Notes IV
General Equilibrium and Welfare Properties

In this lecture we consider a general model of a private ownership economy, i.e., a market economy in which a consumer's wealth is derived from endowments and from claims to profit shares of firms. Denote the claim of consumer i to a share of (the profits of) firm j by θ_{ij}. Then, $\theta_{ij} \in [0,1]$, and $\sum_{i=1}^{I} \theta_{ij} = 1$ for all $j = 1, ..., J$. The notes are based on MWG, chapter 16.

1 Basic Model

$I > 0$, $J > 0$, $L > 0$, all of them are finite. Consumers are characterized by consumption sets $X_i \subset \mathbb{R}^L$, and preferences $\succsim_i$. Each firm is characterized by a production set $Y_j \subset \mathbb{R}^L$. Initial endowments are given by a vector $\bar{\omega} = (\bar{\omega}_1, ..., \bar{\omega}_L) \in \mathbb{R}^{LI}$.

Definition 1 *An economy, $\mathcal{P}$, is defined by*
$\mathcal{P} \equiv (\{X_i, \succsim_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{\bar{\omega}_i, \theta_{i1}, ..., \theta_{iJ}\}_{i=1}^{I})$.

Definition 2 *An* allocation *(x,y) is a consumption vector x_i for each consumer $i = 1, ..., I$, and a production vector y_j for each firm $j = 1, ..., J$. An allocation is* feasible *if $\sum_i x_{li} = \bar{\omega}_l + \sum_j y_{l,j}$ for every good l. The set of feasible allocations is denoted by A, where $A = \{(x,y) : \sum_i x_{li} = \bar{\omega}_l + \sum_j y_{l,j}\}$.*

Be aware that the set A lives in $\mathbb{R}^{L(I+J)}$ space! Notice the following: The set A is closed and bounded (i.e. *compact* under the Euclidean topology) if: every X_i is closed and bounded below; every Y_j is closed, convex, admits the possibility of inaction ($0 \in Y_j$), satisfies the no-free-lunch property (there is no $y_j \in Y_j$ for which $y_j \gg 0$), and satisfies irreversibility (for any $y_j \neq 0$, $y_j \in Y_j$ implies $-y_j$ is not an element of Y_j). Please study Appendix A in MWG (p. 573 f) carefully to understand the argument.

A feasible allocation is *Pareto efficient*[1] if there is no other allocation $(x', y') \in A$ that Pareto dominates it. I.e., if there is no feasible allocation

[1] "(Pareto) efficient" is synonymous to "Pareto optimal". However, you should not confuse *Pareto optimality* with *(social) optimality*, which is a pretty different concept. In order to clearly distinguish Pareto efficiency from social optimality, we will refer to Pareto efficiency (or Pareto optimality) as "efficiency," and we'll refer to social optimality as "optimality."

(x', y') such that $x'_i \succsim x_i$ for all i, and $x'_i \succ x_i$ for some i. Notice that no price vector is involved in defining Pareto efficiency.

Now we'll define a concept, which we'll employ over and over again...

Definition 3 (Price Equilibrium with Transfers)
Given an economy $\mathcal{P}$, a price vector p and an allocation (x^*, y^*) constitute a price equilibrium with transfers (PET) if there is an assignment of wealth levels $(w_1, ..., w_I)$ with $\sum_i w_i = \sum_i p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that
(i) for every j, y_j^* maximizes profits in Y_j,
(ii) for every i, x_i^* is maximal for $\succsim_i$ in the respective budget set,
(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Please do not hesitate to think about this definition for a while (until tomorrow). In short, a PET comes from optimization and equilibrium.

Query. Is a Walrasian equilibrium a PET?

Proposition 1 (First Fun Theorem of Welfare Economics) *If preferences are locally nonsatiated, and if (p, y^*, x^*) is a PET, then allocation (y^*, x^*) is Pareto efficient. In particular, any Walrasian equilibrium is Pareto efficient.*

Query. In Proposition 1 we (only) need that $\succsim_i$ be locally nonsatiated. Show that strong monotonicity implies monotonicity implies local nonsatiation.

