = \begin{cases} 1 & \text{if } 0 \le x < 0.5, \\ 0 & \text{if } 0.5 < x \le 1, \end{cases} \tag{5.4}$$

and the viscosity coefficient is $\nu = 0.25$.

First we carry out a grid convergence study using $\lambda = 1$ with all other specifications as in Table 3. The results are shown in Table 5. Note, in particular, that the difference between $J$ and $V$ decreases as the grid is refined. In the following tests, we will take the grid system of $24 \times 16 \times 4 \times 4$ for the polyhedron, which was also taken for the case with the continuous initial condition.

**Table 4**

*Comparisons of results from feedback and open loop design: Continuous initial condition.*

| | Feedback | Open loop |
|---|---|---|
| J w opt. control | 0.00766 | 0.00681 |
| J w/o control | 0.01818 | 0.01812 |
| Value function | 0.00735 | |

**Table 5**

*Grid convergence study: The difference between optimal cost functional and value function decreases as the grid of the polyhedron is fined.*

| | $4 \times 4 \times 4 \times 4$ | $16 \times 12 \times 4 \times 4$ | $24 \times 16 \times 4 \times 4$ | $32 \times 16 \times 5 \times 5$ |
|---|---|---|---|---|
| J w opt. control | 0.0320 | 0.0319 | 0.0319 | 0.0318 |
| J w/o control | 0.1267 | 0.1267 | 0.1267 | 0.1267 |
| Value function | 0.0272 | 0.0354 | 0.0336 | 0.0324 |
| Comp. time (units) | 0.08 | 1 | 2 | 6 |
| Error: V and J | -15% | 11 % | 5 % | 2% |

The evolutions of the state are depicted in Figure 3 for the uncontrolled and controlled cases. Furthermore, as observed in the discussion of the results in Figure 4, the feedback control agrees well with the open loop design result. The computational results are summarized in Table 6. Again we can claim good agreement between the optimal cost functional and the value functional based on the reduced order calculations.

Let us turn to the effect of noise. First random noise is imposed on the initial condition. The open loop design fails to drive the system to zero, if uniform noise in $[-9, 9]$ is added to the initial condition. The feedback design, however, can still generate an acceptable result, as shown in Figure 5. Another test considered here is to impose random noise on the right-hand side of the Burgers equation (5.2a). The controlled states with random uniform random noise in $[-0.25, 0.25]$ (constant w.r.t. $t$) are displayed in Figure 6, respectively, for feedback and open loop design. Comparing the controlled states at $t = 5$ the feedback result is clearly better than the open loop one. The reader will note a drift in the controlled solution, to a value below 0, for the specific realization of the random numbers for this numerical run. Let us point out here the behavior of the uncontrolled Burgers equation with Neumann boundary conditions and random forcing with zero mean: the solution tends to be constant w.r.t. $x$ with the constant depending on the mean of the concrete realization of the set of random numbers (which happens to be negative for the numerical example depicted in Figure 6).

These comparisons confirm that the reduced-order HJB-based closed loop control design is effective in the presence of noise in the system dynamics.

**6. Conclusion.** This paper deals with nonlinear feedback design for evolution problems. The feedback gain is obtained as the solution of the discrete HJB equation. Since the spatial dimension for the HJB equation depends on the number of spatial grid points used in the numerical scheme for the evolution problem, the size of the HJB equation is numerically infeasible if, e.g., finite element or finite difference approximations are used. Here reduced-order modeling with POD is applied for the spatial discretization of the dynamical system resulting in a low-dimensional HJB equation, which can be solved by a fixed-point–type algorithm. To

