# Notes VI
# Existence, Once Again

In this lecture we give a more general existence proof compared to the ones given in Notes V. In particular, we do not restrict ourselves to exchange economies but consider production economies as well. The notes are based on MWG, chapter 17 (appendix).

## 1 The Starting Point

$I > 0$, $J > 0$, $L > 0$, all of them are finite.

**Definition 1** *An economy with production is defined by* $\mathcal{P} \equiv (\{X_i, \succsim_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}, ..., \theta_{iJ})\}_{i=1}^I)$.

In $\mathcal{P}$, each household's wealth is given by $w_i \equiv p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$. Household $i$'s budget set is $B_i = \{x_i \in X_i : p \cdot x_i \leq w_i\}$.

We now consider a number of differing types of equilibria, each of which is going to play a key role for the existence proof.

**Definition 2 (Walrasian Equilibrium)**
*In* $\mathcal{P}$*, a* Walrasian equilibrium *is an allocation* $(x^*, y^*)$ *and a nonzero price vector* $p = (p_1, ..., p_L)$ *if:*
*(i) For every* $j$*,* $y_j^* \in Y_j$ *maximizes profits in* $Y_j$*:* $p \cdot y_j^* \geq p \cdot y_j$ *for all* $y_j \in Y_j$*.*
*(ii) For every* $i$*,* $x_i^* \in X_i$ *is maximal for* $\succsim_i$ *in the respective budget set* $B_i$*.*
*(iii)* $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$ *, where* $\bar{\omega} = \sum_i \omega_i$ *is the exogenous, aggregate endowment vector.*

**Definition 3 (Walrasian Quasiequilibrium)**
*A* Walrasian quasiequilibrium *is a* Walrasian equilibrium with condition (ii), above, replaced *by:*
*(ii') For every* $i$*,* $p \cdot x_i^* \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$*, and if* $x_i \succ_i x_i^*$ *then* $p \cdot x_i \geq w_i$*.*

Notice that (ii') implies expenditure minimization on the no-worse-than set $\{x_i \in X_i : x_i \succsim_i x_i^*\}$, but not necessarily utility maximization on the budget set $B_i$. Moreover, every Walrasian equilibrium is a Walrasian quasiequilibrium, but not vice versa.

**Definition 4 (Free Disposal Quasiequilibrium)**
*A* free disposal quasiequilibrium *is a* Walrasian quasiequilibrium with condition (iii), above, replaced *by:*
*(iii')* $\sum_i x_i^* \leq \bar{\omega} + \sum_j y_j^*$ *and* $p \cdot (\sum_i x_i^* - \bar{\omega} - \sum_j y_j^*) = 0$ .

Here, the feasibility condition (iii') is changed. Free goods (with a respective price of zero) are allowed to be there in excess supply and can be disposed for free. Notice that every Walrasian quasiequilibrium is a free disposal quasiequilibrium, but not vice versa.

**Definition 5 (Truncated Free Disposal Quasiequilibrium)**
*A* truncated free disposal quasiequilibrium *is a* free disposal quasiequilibrium with truncated production and consumption spaces $\hat{Y}_j$, $\hat{X}_i$, *defined as follows:* $\hat{X}_i \equiv \{x_i \in X_i : |x_{li}| \leq r \text{ for all } l\}$, *and* $\hat{Y}_j \equiv \{y_j \in Y_j : |y_{lj}| \leq r \text{ for all } l\}$.

Here all consumption and production spaces are truncated, so $\hat{Y}_j$, $\hat{X}_i$ are closed and bounded (hence compact) spaces.

