WENDNER: INTERMEDIATE MICROECONOMICS

Inverse Functions

1. Motivation

Consider two functions $f : Y \to \mathbb{R}$ and $g : X \to Y$. The *composition* of these two functions is denoted by $(f \circ g)(x)$, read "f of g of x," so $(f \circ g)(x) = f(g(x))$. E.g., suppose $f(x) = x^2$, and $g(x) = 3x + 5$, then $(f \circ g)(x) = (3x+5)^2$! Or, when $f(x) = x^2+3x+10$, and $g(x) = 6x+5$, then $(f \circ g)(x) = (6x+5)^2 + 3(6x+5) + 10$ — you simply apply the function g to any variable x in $f(x)$.

Now, there are some special kinds of functions for which the following holds:

$$(f \circ g)(x) = (g \circ f)(x) = x\,. \tag{1}$$

E.g., suppose $f(x) = 2x + 5$, and $g(x) = (x - 5)/2$. Then, $(f \circ g)(x) = x$! I.e., $f(10) = 25$, and $g(25) = 10$, or $g(15) = 5$ and $f(5) = 15$.

Function pairs that exhibit this behavior are called *inverse functions*.

2. One to one functions

Under which conditions exists such a pair of functions? Or, asked differently, given a function f, under which conditions can we be sure that an inverse function g exists? The property to look for is strict monotonicity, or *one to one*. A function f is one to one of no two different $x \in X$ produce the same value $f(x)$.[1] All right, let's consider an example. Suppose, $X = \mathbb{R}$, and $f(x) = x^2$. is this function one to one? Answer: no! Why? Consider $y = f(x) = 1$. For this specific $y = 1$, both $x = -1$ and $x = 1$ satisfy $f(x) = 1$, thus, there are *two (not only one)* $x \in X$ such that $f(x) = 1$. So, $f(x) = x^2$ is not one to one. But how about $X = \mathbb{R}$, and $f(x) = 2x$. Here for every value $y = f(x)$ there exists just one $x \in X$ such that $f(x) = y$, thus, $f(x) = 2x$ is one to one.

3. Inverse functions

Only one to one functions can be inverse functions. As mentioned above, a pair of inverse functions satisfies $(f \circ g)(x) = (g \circ f)(x) = x$. The inverse

[1] Consider a function $f : X \to Y$. The function f is one to one if for every $y \in Y$, the set $\{x \in X | y = f(x)\}$ is a singleton.

function is usually labeled by the "exponent -1."

$$(f \circ g)(x) = x \Leftrightarrow g(x) = f^{-1}(x) \quad \text{or} \quad f(x) = g^{-1}(x) \tag{2}$$

Notice that we generally have $g(x) \neq 1/f(x)$. In the example above, e.g., we have $1/f(x) = 1/(2x + 5)$ which certainly is not equal to $g(x) = (x - 5)/2$.

4. Finding the inverse function

Start with $f(x) = y$. Replace every y with an x and every x with a y, and solve the resulting equation for y. Finally, replace y by $f^{-1}(x)$.

Example. $5x + 10 = y$. Step 1: $5y + 10 = x$, step 2: $y = (x - 10)/5$, step 3: $f^{-1}(x) = (x - 10)/5$. It can be verified easily that $(f \circ f^{-1})(x) = 5[(x - 10)/5] + 10 = x$.