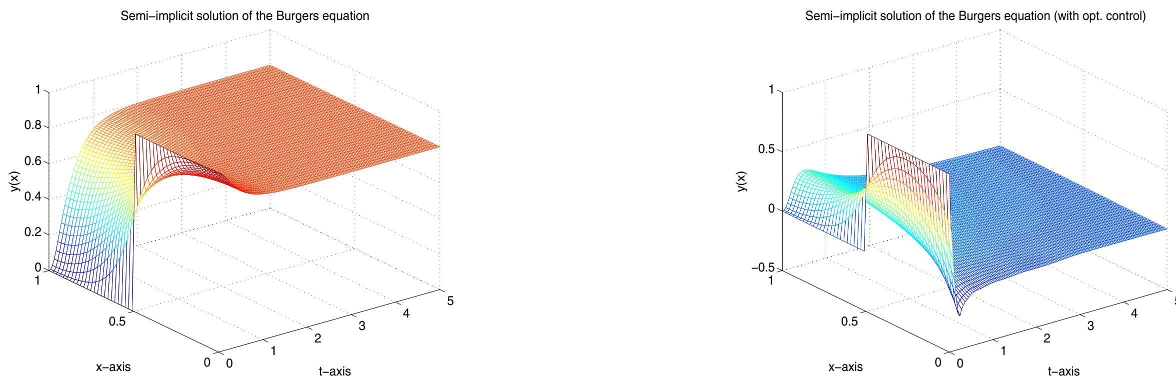


**Figure 3.** *Uncontrolled state (left) and optimal state (right): Discontinuous initial condition.*

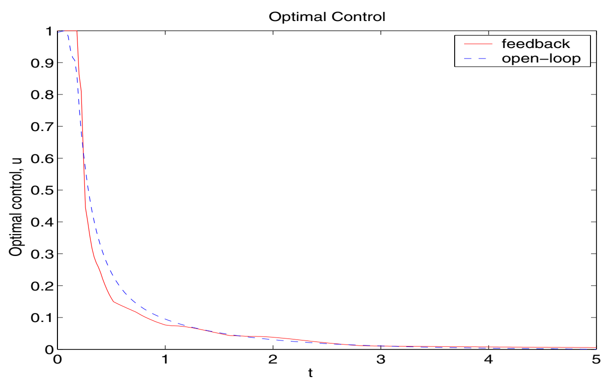


**Figure 4.** *Comparison of optimal control from feedback and open loop design: Discontinuous initial condition.*

accelerate the method both nested iterations and parallelization are utilized. The numerical strategy is illustrated numerically by taking an optimal boundary control problem for the Burgers equation. It turns out that the closed loop control can be computed with reasonable effort. Moreover, the feasibility of the proposed method and the superiority to open loop control is demonstrated by examples including noise in the initial condition and in the forcing function.

## Appendix.

Here we verify the claims made in the second part of section 2. Throughout we assume that $h \in (0, 1]$ and that there exist constants $M, L_1, L_2$ such that

$$hL(y, u) \le M \qquad \text{for all } (y, u) \in \mathbb{R}^n \times U_{\mathsf{ad}}, \tag{A.1}$$

$$|F(y_1, u) - F(y_2, u)| \le L_1 |y_1 - y_2| \qquad \text{for all } y_1, y_2 \in \mathbb{R}^n, u \in U_{\mathsf{ad}}, \tag{A.2}$$

$$|L(y_1, u) - L(y_2, u)| \le L_2 |y_1 - y_2| \qquad \text{for all } y_1, y_2 \in \mathbb{R}^n, u \in U_{\mathsf{ad}}. \tag{A.3}$$

We note that (A.2) and (A.3) are not required for Proposition A.1. Recall that $\beta = e^{-\lambda h}$ for fixed $\lambda > 0$.

**Table 6**

*Comparisons of results from feedback and open loop design: Discontinuous initial condition.*

| | Feedback | Open-loop |
|---|---|---|
| J w. opt. control | 0.0370 | 0.0353 |
| J w/o control | 0.1258 | 0.1258 |
| Value function | 0.0372 | |