Finally, and with less rigor, we define the *Market Game*. As for every game, we need to specify who the players are, what their strategy sets and strategies are (I don't comment on the payoffs here).
*1. Players:* $I$ consumers, $J$ firms, and 1 market agent (Walrasian auctioneer, if you like) who determines prices.
*2. Strategy sets:* $\hat{X}_i$ for consumers, $\hat{Y}_j$ for firms, and the price simplex for the market agent: $\Delta = \{p \in \mathbb{R}^L : p_l \geq 0, \text{ and } \sum_l p_l = 1\}$.
*3. Strategies:* (a) Consumers: choose $x_i' \in \hat{X}_i$ such that (i) $p \cdot x_i' \leq w_i(p, y)$, (ii) $x_i' \succsim_i x_i''$ for all $x_i'' \in \hat{X}_i$ satisfying $p \cdot x_i'' < w_i(p, y)$. (b) Producers: choose production plans $y_j' \in \hat{Y}_j$ that maximize profits in $\hat{Y}_j$ (given $p$). (c) Market Agent: chooses $p \in \Delta$ such as to maximize the value of excess demand: $(\sum_i x_i - \sum_i \omega_i - \sum_j y_j) \cdot p$.
Denote the best-response correspondences by $\tilde{x}_i(x, y, p) \in \hat{X}_i$, $\tilde{y}_i(x, y, p) \in \hat{Y}_j$, $\tilde{p}(x, y, p) \in \Delta$.

Two points merit commentation. First, consumers are not maximizing utility, they are minimizing expenditure — hence, at the level of the market game, we are going for a quasiequilibrium rather than for an equilibrium. Second, the market agent attaches the highest price (=1) to that commodity whose excess demand is the largest.

## 2 Mathematical Preliminaries

Of course, we already had a look at these concepts before. I shortly replicate the discussion for your convenience.

*Compactness of a set in* $\mathbb{R}^N$. The set $A \subset \mathbb{R}^N$ is *compact* if it is closed and bounded relative to $\mathbb{R}^N$.
Consider all converging sequences $\{x_i^n\}_{n=1}^{\infty}$ in $A$ (i.e., every element of a sequence $\{x_i^n\}_{n=1}^{\infty}$ belongs to the set $A$), and denote the respective limit point by $x_i$. The set $A \subset \mathbb{R}^N$ is *closed* if for every converging sequence in $A$, $x_i \in A$.
The set $A \subset \mathbb{R}^N$ is *bounded* if there is $r \in \mathbb{R}$ such that for all $y \in A$ we have: $\|y\| < r$.[1] Clearly, any continuous function defined on a compact set attains a maximum.

*Continuity of Preferences.* Suppose there is a sequence $\{x_i^n\}_{n=1}^{\infty} \in A$ that converges to $x_i \in A$, such that for all $n$: $x_i^n \succsim_i x_i'$. Then continuity of $\succsim_i$ implies that $x_i \succsim_i x_i'$.

*Strict Convexity of Preferences.* Suppose there is a sequence $\{x_i^n\}_{n=1}^{\infty} \in A$, such that for all $n$: $x_i^n \succsim_i x_i^*$ . Then, for any $\alpha \in (0,1)$ we have: $(\alpha\, x_i^n + (1-\alpha)\, x_i^*) \succ_i x_i^*$.
If, in addition, (i) the sequence $\{x_i^n\}_{n=1}^{\infty} \in A$ converges to $x_i \in A$, and (ii) $\succsim_i$ is continuous, we know that $(\alpha\, x_i + (1-\alpha)\, x_i^*) \succ_i x_i^*$.

## 3 Structure of the Existence Proof

Several steps are involved for proofing existence of a Walrasian equilibrium, which is the ultimate objective. Each of these steps works only when certain assumptions (like continuity of preferences etc.) are satisfied. In the following, we consider the proof strategy, consisting of 3 parts (9 steps). In the presentation, we go from the last step to the first step.

**Part 3 of Proof: Walrasian Equilibrium**
**Part 2 of Proof: Solution of the Market Game**
**Part 1 of Proof: Nonemptiness of Best Response Correspondences**

[1] Notice that $r \in \mathbb{R}$ implies $r < \infty$!

**Part 3 of Proof: Walrasian Equilibrium**

- Walrasian equilibrium (To show existence of a WE is the final step.)