Second Fun Theorem of Welfare Economics. We'll progress in two steps. First, we'll state a result showing that under convexity assumptions, every Pareto efficient allocation can be supported by a price vector as a *price quasiequilibrium with transfers* (PQET). Second, we show that under a rather innocent condition ("cheaper consumption condition"), every PQET is a PET.

PQET. Define: (ii'): for every i, x_i^* minimizes expenditure on the set $\{x_i \in X_i : x_i \succsim_i x_i^*\}$, which is the no-worse-than set with respect to x_i^*. Then, Definition 3, where (ii) is replaced by (ii') defines a PQET. (Again, feel free to rest for a day or two, to contemplate about the meaning of a PQET.)

Clearly every PET is a PQET (however, not vice versa), as utility maximization (a consumption bundle being maximal on the budget set) always implies expenditure minimization (a consumption bundle minimizing expenditure on the no-worse-than set). However, the converse may fail to hold if some prices are zero and/or some preferences are not monotone (see Figure 16.D.2 in MWG, p.555).

Proposition 2 (Second Fun Theorem of Welfare Economics) *Consider an economy* $\mathcal{P}$ *where every production set* Y_j *is convex, every preference relation is convex and locally nonsatiated, and every consumption set is convex. Then, for every Pareto efficient allocation* (x^*, y^*) *there exists a price vector* p *such that* (x^*, y^*, p) *is a PQET.*

Please study the separating hyperplane theorem before working through the proof of Proposition 2 (in MWG, p. 552 ff.).

Proposition 3 (Cheaper Consumption Condition) X_i *is convex and* $\succsim_i$ *is continuous. Suppose* $x_i \succ_i x_i^*$ *implies* $p \cdot x_i \geq w_i$*. Then, if there is a consumption vector* $x_i' \in X_i$ *such that* $p \cdot x_i' < w_i$ *(i.e.,* x_i' *is a "cheaper consumption vector" for* (p, w_i)*), it holds that* $x_i \succ_i x_i^*$ *implies* $p \cdot x_i > w_i$*.*

Corollary 1 *Suppose that for every* i*,* X_i *is convex,* $0 \in X_i$*, and* $\succsim_i$ *is continuous. If* $w \gg 0$*, then any PQET is a PET.*

So, Corollary 1 tells us that if $w \gg 0$, utility maximization is equivalent to expenditure minimization (or alternatively, a PQET is equivalent to a PET). So, under the assumptions of Proposition 2, for every Pareto efficient allocation (x^*, y^*) there exists a price vector p such that (x^*, y^*, p) is a PET.

2 Pareto Efficiency and Optimality

Consider an economy $(\{X_i, \succsim_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$. Each feasible allocation (x, y) gives rise to a utility vector $u_i(x_i)$ for $i = 1, ..., I$. The utility possibility set, U, provides all attainable vectors of utility levels: $U = \{(u_1, ..., u_I) \in \mathbb{R}^I :$ there exists a feasible allocation (x, y) such that $u_i \leq u_i(x_i),\ i = 1, ..., I\}$. The Pareto frontier, UP, is the set of all vectors in U that are not Pareto dominated: $UP = \{u \in U :$ there is no $u' \in U$ such that $u_i' \geq u_i$ for all i and $u_i' > u_i$ for some $i\}$. Consequently, a feasible allocation (x, y) is Pareto efficient if and only if $u(x) \in UP$.

Besides efficiency, another *normative* criterion is "optimality." Before discussing optimality, we need to define the concept of a social welfare function (SWF).