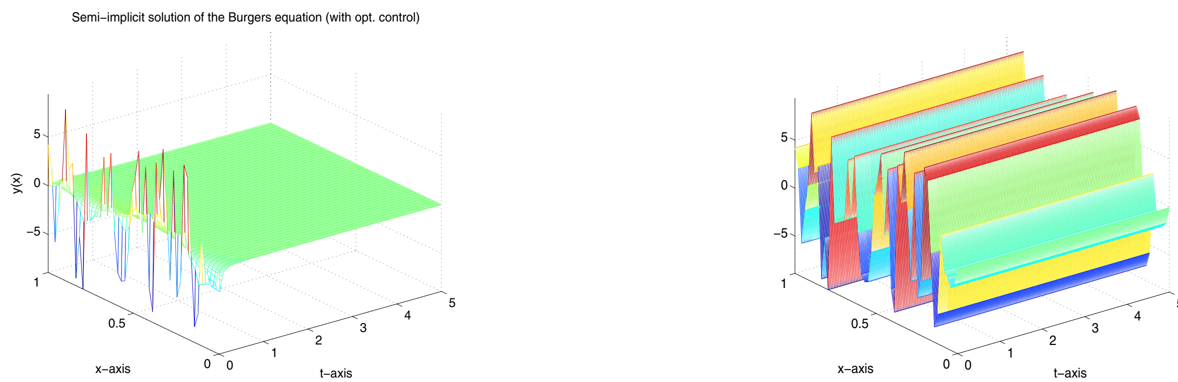


**Figure 5.** *Optimal state with random noise* (9.0) *in the initial condition: Feedback design (left) and open loop design (right).*

Proposition A.1. *The discrete minimal value function $v_h$ is the unique solution of*

$$v_h(y_\circ) = \inf_{u \in U_{\text{ad}}} \left\{ \frac{h}{2}\big(L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)\big) + \beta v_h(y_\circ + hF(y_\circ, u)) \right\}. \quad \text{(A.4)}$$

*Moreover,* $|v_h(y_\circ)| \le M(\frac{1}{1-\beta} - \frac{1}{2})$ *for all* $y_\circ \in \mathbb{R}^n$.

*Proof.* For $u_h = \{u_0, u_1, \dots\} \in \mathcal{U}^h_{\text{ad}}$, set $\bar{u}_h = \{u_1, u_2, \dots\}$, and denote by $y_h = \{y_j(y_\circ, u_h)\}_{j=1}^{\infty}$ the corresponding solution to (2.9). Then

$$y_{j+1}(y_\circ, u_h) = y_j(y_1, \bar{u}_h) \quad \text{for } j \ge 0,$$

where $y_1 = y_\circ + hF(y_\circ, u_0)$. It follows that

$$J_h(y_\circ, u_h) = \frac{h}{2}\big(L(y_\circ, u_0) + \beta L(y_1, u_0)\big) + \beta J_h(y_1, \bar{u}_h), \quad \text{(A.5)}$$

and consequently

$$v_h(y_\circ) \ge \inf_{u \in U_{\text{ad}}} \left\{ \frac{h}{2}\big(L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)\big) + \beta v_h(y_\circ + hF(y_\circ, u)) \right\}.$$

Conversely, let $u \in U^h_{\text{ad}}$ and $\varepsilon > 0$ be arbitrary. Then there exists $u^\varepsilon_h \in \mathcal{U}_{\text{ad}}$ such that

$$v_h(y_\circ + hF(y_\circ, u)) \ge J_h(y_\circ + hF(y_\circ, u), u^\varepsilon_h) - \varepsilon.$$

Using (A.5), we have

$$\beta v_h(y_\circ + hF(y_\circ, u)) \ge J(y_\circ, \hat{u}^\varepsilon_h) - \frac{h}{2}\big(L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)\big) - \beta\varepsilon,$$

