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An embedding domain approach for a class of 2-d shape optimization problems: mathematical analysis ☆

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Abstract

This contribution combines a shape optimization approach to free boundary value problems of Bernoulli type with an embedding domain technique. A theoretical framework is developed which allows to prove continuous dependence of the primal and dual variables in the resulting saddle point problems with respect to the domain. This ensures the existence of a solution of a related shape optimization problem in a sufficiently large class of admissible domains.

Keywords: Shape optimization; Embedding domain technique; Bernoulli problem

1. Introduction

We are concerned with the problem of finding in a given class of domains an optimal member ω^* which minimizes the distance of the flux of the system state u to a desired

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constant flux in its natural norm such that u is the solution of a Dirichlet problem on ω^*. Shape optimization of this type arises, for example, in an optimal control approach to free-boundary value problems of Bernoulli type which serve as mathematical models for problems in ideal fluid dynamics, optimal insulation and electrochemistry [1,4].

This problem was recently considered by the authors in [6]. Representing the dual norm of the flux by the H^1-norm of the solution of an auxiliary transmission problem existence of a solution to the shape optimization problem above was established. By the use of the transmission problem the delicate investigation of the continuous dependence of the normal flux on the boundary of the domain could be avoided. The state equation was numerically solved by an embedding domain technique based on boundary Lagrange multipliers.

Roughly speaking the idea of embedding domain techniques is to extend the state equation to a larger domain with a simple geometry. The original Dirichlet boundary conditions thus become conditions on internal curves which are imposed by Lagrange multipliers. The advantage of such an approach is that due to the simple geometry of the larger domain the extended state equation can be solved more efficiently on a fixed structured grid. This considerably accelerates global optimization methods which typically need a large number of evaluations of the cost functional. Moreover, the extension can be arranged in such a way that the Lagrange multiplier concentrated on a boundary component where homogeneous boundary conditions are prescribed coincides with the normal flux of the original state.

It is the purpose of this note to demonstrate that the fictitious domain approach can serve as a framework for analyzing shape optimization problems. In particular it provides a tool for describing continuous dependence of the states and the Lagrange multipliers with respect to varying domains. This implies existence of a solution to the shape optimization problem. We recall that the Lagrange multipliers are sensitivity measures of the cost with respect to the control variable ω. To our knowledge this result and this approach are new. In [6] we use the embedding domain technique as a computational tool only. The basic features of our analysis are the following: at first we construct a C^1-diffeomorphism of a uniform tubular neighborhood of the boundary of any feasible domain onto a rectangular strip. Here we use the C^2-regularity for the boundaries. Next we build a family of uniformly bounded extension operators which extend periodic functions defined on the boundary of a feasible domain into a tubular neighborhood. These tools will allow us to compare functions which are defined on different domains. Applying our results to the Bernoulli problem this assumption is acceptable since it is known that a C^1 free boundary for the Bernoulli problem in 2 dimensions is in fact analytic; see the discussion in [5]. A numerical realization of our approach is discussed in [6].

The outline of the paper is as follows. In Section 2 we describe the shape optimization problem and the class of feasible domains. The fictitious domain formulation and some basic facts about periodic Sobolev spaces are recalled in Section 3. The continuous dependence of the solution as well as the Lagrange multipliers is discussed in Section 4. The verification of some technical results is deferred to Appendix A.

2. Formulation of the problem

In this note we consider the following shape optimization problem:

$$\min_{\omega\in\mathcal{O}} \frac{1}{2}\left|\frac{\partial u(\omega)}{\partial \nu} - L\right|^2_{H^{-1/2}(\Gamma_f)} \tag{1}$$

$$\begin{aligned} \text{subject to } \Delta u &= 0 \quad \text{in } \omega, \\ u &= 0 \quad \text{on } \Gamma_f, \\ u &= c \quad \text{on } \Gamma_0. \end{aligned} \tag{2}$$

Above $\omega \subset \mathbb{R}^2$ is a doubly connected domain with boundary $\partial\omega = \Gamma_f \cup \Gamma_0$, where Γ_0 is the fixed, given component of the boundary and Γ_f the free component. The fixed boundary component Γ_0 may be empty. Furthermore, ν indicates the outward normal unit vector to ω, $\mathcal{O}$ describes the set of admissible domains and L and c are appropriately chosen constants; see below. This optimization problem is motivated by the Bernoulli free-boundary value problem. A survey of this problem can be found in [5]. If Γ_f is exterior to Γ_0 the *exterior* Bernoulli problem is defined as

$$\begin{aligned} &\text{Find } (\omega^*, u) \in \mathcal{O} \times H^1(\omega^*) \\ &\text{such that } \Delta u = 0 \quad \text{in } \omega^*, \\ &\qquad u = 0 \quad \text{on } \Gamma_f, \\ &\qquad u = 1 \quad \text{on } \Gamma_0, \\ &\qquad \frac{\partial u}{\partial \nu} = L \quad \text{on } \Gamma_f. \end{aligned} \tag{3}$$

It is known that (3) has a solution (ω^*, u) if $L < 0$ and Γ_0 is Lipschitz continuous [3]. In the *interior* Bernoulli problem Γ_f is interior to Γ_0, $u = 0$ on Γ_0, $u = 1$ on Γ_f and $L > 0$. Substituting $u - 1$ for u we may without loss of generality assume $u = 0$ on Γ_f. A solution to the shape optimization problem (1) with vanishing cost leads to a solution of the Bernoulli problem and conversely.

The description of the admissible topologies is a consequence of the fact that we utilize results in [7] on 2π-periodic functions. In particular we restrict ourselves to 2d-domains. We assume that the free boundary component is contained in a set $\mathcal{S}$ of parametrized curves $\gamma: [0, 2\pi] \to \mathbb{R}^2$. We shall denote by Γ_γ the curve represented by γ and by ω_γ the domain bounded by Γ_0 and $\Gamma_f = \Gamma_\gamma$. From now on we shall write Γ_γ instead of Γ_f. Hence, $\omega \in \mathcal{O}$ if and only if $\omega = \omega_\gamma$ for some $\gamma \in \mathcal{S}$. Let $C^k_{2\pi}$ be the space of restrictions to $[0, 2\pi]$ of the subspace of 2π-periodic functions in $C^k(\mathbb{R}, \mathbb{R}^2)$, $k \in \mathbb{N}$. We assume $\gamma \in \mathcal{S}$ if and only if γ satisfies

($\mathcal{S}1$) $\gamma \in C^2_{2\pi}$.

($\mathcal{S}2$) There exist positive constants α, γ_1, γ_2 such that

$$\left|\dot\gamma(t)\right| \geqslant \alpha \quad \text{for all } t \in [0, 2\pi],$$

$$|\dot\gamma|_\infty \leqslant \gamma_1, \qquad |\ddot\gamma|_\infty \leqslant \gamma_2.$$

($\mathcal{S}3$) γ represents a positively oriented closed curve.

($\mathcal{S}4$) $\bar\omega_\gamma \subset \Omega = (-1, 1)^2$.

($\mathcal{S}5$) There exists a positive constant d such that

$$\operatorname{dist}(\Gamma_0, \Gamma_\gamma) \geqslant d, \qquad \operatorname{dist}(\Gamma_\gamma, \partial\Omega) \geqslant d.$$

($\mathcal{S}6$) There is a constant $h > 0$ which does not depend on γ such that for every $t \in [0, 2\pi]$ there are two open discs B_i and B_o of radius h which satisfy $B_i \subset \omega_\gamma$, $B_o \subset \Omega \setminus \bar{\omega}_\gamma$ and $\gamma(t) \in \bar{B}_i \cap \bar{B}_o$.

In ($\mathcal{S}2$) above $|\cdot|_\infty$ denotes the supremum norm. As a consequence of ($\mathcal{S}2$) we note that every parametrization in $\mathcal{S}$ is regular, i.e., the tangent vector is defined everywhere. Assumptions ($\mathcal{S}1$) and ($\mathcal{S}3$) ensure that every element of $\mathcal{S}$ represents a closed curve with a fixed orientation. Let us briefly discuss ($\mathcal{S}6$): choose $\gamma \in \mathcal{S}$, $t \in [0, 2\pi]$ and let x_i be the center of the ball B_i in ($\mathcal{S}6$). Then $\gamma(t) - x_i = -h\nu(t)$ follows from the observation $\gamma(t) \in \operatorname{argmin}\{|\gamma(\tau) - x_i|^2\colon\ \tau \in [0, 2\pi]\}$. This entails $\gamma(t) - \eta h\nu(t) \in \omega_\gamma$ and by a similar reasoning $\gamma(t) + \eta h\nu(t) \in \Omega \setminus \bar{\omega}_\gamma$ for $\eta \in (0, 1)$, where $\nu(t)$ denotes the exterior normal unit vector to Γ_γ at $\gamma(t)$. Hence assumption ($\mathcal{S}6$) implies the existence of a tubular neighborhood $\tilde{D}_\gamma$ of Γ_γ such that

$$\begin{aligned}
&\tilde{D}_\gamma = \bigl\{x \in \Omega\colon\ \operatorname{dist}(x, \Gamma_\gamma) < h\bigr\} = \tilde{D}_\gamma^+ \cup \tilde{D}_\gamma^-,\\
&\tilde{D}_\gamma^\pm = \bigl\{\gamma(t) \pm h\eta\nu(t),\ \eta \in [0, 1),\ t \in [0, 2\pi]\bigr\},\\
&\tilde{D}_\gamma^+ \subset \Omega \setminus \omega_\gamma, \qquad \tilde{D}_\gamma^- \subset \bar{\omega}_\gamma.
\end{aligned} \tag{4}$$

Note that the width h of the tube may be chosen independently of $\gamma \in \mathcal{S}$. As a consequence $\{\Gamma_\gamma\colon\ \gamma \in \mathcal{S}\}$ is a family of simple closed curves. We remark that in view of the regularity results in [2] the family $\mathcal{S}$ contains the free boundary for the exterior problem if the fixed domain is star shaped.