$\nwarrow$

If: *cheaper consumption condition*, then: WQE $\Rightarrow$ WE [step 9]

$\nearrow$

- Walrasian quasiequilibrium

$\nwarrow$

If: *1 firm satisfies free disposal*, then: FDQE $\Rightarrow$ WQE [step 8]

$\nearrow$

- free disposal quasiequilibrium

$\nwarrow$

If: $X_i$, $Y_j$ *convex*, then: TFDQE $\Rightarrow$ FDQE [step 7]

$\nearrow$

- truncated free disposal quasiequilibrium.

**Part 2 of Proof: Solution of the Market Game**

- Truncated free disposal quasiequilibrium

$$\nwarrow$$

If: $\succsim_i$ *are locally nonsatiated*, then a solution to the market game is a TFDQE [step 6]

$$\nearrow$$

- Solution of the Market Game

$$\uparrow$$

Solution *exists* if best response correspondences are:
*nonempty, convex valued, upper hemicontinuous.* [step 5]

$$\uparrow$$

*Nonemptiness* follows from (i) nonemptiness of the set of feasible allocations/of the strategy sets, and from (ii) continuity of $\succsim_i$ (see Part 1 of Proof)

*Convex valuedness* follows from convexity of $\succsim_i$, $X_i$, and $Y_j$ [step 4]

*Upper hemicontinuity* follows from continuity of $\succsim_i$ [step 3] .

**Part 1 of Proof:**
**Nonemptiness of Best Response Correspondences**

- Nonemptiness of best response correspondences is ensured if the the set of feasible allocations, $A$, is (i) compact, and (ii) nonempty.

$$\uparrow$$

*Compactness of* $A$ follows from:
Every $X_i$ is closed, bounded below. Every $Y_j$ is closed. Moreover, the aggregate $Y$ is convex, admits the possibility of inaction, satisfies the no-free-lunch property and is irreversible. [step 2]

*Nonemptiness of* $A$ follows from:
$-\mathbb{R}^L_+ \subset Y$, and for all $i$ there exists $\hat{x}_i \in X_i$ such that $\sum_i \hat{x}_i \leq \bar{\omega}$. [step 1]

# 4 Existence of Walrasian Equilibrium

**Theorem 1** *Suppose that for an economy $\mathcal{P}$ the following assumptions (i) to (iii) hold. Then a Walrasian equilibrium exists.*

*(i) For every $i$, $X_i \subset \mathbb{R}^L$ is closed and convex; $\succsim_i$ is rational, continuous, locally nonsatiated, and convex; $\omega_i \geq \hat{x}_i \in X_i$;*
*(ii) Every $Y_j \subset \mathbb{R}^L$ is closed, convex, includes the origin, and satisfies free disposal;*
*(iii) The set of feasible allocations is compact (i.e.: for all $i$ there exists $\hat{x}_i \in X_i$ such that $\sum_i \hat{x}_i \leq \bar{\omega}$, all $X_i$ are bounded below, all $Y_j$ satisfy the no free lunch condition as well as irreversibility).*

# 5 Sketches of Proofs for each Step

### Step 1 (Nonemptiness of $A$)

The set of feasible allocations, $A$, is nonempty if $-\mathbb{R}^L_+ \subset Y$, and for all $i$ there exists $\hat{x}_i \in X_i$ such that $\sum_i \hat{x}_i \leq \bar{\omega}$. Certainly, $\sum_i \hat{x}_i - \bar{\omega} \in -\mathbb{R}^L_+ \subset Y$. Thus, $\hat{x}_i$ exists, and $\hat{x}_i$ and $\sum_i \hat{x}_i - \bar{\omega}$ are feasible. So, $A$ is nonempty. Notice that the requirements are: (i) $0 \in Y$ (possibility of inaction), and (ii) $0 - \mathbb{R}^L_+ = -\mathbb{R}^L_+ \subset Y$ (free disposal).[2]

### Step 2 (Compactness of $A$)

The set $A$ needs to be closed and bounded. Closedness follows from the fact that all individual production and consumption sets are closed (and the fact that the sum of closed sets is closed).
Boundedness: If all production sets are bounded, the consumption sets are bounded as well, because (i) for all $l$, $\sum_i x_{li} \leq \sum_j y_{lj} + \sum_i \omega_{li}$, and (ii) all $X_i$ are bounded below by assumption. So we need to argue that all $Y_j$ are bounded above. Suppose not, then there is a $Y_j$ that contains a nonnegative vector (production plan). This, however, is precluded by the no-free-lunch property.