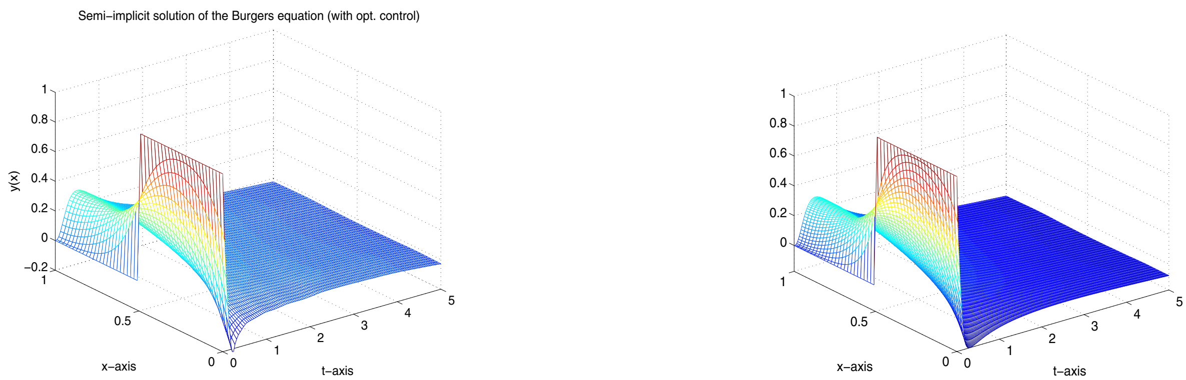


**Figure 6.** *Optimal state with random noise* (0.25) *in the RHS: Feedback design (left) and open loop design (right).*

where $\hat{u}_h^\varepsilon = \{u, u_0^\varepsilon, u_1^\varepsilon, \dots\}$. This implies that

$$v_h(y_\circ) \le \frac{h}{2}(L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)) + \beta v_h(y_\circ + hF(y_\circ, u)). \tag{A.6}$$

Hence, $v_h$ satisfies (A.4).

Turning to uniqueness, assume that $v_h$ and $w_h$ are two solutions to (A.4) and choose $\varepsilon > 0$ arbitrarily. Then for every $y \in \mathbb{R}^n$ there exists $u^\varepsilon = u^\varepsilon(y) \in U_{\mathsf{ad}}$ such that

$$v_h(y) \ge \frac{h}{2}(L(y, u^\varepsilon) + \beta L(y + hF(y, u^\varepsilon), u^\varepsilon)) + \beta v_h(y + hF(y, u^\varepsilon)) - \varepsilon,$$

and

$$w_h(y) \le \frac{h}{2}(L(y, u^\varepsilon) + \beta L(y + hF(y, u^\varepsilon), u^\varepsilon)) + \beta w_h(y + hF(y, u^\varepsilon)).$$

Consequently,

$$\sup_{y \in \mathbb{R}^n}(w_h(y) - v_h(y)) \le \beta \sup_{y \in \mathbb{R}^n}(w_h(y) - u_h(y)) + \varepsilon.$$

This estimate also holds with the roles of $v_h$ and $w_h$ exchanged and hence $v_h = w_h$. Moreover by definition of $J_h(y_\circ, u_h)$ and $|\beta| < 1$ we have

$$|v_h(y_\circ)| \le \frac{M}{2}\left(1 + 2\sum_{j=1}^{\infty}\beta^j\right) = M\left(\frac{1}{1-\beta} - \frac{1}{2}\right). \qquad \blacksquare$$

Next, continuity of the discrete minimal value functionals is addressed.

Proposition A.2. *For every $h \in (0, 1]$ the minimal value functional is uniformly continuous.*

*Proof.* Choose $\epsilon$ arbitrarily and determine $k$ such that $2M\sum_{j=k+1}^{\infty}\beta^j < \epsilon$. For every $\bar{y} \in \mathbb{R}^n$ there exists $u^\epsilon \in \mathcal{U}_{\mathsf{ad}}^h$ such that $v_h(\bar{y}) \ge J(\bar{y}, u^\epsilon) - \epsilon$. Consequently,

$$v_h(y) - v_h(\bar{y}) \le J(y, u^\epsilon) - J(\bar{y}, u^\epsilon) + \epsilon \quad \text{for every } y \in \mathbb{R}^n,$$

and by (A.1)

$$\begin{aligned}v_h(y) - v_h(\bar{y}) \le \frac{1}{2}\left(hL(y, u_0^\epsilon) - hL(\bar{y}, u_0^\epsilon)\right) \\ + \frac{h}{2}\sum_{j=1}^{k}\beta^j |L(y_j(y, u^\epsilon), u_{j-1}^\epsilon) - L(y_j(\bar{y}, u^\epsilon), u_{j-1}^\epsilon)| \\ + \frac{h}{2}\sum_{j=1}^{k}\beta^j |L(y_j(y, u^\epsilon), u_j^\epsilon) - L(y_j(\bar{y}, u^\epsilon), u_j^\epsilon)| \\ + 2M\sum_{j=k+1}^{\infty}\beta^j + \epsilon.\end{aligned}$$