Existence of a solution to (1) usually is derived from some continuity of the cost-functional. This requires that a statement like "$u(\omega_\gamma)$ converges to $u(\omega_{\bar{\gamma}})$ as $\gamma \to \bar{\gamma}$" makes sense. Since the domain of definition of $u(\omega_\gamma)$ depends on γ this amounts to comparing elements of different function spaces. We circumvent the ensuing difficulties by a fictitious domain framework which provides a natural concept for such a convergence. We observe that for the verification of Theorem 2 below only H^1-regularity of the state is required. Therefore, the results of this paper can be readily extended to a general uniformly elliptic second order operator with L^∞ coefficients and an inhomogeneous forcing term in $H^{-1}(\Omega)$. In this case the normal derivative in the cost functional should be replaced by the conormal derivative. This generalization is useful in situations where the continuous dependence of the flux on the domain cannot be argued from the regularity of the state.

3. Reformulation of the problem

3.1. Fictitious domain formulation

It is well known that for any $\omega \in \mathcal{O}$ the state constraint defined by (2) has a unique solution which is at the same time the unique minimizer of

$$\begin{aligned}
&\min_{v \in K} \frac{1}{2}|\nabla v|_\omega^2,\\
&K = \bigl\{v \in H^1(\omega)\colon\ v|_{\Gamma_0} = c,\ v|_{\Gamma_\gamma} = 0\bigr\}.
\end{aligned}$$

Let us replace this constrained optimization problem by the following equivalent problem:

$$\min_{\hat v\in\hat K}\frac{1}{2}|\nabla\hat v|^2_{\Omega},$$
$$\hat K=\{\hat v\in H^1_0(\Omega):\ \hat v|_{\Gamma_0}=c,\ \hat v|_{\Gamma_\gamma}=0\}. \tag{5}$$

We endow the space $H^1_0(\Omega)$ with the norm

$$|\hat v|_{H^1(\Omega)}=(\nabla\hat v,\nabla\hat v)^{1/2}_{\Omega},\quad \hat v\in H^1_0(\Omega).$$

Clearly, (5) has a unique solution $\hat u\in\hat K$ which is characterized by

$$(\nabla\hat u,\nabla\hat v)_{\Omega}=0\quad\text{for all }\hat v\in\{\hat v\in H^1_0(\Omega):\ \hat v|_{\Gamma_0}=0,\ \hat v|_{\Gamma_\gamma}=0\}. \tag{6}$$

Since the constrained variational problem (6) is defined in the fixed domain Ω the original boundary conditions have to be interpreted in the sense of internal traces. It is easy to see that $u=\hat u|_{\omega}$ solves (2). Let $H^{-1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_\gamma)$ be spaces of Lagrange multipliers. Then the necessary optimality conditions for (5) are given by

$$\begin{aligned}
&\text{Find }(\hat u,\lambda_\gamma,\lambda_0)\in H^1_0(\Omega)\times H^{-1/2}(\Gamma_\gamma)\times H^{-1/2}(\Gamma_0)\text{ such that}\\
&\quad(\nabla\hat u,\nabla\hat v)_{\Omega}-\langle\lambda_0,\tau_0\hat v\rangle_{\Gamma_0}-\langle\lambda_\gamma,\tau_\gamma\hat v\rangle_{\Gamma_\gamma}=0,\quad \hat v\in H^1_0(\Omega)\\
&\quad\langle\mu_0,\tau_0\hat u\rangle_{\Gamma_0}+\langle\mu_\gamma,\tau_\gamma\hat u\rangle_{\Gamma_\gamma}=\langle\mu_0,g\rangle_{\Gamma_0},\\
&\qquad(\mu_\gamma,\mu_0)\in H^{-1/2}(\Gamma_\gamma)\times H^{-1/2}(\Gamma_0),
\end{aligned} \tag{7}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_\gamma}$ and $\langle\cdot,\cdot\rangle_{\Gamma_0}$ denote the duality pairings between $H^{-1/2}(\Gamma_\gamma)$ and $H^{1/2}(\Gamma_\gamma)$, respectively, $H^{-1/2}(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$ and $g=c$ on Γ_0. In addition, $\tau_0\hat v=\hat v|_{\Gamma_0}$ and $\tau_\gamma\hat v=\hat v|_{\Gamma_\gamma}$ are the traces of $\hat v$ on Γ_0 and Γ_γ, respectively. System (7) has a unique solution $(\hat u,\lambda_\gamma,\lambda_0)$. It is readily seen that $u=\hat u|_{\omega}$ solves (2) and $\hat u|_{B_\gamma}=0$, where B_γ denotes the connected component of $\Omega\setminus\bar\omega$ adjacent to Γ_γ. As a consequence λ_γ coincides with $\partial u/\partial\nu$ in $H^{-1/2}(\Gamma_\gamma)$; see, e.g., [9]. Hence, the shape optimization problem (1) may be equivalently formulated as

$$\min_{\omega\in\mathcal O}\frac{1}{2}|\lambda_\gamma-L|^2_{H^{-1/2}(\Gamma_\gamma)}, \tag{8}$$

where λ_γ is the second component of the solution of (7).

We now discuss the equivalence between the parametrization of the free boundary by means of $\gamma\in\mathcal S$ and 2π-periodic functions on $[0,2\pi]$.

3.2. Periodic Sobolev spaces

Let $L^2_{2\pi}$ denote the closure of the space of continuous 2π-periodic functions with respect to the norm in $L^2(0,2\pi)$. Following [7, Chapter 8] we define the periodic Sobolev space

$$H^{1/2}_{2\pi}=\{\phi\in L^2_{2\pi}:\ |\phi|_{0,1/2}<\infty\},$$

where the norm $|\cdot|_{0,1/2}$ is given in terms of the Fourier coefficients a_m of ϕ with respect to $\{e^{imt}\}_{m\in\mathbb{Z}}$ by

$$|\phi|_{0,1/2} = \left(\sum_{m=-\infty}^{\infty} (1+m^2)^{1/2}|a_m|^2\right)^{1/2}. \tag{9}$$

It is shown in [7] that for continuously differentiable 2π-periodic functions ϕ this norm is equivalent to

$$|\phi|_{1/2,2\pi} = \left(|\phi|^2_{L^2_{2\pi}} + \int_0^{2\pi}\int_0^{2\pi} \frac{|\phi(t)-\phi(s)|^2}{|\sin((t-s)/2)|^2}\,dt\,ds\right)^{1/2}. \tag{10}$$

Furthermore, if the curve Γ is parametrized by some $\gamma \in \mathcal{S}$ one can define the space

$$H_p^{1/2}(\Gamma) = \left\{\varphi \in L^2(\Gamma):\ \varphi\circ\gamma \in H_{2\pi}^{1/2}\right\}$$

which is endowed with the norm $|\varphi|_{1/2,p} = |\varphi\circ\gamma|_{1/2,2\pi}$. In addition there is also the standard Sobolev space $H^{1/2}(\Gamma)$ the norm of which can be intrinsically expressed as

$$|\varphi|_{1/2} = \left(|\varphi|^2_{L^2(\Gamma)} + \int_\Gamma\int_\Gamma \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^2}\,d\Gamma_x\,d\Gamma_y\right)^{1/2}. \tag{11}$$

This is equivalent to

$$|\varphi|_{1/2,\gamma} = \left(\int_0^{2\pi} |\varphi\circ\gamma|^2|\dot\gamma|\,dt + \int_0^{2\pi}\int_0^{2\pi} \frac{|\varphi\circ\gamma(t)-\varphi\circ\gamma(s)|^2}{|\gamma(t)-\gamma(s)|^2}\big|\dot\gamma(t)\big|\big|\dot\gamma(s)\big|\,dt\,ds\right)^{1/2}, \tag{12}$$

where the notation $|\cdot|_{1/2,\gamma}$ refers to the particular parametrization of Γ used to represent the norm.

Next we turn to the relation among the spaces $H_{2\pi}^{1/2}$, $H_p^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Lemma 1. *Let Γ_γ be a plane curve parametrized by some $\gamma \in \mathcal{S}$. Then the spaces $H^{1/2}(\Gamma_\gamma)$ and $H_p^{1/2}(\Gamma_\gamma)$ coincide as sets and are topologically equivalent. Moreover, the equivalence is uniform with respect to $\gamma \in \mathcal{S}$.*

The proof of this lemma is given in Appendix A. As a consequence the identity $i_\gamma : H^{1/2}(\Gamma_\gamma) \to H_p^{1/2}(\Gamma_\gamma)$ is an isomorphism. The operator $\mathcal{J}_\gamma : H_p^{1/2}(\Gamma_\gamma) \to H_{2\pi}^{1/2}$ given by $\mathcal{J}_\gamma(\varphi) = \varphi\circ\gamma$ is an isometry. In fact, by the definition of the space $H_p^{1/2}(\Gamma_\gamma)$ it is clear that $\mathcal{J}_\gamma$ is an embedding of $H_p^{1/2}(\Gamma_\gamma)$ into $H_{2\pi}^{1/2}$. Since for any $\chi \in H_{2\pi}^{1/2}$ the function $\varphi = \chi\circ\gamma^{-1}$ is an element of $H_p^{1/2}(\Gamma_\gamma)$ we find that $\mathcal{J}_\gamma$ is surjective. Hence the spaces $H_{2\pi}^{1/2}$, $H_p^{1/2}(\Gamma_\gamma)$ and $H^{1/2}(\Gamma_\gamma)$ are homeomorphic.