A social welfare function $W(u_1, ..., u_I)$ assigns a social utility value to every vector from the utility possibility set. A special case of a W is the *linear* social welfare function: $W(u_1, ..., u_I) = \sum_i \lambda_i u_i$, for some constant welfare weights $\lambda = (\lambda_1, ..., \lambda_I)$. In the following, we'll focus on this specification of a social welfare function: $W(u) = \lambda \cdot u$.

Denote the solution of the social welfare maximization problem (SWMP) by $u^* = \max_{u \in U} \lambda \cdot u$. u^* is considered *optimal* according to the specific SWF. Certainly, different SWF (e.g., different welfare weights) may give raise to different solutions to the SWMP.

There exist relations between the normative notions of Pareto efficiency and (social) optimality. If $\lambda >> 0$, $u^* \in UP$. This result resembles the First Fun Theorem. If $\lambda >> 0$, the solution to a SWMP is Pareto efficient.

There is a relation to the Second Fun Theorem as well. If U is convex, then for any $\tilde{u} \in UP$ there is a vector of welfare weights $\lambda \geq 0$ such that $\tilde{u}$ is a solution to the SWMP.

3 First Order Conditions and Pareto Efficiency

Suppose, preferences $\succsim_i$ can be represented by utility functions that are twice continuously differentiable and are strongly monotone: $\nabla u_i(x_i) \gg 0$ at all x_i. All firms' production frontiers are twice continuously differentiable, with $\nabla F_j(y_j) \gg 0$ for all $y_j \in \mathbb{R}^L$.

For a Pareto efficient allocation, we need to maximize utility of a household — without loss of generality, utility of household $i = 1$ — subject to the constraint that utility of all other households are equal to some predetermined level $\bar{u}_i$ (or, more precisely, must not be smaller than some predetermined level $\bar{u}_i$). Formally:

$$\begin{aligned}
&\max \; u_1(x_{11}, ..., x_{L1}) \\
&\text{s.t. } u_i(x_{1i}, ..., x_{Li}) \geq \bar{u}_i \,, \\
&\sum_i x_{li} \leq \bar{\omega}_l + \sum_j y_{lj} \,, \\
&F_j(y_{1j}, ..., y_{Lj}) \leq 0 \,.
\end{aligned}$$

Denote the Lagrange multipliers of these restrictions by $(\delta_i, \mu_l, \gamma_j) \gg 0$ for $i = 2, ..., I$, $l = 1, ..., L$, $j = 1, ..., J$ respectively, and define $\delta_1 = 1$. So, δ_i

represents the gain in household 1's utility upon a reduction of $\bar{u}_i$ by one unit (so, it is an "exchange rate" between consumer i's and consumer 1's utilities). Next, μ_l shows the rise in household 1's utility upon a rise in (a) the initial endowment of some good l, and (b) the rise of production of good l (so, it represents consumer 1's shadow price of good l [= "price" of good l in terms of consumer 1's utility]). Finally, γ_j represents the shadow cost to household 1 of a tightening of firm j's production constraint (so, it converts a change in F_j into units of consumer 1's utility), by, e.g., a rise in production of y_{lj}.
First order conditions:

$$\text{for all } x_{li} > 0 : \delta_i \left(\partial\, u_i / \partial\, x_{li}\right) = \mu_l \,, \tag{1}$$
$$\mu_l = \gamma_j \left(\partial\, F_j / \partial\, y_{lj}\right) . \tag{2}$$

First order condition (1) considers a marginal increase in the available resource x_l (either by an increase of $\bar{\omega}_l$ or by an increase of production of good l). If the additional amount of x_l is given to consumer 1, its utility rises by μ_l. If the additional amount of x_l is given to consumer i, its utility rises by $(\partial\, u_i / \partial\, x_{li})$. So, for given $\bar{u}_i$, we can "relax" consumer i's utility constraint by $(\partial\, u_i / \partial\, x_{li})$ units. This, however, translates to an increase in consumer 1's utility by $\delta_i \left(\partial\, u_i / \partial\, x_{li}\right)$ units. In optimum, both consumer 1's utility gains must be equal.[2]

Next, first order condition (2) deals with an increase in production of good l by firm j, y_{lj}, and passing on the additional amount of good l to consumer 1. In doing so, consumer 1's utility rises by μ_l. However, at the same time, firm j must produce less of all other outputs, which incurs losses for consumer 1 in terms of $\gamma_j \left(\partial\, F_j / \partial\, y_{lj}\right)$. In optimum, FOC (2) must hold, otherwise consumer 1's utility could be increased by a change in the production plan of firm j.