By (A.2) and (A.3), therefore,

$$v_h(y) - v_h(\bar{y}) \le L_2\,|y - \bar{y}|\sum_{j=0}^{k}(1 + L_1)^j + 2\epsilon.$$

Interchanging the roles of $y$ and $\bar{y}$ the desired conclusion follows. ■

Proposition A.3. *Every selection of controls*

$$u_j^* \in S_h(y_j^*) = \underset{u \in U_{\mathsf{ad}}}{\operatorname{argmax}}\left\{-\frac{h}{2}(L(y_j^*, u) + \beta L(y_j^* + hF(y_j^*, u), u)) - \beta v_h(y_j^* + hF(y_j^*, u))\right\}$$

*with $y_0^* = y_\circ$ and $\{y_j^*\}_{j=1}^\infty$ defined by (2.9) is an optimal feedback control.*

*Proof*. Since $U_{\mathsf{ad}}$ is closed and bounded, the mapping $S_h : \mathbb{R}^n \to \mathbb{R}$ is well defined. By (A.5) and the definitions of $\{u_j^*\}_{j=0}^\infty$ and $\{y_j^*\}_{j=0}^\infty$, we have

$$v(y_j^*) = \frac{h}{2}(L(y_j^*, u_j^*) + \beta L(y_{j+1}^*, u_j^*)) + \beta v(y_{j+1}^*)$$

for $j = 0, 1, \ldots$. This implies

$$\begin{aligned}&\sum_{j=0}^{\infty}\beta^j(v(y_j^*) - v(y_{j+1}^*)) \\ &\quad= \frac{h}{2}\sum_{j=0}^{\infty}\beta^j(L(y_j^*, u_j^*) + \beta L(y_{j+1}^*, u_j^*)) \\ &\quad= \frac{h}{2}\left(L(y_\circ, u_0^*) + \sum_{j=1}^{\infty}\beta(L(y_j^*, u_j^*) + L(y_j^*, u_{j-1}^*))\right) = J_h(y_\circ, u_h^*),\end{aligned} \tag{A.7}$$

and consequently $v(y_\circ) = J_h(y_\circ, u_h^*)$ with $u_h^* = \{u_0^*, u_1^*, \ldots\} \in \mathcal{U}_{\mathsf{ad}}^h$. ■

Proposition A.4. *For every compact set $K \subset U_{\mathsf{ad}}$ we have*

$$\lim_{h \to 0^+}\sup_{y_\circ \in K}|v_h(y_\circ) - v(y_\circ)| = 0,$$

*where $v$ is the unique viscosity solution to (HJB).*

*Proof.* The existence of a unique viscosity solution is verified in [4, Theorem III.2.2.], for example. For the convergence result we can proceed as in [4, Theorem VI.1.1.] provided that we verify that $v_h(y_\circ)$ is uniformly bounded w.r.t. $y_\circ$ and $h \in (0, \min(1, 2/\lambda))$; more precisely, we show that

$$\sup\left\{|v_h(y_\circ)| \;:\; y_\circ \in \mathbb{R}^n \text{ and } h \in (0, \min(1, \lambda)]\right\} \le \frac{2M}{\lambda}, \tag{A.8}$$

and that the functions $\underline{v}$ and $\bar{v}$ defined by

$$\underline{v}(y) = \liminf_{(x,h)\to(y,0^+)} v_h(x), \quad \bar{v}(y) = \limsup_{(x,h)\to(y,0^+)} v_h(x) \tag{A.9}$$

are a viscosity supersolution and a viscosity subsolution to (HJB), respectively. To verify (A.8) note that $2M/\lambda$ is a supersolution to (HJB$_h$); i.e., for every $\epsilon > 0$ and $y_\circ \in \mathbb{R}^n$, there exists $u^\epsilon = u^\epsilon(y_o) \in U_{\mathsf{ad}}$ such that

$$\frac{2M}{\lambda} \ge \frac{h}{2}(L(y_\circ, u^\epsilon) + \beta L(y_\circ + hF(y_\circ, u^\epsilon), u^\epsilon)) + \frac{2\beta M}{\lambda} - \epsilon. \tag{A.10}$$