Recall that $\tau_\gamma : H^1(\Omega) \to H^{1/2}(\Gamma_\gamma)$ denotes the trace operator onto Γ_γ and define $\mathcal{T}_\gamma : H^1(\Omega) \to H_{2\pi}^{1/2}$ by

$$\mathcal{T}_\gamma := \mathcal{J}_\gamma \circ i_\gamma \circ \tau_\gamma,$$

i.e., $\mathcal{T}_\gamma$ maps traces on Γ_γ to periodic functions on $[0, 2\pi]$. Then $\mathcal{T}_\gamma \in \mathcal{L}(H^1(\Omega), H^{1/2}_{2\pi})$ and in view of the preceding discussion surjectivity of $\mathcal{T}_\gamma$ follows from the surjectivity of τ_γ. Below we utilize the notation $\langle\cdot,\cdot\rangle_{2\pi}$, $\langle\cdot,\cdot\rangle_p$ and $\langle\cdot,\cdot\rangle_{\Gamma_\gamma}$ to indicate the duality pairings in $H^{1/2}_{2\pi}$, $H^{1/2}_p(\Gamma_\gamma)$ and $H^{1/2}(\Gamma_\gamma)$, respectively. For $\lambda_\gamma \in H^{-1/2}(\Gamma_\gamma)$ and $\varphi \in H^{1/2}(\Gamma_\gamma)$ we obtain

$$\begin{aligned}\langle\lambda_\gamma, \varphi\rangle_{\Gamma_\gamma} &= \left\langle i_\gamma^{-*}\lambda_\gamma, i_\gamma\varphi\right\rangle_p = \left\langle \mathcal{J}_\gamma^{-*} i_\gamma^{-*}\lambda_\gamma, \mathcal{J}_\gamma i_\gamma \varphi\right\rangle_{2\pi}\\ &= \langle\tilde\lambda_\gamma, \mathcal{J}_\gamma i_\gamma\varphi\rangle_{2\pi} = \langle\tilde\lambda_\gamma, \varphi\circ\gamma\rangle_{2\pi},\end{aligned}$$

where we have set

$$\tilde\lambda_\gamma := \mathcal{J}_\gamma^{-*} i_\gamma^{-*}\lambda_\gamma.$$

In particular this implies

$$\langle\lambda_\gamma, \tau_\gamma\hat v\rangle_{\Gamma_\gamma} = \langle\tilde\lambda_\gamma, \mathcal{J}_\gamma i_\gamma \tau_\gamma \hat v\rangle_{2\pi} = \langle\tilde\lambda_\gamma, \mathcal{T}_\gamma \hat v\rangle_{2\pi}$$

for all $\hat v \in H^1(\Omega)$. The norms of λ_γ and $\tilde\lambda_\gamma$ are equivalent uniformly with respect to $\gamma \in \mathcal{S}$. Moreover, a functional induced by a constant L transforms according to

$$\langle L, \varphi\rangle_{\Gamma_\gamma} = L\int_0^{2\pi} \left|\dot\gamma(t)\right| \varphi\circ\gamma\, dt = \left\langle L|\dot\gamma|, \varphi\circ\gamma\right\rangle_{2\pi}.$$

This discussion shows that the optimization problem (7), (8) may be replaced by

$$\min_{\gamma\in\mathcal{S}_\rho} J(\gamma) := \frac{1}{2}\left|\tilde\lambda_\gamma - L|\dot\gamma|\right|^2_{H^{-1/2}_{2\pi}}, \tag{13}$$

where $(\hat u, \tilde\lambda_\gamma, \lambda_0) \in H^1_0(\Omega) \times H^{-1/2}_{2\pi} \times H^{-1/2}(\Gamma_0)$ satisfies

$$\begin{aligned}&(\nabla\hat u, \nabla\hat v)_\Omega - \langle\lambda_0, \tau_0\hat v\rangle_{\Gamma_0} - \langle\tilde\lambda_\gamma, \mathcal{T}_\gamma\hat v\rangle_{2\pi} = 0, \quad \hat v \in H^1_0(\Omega)\\ &\langle\mu_0, \tau_0\hat u\rangle_{\Gamma_0} + \langle\tilde\mu, \mathcal{T}_\gamma\hat v\rangle_{2\pi} = \langle\mu_0, g\rangle_{\Gamma_0}, \quad (\tilde\mu, \mu_0) \in H^{-1/2}_{2\pi} \times H^{-1/2}(\Gamma_0),\end{aligned} \tag{14}$$

and $\mathcal{S}_\rho \subset \mathcal{S}$ will be specified later. The periodic Sobolev spaces were introduced to be able to analyze the dependence on $\gamma \in \mathcal{S}$ of the boundary terms in (7) which represent the free boundary Γ_γ. Boundary terms defined on Γ_0 can be discussed using the standard spaces.

4. Continuous dependence

The main contribution of the paper is the following theorem on continuous dependence of the solution of (14) on $\gamma \in \mathcal{S}$ which implies existence of a solution to the shape optimization problem (13).

Theorem 2. *Assume $\gamma_n \to \gamma$ in $C^1([0, 2\pi], \mathbb{R}^2)$, $\gamma_n, \gamma \in \mathcal{S}$ and let $(\hat u_n, \tilde\lambda_n, \lambda_{0n}) \in H^1_0(\Omega) \times H^{-1/2}_{2\pi} \times H^{-1/2}(\Gamma_0)$ be the solution of* (14) *corresponding to γ_n. Then $\lim_{n\to\infty} \hat u_n = \hat u$ strongly in $H^1_0(\Omega)$, $\lim_{n\to\infty}\tilde\lambda_n = \tilde\lambda$ weakly in $H^{-1/2}_{2\pi}$,* $\lim_{n\to\infty}\lambda_{0n} = \lambda_0$

strongly in $H^{-1/2}(\Gamma_0)$ *and* $(\hat{u}, \tilde{\lambda}, \lambda_0)$ *is the unique solution of* (14) *corresponding to* γ. *In addition, if* $\gamma_n \to \gamma$ *in* $C^2([0, 2\pi], \mathbb{R}^2)$, *then* $(\tilde{\lambda}_n)$ *converges to* $\tilde{\lambda}$ *strongly in* $H_{2\pi}^{-1/2}$.

Corollary 3. *The functional* J *defined in* (13) *attains its minimum in compact subsets of* $\mathcal{S}$.

To provide an example we mention that it can be shown that $\mathcal{S}_\rho = \{\gamma \in \mathcal{S}\colon |\ddot{\gamma}(t) - \ddot{\gamma}(s)| \leqslant \rho|t - s|,\ t, s \in [0, 2\pi]\}$ is compact in $C^2([0, 2\pi], \mathbb{R}^2)$ for every $\rho > 0$.

The proof of Theorem 2 is decomposed into several steps. At first we establish a uniform bound for the operators $\mathcal{T}_\gamma$.

Lemma 4. *The family of trace operators* $\{\mathcal{T}_\gamma\colon \gamma \in \mathcal{S}\}$ *is uniformly bounded with respect to* γ *in* $\mathcal{L}(H_0^1(\Omega), H_{2\pi}^{1/2})$.

Proof. In order to obtain a uniform bound on $\{\mathcal{T}_\gamma\}$ we analyze the proof of the trace theorem [7, Theorem 8.15] with a slight modification to take into account that Γ_γ is in the interior of Ω. The basic step in this proof is to establish a diffeomorphism between $Q = (0, 2\pi) \times (-1, 1)$ and the cut tubular neighborhood D_γ which is $\tilde{D}_\gamma$, defined in (4), cut at $t = 0$,

$$D_\gamma = \bigl\{\gamma(t) + h\eta\nu(t),\ \eta \in (-1, 1),\ t \in (0, 2\pi)\bigr\}. \tag{15}$$

Define the map $S_\gamma : Q \to D_\gamma$ by

$$S_\gamma(t, \eta) = \gamma(t) + h\eta\nu(t), \quad (t, \eta) \in Q. \tag{16}$$

For any $x \in \tilde{D}_\gamma$ let $p(x)$ denote the orthogonal projection of x onto Γ_γ, i.e., $p(x)$ minimizes

$$d(t) = \bigl|x - \gamma(t)\bigr|^2$$

over $t \in [0, 2\pi]$. The estimate

$$\ddot{d}(t) = 2|\dot{\gamma}|^2 - 2\bigl(x - \gamma(t), \ddot{\gamma}(t)\bigr) \geqslant 2\bigl(\alpha^2 - \bigl|x - \gamma(t)\bigr|\bigl|\ddot{\gamma}(t)\bigr|\bigr) \geqslant 2(\alpha^2 - h\gamma_2)$$

shows that $\ddot{d}(t) > \alpha^2$ holds in $\{t \in (0, 2\pi)\colon |x - \gamma(t)| < h\}$ for h sufficiently small (h may be chosen independently of x) which implies the uniqueness of the projection $p(x)$. Therefore, there is a unique $t^* \in [0, 2\pi)$ such that

$$x - \gamma(t^*) = \ell\nu(t^*)$$

with

$$\ell = \begin{cases} -|x - \gamma(t^*)| & x \in \omega_\gamma \cap D_\gamma, \\ |x - \gamma(t^*)| & x \in (\Omega \setminus \omega_\gamma) \cap D_\gamma. \end{cases}$$

This shows that any $x \in D_\gamma$ may be represented as

$$x = \gamma(t) + h\eta\nu(t)$$

with $(t, \eta) \in Q$ uniquely defined. Hence S_γ is bijective. Since

$$\det DS_\gamma(t, \eta) = -h\sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2} + h^2\eta(\dot{\nu}_1\nu_2 - \nu_1\dot{\nu}_2),$$

one concludes that $\det DS_\gamma(t,\eta) \neq 0$ on Q for h sufficiently small. Thus S_γ defines a C^1-diffeomorphism of Q onto D_γ.

