Phase Diagram Analysis & Dynamic Behavior

1 Phase Diagrams Autonomous Systems in 2 Dimensions

$\max \int_{t0}^{t1} f(x,y)\,, \quad \text{s.t.}\ \dot{y} = g(x,y) \quad (*)$

$$H(\lambda, x, y) = f(x,y) + \lambda\, g(x,y)$$

$$\frac{\partial H}{\partial x} = 0 \;\Rightarrow\; \lambda = -f_x/g_x \qquad (1)$$

$$\Rightarrow\; \dot{\lambda}(\dot{x}, \dot{y}, x, y) = -(\dot{f_x/g_x}) \qquad (2)$$

$$-\frac{\partial H}{\partial y} = \dot{\lambda} = -f_y - \lambda\, g_y \qquad (3)$$

$\rightarrow$ solve (2) for $\dot{x}$, considering (1) – (3): $\dot{x} = h(x,y)$ $(**)$

dynamic system I in (y, x) space:

$$\dot{y}(t) = g(y(t), x(t))$$
$$\dot{x}(t) = h(y(t), x(t))$$

equivalently, dynamic system II in (x, y) space:

$$\dot{x}(t) = h(x(t), y(t))$$
$$\dot{y}(t) = g(x(t), y(t))$$

- demarcation lines

  $\rightarrow \dot{y} = 0$- line: $0 = g(x, y)$

  $\rightarrow \dot{x} = 0$- line: $0 = h(x, y)$

  $\Rightarrow$ 4 regions in phase space

- movement off demarcation lines: given by signs of

$$\partial\dot{y}/(\partial x)|_{y^*}$$
$$\partial\dot{x}/(\partial x)|_{x^*}$$

  $\rightarrow$ arrows indicating the direction of trajectories

  ... paths of (x, y) in phase diagram over time

- equilibrium: $\dot{x} = \dot{y} = 0$

- dynamic behavior

  specific movement of trajectories out of equilibrium

  → ...can be identified by

  – phase diagram analysis

  – nature of eigenvalues of Jacobian

- Jacobian of dynamic system I:

$$J_I = \begin{pmatrix} g_y & g_x \\ h_y & h_x \end{pmatrix}$$

- Jacobian of dynamic system II:

$$J_{II} = \begin{pmatrix} h_x & h_y \\ g_x & g_y \end{pmatrix}$$

- J_I and J_{II} deliver same dynamic behavior!

2 Types of Dynamic Behavior "close" to Equilibrium

- stable node (sink)
- unstable node (source)
- saddle (point) → unstable equilibrium
- stable spiral point (stable focus)
- unstable spiral point (unstable focus)
- center → stable but not asymptotically stable

→ other behaviors exist (limit cycles, chaotic behavior)

3 Eigenvalues & Dynamic Behavior

- Jacobian of dynamic system: J_I

- determinant: $\mathrm{det}J = g_y\, h_x - h_y\, g_x$

- trace: $\mathrm{tr}J = g_y + h_x$

- eigenvalues (e_1, e_2): $e = 1/2\,[tr \pm \sqrt{tr^2 - 4det}]$

- relationships

$$\mathrm{tr}J = e_1 + e_2$$

$$\mathrm{det}J = e_1\, e_2$$

$$\mathrm{det}J < 0 \Leftrightarrow e_1 < 0 < e_2$$

$tr^2 - 4det < 0 \Leftrightarrow e_1, e_2$ are complex conjugate
$e = \alpha \pm \beta\, i,\ \alpha = tr/2$

- stable node (sink): $e_1 < 0$, $e_2 < 0$ (both e real)
  $\rightarrow \det J > 0$

- unstable node (source): $e_1 > 0$, $e_2 > 0$ (both e real)
  $\rightarrow \det J > 0$

- saddle (point): $e_1 < 0$, $e_2 > 0$ (both e real)
  $\rightarrow \det J < 0$

- stable spiral point (stable focus):
  e complex conjugate
  $\alpha < 0$, $\beta > 0$ (clockwise movement)
  $\beta < 0$ (counterclockwise movement)

- unstable spiral point (unstable focus):
  e complex conjugate
  $\alpha > 0$, $\beta > 0$ (clockwise movement)
  $\beta < 0$ (counterclockwise movement)

- center: $\alpha = 0$, $\beta > 0$ (clockwise movement)
  $\beta < 0$ (counterclockwise movement)

- Foci and centers

complex conjugate eigenvalues

- if ${}_1e_2 = \alpha \pm \beta\, i$ the Jacobian can be expressed by a matrix J' that exhibits the same properties as J_I or J_{II}:

$$J' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

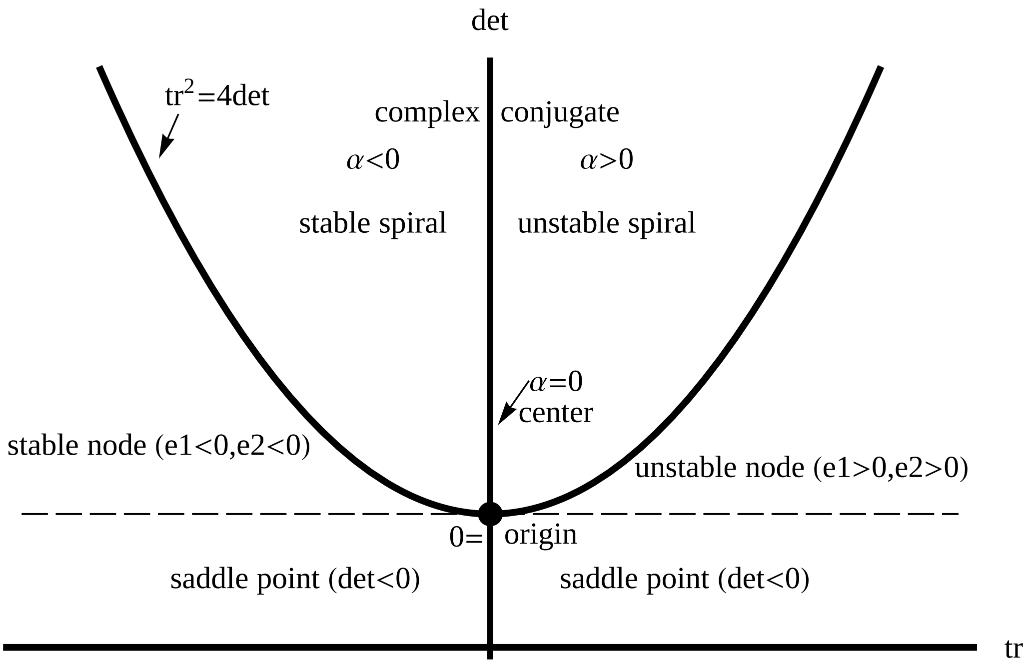

Figure 1: Dynamic Behavior