To verify (A.10) we infer from (A.1) and $\beta \le 1$ that

$$\frac{h}{2}(L(y_\circ, u^\epsilon) + \beta L(y_\circ + hF(y_\circ, u^\epsilon), u^\epsilon)) + \frac{2\beta M}{\lambda} \le \frac{M}{\lambda}\,(h\lambda + 2\beta). \tag{A.11}$$

Utilizing $\beta = e^{-\lambda h} \le 1 - \lambda h/2$ for $h \le 2/\lambda$, we find

$$h\lambda + 2\beta \le h\lambda + 2\left(1 - \frac{h\lambda}{2}\right) = 2$$

so that (A.11) implies (A.10). Since $v_h$ is a solution to (HJB), we have

$$v_h(y_\circ) \le \frac{h}{2}(L(y_\circ, u^\epsilon) + \beta L(y_\circ + hF(y_\circ, u^\epsilon), u^\epsilon)) + \beta v_h(y_\circ + F(y_\circ, u^\epsilon)). \tag{A.12}$$

Combining (A.10) and (A.12), we conclude

$$\sup_{y_\circ\in\mathbb{R}^n}\left(v_h(y_\circ) - \frac{2M}{\lambda}\right) \le \beta \sup_{y_\circ\in\mathbb{R}^n}\left(v_h(y_\circ) - \frac{2M}{\lambda}\right) + \epsilon$$

so that $\sup_{y_\circ\in\mathbb{R}^n}(v_h(y_\circ) - 2M/\lambda) \le 0$. Similarly $-2M/\lambda$ is a subsolution of (HJB$_h$). This implies that $\sup_{y_\circ\in\mathbb{R}^n}(-v_h(y_\circ) - 2M/\lambda) \le 0$ and hence (A.8) follows.

To show that $\underline{v}$ is a viscosity supersolution of (HJB), choose $\phi \in C^1(\mathbb{R}^n)$ and let $y_1$ be a strict minimum of $\underline{v} - \phi$ in the closed ball $\bar{B}(y_1, r), r > 0$. Then (see [4, Lemma V.1.9.]) there exist sequences $\{y_n\}_{n=0}^\infty$ in $\bar{B}(y_1, r)$ and $h_n \to 0^+$ such that

$$(v_{h_n} - \phi)(y_n) = \min_{s\in\bar{B}(y_1,r)}(v_{h_n} - \phi)(s),\; y_n \to y_1,\; v_{h_n}(y_n) \to \underline{v}(y_1). \tag{A.13}$$

Since $v_h$ satisfies (HJB$_h$) we have

$$(1-\beta)v_{h_n}(y_n) - \frac{h_n}{2}\left(L(y_n,u_n) + \beta L(y_n + h_n F(y_n,u_n),u_n)\right)$$
$$+\beta(v_{h_n}(y_n) - \phi(y_n)) - \beta\left(v_{h_n}(y_n + h_n F(y_n,u_n)) - \phi(y_n + h_n F(y_n,u_n))\right)$$
$$+\beta\left(\phi(y_n) - \phi(y_n + h_n F(y_n,u_n))\right) = 0.$$

By (A.13) we have for all $n$ sufficiently large that

$$(1-\beta)v_{h_n}(y_n) - \frac{h_n}{2}\left(L(y_n,u_n) + \beta L(y_n + h_n F(y_n,u_n),u_n)\right)$$
$$+\beta\left(\phi(y_n) - \phi(y_n + h_n F(y_n,u_n))\right) \geq 0.$$

Dividing by $h_n$ and passing to the limit on a subsequence, we obtain

$$\lambda \underline{v}(y_1) - L(y_1,\bar{u}) - \nabla\phi(y_1)\cdot F(y_1,\bar{u}) \geq 0$$

for some $\bar{u} \in U_{\mathsf{ad}}$. Hence $\underline{v}$ is a viscosity supersolution for (HJB). Similarly $\bar{v}$ is a viscosity subsolution. This concludes the proof. ■

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