Choose $u \in C^1(\tilde{D}_\gamma)$ such that u vanishes on $\{\gamma(t) \pm h\nu(t)\colon\ t \in (0,2\pi)\}$. Arguing as in [7, p. 121] one obtains

$$|\mathcal{T}_\gamma u|_{H_{1/2,2\pi}} = \big|(u \circ S_\gamma)(\cdot\,,0)\big|_{H_{1/2,2\pi}} \leqslant \frac{1}{\sqrt{2\pi}}|u \circ S_\gamma|_{H^1(Q)} \leqslant C|u|_{H^1(D_\gamma)}, \tag{17}$$

where the constant C depends on a bound for $|\det DS_\gamma|^{-1}_{L^\infty(Q)}$ which is uniform in $\gamma \in \mathcal{S}$ (see (22)). Finally, the above estimate (17) can be extended to arbitrary $u \in C^1(\bar{\Omega})$. Indeed, choose a function $f \in C^1(\mathbb{R}, \mathbb{R}^+)$ satisfying

$$f(0) = 1, \qquad f \geqslant 0 \quad \text{on } [0,\infty), \qquad f = 0 \quad \text{on } (-\infty,-1] \cup [1,\infty)$$

and define

$$g_\gamma(y) = \begin{cases} f(\pi_2 S_\gamma^{-1}(y)) & y \in D_\gamma, \\ 0 & \text{else}, \end{cases}$$

where π_2 is the canonical projection of $\mathbb{R}^2$ onto the second coordinate. By continuity g_γ can be extended uniquely to an element of $C^1(\bar{\Omega})$ denoted by the same symbol. Note that g_γ depends on γ because D_γ and S_γ do. By construction $ug_\gamma \in C^1(\bar{\Omega})$ satisfies

$$ug_\gamma = 0 \quad \text{in } \bar{\Omega} \setminus D_\gamma, \qquad ug_\gamma = u \quad \text{on } \Gamma_\gamma.$$

Applying (17) to ug_γ one obtains

$$\begin{aligned} |\mathcal{T}_\gamma u|_{H_{1/2,2\pi}} &= \big|\mathcal{T}_\gamma(ug_\gamma)\big|_{H_{1/2,2\pi}} \leqslant C|ug_\gamma|_{H^1(D_\gamma)} \\ &\leqslant \tilde{C}\big(|f|_\infty + |f'|_\infty\big)|u|_{H^1(D_\gamma)} \leqslant \tilde{C}\big(1 + |f'|_\infty\big)|u|_{H^1(\Omega)}. \end{aligned}$$

This completes the proof of the lemma. □

Next we show a basic extension result which is of interest for itself.

Lemma 5. *There exists a continuous linear extension operator $\mathcal{E}_\gamma : H^{1/2}_{2\pi} \to H^1_0(\Omega)$ such that $\mathcal{T}_\gamma\mathcal{E}_\gamma\varphi = \varphi$ holds for all $\varphi \in H^{1/2}_{2\pi}$ and $|\mathcal{E}_\gamma|$ is uniformly bounded for $\gamma \in \mathcal{S}$.*

Proof. By Lemma 10 and the discussion preceding it there is an extension operator $\mathcal{E} : H^{1/2}_{2\pi} \to H^1(Q)$, $Q = (0,2\pi) \times (-1,1)$, satisfying $(\mathcal{E}\varphi)(\cdot\,,0) = \varphi$, $(\mathcal{E}\varphi)(\cdot\,,\pm 1) = 0$ and $(\mathcal{E}\varphi)(0,\cdot) = (\mathcal{E}\varphi)(2\pi,\cdot)$; see also Remark 11. Recall the cut tubular neighborhood D_γ of Γ_γ defined in (15) and the diffeomorphism $S_\gamma : Q \to D_\gamma$ introduced in (16). Let $H^1_p(Q) = \{v \in H^1(Q)\colon\ v(0,\cdot) = v(2\pi,\cdot)\}$, define $\mathcal{D}_\gamma : H^1_p(Q) \to H^1(D_\gamma)$ by

$$\mathcal{D}_\gamma u := u \circ S_\gamma^{-1},$$

and set

$$\mathcal{E}_\gamma\varphi := \widetilde{\mathcal{D}_\gamma\mathcal{E}\varphi},$$

where $\tilde{}$ indicates the extension by zero from $\bar{D}_\gamma$ to Ω. Because of

$$\mathcal{T}_\gamma \mathcal{E}_\gamma \varphi = \mathcal{J}_\gamma \circ i_\gamma \circ \tau_\gamma \widetilde{\mathcal{D}_\gamma \mathcal{E}\varphi} = (\widetilde{\mathcal{D}_\gamma \mathcal{E}\varphi}) \circ \gamma = (\mathcal{E}\varphi)\big(S_\gamma^{-1}(\gamma)\big) = (\mathcal{E}\varphi)(\cdot\,, 0) = \varphi,$$

$\mathcal{E}_\gamma$ satisfies the desired extension property. The uniform boundedness of $\mathcal{E}_\gamma$ will follow from a uniform bound for $\mathcal{D}_\gamma$. Consider

$$\begin{aligned}
|\nabla_{xy}\mathcal{D}_\gamma u|^2_{L^2(D_\gamma)} &= \int_{D_\gamma} \big|\big(\nabla_{t\eta} u \circ S_\gamma^{-1}\big)^T (DS_\gamma)^{-1} \circ S_\gamma^{-1}\big|^2\,dx\,dy \\
&= \int_Q \big|(\nabla_{t\eta} u)^T (DS_\gamma)^{-1}\big|^2 \big|\det(DS_\gamma)\big|\,dt\,d\eta \\
&\leqslant \int_Q |\nabla_{t\eta} u|^2 \big|(DS_\gamma)^{-1}\big|_F^2 \big|\det(DS_\gamma)\big|\,dt\,d\eta,
\end{aligned}$$

where $|\cdot|_F$ denotes the Frobenius norm. In view of

$$\begin{aligned}
\big|(DS_\gamma)^{-1}\big|_F^2 \big|\det(DS_\gamma)\big| &= \frac{1}{|\det(DS_\gamma)|} \left| \begin{pmatrix} h\nu_2 & -h\nu_1 \\ -\dot\gamma_2 - h\eta\dot\nu_2 & \dot\gamma_1 + h\eta\dot\nu_1 \end{pmatrix} \right|_F^2 \\
&= \frac{1}{|\det(DS_\gamma)|} \big(h^2 + |\dot\gamma + h\eta\dot\nu|^2\big)
\end{aligned}$$

and

$$\begin{aligned}
\big|\det(DS_\gamma)\big| &= h\big|-|\dot\gamma| + h\eta(\dot\nu_1\nu_2 - \nu_1\dot\nu_2)\big| \\
&\geqslant h\big(|\dot\gamma| - h|\dot\nu|\big) \geqslant h\big(\alpha - h|\dot\nu|\big) \geqslant \frac{\alpha}{2}h
\end{aligned} \tag{18}$$

which by ($\mathcal{S}2$) holds for h sufficiently small independently of $\gamma \in \mathcal{S}$. Using a uniform bound for $|\dot\gamma|$ and $|\dot\nu|$ one obtains

$$|\nabla_{xy}\mathcal{D}_\gamma u|_{L^2(D_\gamma)} \leqslant c|\nabla_{t\eta} u|_{L^2(Q)},$$

where the constant c is independent of $\gamma \in \mathcal{S}$. Since $|\mathcal{D}_\gamma u|_{L^2(D_\gamma)}$ can be estimated similarly, the desired uniform bound for $\mathcal{D}_\gamma$ follows. $\square$

Remark 6. The construction of the extension operator $\mathcal{E}_\gamma$ shows that $\operatorname{supp}\mathcal{E}_\gamma\varphi \subset \bar{D}_\gamma$.

Lemmas 4 and 5 entail uniform a priori bounds on the solution of (14).

Proposition 7. *The set of solutions* $\{(\hat u_\gamma, \tilde\lambda_\gamma, \lambda_{0\gamma})\colon \gamma \in \mathcal{S}\}$ *of* (14) *is bounded in* $H_0^1(\Omega) \times H_{2\pi}^{-1/2} \times H^{-1/2}(\Gamma_0)$.

Proof. Let $\hat v^* \in H_0^1(\Omega)$ be such that $\hat v^* = c$ on Γ_0 and $\hat v^* = 0$ on Γ_γ for every $\gamma \in \mathcal{S}$. Such a function can be constructed independently of $\gamma \in \mathcal{S}$ by ($\mathcal{S}5$). Then from (5) we derive the bound

$$|\hat u_\gamma|_{H^1(\Omega)} \leqslant |\hat v^*|_{H^1(\Omega)}$$

for all $\gamma \in \mathcal{S}$. Choose $\varphi \in H^{1/2}(\Gamma_0)$, extend φ to $\hat{v} \in H^1(\Omega)$ such that $\operatorname{supp}\varphi \subset \{x \in \Omega\colon \operatorname{dist}(x, \Gamma_0) < d/2\}$ and insert $\hat{v}$ in the first equation of (14). This results in

$$\begin{aligned}\langle \lambda_{0\gamma}, \varphi\rangle_{\Gamma_0} &= \langle \lambda_{0\gamma}, \tau_0\hat{v}\rangle_{\Gamma_0} = (\nabla\hat{u}_\gamma, \nabla\hat{v})_\Omega \leqslant |\hat{u}_\gamma|_{H^1(\Omega)}|\hat{v}|_{H^1(\Omega)}\\ &\leqslant k|\hat{u}_\gamma|_{H^1(\Omega)}|\varphi|_{1/2,\Gamma_0}\end{aligned}$$

which implies that

$$|\lambda_{0\gamma}|_{H^{-1/2}(\Gamma_0)} \leqslant k|\hat{u}_\gamma|_{H^1(\Omega)} \leqslant k|\hat{v}^*|_{H^1(\Omega)},$$

where k denotes a suitable embedding constant. In view of Lemma 5, Remark 6 and (14) one obtains for $\varphi \in H^{1/2}_{2\pi}$,

$$\bigl|\langle\tilde{\lambda}_\gamma, \varphi\rangle_{2\pi}\bigr| = \bigl|\langle\tilde{\lambda}_\gamma, \mathcal{T}_\gamma\mathcal{E}_\gamma\varphi\rangle_{2\pi}\bigr| = \bigl|(\nabla\hat{u}_\gamma, \nabla\mathcal{E}_\gamma\varphi)_\Omega\bigr| \leqslant |\mathcal{E}_\gamma||\hat{u}_\gamma|_{H^1(\Omega)}|\varphi|_{1/2,2\pi}$$

which implies the desired a priori bound for $\tilde{\lambda}_\gamma$ using the bounds for $\hat{u}_\gamma$ and $\mathcal{E}_\gamma$. □

The proof of the next two results is deferred to Appendix A.