[2] Remember, we defined the set of feasible allocations as $A = \{(x, y) \in X_1 \times ... \times X_I \times Y_1 \times ... \times Y_J : \sum_i x_i = \bar{\omega} + \sum_j y_j\} \subset \mathbb{R}^{L(I+J)}$.

**Step 3 (UHC of Best Response Correspondences)**

As $A$ is bounded, we first restrict our attention to the truncated economy. Consider the best response correspondence $\tilde{x}_i(x, y, p)$, and four converging sequences: $p^n \to p$, $y^n \to y$, $x^n \to x$, $x_i'^n \to x_i'$, such that $x_i'^n \in \tilde{x}_i(x^n, y^n, p^n)$. So, $x_i'^n$ is an expenditure minimizing consumption bundle. Then, $\tilde{x}_i(x, y, p)$ is uhc if $x_i' \in \tilde{x}_i(x, y, p)$.
As $p^n \cdot x_i'^n \leq w_i(p^n, y^n)$, continuity (of the dot product) implies $p \cdot x_i' \leq w_i(p, y)$. Consider any $x_i'' \in \hat{X}_i$ for which $x_i'' \succ x_i'$. By continuity of $\succsim_i$, for $n$ large enough, $x_i'' \succ x_i'^n$. Thus, $p^n \cdot x_i'' \geq w_i(p^n, y^n)$. Otherwise there is a cheaper consumption bundle that gives more utility as $x_i'^n$ — but then $x_i'^n$ is not an element of the best response correspondence.
In the limit, as $n \to \infty$, $p \cdot x_i'' \geq w_i(p, y)$. Thus, any consumption bundle which is strictly preferred does not cost less than $x_i'$, so $x_i'$ is an expenditure minimizing consumption bundle, and therefore $x_i' \in \tilde{x}_i(x, y, p)$.
Notice, we need continuity of $\succsim_i$ to establish uhc of the the best response correspondences. Similar arguments can be found for $\tilde{p}$ and $\tilde{y}$.

**Step 4 (Convex Valuedness of Best Response Correspondences)**

For any two elements of the best response correspondence, $x_i, x_i'$, any linear combination, $x_{i\alpha} = \alpha\, x_i + (1-\alpha)x_i'$ is (i) affordable, (ii) at least as good (by convexity of preferences). For any cheaper consumption bundle $x_i''$, it holds that $x_i \succsim_i x_i''$ (otherwise, $x_i$ would not be an element of the best response correspondence), hence, $x_{i\alpha} \succsim_i x_i''$. Thus all $x_{i\alpha} \in \tilde{x}_i(x, y, p)$.

So, by steps 1 to 4, the best response correspondences are nonempty, convex valued, and uhc.
Nonemptiness follows from step 1. For $\tilde{p}(.)$, $\tilde{x}(.)$, and $\tilde{y}(.)$, we are maximizing continuous functions on nonempty, compact sets — so, by the Weierstrass Theorem, there are solutions to the maximization problems, and so the best response correspondences are nonempty.

**Step 5 (Existence of a Fixed Point)**

A solution to the market game exists, if we are having a fixed point: $x_i^* \in \tilde{x}_i(x^*, y^*, p)$ for all $i$, $y_j^* \in \tilde{y}_j(x^*, y^*, p)$ for all $j$, and $p \in \tilde{p}(x^*, y^*, p)$. Step 6 shows that such a fixed point actually is a solution.

Considering the best response correspondences jointly, we define a correspondence $\Psi$ from $X_1 \times ... \times X_I \times Y_1 \times ... \times Y_J \times \Delta$ to itself. Notice that $\Psi$ is an uhc correspondence, and $X_1 \times ... \times X_I \times Y_1 \times ... \times Y_J \times \Delta$ is a nonempty, compact, and convex set. Thus, by Kakutani's Fixed Point Theorem, there is a fixed point, i.e., there is $(x^*, y^*, p^\star)$ such that $(x^*, y^*, p^*) = \Psi(x^*, y^*, p^*)$.