[2]Otherwise, if $\delta_i \left(\partial\, u_i / \partial\, x_{li}\right) > \mu_l$, one could pass on the additional amount of x_l to consumer i. Then one could lower consumer i's utility constraint by more than what amounts to consumer 1's shadow price of x_l, and consumer 1 could do better (which contradicts (1) being a solution to the optimization problem.

Notice that (1) and (2) also imply a different set of first order conditions:

$$\frac{\partial u_i/\partial x_{li}}{\partial u_i/\partial x_{l'i}} = \frac{\partial u_{i'}/\partial x_{li'}}{\partial u_{i'}/\partial x_{l'i'}}, \tag{3}$$

$$\frac{\partial F_j/\partial y_{lj}}{\partial F_j/\partial y_{l'j}} = \frac{\partial F_{j'}/\partial y_{lj'}}{\partial F_{j'}/\partial y_{l'j'}}, \tag{4}$$

$$\frac{\partial u_i/\partial x_{li}}{\partial u_i/\partial x_{l'i}} = \frac{\partial F_j/\partial y_{lj}}{\partial F_j/\partial y_{l'j}}. \tag{5}$$

FOC (3) says that that the marginal rate of substitution between two goods, l and l', is equal among consumers. FOC (4) requires the marginal rate of transformation of l for l' to be equal between any firms j and j'. Finally, FOC (5) says that, a Pareto efficient allocation is characterized by equality between the marginal rate of substitution of l for l' (for any i) and the marginal rate of transformation of l for l' (for any j).

4 More on the Basics...

Nonconvex Production Technologies and Marginal Cost Pricing

Obviously, the Second Fun Theorem runs into trouble when production sets are not convex. However, there is a (weaker) result, paralleling the Second Fun Theorem, for nonconvex production sets.

Proposition 4 (Marginal Cost Price Equilibrium with Transfers) *Suppose, the basic assumptions of the previous section hold, except for convexity of production sets. Suppose further that appropriate differentiability assumptions hold. Then, if* (x^*, y^*) *is Pareto efficient, there exists a price vector* $p = (p_1, ..., p_L)$ *and wealth levels* $(w_1, ..., w_I)$ *with* $\sum_i w_i = p\cdot\bar{\omega} + \sum_j p\cdot y_j^*$ *such that:*
(i) For any firm j*:* $p = \alpha_j \nabla F_j(y_j^*)$ *for some* $\alpha_j > 0$ *,*
(ii) for any i*,* x_i^* *is maximal for* $\succsim_i$ *in the budget set* $\{x_i \in X_i : p \cdot x_i \leq w_i\}$ *,*
(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$ *.*

Notice, that the definition of a marginal cost price equilibrium with transfers closely resembles that of a PET (with the exception of (i)).

Query: Suppose there is just one input and one output. What exactly is (i) saying in this case?

The Set of Feasible Allocations

Suppose that (i) every X_i is closed, and bounded below, and (ii) every Y_j is closed, and (iii) Y is convex, admits the possibility of inaction, satisfies the free lunch property, and is irreversible.
Then, the set of feasible allocations is closed and bounded. Moreover, it is nonempty if we allow for free disposal. [3]

[3] To be precise: ... and, in addition, we can choose $\hat{x} \in X_i$ for every i such that $\sum_i \hat{x}_i \leq \bar{\omega}$.