Lemma 8. *Let $\gamma_n \to \gamma$ in $C([0, 2\pi], \mathbb{R}^2)$, $\gamma_n, \gamma \in \mathcal{S}$ and let $(\hat{u}_n)$ be any sequence in $H^1_0(\Omega)$ satisfying $\tau_0\hat{u}_n = g$ and $\tau_{\gamma_n}\hat{u}_n = 0$. If $\hat{u}_n$ converges weakly to $\hat{u}$ in $H^1_0(\Omega)$ then $\tau_0\hat{u} = g$ and $\tau_\gamma\hat{u} = 0$.*

Lemma 9. *Let $\gamma_n \to \gamma$ in $C^1([0, 2\pi], \mathbb{R}^2)$, $\gamma_n, \gamma \in \mathcal{S}$. Then $\mathcal{T}_{\gamma_n}$ converges strongly to $\mathcal{T}_\gamma$.*

Now we are ready to enter the proof of Theorem 2.

Proof of Theorem 2. By Lemma 7 one can extract a subsequence (again denoted by $((\hat{u}_n, \tilde{\lambda}_n, \lambda_{0n}))$) converging weakly to $(\hat{u}, \tilde{\lambda}, \lambda_0) \in H^1_0(\Omega) \times H^{-1/2}_{2\pi} \times H^{-1/2}(\Gamma_0)$. In view of Lemma 8 the second equation in (14) is satisfied. For fixed $\hat{v} \in H^1_0(\Omega)$ one obtains using the first equation in (14),

$$\begin{aligned}\lim_{n\to\infty}\langle\tilde{\lambda}_n, \mathcal{T}_{\gamma_n}\hat{v}\rangle_{2\pi} &= \lim_{n\to\infty}\bigl((\nabla\hat{u}_n, \nabla\hat{v})_\Omega - \langle\lambda_{0n}, \tau_0\hat{v}\rangle_{\Gamma_0}\bigr)\\ &= (\nabla\hat{u}, \nabla\hat{v})_\Omega - \langle\lambda_0, \tau_0\hat{v}\rangle_{\Gamma_0}.\end{aligned}\tag{19}$$

Because of

$$\langle\tilde{\lambda}_n, \mathcal{T}_{\gamma_n}\hat{v}\rangle_{2\pi} = \langle\tilde{\lambda}_n, \mathcal{T}_{\gamma_n}\hat{v} - \mathcal{T}_\gamma\hat{v}\rangle_{2\pi} + \langle\tilde{\lambda}_n, \mathcal{T}_\gamma\hat{v}\rangle_{2\pi},$$

Lemma 9 entails

$$\lim_{n\to\infty}\langle\tilde{\lambda}_n, \mathcal{T}_{\gamma_n}\hat{v}\rangle_{2\pi} = \langle\tilde{\lambda}, \mathcal{T}_\gamma\hat{v}\rangle_{2\pi},$$

which combined with (19) shows that $(\hat{u}, \tilde{\lambda}, \lambda_0)$ satisfies also the first equation in (14). By uniqueness of the solution the original sequence converges weakly to $(\hat{u}, \tilde{\lambda}, \lambda_0)$. In view of (14) we have

$$\begin{aligned}&(\nabla\hat{u}_n, \nabla\hat{v})_\Omega - \langle\lambda_{0n}, \tau_0\hat{v}\rangle_{\Gamma_0} - \langle\tilde{\lambda}_n, \mathcal{T}_{\gamma_n}\hat{v}\rangle_{2\pi} = 0,\\ &(\nabla\hat{u}, \nabla\hat{v})_\Omega - \langle\lambda_0, \tau_0\hat{v}\rangle_{\Gamma_0} - \langle\tilde{\lambda}, \mathcal{T}_\gamma\hat{v}\rangle_{2\pi} = 0.\end{aligned}\tag{20}$$

Inserting $\hat{v} = \hat{u}_n$ in the former and $\hat{v} = \hat{u}$ in the latter and using the second equation in (14) yields

$$|\nabla \hat{u}_n|^2_\Omega = \langle \lambda_{0n}, g\rangle_{\Gamma_0}, \qquad |\nabla \hat{u}|^2_\Omega = \langle \lambda_0, g\rangle_{\Gamma_0},$$

which entails $\lim_{n\to\infty} |\hat{u}_n|_{H^1(\Omega)} = |\hat{u}|_{H^1(\Omega)}$. Together with weak convergence this implies strong convergence of $\hat{u}_n$ to $\hat{u}$.

For any $\varphi \in H^{1/2}(\Gamma_0)$ let $\hat{v} \in H^1_0(\Omega)$ be such that $\tau_0\hat{v} = \varphi$ and $\hat{v}$ vanishes outside of a sufficiently small neighborhood of Γ_0. Insert $\hat{v}$ in (20) to obtain

$$\begin{aligned}\bigl|\langle \lambda_{0n} - \lambda_0, \varphi\rangle_{\Gamma_0}\bigr| &= \bigl|\bigl(\nabla(\hat{u}_n - \hat{u}), \nabla\hat{v}\bigr)_\Omega\bigr| \\ &\leqslant |\hat{u}_n - \hat{u}|_{H^1(\Omega)}|\hat{v}|_{H^1(\Omega)} \leqslant k|\hat{u}_n - \hat{u}|_{H^1(\Omega)}|\varphi|_{1/2,\Gamma_0}\end{aligned}$$

which implies convergence of λ_{0n} to λ_0 in $H^{-1/2}(\Gamma_0)$.

In order to obtain strong convergence of $\tilde{\lambda}_n$ to $\tilde{\lambda}$ in $H^{-1/2}_{2\pi}$ we assume $\gamma_n \to \gamma$ in $C^2([0,2\pi],\mathbb{R}^2)$. Observe that for every $\varphi \in H^{1/2}_{2\pi}$ the extensions $\mathcal{E}_{\gamma_n}\varphi$, respectively, $\mathcal{E}_\gamma\varphi$ vanish in a neighborhood of Γ_0. Therefore, using (20) we obtain for $\varphi \in H^{1/2}_{2\pi}$,

$$\begin{aligned}\langle \tilde{\lambda}_n - \tilde{\lambda}, \varphi\rangle_{2\pi} &= \langle \tilde{\lambda}_n, \mathcal{T}_{\gamma_n}\mathcal{E}_{\gamma_n}\varphi\rangle_{2\pi} - \langle \tilde{\lambda}, \mathcal{T}_\gamma\mathcal{E}_\gamma\varphi\rangle_{2\pi} \\ &= (\nabla\hat{u}_n, \nabla\mathcal{E}_{\gamma_n}\varphi)_\Omega - (\nabla\hat{u}, \nabla\mathcal{E}_\gamma\varphi)_\Omega \\ &= \bigl(\nabla(\hat{u}_n - \hat{u}), \nabla\mathcal{E}_{\gamma_n}\varphi\bigr)_\Omega + \bigl(\nabla\hat{u}, \nabla(\mathcal{E}_{\gamma_n} - \mathcal{E}_\gamma)\varphi\bigr)_\Omega.\end{aligned}$$

Below we shall denote by C a generic positive constant which does not depend on γ and γ_n. Lemma 5 entails the estimate

$$\bigl|\bigl(\nabla(\hat{u}_n - \hat{u}), \nabla\mathcal{E}_{\gamma_n}\varphi\bigr)_\Omega\bigr| \leqslant C|\hat{u}_n - \hat{u}|_{H^1(\Omega)}|\varphi|_{1/2,2\pi}. \tag{21}$$

