Optimal Control Theory: Ingredients and the Standard Problem

"We are taught early in life about the necessity to plan ahead. Present decisions affect future events by making certain opportunities available, by precluding others, and by altering the costs of still others."

1 Ingredients

- control variable $x(t)$
- control set $X \subseteq \mathbb{R}$
- state variable:
  - $\dot{y}(t) = g(t, x(t), y(t))$ – equation of motion
  - $y(t_0) = y_0$ – initial condition
- period $t \in [t_0, t_1]$
- terminal condition
  - $y(t_1) = y_1$
  - $y(t_1) = \text{free}$
  - $y(t_1) \geq y_1$ (truncated)
- objective function $\int_{t_0}^{t_1} f(t, x(t), y(t))\, dt$

- formulate problem

$$\max \int_{t_0}^{t_1} f(t, x(t), y(t))\, dt \qquad (*)$$

$$\left.\begin{aligned} & x(t) \in X \subseteq \mathbb{R} \\ & \dot{y}(t) = g(t, x(t), y(t)) \\ & y(t_0) = y_0 \\ & (i)\, y(t_1)\, \text{free}\,, \quad (ii)\, y(t_1) \geq y_1\,, \quad (iii)\, y(t_1) = y_1 \end{aligned}\right\} (**)$$

- admissible pair $[(x(t), y(t)]$ satisfies $(**)$

- optimal pair $[(x(t), y(t)]$ is admissible and maximizes $(*)$

- Hamiltonian function

$$H(t, x(t), y(t), \lambda(t)) = f(t, x(t), y(t)) + \lambda(t)\, g(t, x(t), y(t))$$

- solve problem (necessary conditions)

set up Hamiltonian H

(1) choose $x(t)$ that maximizes H

$\longrightarrow$ interior solution: $\partial H/[\partial x(t)] = 0$

(2) $-\partial H/[\partial y(t)] = \dot{\lambda}(t)$

(3) transversality condition

(i) $\lambda(t_1) = 0$

(ii) $\lambda(t_1) \geq 0$ and $\lambda(t_1) = 0$ if $y(t_1) > y_1$

(iii) $y(t_1) = y_1$

- Mangasarian sufficient conditions

Suppose $[(x(t), y(t)]$ satisfies (1) – (3), the control set X is convex, and H is concave in $[(x(t), y(t)]$ for all $t \in [t_0, t_1]$. Then $[(x(t), y(t)]$ is an optimal pair.

Points to note

- boundary vs interior solutions
- the two effects of $\triangle\, x(t)$
- interpretation of $\lambda(t)$

  $\rightarrow$ terminal condition (ii) and transversality condition (binding vs nonbinding case)
- extensions of standard problem
  - t_1 variable (optimal t_1 to be found)
  - many control & state variables
  - current value formulations
  - $t_1 \rightarrow \infty$
- explicit solutions vs phase diagram analysis