**Step 6 (A Fixed Point is a TFDQE)**

We only need to establish property (iii'), as (i) and (ii') are satisfied by definition of the market game. Thus, we need to show that

$$\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^* \leq 0 \quad \text{and } p \cdot \left( \sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^* \right) = 0\,.$$

From the feasibility condition for each consumer it follows that $p \cdot (\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*) \leq 0$. This implies that $\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^* \leq 0$. Otherwise there would be one commodity, say $l$, in excess demand. The market agent would then set $p_l = 1$ and all $p_k = 0$ for all $k \neq l$. Consequently, $p \cdot (\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*)$ would exceed zero. Thus, $(x^*, y^*) \in A$, and as $A$ is bounded, $x_{li}^* < r$ for all $i$ and all $l$. By local nonsatiation, all individual budget constraints are satisfied with equality. Thus, $p \cdot \left(\sum_i x_i^* - \sum_i \omega_i - \sum_j y_j^*\right) = 0$. So a fixed point is a TFDQE.

**Step 7 (A TFDQE is a FDQE)**

This result follows from convexity of $\succsim_i$. As all $(x^*, y^*) \in A$, and as $A$ is bounded, we know that $x_{li}^* < r$ for all $l$ and $i$.
Proof strategy: We know that in the (nontruncated) FDQE, (ii') definitely holds (by definition). (TFDQE $\Rightarrow$ FDQE) $\Leftrightarrow$ ($\neg$ FDQE $\Rightarrow$ $\neg$ TFDQE). So the proof by the contrapositive says: suppose (ii') does not hold in the **non**truncated equilibrium (so FDQE does not hold), then (ii') does not hold in the truncated equilibrium (so TFDQE does not hold). This statement is equivalent to TFDQE $\Rightarrow$ FDQE.
If (ii') does not hold in the **non**truncated economy, then there exists an $x_i \in X_i$ such that $x_i \succ_i x_i^*$, and $p \cdot x_i < w_i$. Now, consider a sequence: $x_i^n = (1 - 1/n)x_i^* + (1/n)\, x_i$. It certainly holds that $p \cdot x_i^n < w_i$ for all $n$. By convexity of preferences, $x_i^n \succsim_i x_i^*$. For $n$ large enough, $|x_{li}^n| < r$ (and we are back in the truncated economy). Then, by local nonsatiation, there is an

$x'_i \in \hat{X}_i$ such that $x'_i \succ_i x^n_i$ and still $p \cdot x'_i < p \cdot w_i$. But then, we have $x'_i \in \hat{X}_i$ and $x'_i \succ_i x^n_i \succ_i x^*_i$. Thus, in the truncated economy, $x^*_i$ fails to satisfy (ii'), as we found a strictly preferred, cheaper consumption bundle. Consequently, $(x^*, y^*)$ is not in TFDQE. So we are done here.

**Step 8 (A FDQE is a WQE)**

Suppose there is one firm, say firm 1, for which its production set $Y_1$ satisfies the free disposal property. If $(x^*_1, ..., x^*_I, y^*_1, ..., y^*_J, p)$ is a FDQE (with some goods in excess supply), then there is a ${y^*}'_1 \leq y^*_1$ such that $(x^*_1, ..., x^*_I, {y^*}'_1, y^*_2, ..., y^*_J, p)$ is a WQE .

**Step 9 (A WQE is a WE)**

A WQE satisfies the *cheaper consumption (CC) condition* for consumer $i$ if there exists an $x_i \in X_i$: $p \cdot x_i < p \cdot w_i$.

Consider the WQE $(x^*, y^*, p)$. Any consumer who satisfies the CC property at $(x^*, y^*, p)$ is preference maximizing in her budget set (in addition to expenditure minimizing in her no-worse-than set). Hence, if the CC property holds for all $i \in I$, $(x^*, y^*, p)$, a WQE is also a WE.

(Now I am pretty exhausted ... so, there is no bonus stuff today!)