By Remark 6 we obtain

$$\begin{aligned}\bigl(\nabla\hat{u}, \nabla(\mathcal{E}_{\gamma_n} - \mathcal{E}_\gamma)\varphi\bigr)_\Omega &= \int_{D_{\gamma_n}} \nabla\hat{u}^T\nabla\mathcal{E}_{\gamma_n}\varphi\,dx\,dy - \int_{D_\gamma} \nabla\hat{u}^T\nabla\mathcal{E}_\gamma\varphi\,dx\,dy \\ &= \int_{S_{\gamma_n}(Q)} \nabla\hat{u}^T\nabla\bigl((\mathcal{E}\varphi)\circ S^{-1}_{\gamma_n}\bigr)\,dx\,dy - \int_{S_\gamma(Q)} \nabla\hat{u}^T\nabla\bigl((\mathcal{E}\varphi)\circ S^{-1}_\gamma\bigr)\,dx\,dy,\end{aligned}$$

where $S_\gamma : Q \to D_\gamma$ is the diffeomorphism defined in (16) and $\mathcal{E} : H^{1/2}_{2\pi} \to H^1(Q)$ is the extension operator introduced in Lemma 5. A short calculation shows

$$\begin{aligned}&\int_{S_\gamma(Q)} \nabla\hat{u}^T\nabla\bigl((\mathcal{E}\varphi)\circ S^{-1}_\gamma\bigr)\,dx\,dy \\ &\quad= \int_{S_\gamma(Q)} \nabla\hat{u}^T(DS_\gamma)^{-T}\circ S^{-1}_\gamma(\nabla\mathcal{E}\varphi)\circ S^{-1}_\gamma\,dx\,dy \\ &\quad= \int_Q \nabla\hat{u}^T\circ S_\gamma(DS_\gamma)^{-T}(\nabla\mathcal{E}\varphi)|\det DS_\gamma|\,dt\,d\eta\end{aligned}$$

and analogously for γ replaced by γ_n. This results in

$$\begin{aligned}
&\big(\nabla\hat{u}, \nabla(\mathcal{E}_{\gamma_n} - \mathcal{E}_\gamma)\varphi\big)_\Omega \\
&\quad = \int_Q \nabla\hat{u}^T \circ S_{\gamma_n}(DS_{\gamma_n})^{-T}(\nabla\mathcal{E}\varphi)|\det DS_{\gamma_n}|\,dt\,d\eta \\
&\qquad - \int_Q \nabla\hat{u}^T \circ S_\gamma(DS_\gamma)^{-T}(\nabla\mathcal{E}\varphi)|\det DS_\gamma|\,dt\,d\eta \\
&\quad = \int_Q \nabla\hat{u}^T \circ S_{\gamma_n}(DS_{\gamma_n})^{-T}(\nabla\mathcal{E}\varphi)\big(|\det DS_{\gamma_n}| - |\det DS_\gamma|\big)\,dt\,d\eta \\
&\qquad + \int_Q \nabla\hat{u}^T \circ S_{\gamma_n}(DS_{\gamma_n})^{-T}(DS_\gamma - DS_{\gamma_n})(DS_\gamma)^{-1}(\nabla\mathcal{E}\varphi)|\det DS_\gamma|\,dt\,d\eta \\
&\qquad + \int_Q (\nabla\hat{u}^T \circ S_{\gamma_n} - \nabla\hat{u}^T \circ S_\gamma)(DS_\gamma)^{-T}(\nabla\mathcal{E}\varphi)|\det DS_\gamma|\,dt\,d\eta \\
&\quad \equiv I_{1n} + I_{2n} + I_{3n}.
\end{aligned}$$

Using $(\mathcal{S}2)$ and Lemma 5 one obtains the estimate

$$\begin{aligned}
|I_{1n}| &\leqslant |\det DS_{\gamma_n} - \det DS_\gamma|_{L^\infty(Q)} \big||DS_{\gamma_n}^{-T}|_F\big|_{L^\infty(Q)} \int_Q |\nabla\hat{u}^T \circ S_{\gamma_n}||\nabla\mathcal{E}\varphi|\,dt\,d\eta \\
&\leqslant C|\det DS_{\gamma_n} - \det DS_\gamma|_{L^\infty(Q)}|\nabla\hat{u}^T \circ S_{\gamma_n}|_{L^2(Q)}|\varphi|_{1/2,2\pi}.
\end{aligned}$$

Above we used the fact that $\big||DS_{\gamma_n}^{-T}|_F\big|_{L^\infty(Q)}$ can be bounded independently of n which is shown in the proof of Lemma 5. By (18) one finds

$$\left(\int_Q |\nabla\hat{u}^T \circ S_{\gamma_n}|^2\,dt\,d\eta\right)^{1/2} \leqslant \frac{1}{|\det DS_{\gamma_n}|_{L^\infty(Q)}}|\hat{u}|_{H^1(\Omega)} \leqslant C|\hat{u}|_{H^1(\Omega)}, \tag{22}$$

which implies

$$|I_{1n}| \leqslant C|\det DS_{\gamma_n} - \det DS_\gamma|_{L^\infty(Q)}|\varphi|_{1/2,2\pi}. \tag{23}$$

A similar argument leads to

$$|I_{2n}| \leqslant C\big||DS_{\gamma_n} - DS_\gamma|_F\big|_{L^\infty(Q)}|\varphi|_{1/2,2\pi}. \tag{24}$$

In order to estimate I_{3n} choose an arbitrary $\varepsilon > 0$ and $\hat{w} \in C_0^\infty(\Omega)$ such that $|\hat{u} - \hat{w}|_{H^1(\Omega)} < \varepsilon$. By an easy argument one derives

$$|\nabla\hat{u}^T \circ S_{\gamma_n} - \nabla\hat{u}^T \circ S_\gamma|_{L^2(Q)} \leqslant C\big(\varepsilon + |\nabla\hat{w}^T \circ S_{\gamma_n} - \nabla\hat{w}^T \circ S_\gamma|_{L^2(Q)}\big)$$

which in turn results in

$$|I_{3n}| \leqslant C\big(\varepsilon + |\nabla\hat{w}^T \circ S_{\gamma_n} - \nabla\hat{w}^T \circ S_\gamma|_{L^2(Q)}\big)|\varphi|_{1/2,2\pi}. \tag{25}$$

Combining the estimates (23)–(25) one eventually obtains

$$\left|\left(\nabla\hat{u}, \nabla(\mathcal{E}_{\gamma_n} - \mathcal{E}_\gamma)\varphi\right)_\Omega\right| \leqslant \left(c(\gamma_n,\gamma) + C\varepsilon\right)|\varphi|_{1/2,2\pi}, \tag{26}$$

where $c(\gamma_n,\gamma)$ vanishes as $\gamma_n \to \gamma$ in $C^2([0,2\pi],\mathbb{R}^2)$. Since ε was chosen arbitrarily the claim follows from the estimates (21) and (26). $\square$

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Appendix A

Proof of Lemma 1. We prove a slightly stronger result. Let $C^{1,1}_{2\pi}$ be the space of restrictions to $[0,2\pi]$ of the subspace of 2π-periodic functions in $C^{1,1}(\mathbb{R},\mathbb{R}^2)$ and define

$$\begin{aligned}\hat{\mathcal{S}} = \big\{\gamma \in C^{1,1}_{2\pi}\colon\ &|\dot\gamma|_\infty \leqslant \gamma_1,\ \left|\dot\gamma(t)\right| \geqslant \alpha,\\ &\left|\dot\gamma(t) - \dot\gamma(s)\right| \leqslant \gamma_2|t-s| \text{ for all } t,s \in [0,2\pi],\\ &\gamma \text{ satisfies } (\mathcal{S}3)\text{–}(\mathcal{S}6)\big\},\end{aligned}$$

where α, γ_1 and γ_2 are given by $(\mathcal{S}2)$. Note that $\hat{\mathcal{S}}$ is compact in $C^1_{2\pi}$ and $\mathcal{S} \subset \hat{\mathcal{S}}$.

Choose $\gamma \in \hat{\mathcal{S}}$. By definition $\varphi \in L^2(\Gamma)$ is an element of $H^{1/2}_p(\Gamma)$ if and only if

$$|\varphi|^2_{1/2,p} = |\varphi\circ\gamma|^2_{1/2,2\pi} = |\varphi\circ\gamma|^2_{L^2_{2\pi}} + \int_0^{2\pi}\!\!\int_0^{2\pi} \frac{|\varphi\circ\gamma(t) - \varphi\circ\gamma(s)|^2}{|\sin((t-s)/2)|^2}\,dt\,ds < \infty.$$

Using the parametrization $\gamma \in \hat{\mathcal{S}}$ we have $\varphi \in H^{1/2}(\Gamma)$ if and only if

$$\begin{aligned}|\varphi|^2_{1/2,\gamma} = &\int_0^{2\pi} |\varphi\circ\gamma|^2|\dot\gamma|\,dt\\ &+ \int_0^{2\pi}\!\!\int_0^{2\pi} \frac{|\varphi\circ\gamma(t) - \varphi\circ\gamma(s)|^2}{|\gamma(t)-\gamma(s)|^2}\left|\dot\gamma(t)\right|\left|\dot\gamma(s)\right|\,dt\,ds < \infty.\end{aligned}$$

Hence the equivalence follows from the assumptions on $\hat{\mathcal{S}}$ which imply the estimate

$$m \leqslant \frac{|\gamma(t)-\gamma(s)|}{|\sin((t-s)/2)|} \leqslant M, \quad 0 \leqslant t,s \leqslant 2\pi,$$

or equivalently

$$m \leqslant \frac{|\gamma(\tau+s)-\gamma(s)|}{|\sin(\tau/2)|} \leqslant M, \quad 0 \leqslant |\tau|, s \leqslant 2\pi, \tag{A.1}$$

for positive constants m, M which are independent of $\gamma \in \hat{\mathcal{S}}$.

