Digression
Important Formulas for Integration and Differential Equations

In the following, we will employ a few formulas regularly. These formulas are not difficult but they are important. An excellent source is Leonard, D. and N. Van Long (1992), Optimal Control Theory and Static Optimization, Cambridge: Cambridge University Press (Ch. 2).

Integration

$$\int f(x)\,dx = F(x) + C, \quad \text{where}\, F'(x) = f(x)\,, \tag{1}$$

$$\int x^a\,dx = \frac{1}{1+a}\,x^{1+a} + C\,, \tag{2}$$

$$\int \frac{1}{x}\,dx = \ln\,x + C\,, \tag{3}$$

$$\int e^{a\,x}\,dx = \frac{1}{a}\,e^{a\,x} + C\,, \tag{4}$$

where C represents an arbitrary constant.

Growth Rates

If $x(t)$ grows at a constant growth rate, γ, then: $x(t_1) = x(t_0)\,e^{\gamma(t_1-t_0)}$.

If $x(t)$ grows at a rate, $\gamma(t)$, then: $x(t_1) = x(t_0)\,e^{\int_{t_0}^{t_1}\gamma(s)\,ds}$.

First-order Linear Differential Equations

Define $\dot{x}(t) \equiv \frac{\partial\,x(t)}{\partial\,t}$. An equation $f(\dot{x}(t), x(t), t) = 0$ that involves the derivative of $x(t)$ with respect to t is called a differential equation. E.g., $\dot{x}(t) + a\,x(t) = b$ is a differential equation. Clearly, an equation that involves $\frac{\partial^m\,x(t)}{(\partial\,t)^m}$ is also a differential equation (DE), but not a first order DE.

A differential equation involving only first order partial derivatives is called a first order differential equation. In case the functional form is linear, the equation is said to be a first order linear differential equation. The following equations are first order linear differential equations.

$$\dot{x}(t) + a\,x(t) = b\,, \tag{5}$$
$$\dot{x}(t) + a\,x(t) = b(t)\,, \tag{6}$$
$$\dot{x}(t) + a(t)\,x(t) = b(t)\,. \tag{7}$$

These first order linear differential equations have the following explicit solutions, where C is an arbitrary constant:

$$x(t) = C\,e^{-a\,t} + b/a\,, \tag{8}$$
$$x(t) = C\,e^{-a\,t} + e^{-a\,t} \int e^{a\,t}\,b(t)\,dt \tag{9}$$
$$x(t) = e^{-\int a(t)\,dt} \left[C + \int e^{\int a(t)\,dt}\,b(\tau)\,d\tau \right]. \tag{10}$$

For $x(t_0) = x_0$, given, we can determine the constants C in (8) to (10):

$$x(t) = \frac{b}{a} + e^{-a(t-t_0)} \left[x_0 - \frac{b}{a} \right], \tag{11}$$
$$x(t) = x_0\,e^{-a(t-t_0)} + \int_{t_0}^{t} b(\tau)\,e^{-a(t-\tau)}\,d\tau\,, \tag{12}$$
$$x(t) = x_0\,e^{-\int_{t_0}^{t} a(s)\,ds} + \int_{t_0}^{t} b(\tau)\,e^{-\int_{\tau}^{t} a(s)\,ds}\,d\tau\,. \tag{13}$$