The proof of (A.1) utilizes the elementary inequality

$$\frac{2}{\pi} \leqslant \frac{\sin x}{\min(x, \pi - x)} < 1, \quad 0 < x < \pi. \tag{A.2}$$

The estimate from above in (A.1) is an easy consequence of (A.2). Next we establish the estimate from below in (A.1). There is $0 < \delta < \pi$ which does not depend on $\gamma \in \hat{\mathcal{S}}$ such that for every $s \in I = [0, 2\pi]$ at least one of the inequalities

$$\left|\dot{\gamma}_1(\xi)\right| \geqslant \frac{\alpha}{4}, \qquad \left|\dot{\gamma}_2(\xi)\right| \geqslant \frac{\alpha}{4}$$

holds for all $\xi \in (s - \delta, s + \delta) \cap I$. This is a consequence of $|\dot{\gamma}(s)| \geqslant \alpha$ for all $s \in [0, 2\pi]$ and the uniform equicontinuity of $\dot{\gamma}$. Next we partition the interval I into

$$I_1 = \left\{ s \in I\colon \left|\dot{\gamma}_1(\xi)\right| \geqslant \frac{\alpha}{4} \text{ for } \xi \in (s - \delta, s + \delta) \cap I \right\},$$
$$I_2 = I \setminus I_1.$$

We distinguish three cases: assume at first $0 < |\tau| < \delta$ and choose $s \in I_1$. The argument for $s \in I_2$ is analogous. Using (A.2) and the properties of $\hat{\mathcal{S}}$ one obtains

$$\begin{aligned}\frac{|\gamma(\tau + s) - \gamma(s)|}{|\sin(\tau/2)|} &> \frac{2}{|\tau|}\left|\gamma(\tau + s) - \gamma(s)\right| \geqslant \frac{2}{|\tau|}\left|\gamma_1(\tau + s) - \gamma_1(s)\right| \\ &= \left|\frac{2}{\tau}\int_s^{\tau+s}\left|\dot{\gamma}_1(t)\right| dt\right| \geqslant \frac{\alpha}{2}.\end{aligned}$$

By a limiting argument this inequality also holds for $\tau = 0$. The same inequality can be established if $2\pi - \delta < |\tau| \leqslant 2\pi$ using the periodicity of γ. If $\delta \leqslant |\tau| \leqslant 2\pi - \delta$ the desired estimate follows from

$$\kappa \leqslant \left|\gamma(\tau + s) - \gamma(s)\right| \leqslant 2\gamma_0 \tag{A.3}$$

for $(|\tau|, s) \in [\delta, 2\pi - \delta] \times I$, where γ_0 and κ are independent of $\gamma \in \hat{\mathcal{S}}$. We only prove the lower bound in (A.3). Assume on the contrary that there are sequences $(\gamma_n) \subset \hat{\mathcal{S}}$, $(|\tau_n|, s_n) \in [\delta, 2\pi - \delta] \times I$ such that

$$\left|\gamma_n(\tau_n + s_n) - \gamma_n(s_n)\right| < \frac{1}{n}, \quad n \in \mathbb{N}.$$

By compactness of $\hat{\mathcal{S}}$ and $[\delta, 2\pi - \delta] \times I$ one can, without loss of generality, assume $\lim_{n\to\infty} \gamma_n = \gamma$ in $C^1_{2\pi}$, $\lim_{n\to\infty} \tau_n = \tau$ and $\lim_{n\to\infty} s_n = s$ with $\gamma \in \hat{\mathcal{S}}$, $(|\tau|, s) \in [\delta, 2\pi - \delta] \times I$. In view of

$$\begin{aligned}\left|\gamma(\tau + s) - \gamma(s)\right| \leqslant{} & \left|\gamma(\tau + s) - \gamma(\tau_n + s_n)\right| + \left|\gamma(\tau_n + s_n) - \gamma_n(\tau_n + s_n)\right| \\ & + \left|\gamma_n(\tau_n + s_n) - \gamma_n(s_n)\right| + \left|\gamma_n(s_n) - \gamma(s_n)\right| \\ & + \left|\gamma(s_n) - \gamma(s)\right| \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

we arrive at $\gamma(\tau + s) = \gamma(s)$ which contradicts the fact that γ defines a simple closed curve. The equivalence of the norms $|\cdot|_{1/2,p}$ and $|\cdot|_{1/2}$ is now an easy consequence of (A.1). □

Proof of Lemma 8. The first statement is a consequence of the fact that $\tau_0 \in \mathcal{L}(H_0^1(\Omega), H^{1/2}(\Gamma_0))$. For the proof of the second statement one defines

$$\tilde{u}_n = \begin{cases} \hat{u}_n & \text{in } \omega_n \cup \bar{\omega}_0, \\ 0 & \text{in } \Omega \setminus (\bar{\omega}_n \cup \bar{\omega}_0), \end{cases} \qquad \tilde{u} = \begin{cases} \hat{u} & \text{in } \omega \cup \bar{\omega}_0, \\ 0 & \text{in } \Omega \setminus (\bar{\omega} \cup \bar{\omega}_0), \end{cases}$$

where $\omega_n := \omega_{\gamma_n}$, $\omega := \omega_\gamma$. In the exterior Bernoulli problem ω_0 is the inner connected component surrounded by Γ_0, in the interior Bernoulli problem ω_0 is the domain bounded by Γ_0 and $\partial\Omega$. Since $\tilde{u}_n \in H_0^1(\Omega)$ for every n and the sequence $(\tilde{u}_n)$ is bounded in $H_0^1(\Omega)$ one can extract a subsequence $(\tilde{u}_{n_k})$ which converges weakly in $H_0^1(\Omega)$, hence strongly in $L^2(\Omega)$ to some function $\tilde{v} \in H_0^1(\Omega)$. Next we shall show that $\tilde{v} = 0$ in $\Omega \setminus (\bar{\omega} \cup \bar{\omega}_0)$. Indeed, choose $x \in \Omega \setminus (\bar{\omega} \cup \bar{\omega}_0)$ and let $N(x) \subset \Omega \setminus (\bar{\omega} \cup \bar{\omega}_0)$ be an arbitrary neighborhood of x. Then uniform convergence of γ_n to γ implies $N(x) \subset \Omega \setminus (\bar{\omega}_n \cup \bar{\omega}_0)$ for n sufficiently large which in turn entails $\tilde{u}_{n_k} = 0$ in $N(x)$ for k sufficiently large. Hence $\tilde{v} = 0$ in $\Omega \setminus (\bar{\omega} \cup \bar{\omega}_0)$. By a similar reasoning one argues $\tilde{v} = \hat{u}$ on $\omega \cup \bar{\omega}_0$. As a consequence we conclude $\tilde{v} = \tilde{u}$ in Ω which by uniqueness of the limit implies $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$ weakly in $H_0^1(\Omega)$. Now the statement follows from $\tau_\gamma \hat{u} = \tau_\gamma \tilde{u} = 0$. □

Proof of Lemma 9. Because of Lemma 4 we may restrict ourselves to $v \in C^\infty(\bar{\Omega})$ and estimate

$$\begin{aligned} |\mathcal{T}_{\gamma_n} v - \mathcal{T}_\gamma v|^2_{1/2,2\pi} &= \int_0^{2\pi} \big(v\big(\gamma(t)\big) - v\big(\gamma_n(t)\big)\big)^2\, dt \\ &\quad + \int_0^{2\pi}\int_0^{2\pi} \frac{(v(\gamma(t)) - v(\gamma_n(t)) - (v(\gamma(\tau)) - v(\gamma_n(\tau))))^2}{(\sin((t-\tau)/2))^2}\, d\tau\, dt \\ &\equiv I_{1n} + I_{2n}. \end{aligned}$$

By Lebesgue's dominated convergence theorem one obtains $\lim_{n\to\infty} I_{1n} = 0$. Define for $(t,\tau) \in [0,2\pi] \times [0,2\pi]$ the sequence of functions

$$h_n(t,\tau) := v\big(\gamma(t)\big) - v\big(\gamma_n(t)\big) - \big(v\big(\gamma(\tau)\big) - v\big(\gamma_n(\tau)\big)\big)$$

and observe that for all $n \in \mathbb{N}$,

$$h_n(t,t) = h_n(0,2\pi) = h_n(2\pi,0) = 0, \quad h_n \in C^1\big([0,2\pi]\big),\ h_n|_{C^1} \leqslant \tilde{c},$$

is satisfied for some $\tilde{c} > 0$. Furthermore, let

$$\varphi_n(t,\tau) = \frac{h_n(t,\tau)}{\sin((t-\tau)/2)},$$

where $(t,\tau) \in U = [0,2\pi]^2 \setminus (\{t = \tau\colon\ \tau \in [0,2\pi]\} \cup \{(0,2\pi)\} \cup \{(2\pi,0)\})$. Note that $\lim_{n\to\infty} \varphi_n(t,\tau) = 0$ holds for every $(t,\tau) \in U$. Next we derive an integrable bound for φ_n^2. For this purpose we introduce for some sufficiently small $\delta > 0$,

$$V_\delta = \big(\big\{(t,\tau) \in [0,2\pi]^2\colon\ |t-\tau| < \delta\big\} \cup B\big((0,2\pi),\delta\big) \cup B\big((2\pi,0),\delta\big)\big) \cap U.$$

Then φ_n is bounded on $U \setminus V_\delta$ uniformly with respect to n. For $(t,\tau) \in V_\delta$ with $|t-\tau| < \delta$ the mean value theorem implies

$$\left|h_n(t,\tau)\right| = \left|h_n(t,\tau) - h_n(t,t)\right| \leqslant |h_n|_{C^1}|\tau - t|$$

and consequently

$$\left|\varphi_n(t,\tau)\right| \leqslant |h_n|_{C^1}\left|\frac{\tau - t}{\sin((t-\tau)/2)}\right| \leqslant 2\tilde{c}\frac{1}{1-\delta^2/24}.$$

It remains to estimate φ_n in $(B((0,2\pi),\delta) \cup B((2\pi,0),\delta)) \cap U$. By symmetry it suffices to provide a bound in $B((2\pi,0),\delta) \cap U$. An argument similar to the preceding one leads to

$$\left|h_n(t,\tau)\right| = \left|h_n(t,\tau) - h_n(2\pi,0)\right| \leqslant |h_n|_{C^1}\left((t-2\pi)^2 + \tau^2\right)^{1/2}$$

for all $(t,\tau) \in B((2\pi,0),\delta) \cap U$ which in turn entails the bound

$$\begin{aligned}\left|\varphi_n(t,\tau)\right| &\leqslant |h_n|_{C^1}\frac{((t-2\pi)^2+\tau^2)^{1/2}}{|\sin((t-\tau)/2)|}\\ &\leqslant \tilde{c}\frac{((t-2\pi)^2+\tau^2)^{1/2}}{2\pi - t + \tau}\frac{2\pi - t+\tau}{\sin((2\pi - t + \tau)/2)}\\ &\leqslant \tilde{c}\frac{2\pi - t + \tau}{\sin((2\pi - t+\tau)/2)} \leqslant \frac{2\tilde{c}}{1-\delta^2/6}.\end{aligned}$$

Since

$$I_{2n} = \int_0^{2\pi}\int_0^t \varphi_n^2(t,\tau)\,dt\,d\tau$$

we obtain $\lim_{n\to\infty} I_{2n} = 0$ applying the dominated convergence theorem again. This completes the proof of the lemma. □

For the sake of completeness we indicate in the following lemma the construction of the periodic extension operator $\mathcal{E}$ referred to in Lemma 5. We utilize the space of 2-periodic functions $H_2^{1/2}$ which is defined as $H_{2\pi}^{1/2}$ with $[0,2\pi]$ replaced by $[-1,1]$. Endowing $H_2^{1/2}$ with the equivalent norm

$$|\varphi|_{1/2,2} = \left(\pi\int_{-1}^{1}\varphi^2(x)\,dx + \pi^2\int_{-1}^{1}\int_{-1}^{1}\frac{|\varphi(t)-\varphi(s)|^2}{|\sin(\pi/2)(t-s)|^2}\,dt\,ds\right)^{1/2},$$

the spaces $H_2^{1/2}$ and $H_{2\pi}^{1/2}$ are isometric.

Lemma 10. *Let $R = (-1,1)^2$, $I = (-1,1)$. Then there exists a continuous linear extension operator $\mathcal{E}$ from $H_2^{1/2}$ into $H^1(R)$ such that $\mathcal{E}u(\cdot,0) = u$.*

Proof. Choose u in $C^\infty(I) \cap H_2^{1/2}$ and set $u = 0$ on $\mathbb{R} \setminus (-4, 4)$. The proof is a modification of Lemma 6.9.1 in [8] to account for the periodicity of u. For all $(x, y) \in R^+$, $R^+ = \{(x, y) \in R\colon\ y > 0\}$ define

$$(\mathcal{E}u)(x, y) = \frac{1}{y} \int\limits_{|x-\xi|<y} \varphi_0\left(\frac{x-\xi}{y}\right) u(\xi)\, d\xi,$$

where $\varphi_0 \in C_0^\infty(\mathbb{R})$ is a mollifying function satisfying $\varphi_0 \geqslant 0$ on $\mathbb{R}$, $\int_{\mathbb{R}} \varphi_0(x)\, dx = 1$ and $\operatorname{supp} \varphi_0 = [-1, 1]$. Then by [8, Theorem 2.5.3] we infer $\mathcal{E}u \in C^\infty(R^+)$ and

$$\lim_{y\to 0} \left|\mathcal{E}u(\cdot\,, y) - u\right|_{L^2(I)} = 0.$$

In view of

$$(\mathcal{E}u)(1, y) = \int_{-1}^{1} \varphi_0(z) u(1 - yz)\, dz = \int_{-1}^{1} \varphi_0(z) u(-1 - yz)\, dz = (\mathcal{E}u)(-1, y),$$

$(\mathcal{E}u)(\cdot\,, y)$ is 2-periodic for all $0 < y \leqslant 1$. Next we estimate $|\mathcal{E}u|_{L^2(R^+)}$. Using Hölder's inequality and Fubini's theorem one finds

$$\begin{aligned}
|\mathcal{E}u|^2_{L^2(R^+)} &= \int_0^1 \int_{-1}^1 \left| \int_{-1}^1 \varphi_0(z) u(x - yz)\, dz \right|^2 dx\, dy \\
&\leqslant \int_0^1 \int_{-1}^1 \int_{-1}^1 \varphi_0(z) u^2(x - yz)\, dz\, dx\, dy \\
&= \int_0^1 \int_{-1}^1 \varphi_0(z) \int_{-1}^1 u^2(x - yz)\, dx\, dz\, dy \\
&= \int_0^1 \int_{-1}^1 \varphi_0(z) \int_{-1-yz}^{1-yz} u^2(\xi)\, d\xi\, dz\, dy \\
&\leqslant \int_0^1 \int_{-1}^1 \varphi_0(z) \int_{-2}^{2} u^2(\xi)\, d\xi\, dz\, dy = 2|u|^2_{L^2(I)}.
\end{aligned}$$

In the last step we used the periodicity of u. Next we turn to the estimate of $|\partial_x \mathcal{E}u|^2_{L^2(R^+)}$. Because of $\varphi_0(-1) = \varphi_0(1) = 0$ we get

$$\partial_x \mathcal{E}u(x, y) = \frac{1}{y^2} \int_{x-y}^{x+y} \varphi_0'\left(\frac{x-\xi}{y}\right) u(\xi)\, d\xi, \quad (x, y) \in R^+.$$

This relation is also valid if one replaces u by $\psi = 1$ on $(-2, 2)$. Then $\mathcal{E}\psi = 1$ on R^+, hence $\partial_x \mathcal{E}\psi = 0$ on R^+ which implies

$$\frac{1}{y^2}\int_{x-y}^{x+y} \varphi_0'\left(\frac{x-\xi}{y}\right) d\xi = \frac{1}{y}\int_{-1}^{1} \varphi_0'(z)\,dz = 0$$

on R^+. As a consequence we obtain

$$\begin{aligned}
\partial_x \mathcal{E}u(x, y) &= \frac{1}{y^2}\int_{x-y}^{x+y} \varphi_0'\left(\frac{x-\xi}{y}\right) u(\xi)\,d\xi - \frac{u(x)}{y}\int_{-1}^{1} \varphi_0'(z)\,dz \\
&= \int_{-1}^{1} \varphi_0'(z)\frac{u(x-yz)-u(x)}{y}\,dz
\end{aligned}$$

and hence

$$\begin{aligned}
|\partial_x \mathcal{E}u|^2_{L^2(R^+)} &= \int_0^1\int_{-1}^1 \left|\int_{-1}^1 \varphi_0'(z)\frac{u(x-yz)-u(x)}{y}\,dz\right|^2 dx\,dy \\
&\leqslant \int_0^1\int_{-1}^1\int_{-1}^1 (\varphi_0')^2(z)\,dz \int_{-1}^1 \frac{|u(x-yz)-u(x)|^2}{y^2}\,dz\,dx\,dy \\
&\leqslant c_1\int_0^1\int_{-1}^1\int_{|\xi-x|<y} \frac{|u(\xi)-u(x)|^2}{y^3}\,d\xi\,dx\,dy.
\end{aligned}$$

The domain of integration in the last integral is given by

$$D = \left\{(x, y, \xi):\ |x| < 1,\ 0 < y < 1,\ |\xi - x| < y\right\}.$$

For fixed $x \in I$ consider the section $D_x = \{(y, \xi):\ (x, y, \xi) \in D\}$, hence

$$D = \bigcup_{|x|<1} D_x.$$

Observe that D_x can be written as

$$\begin{aligned}
D_x = &\left\{(y, \xi):\ x < \xi < x + 1,\ 0 < \xi - x < y < 1\right\} \\
&\cup \left\{(y, \xi):\ x - 1 < \xi < x,\ 0 < x - \xi < y < 1\right\}.
\end{aligned}$$

Therefore Fubini's theorem yields

$$|\partial_x \mathcal{E}u|^2_{L^2(R^+)} \leqslant c_1 \int_{-1}^{1} \int_{x-1}^{x} \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi-x)|^2} \left[\left| \sin\frac{\pi}{2}(\xi-x) \right|^2 \int_{x-\xi}^{1} y^{-3}\,dy \right] d\xi\,dx$$
$$+ c_1 \int_{-1}^{1} \int_{x}^{x+1} \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi-x)|^2} \left[\left| \sin\frac{\pi}{2}(\xi-x) \right|^2 \int_{\xi-x}^{1} y^{-3}\,dy \right] d\xi\,dx.$$

Note that

$$\left| \sin\frac{\pi}{2}(\xi-x) \right|^2 \int_{|x-\xi|}^{1} y^{-3}\,dy \leqslant \frac{1}{\pi},$$

which implies

$$|\partial_x \mathcal{E}u|^2_{L^2(R^+)} \leqslant \frac{c_1}{\pi} \int_{-1}^{1} \left[\int_{x-1}^{x} \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi-x)|^2}\,d\xi + \int_{x}^{x+1} \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi-x)|^2}\,d\xi \right] dx$$
$$\leqslant \frac{c_1}{\pi} \int_{-1}^{1} \int_{-2}^{2} \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi-x)|^2}\,d\xi\,dx$$
$$= \frac{c_1}{\pi} \int_{-1}^{1} \left[\int_{-1}^{1} + \int_{-2}^{-1} + \int_{1}^{2} \right] \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi-x)|^2}\,d\xi\,dx.$$

By the periodicity of u we have, for example,

$$\int_{-2}^{-1} \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi+2-x)|^2}\,d\xi = \int_{-2}^{-1} \frac{|u(\xi+2)-u(x)|^2}{|\sin(\pi/2)(\xi+2-x)|^2}\,d\xi$$
$$= \int_{0}^{1} \frac{|u(\xi)-u(x)|^2}{|\sin(\pi/2)(\xi-x)|^2}\,d\xi$$

which leads to the estimate

$$|\partial_x \mathcal{E}u|^2_{L^2(R^+)} \leqslant \frac{2c_1}{\pi^3} |u|^2_{H_2^{1/2}}.$$

Analogously a similar estimate can be derived for $|\partial_y \mathcal{E}u|^2_{L^2(R^+)}$. A density argument completes the proof that $\mathcal{E}$ extends continuously into R^+. Finally we define $\mathcal{E}u$ by reflexion on $y=0$. □

Remark 11. Multiplying $\mathcal{E}\varphi$ by a function $\chi \in C^\infty(R)$ such that $\chi(\cdot\,,0)=1$, $\chi \geqslant 0$ and $\chi(\cdot\,,\pm 1)=0$ one obtains an extension of φ which vanishes for $y=\pm 1$.

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